Chapter-1 Co-ordinate Systems

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Co-ordinate system: It is the system to locate or determine the location of points in space by labelling the points with different parameters.

A point or vector can be represented in any curvilinear co-ordinate system, which may be orthogonal or nonorthogonal.

→ orthogonal is one in which the co-ordinates are mutually perpendicular.

examples

(a) Cartesian (or rectangular)
(b) circular cylindrical
(c) spherical
(d) elliptic cylindrical
(e) Parabolic cylindrical
(f) Conical
(g) ellipsoidal etc.

Cartesian Co-ordinate Systems (x, y, z)

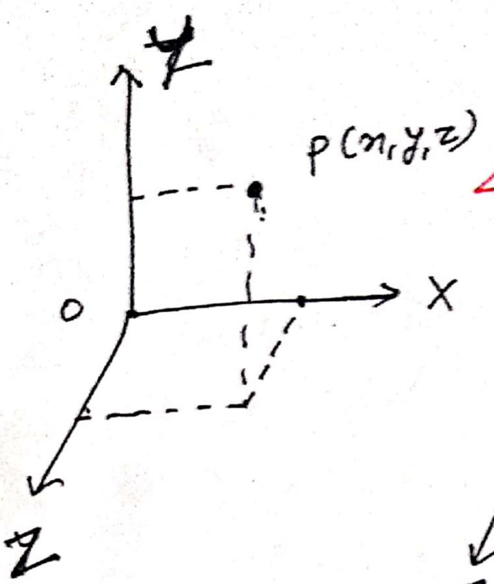
A point P can be represented by (x, y, z). The ranges of the co-ordinate variables x, y and z are

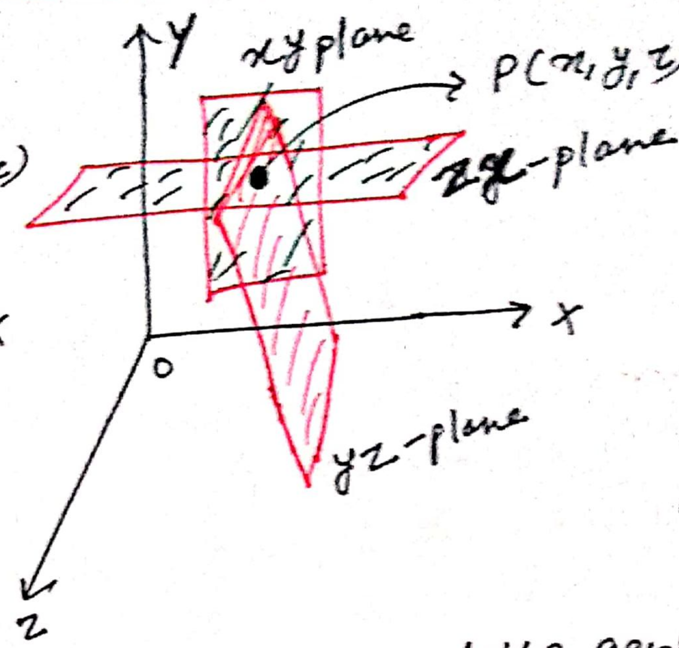
$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

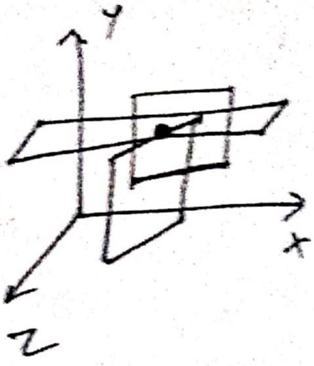
Cartesian Co-ordinate system is the basic and simplest system of representing the point. In this system a point is the intersection of three planes; the xy-plane, yz-plane and zx-plane.

Any vector in Cartesian co-ordinate system is represented by its components along x, y & z directions as:

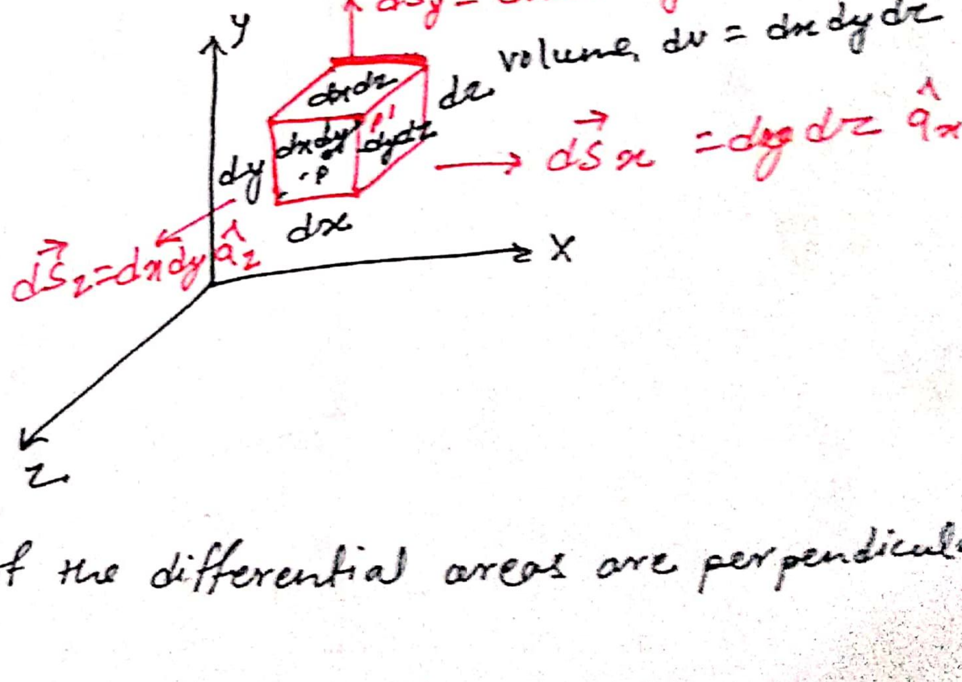
$$\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$



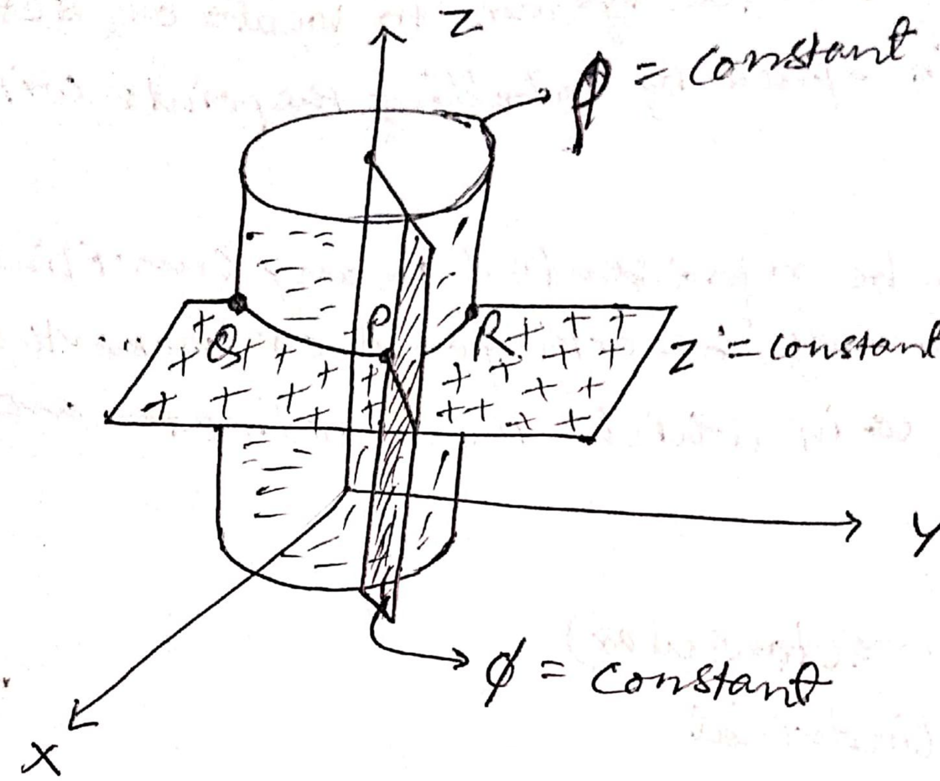


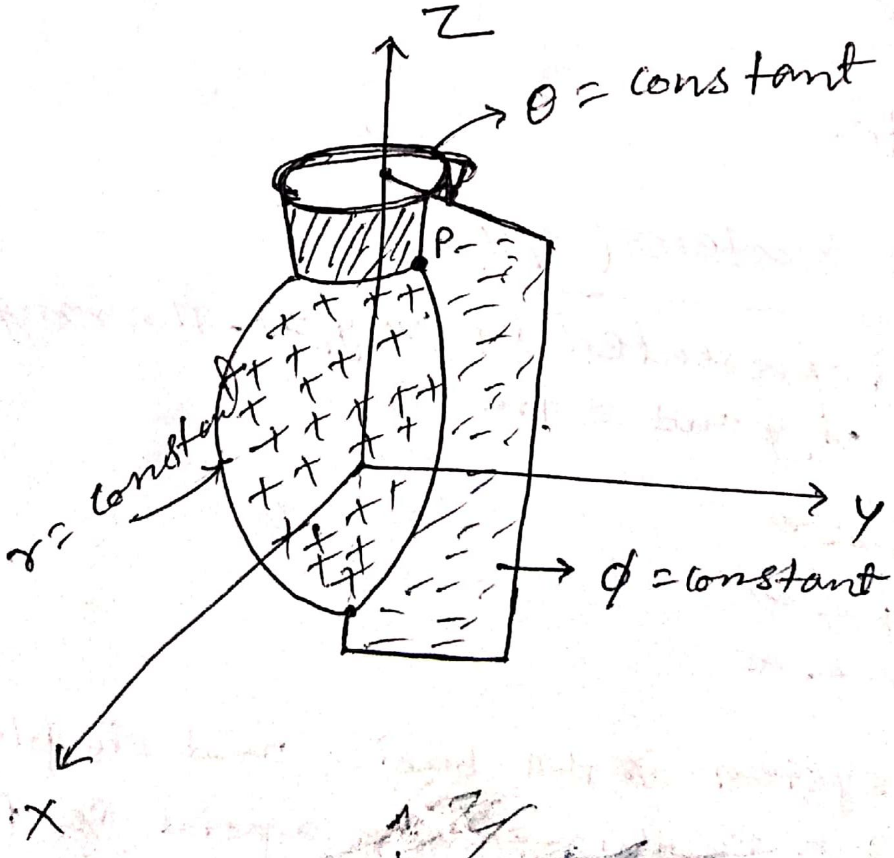


If we visualize three planes intersecting at the general point P (x, y, z), we may increase each co-ordinate value by a differential amount and obtaine three slightly displaced planes intersecting at point $P'(x+dx, y+dy, z+dz)$. The six planes define a parallelopiped whose volume is $dv = dx.dy.dz$; surfaces having differential areas of $dS_x = dydz$, $dS_y = dx.dz$ and $dS_z = dx.dy$. Also, the distance from P to P′ is diagonal of parallelopiped and has $dl = \sqrt{dx^2+dy^2+dz^2}$ length.



The directions of the differential areas are perpendicular to the plane.


Z
ρ = constant
P
Q
R
z = constant
Y
φ = constant
X


Z
θ = constant
P
r = constant
Y
φ = constant
X

Circular Cylindrical Co-ordinate System or Cylindrical Co-ordinate system (ρ, ϕ, z)

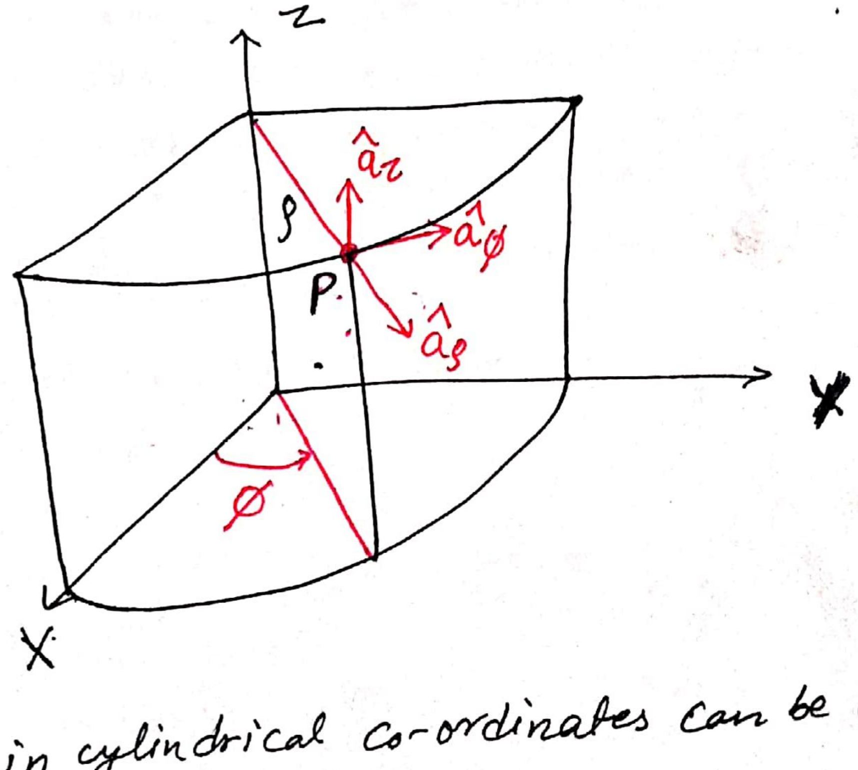
The circular cylindrical co-ordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry. A point P in cylindrical co-ordinates is represented as (ρ, ϕ, z), where ρ - radius of the cylinder passing through P or the radial distance from the z-axis; ϕ is called the azimuthal angle, is measured from the x-axis in the xy-plane; and z is the same as in Cartesian system.

The ranges of the variables are:

$$0 \le \rho \le \infty$$

$$0 \le \phi \le 2\pi$$

$$-\infty < z < \infty$$



A vector $\vec{A}$ in cylindrical co-ordinates can be written as:

$$\vec{A} = (A_\rho, A_\phi, A_z) \quad \text{or} \quad \vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$$

where, $\hat{a}_\rho$, $\hat{a}_\phi$ and $\hat{a}_z$ are unit vectors in the ρ, ϕ & z-directions respectively.

Magnitude of $\vec{A}$ is, $|\vec{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$

Since, $\hat{a}_\rho$, $\hat{a}_\phi$ & $\hat{a}_z$ are mutually perpendicular we have,

$$\hat{a}_\rho \cdot \hat{a}_\rho = \hat{a}_\phi \cdot \hat{a}_\phi = \hat{a}_z \cdot \hat{a}_z = 1$$

$$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0$$

$$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z \qquad \hat{a}_\phi \times \hat{a}_\rho = -\hat{a}_z$$

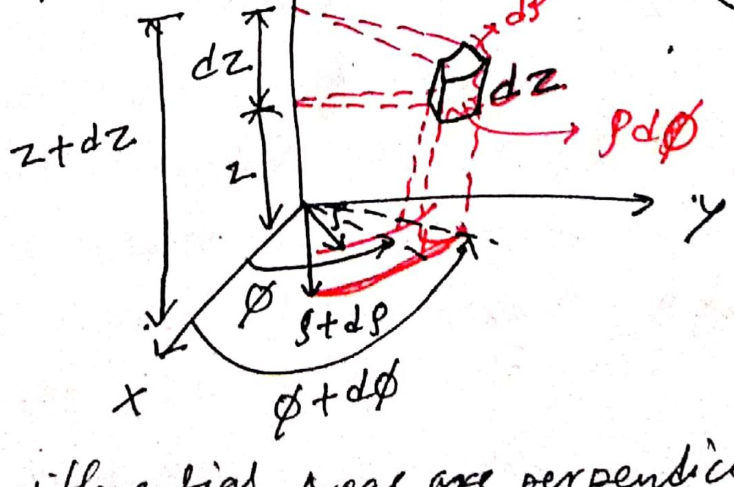
$$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho \qquad \hat{a}_z \times \hat{a}_\phi = -\hat{a}_\rho$$

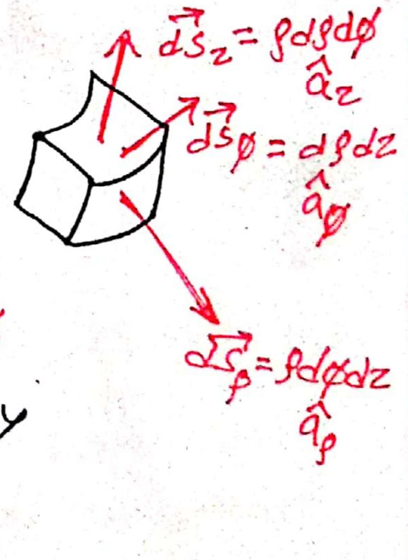
$$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi \qquad \hat{a}_\rho \times \hat{a}_z = -\hat{a}_\phi$$

A differential volume element in cylindrical co-ordinates may be obtained by increasing ρ, ϕ and z by differential increments $d\rho$, $d\phi$ & dz. The two cylinders of radius ρ and $\rho + d\rho$, the two radial planes at angles ϕ and $\phi + d\phi$ and the two planes at z & $z+dz$ now enclose a small volume. As the volume element becomes very small, its shape approaches that of a parallelopiped having sides of length $d\rho$, $\rho d\phi$ and dz. Here, $d\rho$ & dz are dimensionally lengths but $d\phi$ is not; $\rho d\phi$ is the length (arc length). The surface areas are:

$$dS_\rho = \rho\, d\phi\, dz, \quad dS_\phi = d\rho\, dz, \quad dS_z = \rho\, d\rho\, d\phi$$

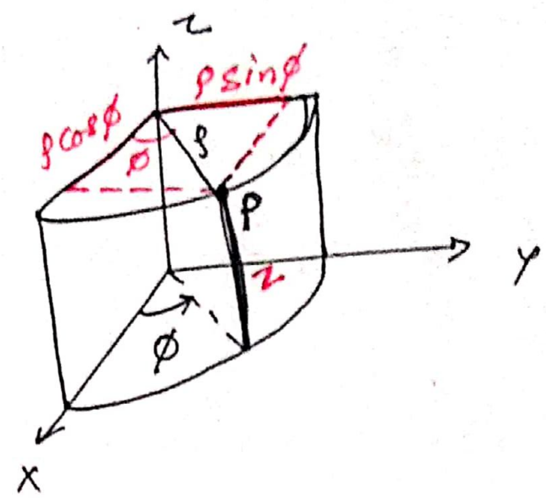
and the volume, $dv = \rho\, d\rho\, d\phi\, dz$





Directions of differential Areas are perpendicular to the plane

(2) The variables of the rectangular and cylindrical co-ordinate systems are easily related to each other.



From figure, $x = \rho\cos\phi$, $y = \rho\sin\phi$ & $z = z$ | x, y & z are Cartesian variables along x, y & z directions.

Also, we can express Cartesian co-ordinates in terms of cylindrical co-ordinates as:

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\& \; z = z$$

Using above, scalar functions in one systems are easily transformed into the other systems.

For a vector function in one co-ordinate system, however requires two steps in order to transform it to another system, because a different set of component vectors is generally required.

Let a rectangular vector be

$$\vec{A}_{rect.} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$$

and the vector in cylindrical co-ordinate system

if $\vec{A}_{cyl} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$.

So, we can see that $A_\rho = \vec{A}_{rect} \cdot \hat{a}_\rho$

$$A_\phi = \vec{A}_{rect} \cdot \hat{a}_\phi$$

$$\& \; A_z = \vec{A}_{rect} \cdot \hat{a}_z = A_z .$$

Expanding we have,

$$A_\rho = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \cdot \hat{a}_\rho = A_x \hat{a}_x \cdot \hat{a}_\rho + A_y \hat{a}_y \cdot \hat{a}_\rho$$

$$A_\phi = (A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z) \cdot \hat{a}_\phi = A_x \hat{a}_x \cdot \hat{a}_\phi + A_y \hat{a}_y \cdot \hat{a}_\phi$$

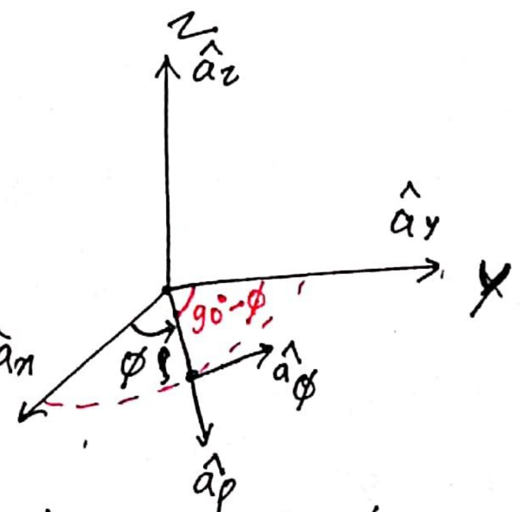
$$\& \; A_z = A_z \hat{a}_z \cdot \hat{a}_z = A_z$$

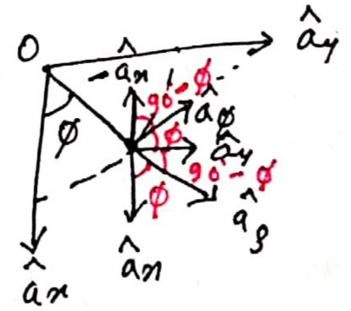
where,

$$\hat{a}_z \cdot \hat{a}_z = 1$$

$$\hat{a}_z \cdot \hat{a}_\rho = \hat{a}_z \cdot \hat{a}_\phi = 0$$

Here, in order to get value of A_ρ & A_ϕ we need to have values of $\hat{a}_x \cdot \hat{a}_\rho$, $\hat{a}_x \cdot \hat{a}_\phi$, $\hat{a}_y \cdot \hat{a}_\rho$ & $\hat{a}_y \cdot \hat{a}_\phi$.





so,

$$\hat{a}_x \cdot \hat{a}_\rho = |\hat{a}_x||\hat{a}_\rho| \cos\phi = \cos\phi$$

$$\hat{a}_y \cdot \hat{a}_\rho = |\hat{a}_y| \cdot |\hat{a}_\rho| \cos(90° - \phi)$$

$$= \sin\phi$$

$$\hat{a}_x \cdot \hat{a}_\phi = -|\hat{a}_x||\hat{a}_\phi| \cos(90° - \phi)$$

or $\cos(90° + \phi)$

$= -\sin\phi$

$$= -\sin\phi$$

$$\hat{a}_y \cdot \hat{a}_\phi = |\hat{a}_y||\hat{a}_\phi| \cos\phi = \cos\phi$$

therefore,

$$A_\rho = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$\& \; A_z = A_z$$

Hence, we can write in tabular form as:

	$\hat{a}_\rho$	$\hat{a}_\phi$	$\hat{a}_z$
$\hat{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\hat{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\hat{a}_z \cdot$	0	0	1

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Also, if we change Cylindrical co-ordinate to Cartesian then we have,

$$A_x = \vec{A}_{cyl.} \cdot \hat{a}_x = A_\rho \hat{a}_\rho \cdot \hat{a}_x + A_\phi \hat{a}_\phi \cdot \hat{a}_x$$

$$A_y = \vec{A}_{cyl.} \cdot \hat{a}_y = A_\rho \hat{a}_\rho \cdot \hat{a}_y + A_\phi \hat{a}_\phi \cdot \hat{a}_y$$

$$\& \ A_z = \vec{A}_{cyl.} \cdot \hat{a}_z = A_z$$

Again, from above figure.

$$\hat{a}_\rho \cdot \hat{a}_x = |\hat{a}_\rho| \cdot |\hat{a}_x| \cdot \cos\phi = \cos\phi$$
$$\hat{a}_\rho \cdot \hat{a}_y = |\hat{a}_\rho| \cdot |\hat{a}_y| \cos(90^\circ - \phi) = \sin\phi$$
$$\hat{a}_\phi \cdot \hat{a}_x = -|\hat{a}_\phi| |\hat{a}_x| \cos(90^\circ - \phi) = -\sin\phi$$
$$\hat{a}_\phi \cdot \hat{a}_y = |\hat{a}_\phi| |\hat{a}_y| \cos\phi = \cos\phi$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

Hence, we can write in tabular form as:

	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{a}_\rho \cdot$	$\cos\phi$	$\sin\phi$	0
$\hat{a}_\phi \cdot$	$-\sin\phi$	$\cos\phi$	0
$\hat{a}_z \cdot$	0	0	1

Therefore,

$$A_x = A_\rho \cos\phi - A_\phi \sin\phi$$

$$A_y = A_\rho \sin\phi + A_\phi \cos\phi$$

$$\& \ A_z = A_z$$

Sin & Cosec +ve

$90° + \theta$, $180° - \theta$

S

$90° - \theta$

All +ve

A

$180° + \theta$, $270° - \theta$

T

Tan & Cot +ve

C

$270° + \theta$, $360° - \theta$

Cos & Sec +ve

$\sin(90° - \theta) = \cos\theta$

$\cos(90° - \theta) = \sin\theta$

$\tan(90° - \theta) = \cot\theta$

$\sin(90° + \theta) = \cos\theta$

$\cos(90° + \theta) = -\sin\theta$

$\tan(90° + \theta) = -\cot\theta$

$\sin(180° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

$\tan(180° - \theta) = -\tan\theta$

Spherical Co-ordinate System (r, θ, ϕ)

The spherical Co-ordinate System is most appropriate when dealing with problems having a degree of spherical symmetry. A point P in spherical Co-ordinate system can be represented as (r, θ, ϕ). r is defined as the distance from the origin to point P or the radius of a sphere centered at the origin and passing through P; θ (called the collatitude) is the angle betn z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical Co-ordinates). The ranges of the variables are:

$$0 \leq r < \infty$$
$$0 \leq \theta \leq \pi$$
$$0 \leq \phi \leq 2\pi$$

A vector $\vec{A}$ in spherical coordinates may be written as:

$$\vec{A} = (A_r, A_\theta, A_\phi) \quad \text{or} \quad A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$$

where, $\hat{a}_r, \hat{a}_\theta$ & $\hat{a}_\phi$ are unit vectors along r, θ & ϕ directions.

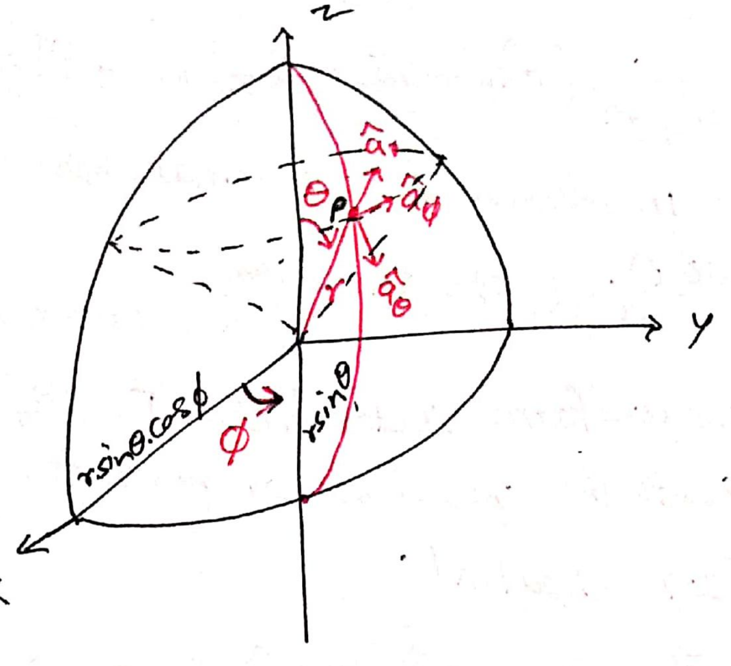
The magnitude of $\vec{A}$ is $|\vec{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$

The unit vectors $\hat{a}_r, \hat{a}_\theta$ & $\hat{a}_\phi$ are mutually orthogonal; $\hat{a}_r$ being directed along the radius or in the direction of increasing r, $\hat{a}_\theta$ in the direction of increasing θ, and $\hat{a}_\phi$ in the direction of increasing ϕ. Thus,

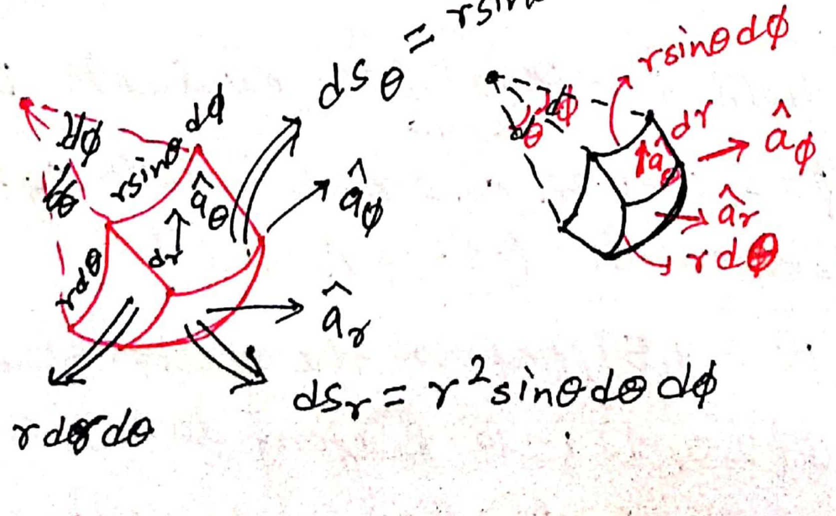
$$\hat{a}_r \cdot \hat{a}_r = \hat{a}_\theta \cdot \hat{a}_\theta = \hat{a}_\phi \cdot \hat{a}_\phi = 1$$

$$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_r = 0$$

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi, \quad \hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r, \quad \hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$$



A differential volume may be constructed in spherical co-ordinates by increasing r, θ & ϕ by dr, $d\theta$ an $d\phi$. The distance between the two spherical surfaces of radius r and $r+dr$ is dr; the distance bet.ⁿ the two cones having generating angles of θ and $\theta + d\theta$ is $rd\theta$; and the distance bet.ⁿ the two radial planes at angles ϕ and $\phi + d\phi$ is found to be $r\sin\theta d\phi$. The surfaces have the areas of $rdrd\theta$, $r\sin\theta\, dr\, d\phi$ and $r^2 \sin\theta\, d\theta\, d\phi$, and the volume is $r^2 \sin\theta\, dr\, d\theta\, d\phi$.



$dS_\theta = r\sin\theta\, dr\, d\phi$

$dS_r = r^2 \sin\theta\, d\theta\, d\phi$

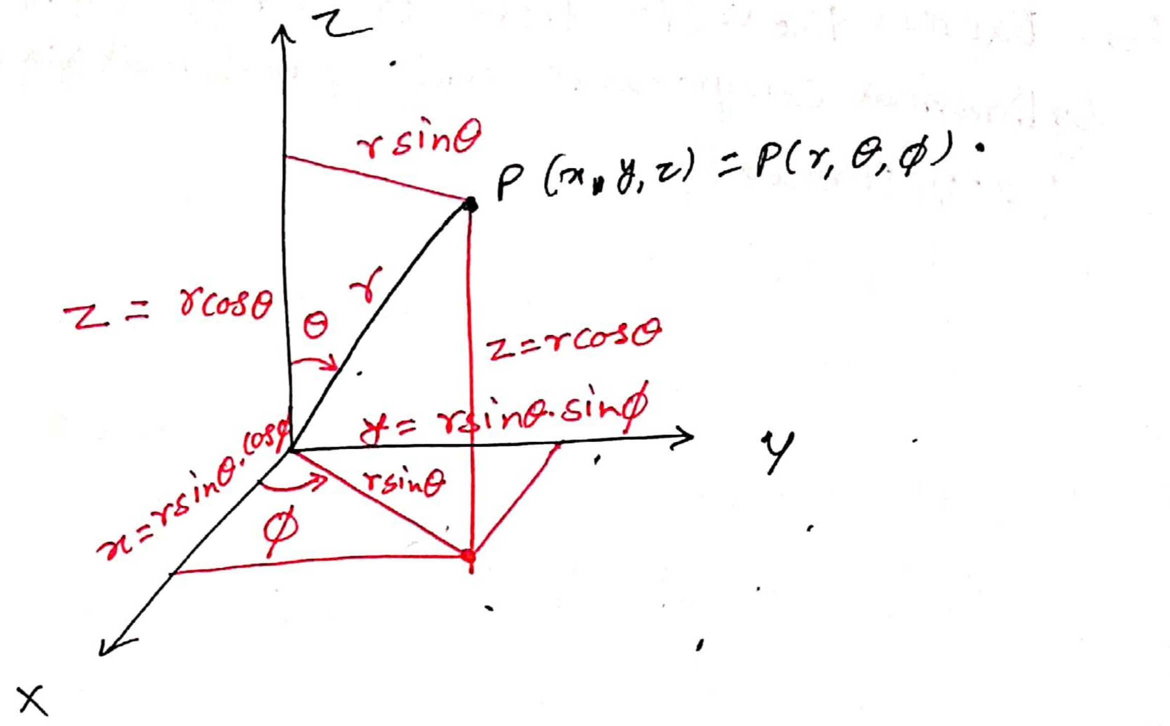
$dS_\phi = r\, dr\, d\theta$

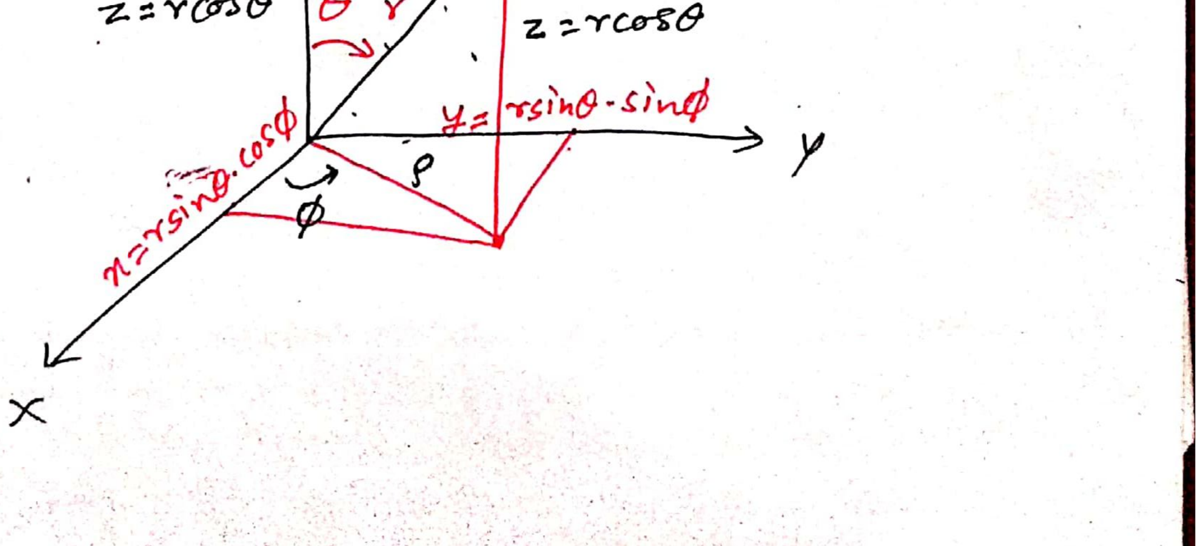
(17) The space variables (x, y, z) in Cartesian Co-ordinates can be related to variables (r, θ, ϕ) of spherical co-ordinate systems as:

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} \quad \& \quad \phi = \tan^{-1}\frac{y}{x}$$

or,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta.\sin\phi \quad \& \quad z = r\cos\theta$$





→ Let the rectangular vector $\vec{A}_{rect.} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$ and in spherical form it can be:

$$\vec{A}_{sphe} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$$

So, we can write

$$A_r = \vec{A}_{rect.} \cdot \hat{a}_r$$

$$A_\theta = \vec{A}_{rect.} \cdot \hat{a}_\theta$$

$$A_\phi = \vec{A}_{rect} \cdot \hat{a}_\phi$$

Expanding we have,

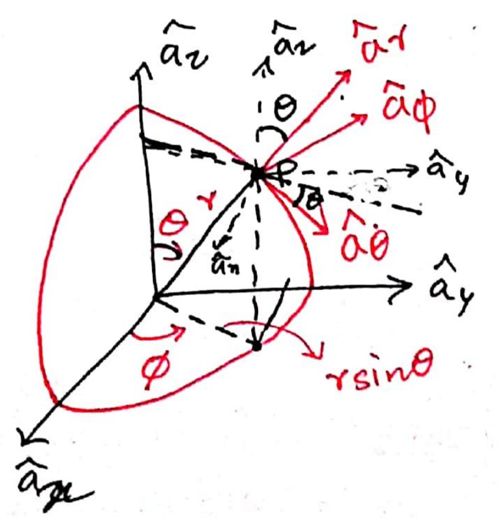
$$A_r = A_x (\hat{a}_x \cdot \hat{a}_r) + A_y (\hat{a}_y \cdot \hat{a}_r) + A_z (\hat{a}_z \cdot \hat{a}_r)$$

$$A_\theta = A_x (\hat{a}_x \cdot \hat{a}_\theta) + A_y (\hat{a}_y \cdot \hat{a}_\theta) + A_z (\hat{a}_z \cdot \hat{a}_\theta)$$

$$A_\phi = A_x (\hat{a}_x \cdot \hat{a}_\phi) + A_y (\hat{a}_y \cdot \hat{a}_\phi) + A_z (\hat{a}_z \cdot \hat{a}_\phi)$$

$$= A_x (\hat{a}_x \cdot \hat{a}_\phi) + A_y (\hat{a}_y \cdot \hat{a}_\phi) + 0 \quad [\hat{a}_z \cdot \hat{a}_\phi = 0]$$

$$= A_x (\hat{a}_x \cdot \hat{a}_\phi) + A_y (\hat{a}_y \cdot \hat{a}_\phi)$$



Here, in order to get values of A_r, A_θ & A_ϕ we need to have values of $\hat{a}_x \cdot \hat{a}_r$, $\hat{a}_y \cdot \hat{a}_r$, $\hat{a}_x \cdot \hat{a}_\theta$, $\hat{a}_y \cdot \hat{a}_\theta$, $\hat{a}_z \cdot \hat{a}_r$, $\hat{a}_z \cdot \hat{a}_\theta$, $\hat{a}_x \cdot \hat{a}_\phi$ & $\hat{a}_y \cdot \hat{a}_\phi$.

(12) So,

$\hat{a}_x \cdot \hat{a}_r = \sin\theta \cdot \cos\phi$

$\hat{a}_y \cdot \hat{a}_r = \sin\theta \cdot \sin\phi$

$\hat{a}_x \cdot \hat{a}_\theta = \cos\theta \cdot \cos\phi$

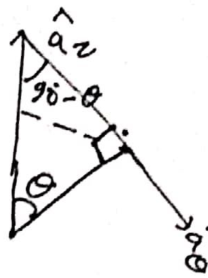
$\hat{a}_y \cdot \hat{a}_\theta = \cos\theta \cdot \sin\phi$

$\hat{a}_x \cdot \hat{a}_\phi = -\sin\phi$

$\hat{a}_y \cdot \hat{a}_\phi = \cos\phi$

$\hat{a}_z \cdot \hat{a}_r = \cos\theta$

$\hat{a}_z \cdot \hat{a}_\theta = -\sin\theta$



Therefore:

$A_r = A_x \sin\theta \cos\phi + A_y \sin\theta \cdot \sin\phi + A_z \cos\theta$

$A_\theta = A_x \cos\theta \cdot \cos\phi + A_y \cos\theta \cdot \sin\phi - A_z \sin\theta$

$A_\phi = -A_x \sin\phi + A_y \cos\phi$

Hence we can write as:

	$\hat{a}_r$	$\hat{a}_\theta$	$\hat{a}_\phi$
$\hat{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{a}_y \cdot$	$\sin\theta \cdot \sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cdot\cos\phi & \cos\theta\cdot\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} .$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Also, ~~for~~ spherical to Cartesian

$\hat{a}_r \cdot \hat{a}_x = \sin\theta \cdot \cos\phi$

$\hat{a}_r \cdot \hat{a}_y = \sin\theta \cdot \sin\phi$

$\hat{a}_r \cdot \hat{a}_z = \cos\theta$

$\hat{a}_\theta \cdot \hat{a}_x = \cos\theta \cdot \cos\phi$

$\hat{a}_\theta \cdot \hat{a}_y = \cos\theta \cdot \sin\phi$

$\hat{a}_\theta \cdot \hat{a}_z = -\sin\theta$

$\hat{a}_\phi \cdot \hat{a}_x = -\sin\phi$

$\hat{a}_\phi \cdot \hat{a}_y = \cos\phi$

	$\hat{a}_x$	$\hat{a}_y$	$\hat{a}_z$
$\hat{a}_r \cdot$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$
$\hat{a}_\theta \cdot$	$\cos\theta \cos\phi$	$\cos\theta \cdot \sin\phi$	$-\sin\theta$
$\hat{a}_\phi \cdot$	$-\sin\phi$	$\cos\phi$	0

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

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For, Cylindrical to spherical

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cdot\cos\phi & \sin\theta\cdot\sin\phi & \cos\theta \\ \cos\theta\cdot\cos\phi & \cos\theta\cdot\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\therefore \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

For spherical to cylindrical

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta\cdot\cos\phi & \cos\theta\cdot\cos\phi & -\sin\phi \\ \sin\theta\cdot\sin\phi & \cos\theta\cdot\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\therefore \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Scalar Transformation

Qn: Express a scalar potential field, $V = x^2 + 2y^2 + 3z^2$ in spherical co-ordinates. Find value of V at a point $P(2, 60°, 90°)$.
[2073 Chaitra]

Soln. Given,

potential field, $V = x^2 + \cancel{2}y^2 + 3z^2$

Converting potential field into spherical co-ordinates

$$V_{sph} = (r\sin\theta\cos\phi)^2 + 2(r\sin\theta\cdot\sin\phi)^2 + 3(r\cos\theta)^2$$

Since, the point $P(2, 60°, 90°)$ is in spherical co-ordinates.

We have,

$$\text{value of } V \text{ at } P = V_{sph}(P) = (2\sin 60°\cdot\cos 90°)^2 + 2(2\sin 60°\cdot\sin 90°)^2 + 3(2\cdot\cos 60°)^2$$

$$= 4\times\frac{3}{4}\times 0 + 2\times 4\times\frac{3}{4}\times 1 + 3\times 4\times\frac{1}{4}$$

$$= 0 + 6 + 3$$

$$= 9.$$

Qn: Convert the vector $\vec{F} = F_x\hat{a}_x + F_y\hat{a}_y + F_z\hat{a}_z$ to ~~both~~ spherical co-ordinate system. (2074 Ashwin)

Cylindrical to spherical & vice versa

$\hat{a}_\rho \cdot \hat{a}_r = \sin\theta = \hat{a}_r \cdot \hat{a}_\rho$

$\hat{a}_z \cdot \hat{a}_r = \cos\theta = \hat{a}_r \cdot \hat{a}_z$

$\hat{a}_\rho \cdot \hat{a}_\phi = 0 = \hat{a}_\phi \cdot \hat{a}_\rho$

$\hat{a}_z \cdot \hat{a}_\phi = 0 = \hat{a}_\phi \cdot \hat{a}_z$

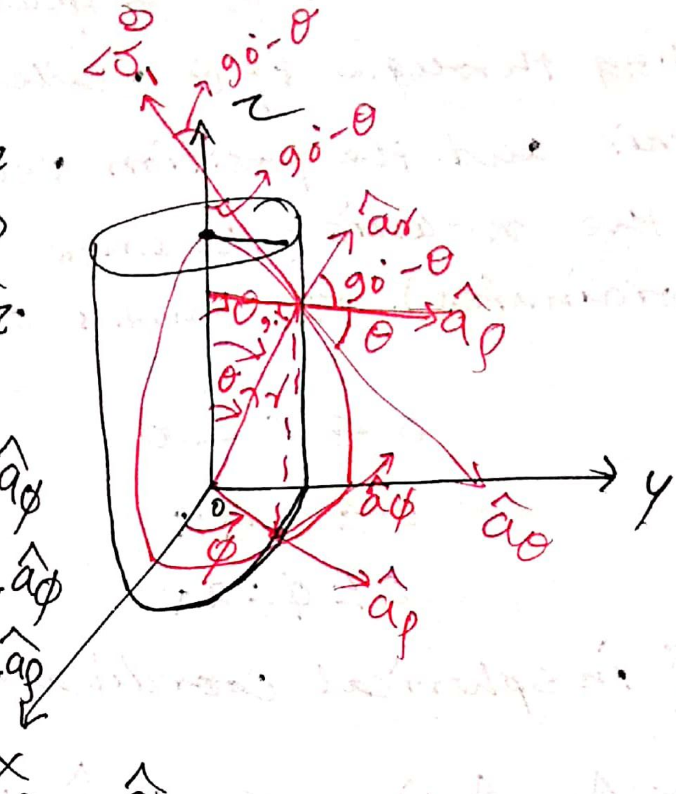
$\hat{a}_\phi \cdot \hat{a}_\phi = 1$

$\hat{a}_\phi \cdot \hat{a}_r = 0 = \hat{a}_r \cdot \hat{a}_\phi$

$\hat{a}_\phi \cdot \hat{a}_\theta = 0 = \hat{a}_\theta \cdot \hat{a}_\phi$

$\hat{a}_\rho \cdot \hat{a}_\theta = \cos\theta = \hat{a}_\theta \cdot \hat{a}_\rho$

$\hat{a}_z \cdot \hat{a}_\theta = -\sin\theta = \hat{a}_\theta \cdot \hat{a}_z$



	$\hat{a}_r$	$\hat{a}_\theta$	$\hat{a}_\phi$
$\hat{a}_\rho$	$\sin\theta$	$\cos\theta$	0
$\hat{a}_\phi$	0	0	1
$\hat{a}_z$	$\cos\theta$	$-\sin\theta$	0

	$\hat{a}_\rho$	$\hat{a}_\phi$	$\hat{a}_z$
$\hat{a}_r$	$\sin\theta$	0	$\cos\theta$
$\hat{a}_\theta$	$\cos\theta$	0	$-\sin\theta$
$\hat{a}_\phi$	0	1	0

Qn: A field vector is given by an expression

$$\vec{A} = \frac{1}{\sqrt{x^2+y^2+z^2}}(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z)$$, transform this vector in cylindrical co-ordinate system at point $(2, 30°, 6)$. [2073 Shrawan]

Qn: Express the uniform vector field $\vec{F} = 5\hat{a}_x$ in ⓐ cylindrical components ⓑ spherical components.

[2072 Chaitra]

Qn: Transform $\vec{A} = 10\hat{a}_x - 8\hat{a}_y + 6\hat{a}_z$; at point P(10, -8, 6) to cylindrical co-ordinate system. (2072 Kartik)

Qn: Transform the vector $4\hat{a}_x - 2\hat{a}_y - 4\hat{a}_z$ into spherical co-ordinates at point P(2, -3, 4). [2071 Chaitra]

Qn: Transform vector field $\vec{A} = r\cos\phi\,\hat{a}_r + z\hat{a}_z$ $(\rho\cos\phi\,\hat{a}_\rho + z\hat{a}_z)$ at point P(1, 30°, 2) in cylindrical co-ordinate system to spherical co-ordinate. [2068 Shrawan]

Qn: Transform vector $\vec{A} = \rho\sin\phi\,\hat{a}_z$ at point (1, 45°, 2) in cylindrical co-ordinate system to a vector in spherical co-ordinate system. [2068 Chaitra]

Qn: Given a vector field $\vec{D} = \frac{x\hat{a}_x + y\hat{a}_y}{x^2+y^2}$, evaluate D at the point where $\rho = 2, \phi = 0.2\pi$ and $z = 5$ in both cylindrical and cartesian components. [2069 Ashad]

Qn: Given a point P(-3, 4, 5), express the vector that extends from point P to Q(2, 0, -1) in ⓐ Rectangular co-ordinates ⓑ Cylindrical co-ordinates ⓒ Spherical co-ordinates. [2069 Chaitra]

Qn: At point $P(-3,-4,5)$, express that vector that extends from P to $Q(2,0,-1)$ in spherical co-ordinates.
[2070 Ashad]

Qn: Transform the vector $\vec{A} = y\hat{a}_x + x\hat{a}_y + z\hat{a}_z$ into cylindrical co-ordinates at a Point $P(2, 45^\circ, 5)$.
[2070 Chaitra]

Qn: Express the vector field, $\vec{G} = (x^2+y^2)^{-1}(x\hat{a}_x + y\hat{a}_y)$ in cylindrical components and cylindrical variables.
[2071 Shrawan]

Examples:

(a) Change $P(5, 2, -6)$ into cylindrical and spherical Co-ordinate systems.

Sol.ⁿ

Given, $P(5, 2, -6)$ in cartesian co-ordinate system

where, $x = 5,\ y = 2\ \&\ z = -6$

So, In cylindrical system

$$\rho = \sqrt{x^2+y^2} = \sqrt{5^2+2^2} = \sqrt{29}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{5}\right)$$

$$\&\ Z = z = -6.$$

Again, in spherical system.

$$r = \sqrt{\rho^2+z^2} = \sqrt{x^2+y^2+z^2} = \sqrt{5^2+2^2+(-6)^2}$$

$$= \sqrt{65}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \boxed{\cancel{\frac{-\sqrt{29}}{6}}}\ \tan^{-1}\left(\frac{-\sqrt{29}}{6}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{5}\right)$$

(b) Change $P(2, 30^\circ, 5)$ into Cartesian & Spherical Co-ordinate systems.

Sol.ⁿ We have, $P(2, 30^\circ, 5)$ in cylindrical co-ordinate system. So

$$\rho = 2$$
$$\phi = 30^\circ \quad \&\ Z = 5$$

(12)

Now, In Cartesian Co-ordinate system

$$x = \rho \cos\phi = 2\cos 30^\circ = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = \rho \sin\phi = 2 \sin 30^\circ = 2 \cdot \frac{1}{2} = 1$$

& $z = z = 5$

Again, in spherical co-ordinate system.

$$r = \sqrt{\rho^2 + z^2} = \sqrt{2^2 + 5^2} = \sqrt{29}$$

$$\theta = \tan^{-1}\left(\frac{\rho}{z}\right) = \tan^{-1}\left(\frac{2}{5}\right)$$

& $\phi = \phi = 30^\circ$

ⓒ Change $P(5, 60^\circ, 30^\circ)$ into Cartesian & Cylindrical co-ordinate systems.

Sol.ⁿ Here, $P(5, 60^\circ, 30^\circ)$ is in spherical co-ordinate system

so, $r = 5$

$\theta = 60^\circ$

& $\phi = 30^\circ$

Now, in Cartesian Co-ordinate system

$$x = r\sin\theta \cdot \cos\phi$$
$$= 5 \cdot \sin 60^\circ \cdot \cos 30^\circ$$
$$= 5 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$$
$$= \frac{15}{4}$$

$$y = r\sin\theta \cdot \sin\phi = 5 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{5\sqrt{3}}{4}$$

& $z = r\cos\theta$
$$= 5 \cdot \cos 60^\circ$$
$$= 5 \cdot \frac{1}{2}$$
$$= \frac{5}{2}$$

Again, In cylindrical co-ordinate system

$$\rho = r\sin\theta = 5 \cdot \sin 60^\circ = \frac{5\sqrt{3}}{2}$$

$$\phi = \phi = 30^\circ$$

$$\& \ z = r\cos\theta = 5 \cdot \cos 60^\circ = \frac{5}{2}$$

Qn: A vector in Cartesian co-ordinate system is $\vec{A} = yz\,\hat{a}_x + xz\,\hat{a}_y + xy\,\hat{a}_z$. Transform this vector to that in cylindrical & spherical co-ordinate systems.

Sol.^{n} Given, $\vec{A}$ in cartesian co-ordinate system as:

$$\vec{A} = yz\,\hat{a}_x + xz\,\hat{a}_y + xy\,\hat{a}_z$$

To change this vector into cylindrical co-ordinate system we need to calculate A_ρ, A_ϕ & A_z; so that we get

$$\vec{A}_{(cyl.)} = A_\rho\,\hat{a}_\rho + A_\phi\,\hat{a}_\phi + A_z\,\hat{a}_z$$

Now,

$$A_\rho = \vec{A} \cdot \hat{a}_\rho = (yz\,\hat{a}_x + xz\,\hat{a}_y + xy\,\hat{a}_z) \cdot \hat{a}_\rho$$

$$= yz\cos\phi + xz\sin\phi + xy \cdot 0$$

$$= yz\cos\phi + xz\sin\phi$$

Again, we have, $y = \rho\sin\phi$ & $x = \rho\cos\phi$

So, $A_\rho = \rho\sin\phi \cdot z\cos\phi + \rho\cos\phi \cdot z\sin\phi$

$$= 2\rho z\sin\phi \cdot \cos\phi = \rho z \sin 2\phi$$

 similarly,

$$A_\phi = \vec{A}\cdot\hat{a}_\phi = (yz\,\hat{a}_x + xz\,\hat{a}_y + xy\,\hat{a}_z)\cdot\hat{a}_\phi$$

$$= yz(-\sin\phi) + xz\cos\phi + xy\cdot 0$$

$$= -yz\sin\phi + xz\cos\phi$$

$$= -\rho\sin\phi\, z\sin\phi + \rho\cos\phi\cdot z\cos\phi$$

$$= \rho z(\cos^2\phi - \sin^2\phi)$$

$$= \rho z\cos 2\phi$$

& $$A_z = \vec{A}\cdot\hat{a}_z = (yz\,\hat{a}_x + xz\,\hat{a}_y + xy\,\hat{a}_z)\cdot\hat{a}_z$$

$$= xy$$

$$= \rho\cos\phi\cdot\rho\sin\phi$$

$$= \rho^2\cos\phi\cdot\sin\phi$$

$$= \frac{\rho^2}{2}\sin 2\phi$$

$$\therefore\ \vec{A}(\text{cyl.}) = \rho z\sin 2\phi\,\hat{a}_\rho + \rho z\cos 2\phi\,\hat{a}_\phi + \frac{\rho^2}{2}\sin 2\phi\cdot\hat{a}_z$$

Again, to change into spherical co-ordinate system we need to calculate A_r, A_θ & A_ϕ; so that we get

$$\vec{A}_{(\text{sphe.})} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$$

By doing similar tasks as above, we get

$$\vec{A}(\text{sphe.}) = 3r^2\sin^2\theta\cdot\sin\phi\cos\phi\cdot\cos\theta\,\hat{a}_r + (2r^2\cos^2\theta\cdot\sin\theta\cdot\sin\phi\cdot\cos\phi - r^2\sin^3\theta\cdot\cos\phi\cdot\sin\phi)\hat{a}_\theta + r^2\sin\theta\cdot\cos\theta\cdot\cos 2\phi\,\hat{a}_\phi.$$

Assignment

(1) Convert $\vec{B} = \frac{yz}{x}\,\hat{a}_x$ into spherical Co-ordinate system.

(ii) Convert $\vec{B} = y\hat{a}_x - x\hat{a}_y + z\hat{a}_z$ into cylindrical Co-ordinate system.

(iii) Convert $\vec{B}\,(\text{cyl.}) = \rho\hat{a}_\rho + \sin\phi\,\hat{a}_\phi + z\hat{a}_z$ into Cartesian Co-ordinate system.

(iv) Convert $\vec{B} = r^2\hat{a}_r + \sin\theta\,\hat{a}_\phi$ into Cartesian Co-ordinate system.

6. **Scalar field** - Region in which scalar point function is defined. Scalar point function assigns values to points in the given region depending on the co-ordinate of the point. eg:- temperature distribution in a room etc.

- **Vector field** - Region in which vector point function is defined. Vector point function assigns values to points in region depending on the co-ordinate of the point. eg:- electric field etc.

Del operator (∇)

The del operator, written ∇, is the vector differential operator. In Cartesian co-ordinates,

$$\nabla = \frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z$$

It is also known as the gradient operator, is not a vector in itself, but when it operates on a scalar function, for eg:- a vector ensues. The operator is useful in defining

1) The gradient of a scalar V, written as ∇V.
2) The divergence of a vector $\vec{A}$, written as $\nabla \cdot \vec{A}$
3) The curl of a vector $\vec{A}$, written as $\nabla \times \vec{A}$
4) The Laplacian of a scalar V, written as $\nabla^2 V$

By using transformation formula we get ∇ operator in cylindrical and co-ordinate systems as:

$$\nabla_{(cyl.)} = \frac{\partial}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a}_\phi + \frac{\partial}{\partial z}\hat{a}_z$$

&

$$\nabla_{(sphe.)} = \frac{\partial}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a}_\phi$$

1) Gradient of a Scalar

The gradient of a scalar field V is a vector that represents both the magnitude and the direction of the maximum space rate of increase of V. The mathematical expression for the gradient can be written as:

$$\boxed{\vec{G} = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z}$$

or

$$\boxed{\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z}$$

for cylindrical co-ordinates,

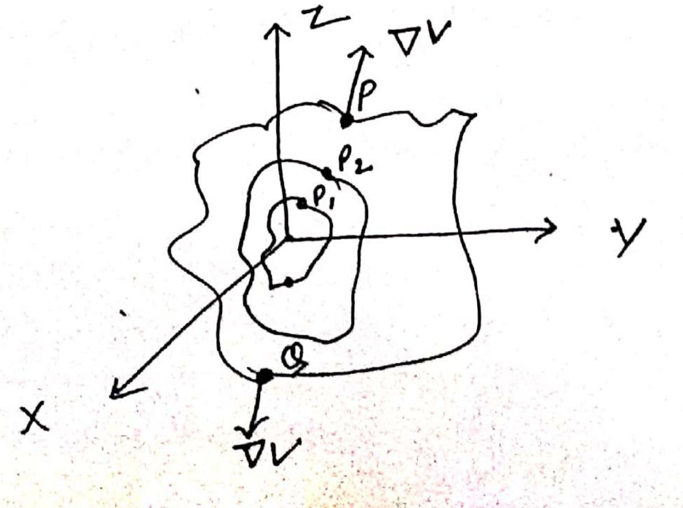
$$\boxed{\nabla V = \frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z}$$

& for spherical co-ordinates,

$$\boxed{\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi}$$

(a) Magnitude of ∇V gives the maximum rate of change in V per unit distance.

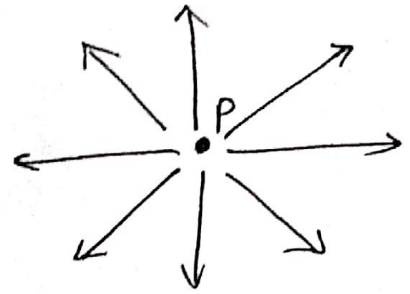
(b) ∇V points in the direction of the max^m rate of change in V.



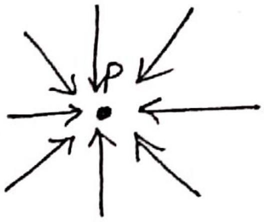
Divergence of a vector

The divergence of $\vec{A}$ at a given point P is the outward flux per unit volume as the volume shrinks about P. i.e $\text{div}\vec{A} = \nabla\cdot\vec{A}$

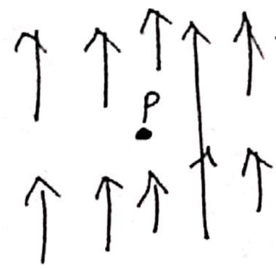
$$= \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\cdot\vec{ds}}{\Delta v}$$



+ve divergence



-ve divergence



zero divergence

Divergence of $\vec{A}$ in cartesian co-ordinates is:

$$\boxed{\text{div.}\vec{A} = \nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

In cylindrical co-ordinates,

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

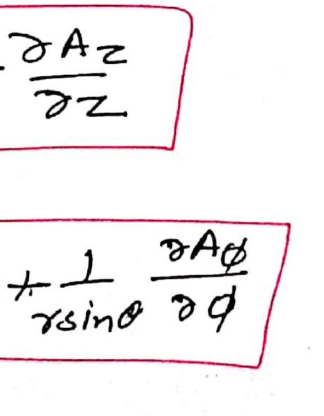
& in spherical co-ordinates,

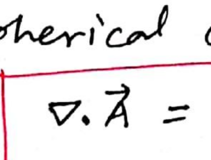
$$\boxed{\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}}$$

Curl of a vector

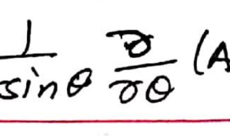
The curl of $\vec{A}$ is an axial (or rotational) vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal dir^n of the area when the area is oriented so as to make a circulation maximum.

i.e $\text{curl}\,\vec{A} = \nabla\times\vec{A} = \left(\lim_{\Delta s \to 0} \frac{\oint_L \vec{A}\cdot\vec{dl}}{\Delta s}\right)_{max} \hat{a}_n$





Curl at P points out of the page



Curl at P is zero.

(20) Curl in Cartesian Co-ordinates is:

$$\nabla \times \vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{a}_z$$

or

$$\nabla \times \vec{A} = \begin{bmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

In cylindrical co-ordinates,

$$\nabla \times \vec{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\hat{a}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\hat{a}_z$$

or

$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{bmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{bmatrix}$$

& in spherical Co-ordinates.

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right]\hat{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(r A_\phi)}{\partial r}\right]\hat{a}_\theta + \frac{1}{r}\left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{a}_\phi$$

or

$$\nabla \times \vec{A} = \begin{bmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta A_\phi \end{bmatrix}$$

10. In general form we can use three arbitrary variables h_1, h_2 & h_3 where h_1, h_2, h_3 has values

(a) $h_1 = h_2 = h_3 = 1$ for Cartesian

(b) $h_1 = 1,\ h_2 = \rho,\ h_3 = 1$ for cylindrical

(c) $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ for Spherical

So, we can write

Gradient of scalar field

$$\nabla V = \frac{1}{h_1}\frac{\partial V}{\partial u}\hat{a}_u + \frac{1}{h_2}\frac{\partial V}{\partial v}\hat{a}_v + \frac{1}{h_3}\frac{\partial V}{\partial w}\hat{a}_w$$

where, u, v & w are co-ordinate axes.

i.e for cartesian $\Rightarrow u = x,\ v = y,\ w = z$

for cylindrical $\Rightarrow u = \rho,\ v = \phi,\ w = z$

for spherical $\Rightarrow u = r,\ v = \theta,\ w = \phi$

Divergence of vector field

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 A_u) + \frac{\partial}{\partial v}(h_1 h_3 A_v) + \frac{\partial}{\partial w}(h_1 h_2 A_w)\right]$$

Curl of vector field

$$\nabla \times \vec{A} = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial v}(h_3 A_w) - \frac{\partial}{\partial w}(h_2 A_v)\right]\hat{a}_u + \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial w}(h_1 A_u) - \frac{\partial}{\partial u}(h_3 A_w)\right]\hat{a}_v + \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u}(h_2 A_v) - \frac{\partial}{\partial v}(h_1 A_u)\right]\hat{a}_w$$

Qns. ① Find the divergence of the vector

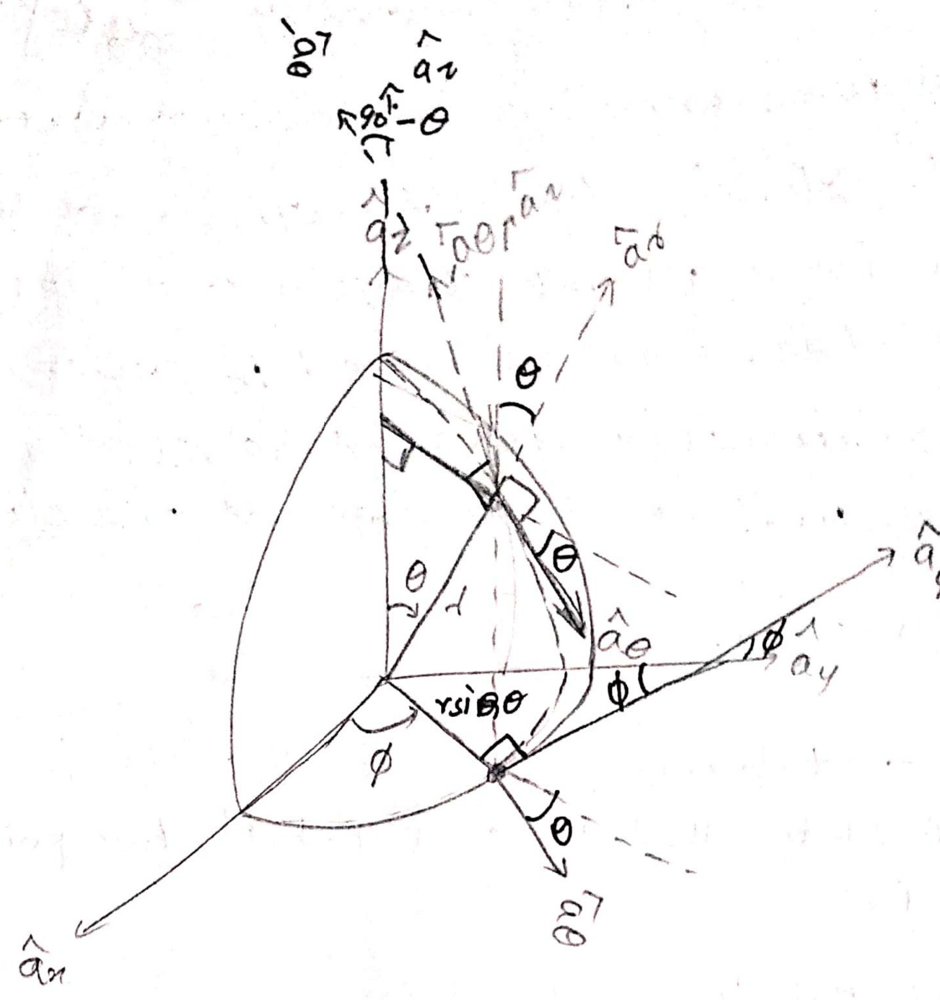
$$\vec{F} = xyz\,\hat{a}_x + x^2yz\,\hat{a}_y + xyz^2\,\hat{a}_z$$

② Find the gradient of the scalar

$$F = r^2 + r\cos\theta \cdot \sin\phi$$

③ Find the curl of the vector

$$\vec{F} = \rho\,\hat{a}_\rho + \rho\sin\phi\,\hat{a}_\phi + z\,\hat{a}_z$$



$$\hat{a}_x \cdot \hat{a}_r = \sin\theta \cdot \cos\phi$$
$$\hat{a}_y \cdot \hat{a}_r = \sin\theta \cdot \sin\phi$$
$$\hat{a}_z \cdot \hat{a}_r = \cos\theta$$
$$\hat{a}_x \cdot \hat{a}_\theta = \cos\theta \cdot \cos\phi$$
$$\hat{a}_y \cdot \hat{a}_\theta = \cos\theta \cdot \sin\phi$$
$$\hat{a}_z \cdot \hat{a}_\theta = -\sin\theta$$
$$\hat{a}_x \cdot \hat{a}_\phi = -\sin\phi$$
$$\hat{a}_y \cdot \hat{a}_\phi = \cos\phi$$
$$\hat{a}_z \cdot \hat{a}_\phi = 0$$