On Numerical Solution of Lorenz’s Equations.

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Table of Contents

Introduction

The equations produced by Lorenz Edward N. in his work on ‘Deterministic Nonperiodic Flow’ (1962) has been solved numerically in this paper. The system of nonlinear coupled differential equations are given by equations 1-3. [1]

$$\frac{dx}{dt} = -\sigma(x-y) \qquad (1)$$

$$\frac{dy}{dt} = -xz + rx - y \qquad (2)$$

$$\frac{dz}{dt} = xy - bz \qquad (3)$$

with $\sigma = 10,\ r = 28\ and\ b = 8/3$.

Explicit Euler is a numerical integration scheme which takes the values of a function at the current iteration step to update the values onto the next one for the approximation of the function around that point. The scheme is given in equation (4) [2].

$$y_{n+1} = y_n + h\,f_n \qquad (4)$$

Where f_n is the differential equation being solved.

Similarly, the Runge-Kutta scheme is derived from expanding the Taylor series upto the fifth order. The result of the derived Runge Kutta scheme is given in equation (5).[3]

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \qquad (5)$$

with

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$$
$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right)$$
$$k_4 = f(x_i + h, y_i + k_3h)$$

The equations in our concern (equations 1-3) are based on variables x,y,z and t. The schemes presented in equations 4 and 5 are therefore expanded to 4 variables during the numerical integration process.

Methodology

Numerical integration via the Euler and Runge-Kutta fourth order schemes respectively are done with the step size of 0.001 and the initial conditions x(0) = y(0)= z(0) = 10 for time ranging from t=0 to t=30. The choice of numerical integrator is Matlab 2019a.

A check to compare the two methods is done by altering the step size. Furthermore, a small analysis is done on the system of equations from Lorenz by checking whether a pattern in his phase portraits is maintained when the initial conditions are changed.

Finally , results are plotted as functions of x(t), y(t) and z(t), and parametric plots of the results are presented for both the Explicit Euler and the Runge-Kutta fourth order (RK4) schemes.

Results and Analysis

Figure 1 shows the x(t), y(t) and z(t) plots for the integration via Euler scheme. The integration is done such that initially, the system is solved numerically for time t= 0 to t=30 and then from 0 to 15 wherein firstly, the x,y,z values at that time are taken and plugged in again back to the Euler scheme as initial conditions for simulation from t = 15 to t=30 seconds. Then the values at t=15 are rounded off to two decimal places and plugged in to the integrator for numerical solution upto t=30s. The same method is done for the Runge-Kutta scheme as well. Parametric plots for both schemes are presented together in section 2 of Results and Analysis.

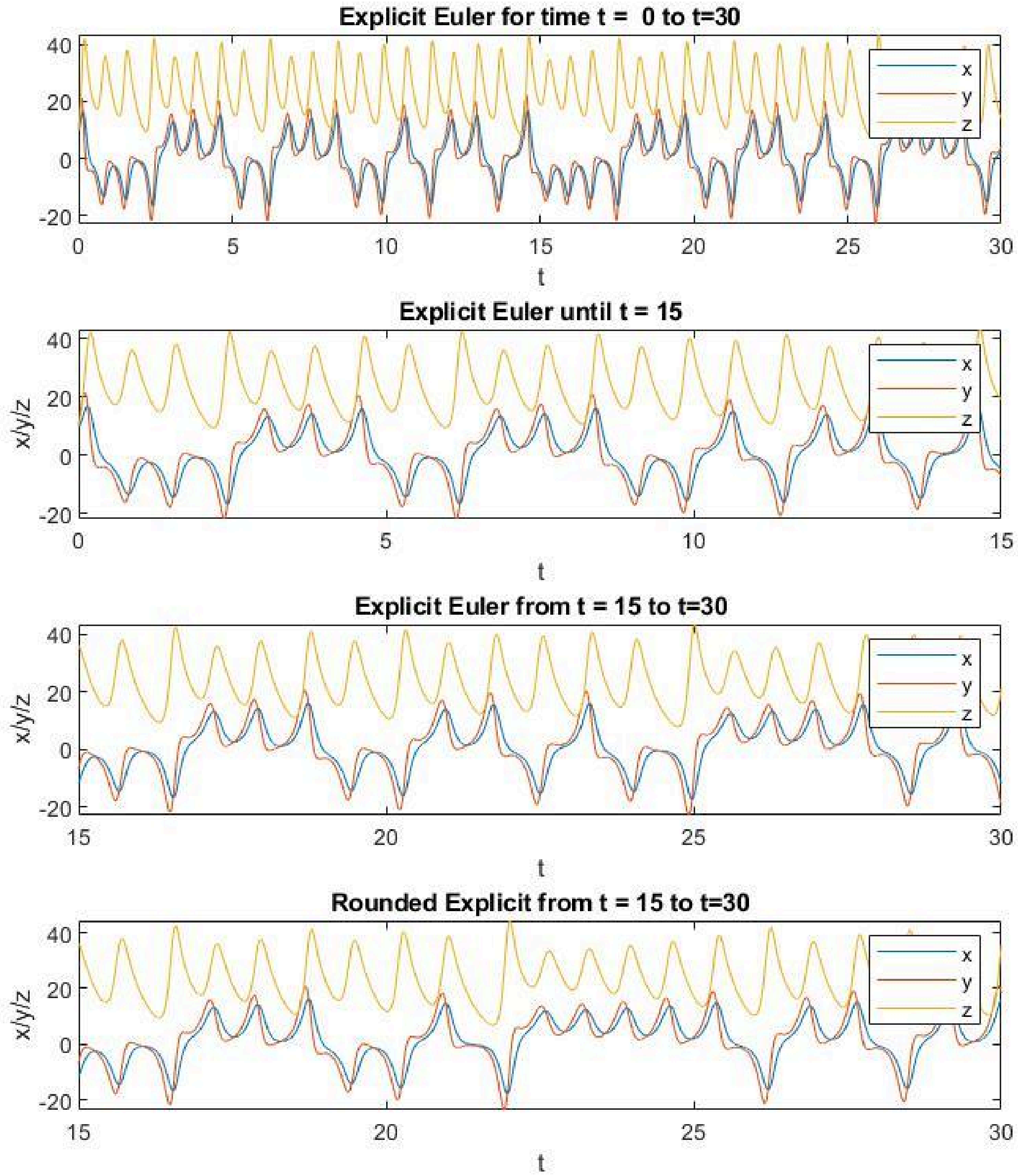

Figure 1: Explicit Euler numerical solution for Lorenz's equations.

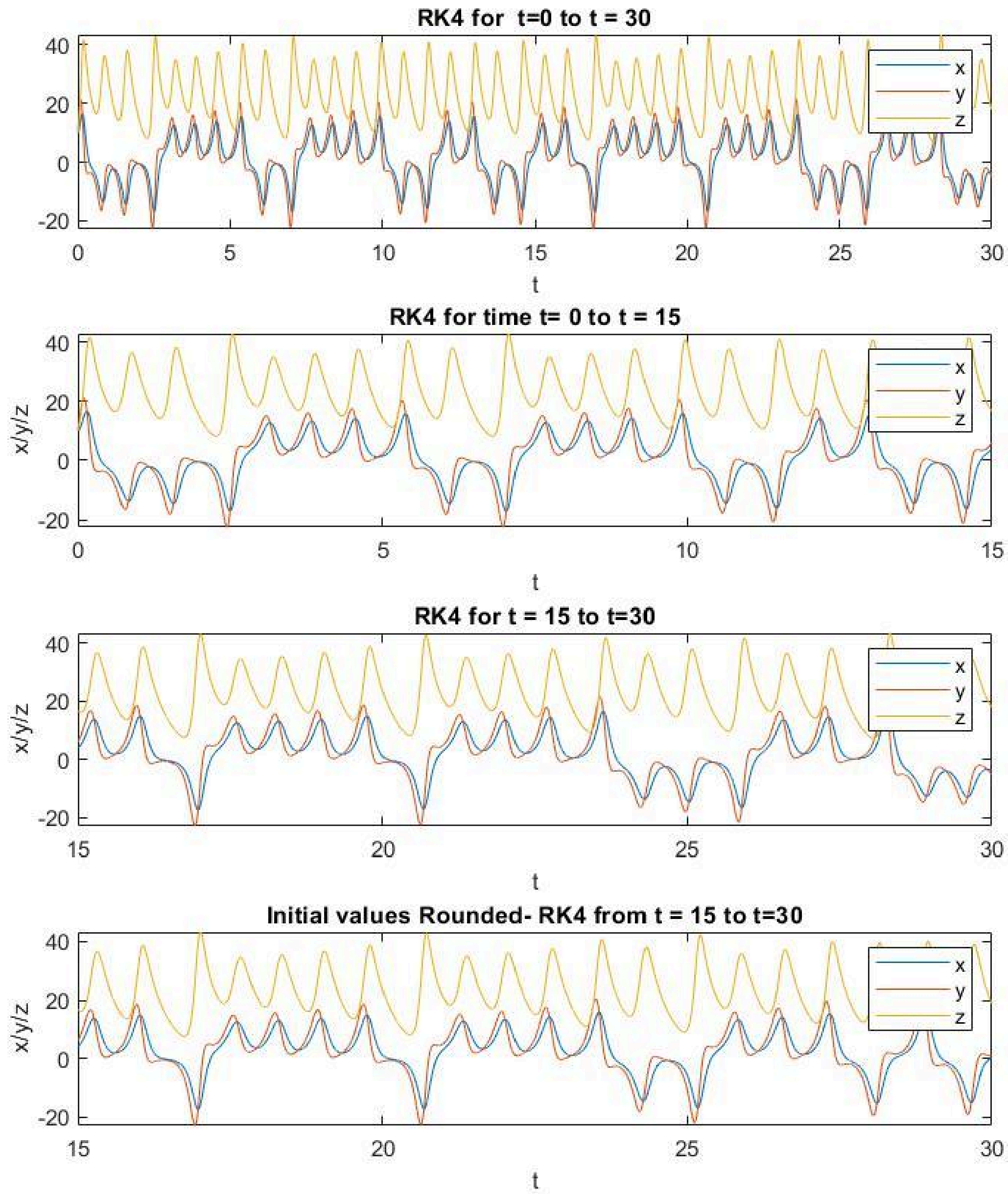

Figure 2: Runge-Kutta fourth order (RK4) numerical solution for Lorenz's equations.

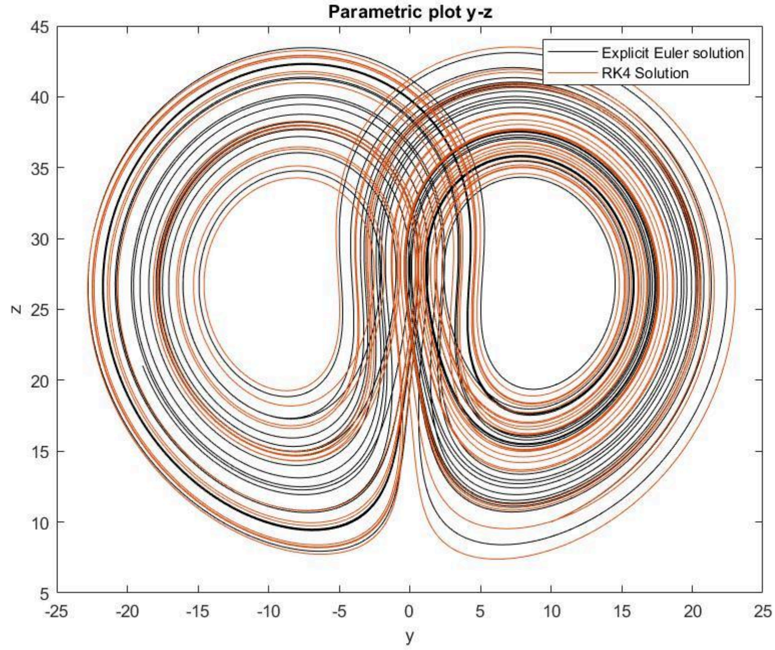

Figure 3: y-z plot of the solution to Lorenz's equations by Euler and Runge-Kutte schemes.

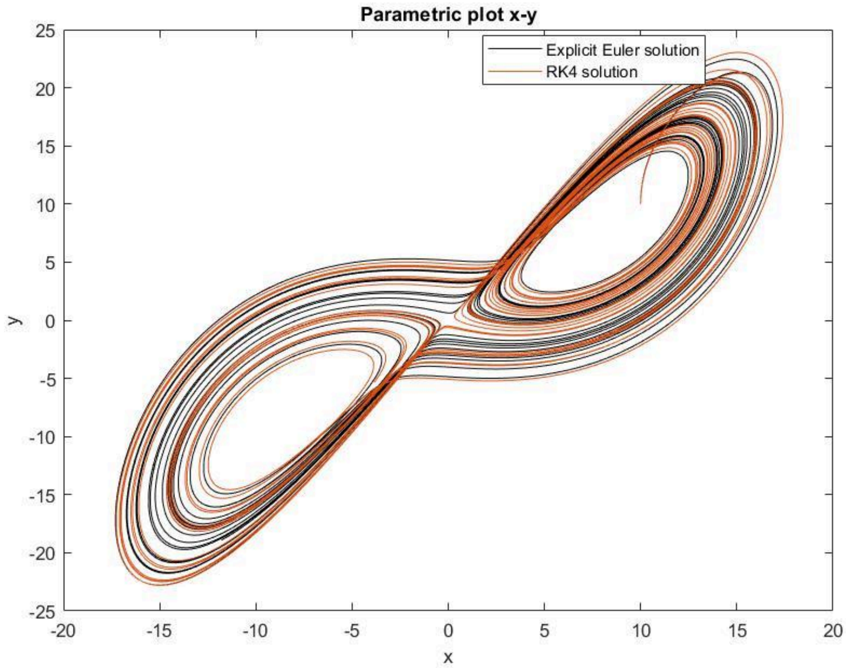

Figure 4: x-y plot of the solution to Lorenz's equations by Euler and Runge-Kutte schemes.

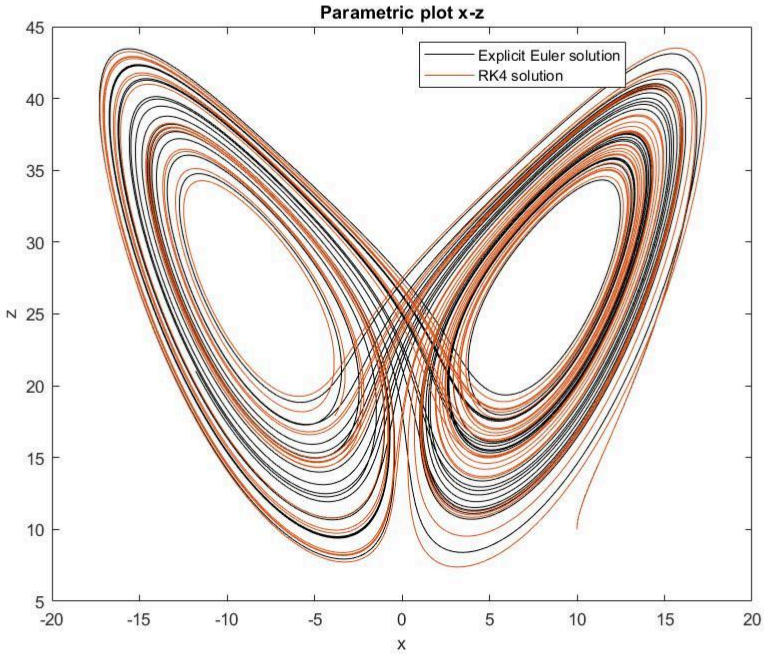

Figure 5: z-x plot of the solution to Lorenz's equations by Euler and Runge-Kutte schemes.

There is a slight change in the results when the value at t=15 is not rounded to that when it is rounded. The solution have similar pattern although not completely the same, upto around t= 25 where both results for RK4 and Euler scheme show a change in pattern in the solution. This gives rise to the idea that the solution may be altered drastically by changing the initial conditions.

When the initial values are altered, the behaviour of the system is chaotic. There is a certain pattern when the initial conditions are kept at the ones used for the numerical solution, however when altered by a factor of 5 i.e. x(0) = y(0)= z(0) = 50, the results are different from the pattern at x(0) = y(0)= z(0) = 10. The results are not deterministic for this system of equations.

Furthermore, on altering the step size, the results of the simulation show a difference between the RK4 and the Euler schemes in that the RK4 scheme keeps stability for a bigger value of the step size(increased from h=0.001 to h=0.01). Given that the derivation of the RK4 scheme comes from the expansion of the Taylor series upto fifth order, this result is expected. When increasing the step size the parametric plot also gives evidence to the idea that the RK4 scheme holds on to stability for a bigger step size compared to the Explicit Euler scheme (see figures below). The Euler scheme does not keep the pattern from the solution when the stepsize was h=0.001. It portrays the idea that there is little change in the values of x,y and z approximately from time periods 5-8 , 16-18 and 24-26, this is not true as shown from previous simulation results. The reason could be because the changes are drastic in that region and a small step size misses to integrate properly.

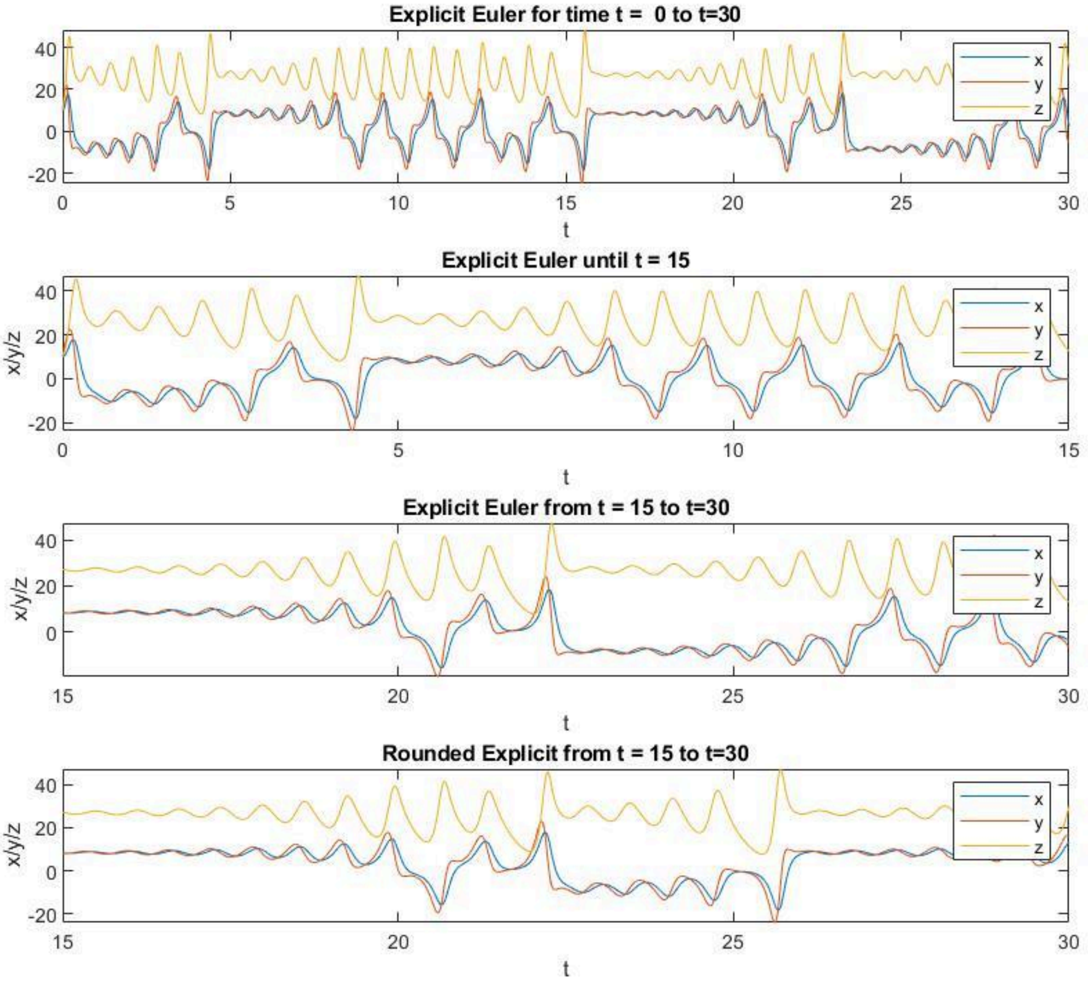

Figure 6: Explicit Euler numerical solution for Lorenz's equations with step size h=0.01.

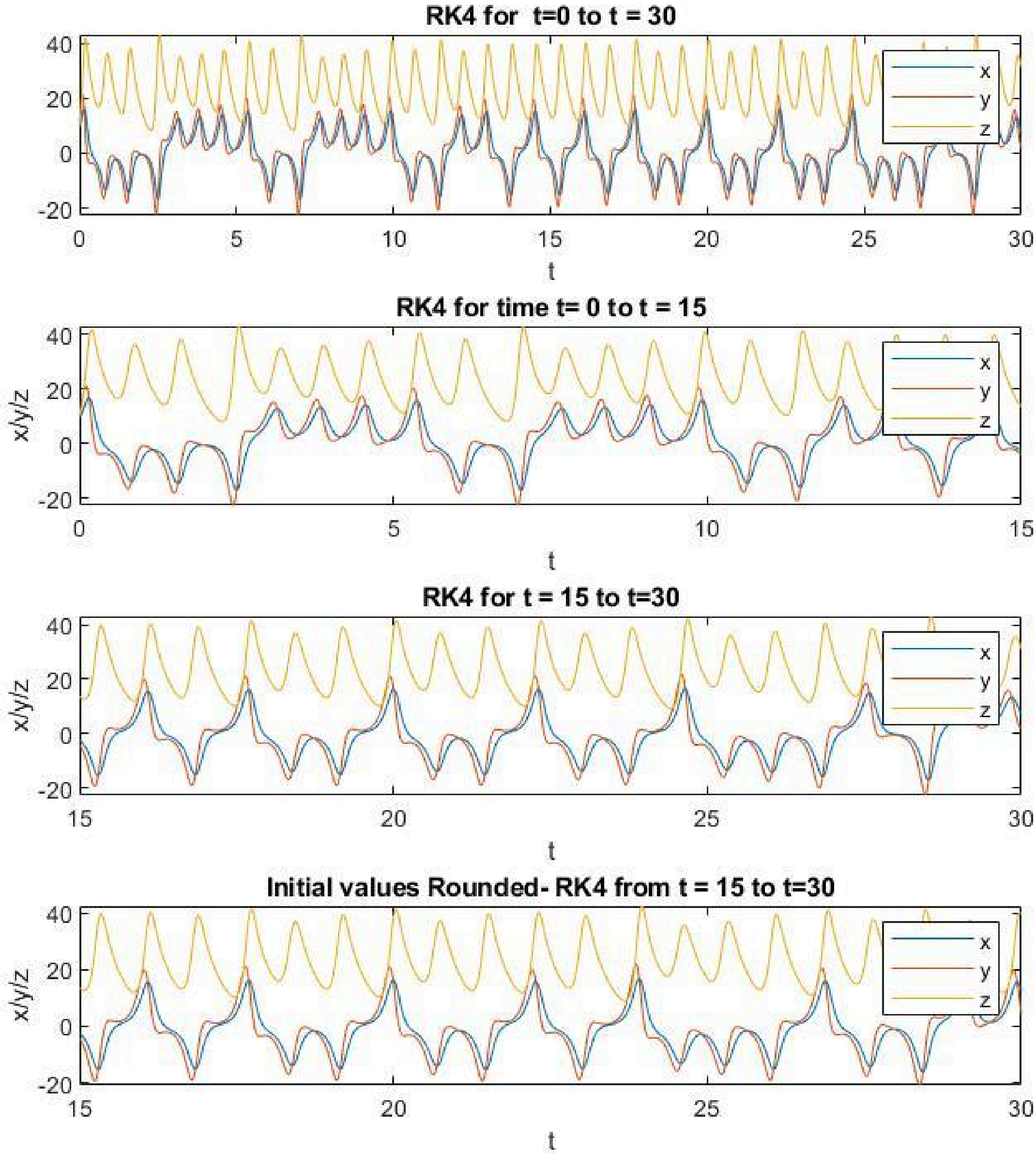

Figure 7: Runge-Kutta fourth order (RK4) numerical solution for Lorenz's equations with step size h= 0.01.

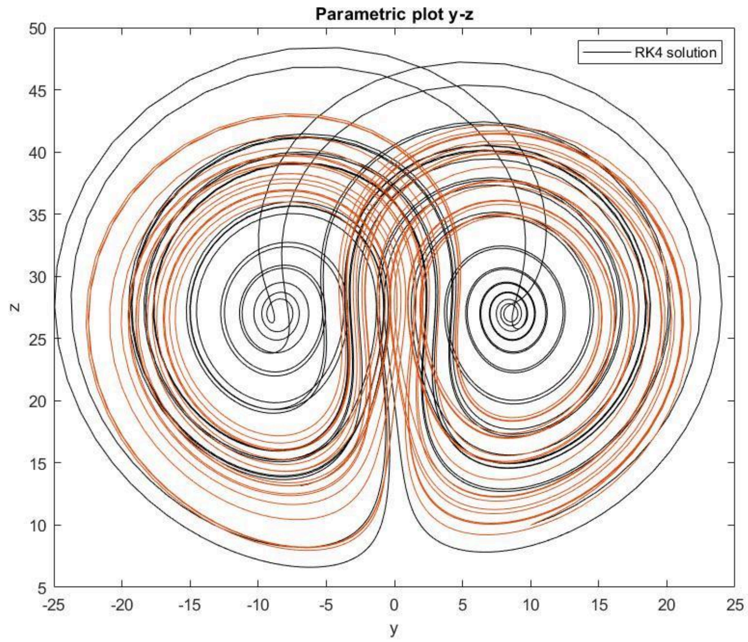

Figure 8: y-z plot with step size h = 0.01.

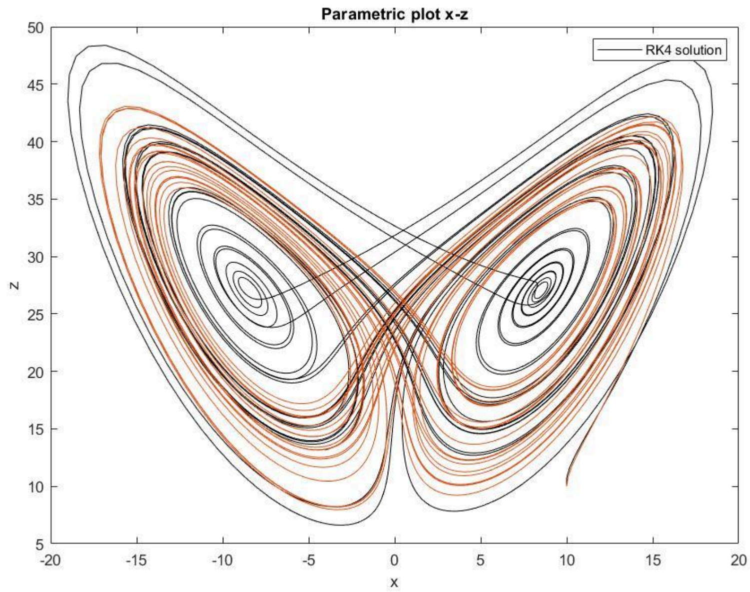

Figure 9: x-z plot with step size h = 0.01.

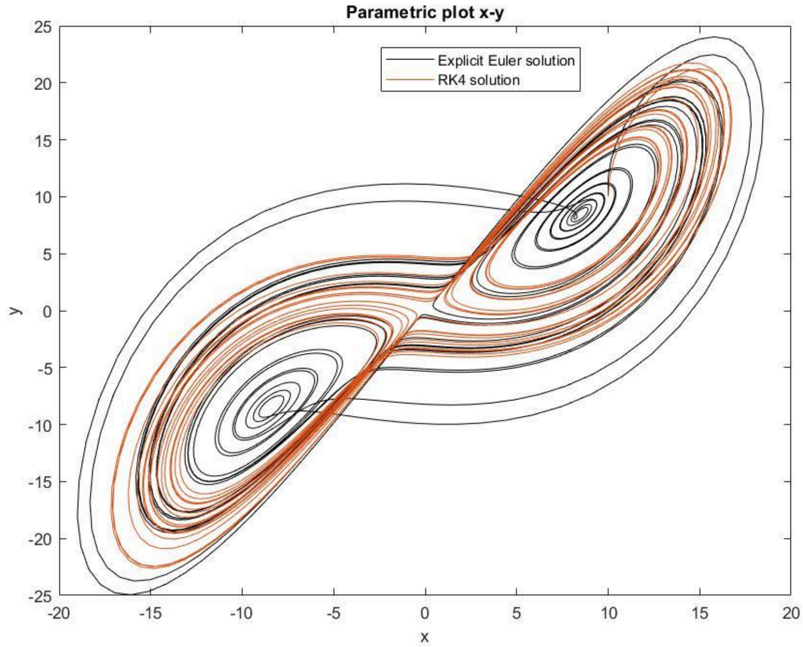

Figure 10: x-z plot with step size h = 0.01.

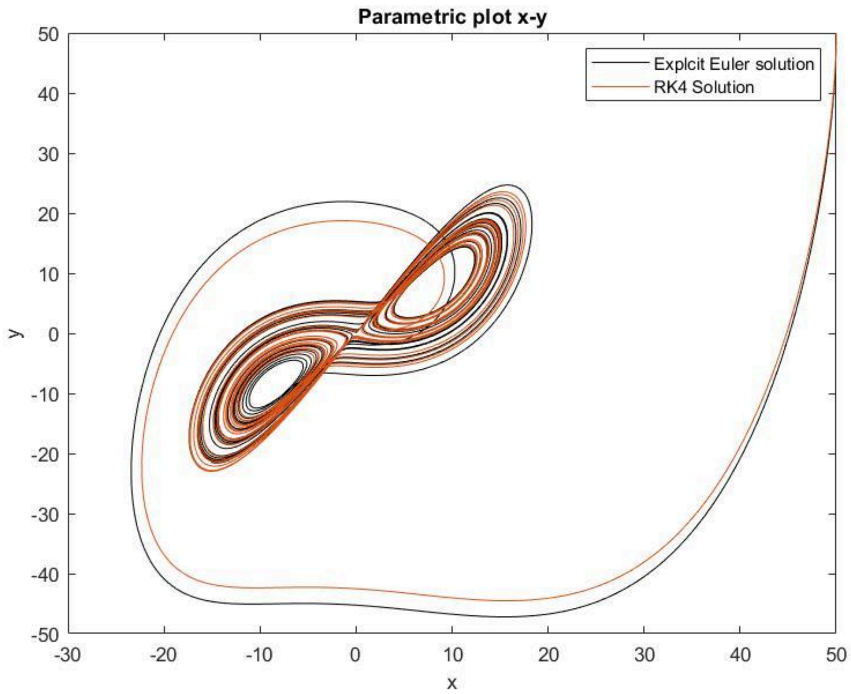

Figure 11: x-y plot with initial conditions x(0) = y(0)= z(0)= 50.

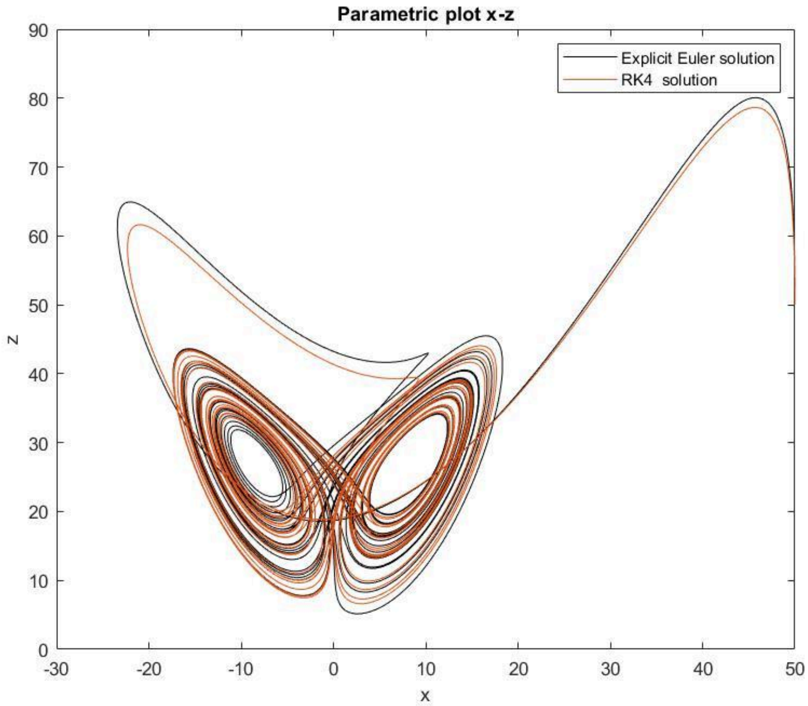

Figure 12: x-z plot with initial conditions x(0) = y(0)= z(0)= 50.

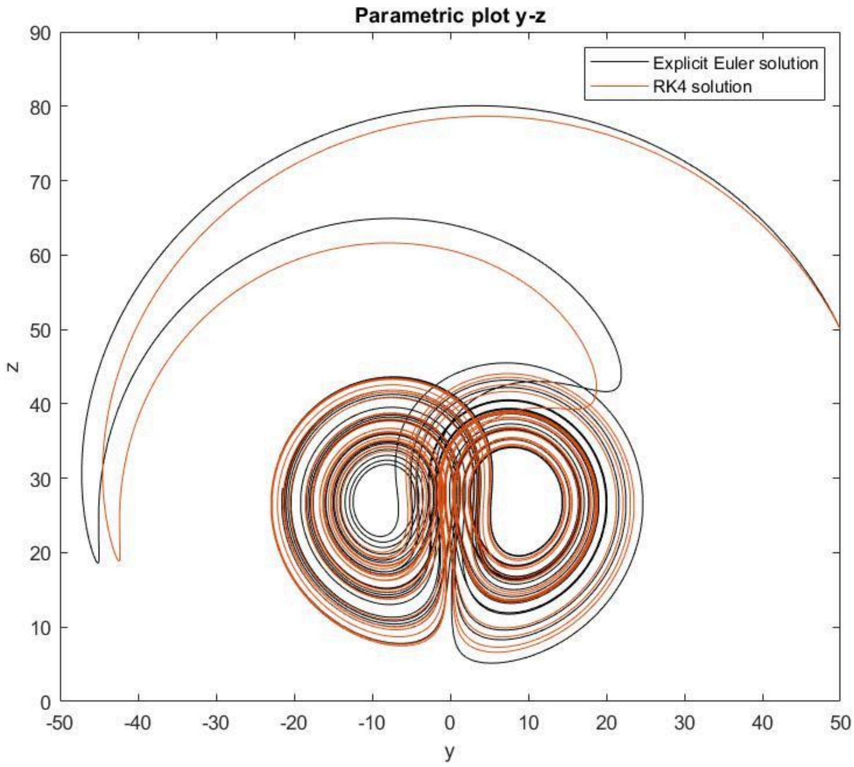

Figure 13: z-y plot with initial conditions x(0) = y(0)= z(0)= 50.

Summary

What we learn from this investigation is that in non linear processes/phenomena such as atmospheric dynamics, it is not deterministic. The system of coupled differential equations is unstable for changes in input parameters. In the real life scenario, when we change input parameters slightly, the system can behave in an unstable manner. For example, a change in atmospheric situation at a certain location, when changed (for example by a natural process such as a hurricane), it is not possible to say deterministically what will happen, but we can only have predictions on the response of the environment.

Since a small change in initial conditions and/or input parameters yield a large change in output, we can use prediction schemes to determine what will happen next, and not say it in absolutes. This can be expanded to the cliche of the butterfly effect (seen like on figure 5), that when a butterfly flaps its wings in one part of the world, we might have something big happening on the other side of the world. Although scientifically absurd, it does make a little sense.

Bibliography

[1] Lorenz, Edward N., ”Deterministic non-periodic flow”, Journal of the Atmospheric Sciences. 20 (2): 130141 (1963)

[2] ‘Runge-Kutta 4th Order Method for Ordinary Differential Equations’. Web. Accessed on 11.02.2019. https://mathforcollege.com/nm/mws/gen/08ode/mws_gen_ode_txt_runge4th.pdf

[3] Paul’s Online Notes. Euler’s Method. Web. Accessed on 11.02.2019. http://tutorial.math.lamar.edu/Classes/DE/EulersMethod.aspx