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Chapter-2 Electric Fields

Coulomb's Law and field Intensity

Coulomb's Law is an experimental law formulated in 1785 by the French Colonel, Charles Augustin de Coulomb. It deals with the force a point charge exerts on another point charge. Charges are measured in coulombs (C). One Coulomb is approximately equivalent to 6×10^{18} electrons; it is a very large unit of charge because one electron charge $e = -1.6019\times10^{-19}$ C.

Coulomb's Law - Statement

Coulomb's law states that the force F betn two point charges Q_1 and Q_2 is:

1. along the line joining them
2. Directly proportional to the product $Q_1 \cdot Q_2$ of the charges
3. Inversely proportional to the square of the distance R betn them.

or

The force betn two charges separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance betn them.

That is;

$$F \propto Q_1 \cdot Q_2 \quad \text{- - - (a)}$$

$$F \propto \frac{1}{R^2} \quad \text{- - - (b)}$$

Combining (a) + (b)

$$F \propto \frac{Q_1 \cdot Q_2}{R^2}$$

$$\therefore F = \frac{k Q_1 Q_2}{R^2} \quad \text{- - - (c)}$$

where k is the proportionality constant. In SI units, charges Q_1 and Q_2 are in coulombs (C), the distance R is in meters (m), and the force F is in newtons (N) so that $k = \frac{1}{4\pi\varepsilon_0}$. The constant ε_0 is known as the permittivity of free space (in farads per meter) and has the value

$$\varepsilon_0 = 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi} \text{ F/m}$$

$$\text{or} \quad k = \frac{1}{4\pi\varepsilon_0} \simeq 9 \times 10^{9} \text{ m/F}$$

Now, The expression (c) becomes

$$\boxed{F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}}$$

If point charges Q_1 & Q_2 are located at points having position vectors $\vec{r}_1$ & $\vec{r}_2$ then the force $\vec{F}_{12}$ on Q_2 due to Q_1 is given by

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \hat{a}_{R_{12}} \quad \text{where, } \vec{R}_{12} = \vec{r}_2 - \vec{r}_1$$

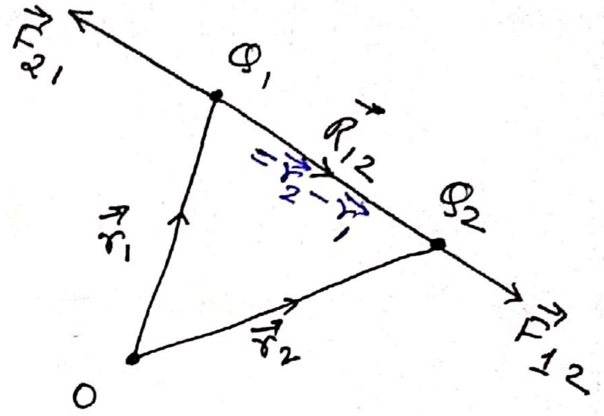
$$\therefore R = |\vec{R}_{12}|$$

and $\hat{a}_{R_{12}} = \dfrac{\vec{R}_{12}}{|\vec{R}_{12}|} = \dfrac{\vec{R}_{12}}{R}$

Substituting in above expression; $\hat{a}_{R_{12}} = \dfrac{\vec{R}_{12}}{R}$

we get,

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \cdot \frac{\vec{R}_{12}}{R}$$



$$\boxed{\therefore \vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^3} \cdot \vec{R}_{12}}$$

or,

$$\vec{F}_{12} = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 |\vec{r}_2 - \vec{r}_1|^3} \cdot (\vec{r}_2 - \vec{r}_1)$$

Again, for the Force on charge Q_1 due to charge Q_2 we can write as:

$$\vec{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \cdot \hat{a}_{R_{21}}$$

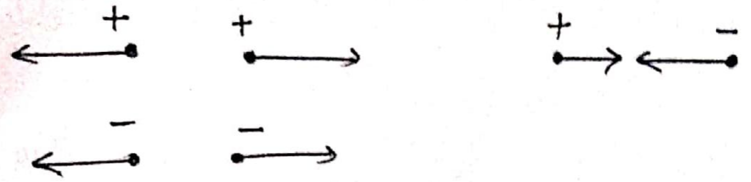
From figure, since $\hat{a}_{R_{21}} = -\hat{a}_{R_{12}}$
& $|\vec{R}_{12}| = |\vec{R}_{21}| = R$

$$\vec{F}_{21} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \cdot \left(\frac{-\hat{a}_{R_{12}}}{R}\right)$$

$$\boxed{\therefore \vec{F}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R^3} \cdot \hat{a}_{R_{12}} = -\vec{F}_{12}}$$

or $\vec{F}_{21} = \dfrac{-Q_1 Q_2}{4\pi\epsilon_0 |\vec{r}_1 - \vec{r}_2|^3} \cdot (\vec{r}_2 - \vec{r}_1)$

Notes:



1. Like charges (charges of same sign) repel each other while unlike charges attract.
2. The distance betn the charged bodies Q_1 & Q_2 must be large compared with the linear dimensions of the bodies i.e Q_1 & Q_2 must be point charges.
3. Q_1 & Q_2 must be static (at rest).
4. The signs of Q_1 & Q_2 must be taken into account.

If we have more than two point charges, we can use the principle of superposition to determine the force on a particular charge. The principle states that if there are N charges $Q_1, Q_2, \dots Q_N$ located respectively, at points with position vectors $\vec{r}_1, \vec{r}_2, \dots \vec{r}_N$, the resultant force $\vec{F}$ on a charge Q located at point $\vec{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \dots Q_N$.

Hence,

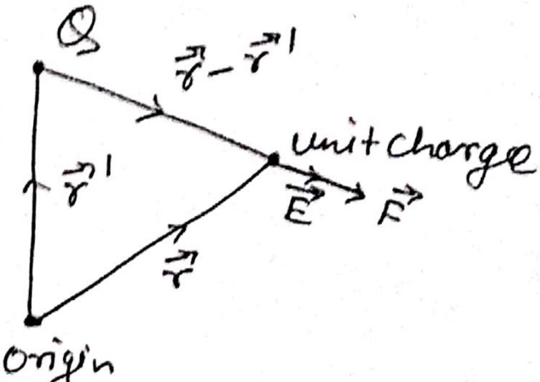
$$\vec{F} = \frac{QQ_1(\vec{r}-\vec{r}_1)}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|^3} + \frac{QQ_2(\vec{r}-\vec{r}_2)}{4\pi\epsilon_0|\vec{r}-\vec{r}_2|^3} + \dots + \frac{Q \cdot Q_N(\vec{r}-\vec{r}_N)}{4\pi\epsilon_0|\vec{r}-\vec{r}_N|^3}$$

or,

$$\boxed{\vec{F} = \frac{Q}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}}$$

The electric field intensity (or electric field strength) $\vec{E}$ is the force per unit charge when placed in the electric field.

Thus, $\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q}$



or simply $\boxed{\vec{E} = \frac{\vec{F}}{Q}}$.

The electric field intensity $\vec{E}$ is in the direction of the force $\vec{F}$ and is measured in newtons/Coulomb or volts/meter.

The electric field intensity at point $\vec{r}$ due to a point charge located at $\vec{r}'$ is readily obtained as:

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2}\hat{a}_R = \frac{Q(\vec{r}-\vec{r}')}{4\pi\epsilon_0 |\vec{r}-\vec{r}'|^3}}$$

Now,

For N point charges $Q_1, Q_2, \ldots Q_N$ located at $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_N$, the electric field intensity at point $\vec{r}$ is obtained as:

$$\vec{E} = \frac{Q_1(\vec{r}-\vec{r}_1)}{4\pi\epsilon_0 |\vec{r}-\vec{r}_1|^3} + \frac{Q_2(\vec{r}-\vec{r}_2)}{4\pi\epsilon_0 |\vec{r}-\vec{r}_2|^3} + \cdots + \frac{Q_3(\vec{r}-\vec{r}_N)}{4\pi\epsilon_0 |\vec{r}-\vec{r}_N|^3}$$

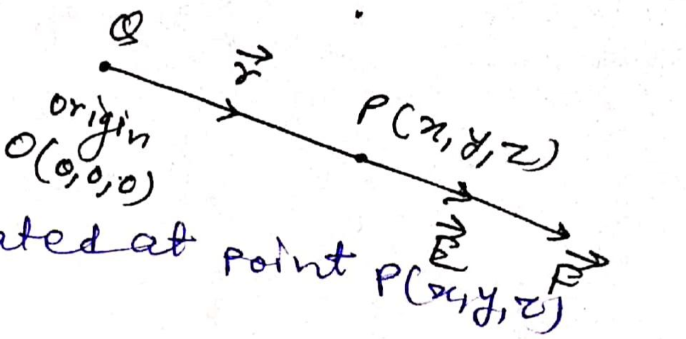
or,

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N} \frac{Q_k(\vec{r}-\vec{r}_k)}{|\vec{r}-\vec{r}_k|^3}}$$

Q: If the point charge Q is located at origin then find the Force and electric field intensity at point P(x,y,z) where; a unit charge is located.

Sol.^{n}

Given,



Point charge is at origin

and a unit charge is at located at point P(x,y,z)

i.e $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

then

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2} \quad \& \quad \hat{a}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z)}{\sqrt{x^2+y^2+z^2}}$$

Now, we have, the force due to point charge Q on the unit charge at point P(x,y,z) is:

$$\vec{F} = \frac{Q \cdot 1}{4\pi\varepsilon_0 r^2} \cdot \hat{a}_r = \frac{Q}{4\pi\varepsilon_0 r^2} \cdot \frac{\vec{r}}{r} = \frac{Q}{4\pi\varepsilon_0 r^3} \cdot \vec{r}$$

$$= \frac{Q}{4\pi\varepsilon_0} \cdot \frac{(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z)}{(x^2+y^2+z^2)^{3/2}}$$

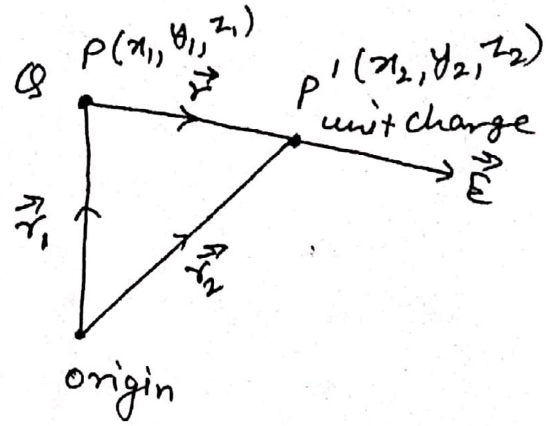
Since, the charge at Point P(x,y,z) is unit charge the above calculated force is the electric field intensity.

$$\therefore \vec{E} = \vec{F} = \frac{Q}{4\pi\varepsilon_0} \cdot \frac{(x\hat{a}_x + y\hat{a}_y + z\hat{a}_z)}{(x^2+y^2+z^2)^{3/2}}$$

...! If the point charge Q is not at origin but at any ...int $P(x_1, y_1, z_1)$ and the unit charge is at point $P'(x_2, y_2, z_2)$ ...en calculate the electric field intensity at P'.

Solⁿ

Given, the charge Q is located at $P(x_1, y_1, z_1)$ & unit charge is located at $P'(x_2, y_2, z_2)$ then the position vectors will be



$$\vec{r}_1 = x_1\hat{a}_x + y_1\hat{a}_y + z_1\hat{a}_z$$

$$\vec{r}_2 = x_2\hat{a}_x + y_2\hat{a}_y + z_2\hat{a}_z$$

& $$\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z$$

Now, The electric field intensity at $P'(x_2, y_2, z_2)$ is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \cdot \frac{\hat{a}_r}{r^2} = \frac{Q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{\cancel{4\pi\epsilon_0}\, r^3}$$

$$\therefore \vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{[(x_2 - x_1)\hat{a}_x + (y_2 - y_1)\hat{a}_y + (z_2 - z_1)\hat{a}_z]}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}}$$

where,

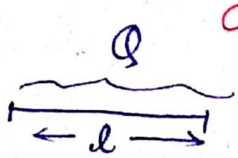
$$r = |\vec{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Charge Densities

(a) Line charge Density - It is the charge per unit length of the straight line charge. Its unit is Coulomb/meter (C/m) and represented by ρ_l.

$Q = \int_L \rho_l \, dl$

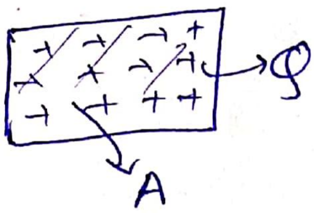
i.e $\rho_l = \dfrac{Q}{l}$



where, Q = total charge on the straight line
l = length of straight line charge.

(b) Surface charge density - It is the charge per unit area of the sheet containing charge. Its unit is Coulomb/m^2 (C/m^2) and represented by ρ_s.

i.e $\rho_s = \dfrac{Q}{A}$



where, Q = total charge on the sheet
A = Area of the sheet

$Q = \int_S \rho_s \, ds$

(c) Volume charge density - It is the charge per unit volume of the space containing charge inside it. Its unit is Coulumb / meter^3 (C/m^3) and represented by ρ_v.

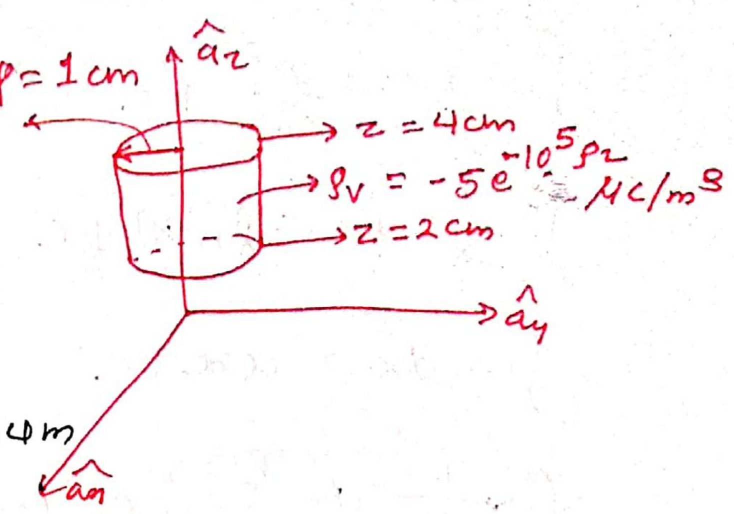
i.e $\rho_v = \lim_{\Delta V \to 0} \dfrac{\Delta Q}{\Delta V}$ $\qquad \rho_v = \dfrac{Q}{V}$

or, $\Delta Q = \rho_v \Delta V$

$\therefore$ Total charge, $Q = \int_{vol.} \Delta Q = \int_{vol.} \rho_v \, dv$

Find the total charge within the volume shown below.

Sol.^{n} We have, the ranges of three parameters in cylindrical co-ordinate system are



$0 \le \rho \le 1\,cm \Rightarrow 0 \le \rho \le 0.01\,m$

$0 \le \phi \le 2\pi$

&: $2\,cm \le z \le 4\,cm \Rightarrow 0.02 \le z \le 0.04\,m$

Now, we can evaluate the total charge within the volume as:

$$Q = \int_{vol.} \rho_v \, dv$$

$$= \int_{vol.} -5e^{-10^5 \rho z} \times 10^{-6} \, dv \qquad [1\mu C = 10^{-6} c]$$

$$= \int_{\rho=0}^{0.01} \int_{\phi=0}^{2\pi} \int_{z=0.02}^{0.04} -5e^{-10^5 \rho z} \times 10^{-6} \, d\rho \, d\phi \, dz$$

$$= \int_{\rho=0}^{0.01} \int_{z=0.02}^{0.04} -5e^{-10^5 \rho z} \times 10^{-6} (2\pi) \, \rho \, d\rho \, dz$$

$$= -10\pi \times 10^{-6} \int_{\rho=0}^{0.01} \int_{z=0.02}^{0.04} e^{-10^5 \rho z} \, \rho \, d\rho \, dz$$

$$= -\pi \times 10^{-5} \int_{\rho=0}^{0.01} \left(\frac{e^{-10^5 \rho z}}{-10^5 \cancel{\rho}} \right)_{0.02}^{0.04} \cancel{\rho} \, d\rho$$

$$= 10^{-10}\pi \int_{\rho=0}^{0.01} \left(e^{-10^5\rho \cdot 0.04} - e^{-10^5\rho\, 0.02}\right) d\rho$$

$$= 10^{-10}\pi \int_{\rho=0}^{0.01} \left(e^{-4000\rho} - e^{-2000\rho}\right) d\rho$$

$$= 10^{-10}\pi \left\{ \left.\frac{e^{-4000\rho}}{(-4000)}\right|_0^{0.01} - \left.\frac{e^{-2000\rho}}{(-2000)}\right|_0^{0.01} \right\}$$

$$= 10^{-10}\pi \left\{ \frac{e^{-40} - e^{0}}{(-4000)} - \frac{e^{-20} - e^{0}}{(-2000)} \right\}$$

$$= -\frac{10^{-13}}{4}\pi \cdot (e^{-40} - 1) + \frac{10^{-13}}{2}\pi (e^{-20} - 1)$$

$$\approx -\frac{10^{-13}}{4}\pi(-1) + \frac{10^{-13}}{2}\pi(-1)$$

$$= 10^{-13}\pi \left[\frac{1}{4} - \frac{1}{2}\right]$$

$$= -\frac{10^{-13}}{4}\pi$$

$$= -\frac{\pi}{40}\ pC$$

$$= -0.0785\ pC$$

$\therefore$ Total charge enclosed in a given volume is

$$Q = -0.0785\ pC.$$

Integration

$$\int x\,dx = \frac{x^2}{2} + c$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int a\,dx = ax + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2}\ln(x^2+a^2) + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int e^x\,dx = e^x + c$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + c$$

$$\int u\,dv = uv - \int v\,du$$

$u(x)$ & $v(x)$ are differentiable functions.

Differentiation

$$\frac{d(a)}{dx} = 0$$

$$\frac{dx}{dx} = 1$$

$$\frac{d(x^2)}{dx} = 2x$$

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d(e^x)}{dx} = e^x$$

$$\frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(x^n)}{dx} = n \cdot x^{n-1}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

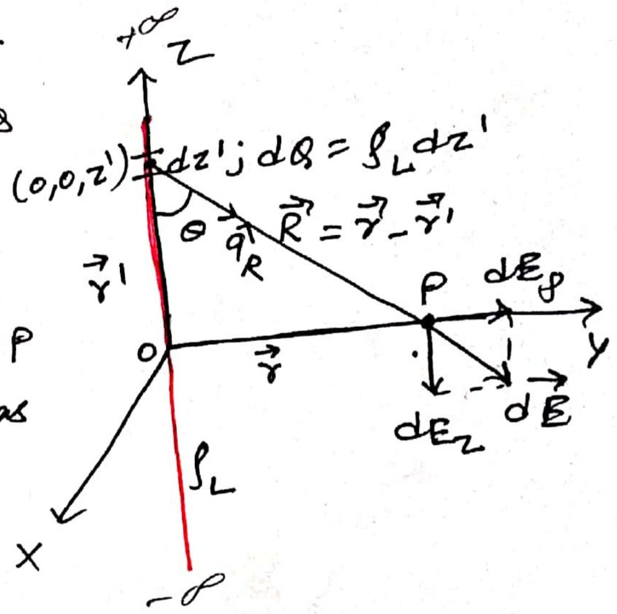
Electric field intensity due to a line charge

Let us Assume a straight-line charge extending along the z-axis in a cylindrical co-ordinate system from $-\infty$ to ∞, as shown in figure. We need to evaluate the electric field intensity $\vec{E}$ at any and every point resulting from a uniform line charge density ρ_L.

If we vary ϕ keeping ρ & z constant the field does not change that means azimuthal symmetry is present.

Again, if we keep ϕ & ρ constant and vary z, the field at arbitrary point P will not be changed. This means it has axial symmetry is present.

But if we change ρ keeping z & ϕ constant the field value changes accordingly. So, it says that the field changes only with the variation of ρ.



Now, from the figure we can see that the field has two components $\vec{E}_z$ & $\vec{E}_\rho$. Since, the line charge is symmetric about origin; the field due to +ve z-direction cancels the field due to -ve z-direction. So, we will have only $\vec{E}_\rho$ component.

Now, let us choose a point P(0,y,0) on the y-axis at which the field is to be determined.

$\therefore$ The incremental field due to the incremental charge $dQ = \rho_L dz'$ at point P is

$$d\vec{E} = \frac{\rho_L dz' (\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}\,'|^3}$$

where, $\vec{r} = y\hat{a}_y = \rho\hat{a}_\rho$

$\vec{r}\,' = z'\hat{a}_z$.

and $\vec{r} - \vec{r'} = \rho \hat{a}_\rho - z'\hat{a}_z$

Therefore, $$d\vec{E} = \frac{\rho_L dz' (\rho \hat{a}_\rho + z'\hat{a}_z)}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

since, only $\vec{E}_\rho$ component is present we may simplify;

$$dE_\rho = \frac{\rho \rho_L dz'}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

and

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho \rho_L dz'}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

Now, Let us assume $z' = \rho \cot\theta$, we have

$$E_\rho = \frac{\rho_L}{4\pi\epsilon_0} \rho \left(\frac{1}{\rho^2} \cdot \frac{z'}{\sqrt{\rho^2 + z'^2}} \right)_{-\infty}^{\infty}$$

and $$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

or, Finally $$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho} \hat{a}_\rho}$$

$z' = \rho \cot\theta$

$z' = -\infty, \theta = \pi$

$z' = \infty, \theta = 0^\circ$

$dz' = -\rho \operatorname{cosec}^2\theta \, d\theta$

$$E_\rho = \frac{\rho^2 \rho_L}{4\pi\epsilon_0} \int_\pi^0 \frac{-\operatorname{cosec}^2\theta \, d\theta}{\rho^3 (1+\cot^2\theta)^{3/2}}$$

$$= \frac{-\rho_L}{4\pi\epsilon_0 \rho} \int_\pi^0 \frac{\operatorname{cosec}^2\theta \, d\theta}{\operatorname{cosec}^3\theta}$$

$$= \frac{-\rho_L}{4\pi\epsilon_0 \rho} \int_\pi^0 \sin\theta \, d\theta$$

$$= \frac{-\rho_L}{4\pi\epsilon_0 \rho} (-\cos\theta) \Big|_\pi^0$$

$$= \frac{\rho_L [\cos 0 - \cos\pi]}{4\pi\epsilon_0 \rho}$$

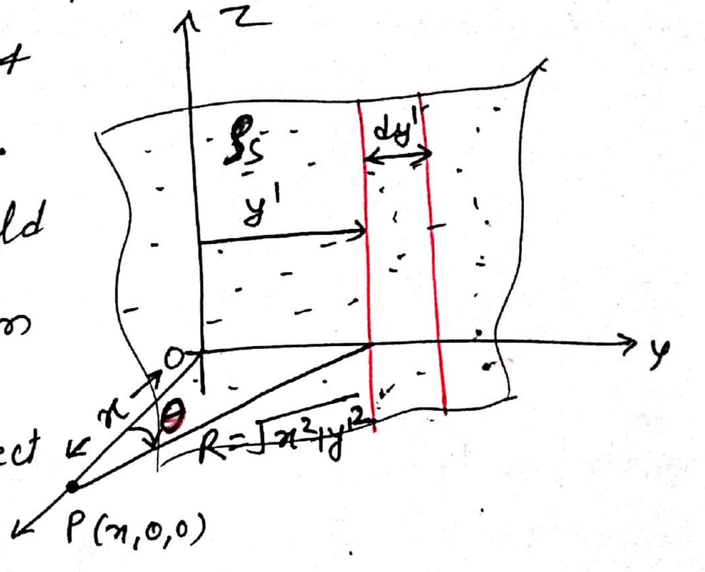
$$= \frac{\rho_L \cdot 2}{4\pi\epsilon_0 \rho}$$

$$E_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}$$

Here, we see that the field falls off inversely with the distance to the charged line, as compared to point charge, where the field decreased with the square of the distance.

Electric field intensity due to sheet charge

Let us consider a sheet of charge, having uniform charge density ρ_s C/m^2 is placed in the yz - plane as shown in figure. As from the symmetry, the field due to the sheet charge at any arbitrary point P does not vary with the variation of z & y. The y & z components of the field at point $P(x,0,0)$ arising from differential elements of charge symmetrically located with respect to the point P will cancel. Hence, only $\vec{E}_x$ component is present.



Let us divide the sheet into differential - width strips, so that we can use the infinite line charge field expression. Here,

the line charge density $\rho_L = \rho_s \, dy'$, and the distance from this line charge to the general point P on the x axis is $R = \sqrt{x^2+y'^2}$.

Then, the field E_x due to differential width strip is

$$dE_x = \frac{\rho_s \, dy'}{2\pi\varepsilon_0 \sqrt{x^2+y^2}} \cdot \cos\theta = \frac{\rho_s}{2\pi\varepsilon_0} \cdot \frac{x}{(x^2+y^2)} \cdot dy'$$

$[dE_x = dE \cos\theta]$

Now, integrating effects of all the strips.

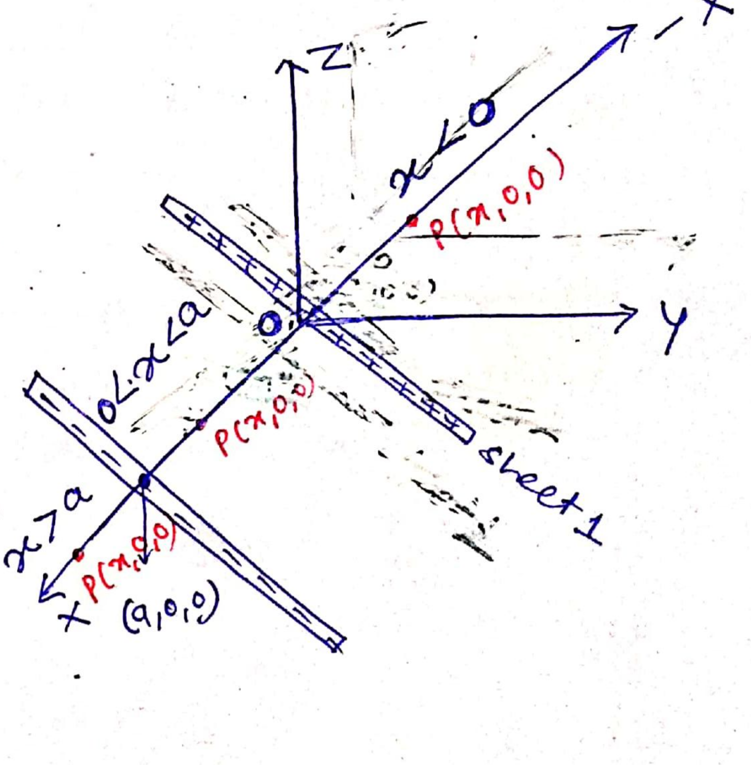
$$E_x = \frac{\rho_s}{2\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{x \, dy'}{x^2+y^2} = \frac{\rho_s}{2\pi\varepsilon_0} \tan^{-1}\frac{y}{x} \Big|_{-\infty}^{\infty} = \frac{\rho_s}{2\varepsilon_0}$$

If the point P were chosen at -ve x-axis the field will be $E_x = -\dfrac{\rho_s}{2\varepsilon_0}$

For the direction we use the normal vector perpendicular to the sheet containing the charge and is directed outward or away from the sheet.

$$\therefore \boxed{\vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_N}$$

If a second infinite sheet of charge, having a negative charge density $-\rho_s$, is located in the plane $x = a$, we may find the total field by adding the contribution of each sheet.



In the region, $x > a$

$$\vec{E}_+ = \frac{\rho_s}{2\varepsilon_0}\hat{a}_x$$

$$\vec{E}_- = -\frac{\rho_s}{2\varepsilon_0}\hat{a}_x$$

$$\therefore \vec{E} = \vec{E}_+ + \vec{E}_- = 0$$

and for $x < 0$,

$$\vec{E}_+ = -\frac{\rho_s}{2\varepsilon_0}\hat{a}_x.$$

$$\vec{E}_- = \frac{-(-\rho_s)}{2\varepsilon_0}\hat{a}_x = \frac{\rho_s}{2\varepsilon_0}\hat{a}_x$$

$$\therefore \vec{E} = \vec{E}_+ + \vec{E}_- = 0$$

and when $0 < x < a$,

$$\vec{E}_+ = \frac{\rho_s}{2\epsilon_0}\hat{a}_x$$

$$\vec{E}_- = -\frac{(-\rho_s)}{2\epsilon_0}\hat{a}_x = \frac{\rho_s}{2\epsilon_0}\hat{a}_x$$

$$\therefore \vec{E} = \vec{E}_+ + \vec{E}_- = \left[\frac{\rho_s}{2\epsilon_0} + \frac{\rho_s}{2\epsilon_0}\right]\hat{a}_x$$

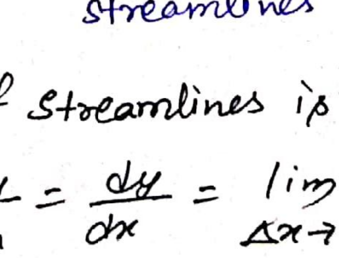
or, $\boxed{\vec{E} = \frac{\rho_s}{\epsilon_0}\hat{a}_x}$

It is the field betn the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation.

Equations of Streamlines (Flux Lines/ Direction Lines)

Streamlines are those lines which are everywhere tangent to $\vec{E}$ around the charge. Streamlines are also called flux lines or direction lines.

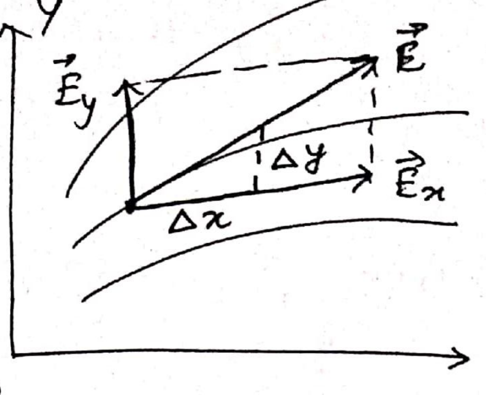
A small positive test charge placed at anypoint in this field and free to move would accelerate in the dirn of the streamline passing through that point.



streamlines

The equation of streamlines is obtained by solving the differential equation $\frac{E_y}{E_x} = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ obtained from the geometry.

Let us consider the field of the uniform line charge with line charge density $\rho_L = 2\pi\varepsilon_0$



then,

$$\vec{E} = \frac{1}{\rho}\hat{a}_\rho \quad \left[\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\cdot\hat{a}_\rho = \frac{2\pi\varepsilon_0}{2\pi\varepsilon_0\rho}\cdot\hat{a}_\rho = \frac{1}{\rho}\hat{a}_\rho\right]$$

In rectangular Co-ordinates; [set $E_z = 0$]

$$\vec{E} = \frac{x}{x^2+y^2}\hat{a}_x + \frac{y}{x^2+y^2}\hat{a}_y$$

$E_x = \vec{E}\cdot\hat{a}_x = \frac{1}{\rho}\hat{a}_\rho\cdot\hat{a}_x$

$= \frac{1}{\rho}\cos\phi = \frac{x}{\rho^2}$

$\therefore E_x = \frac{x}{x^2+y^2}$

& $E_y = \vec{E}\cdot\hat{a}_y = \frac{y}{\rho^2} = \frac{y}{x^2+y^2}$

Thus, from differential Equation

$$\boxed{\frac{dy}{dx} = \frac{E_y}{E_x}} = \frac{y}{x}$$

or, $\frac{dy}{dx} = \frac{y}{x}$

or, $\frac{dy}{y} = \frac{dx}{x}$

or, $\ln y = \ln x + C_1$ or $\ln x + \ln C$

from which the equations of streamlines are obtained as:

$$y = Cx$$

If we want to find the Eq^n of one particular stream-line, say one passing through $P(-2, 7, 10)$, we substitute the co-ordinates of that point into the streamline Eq^n and evaluate C. i.e $7 = C(-2)$

$\therefore C = -3.5$

So, $y = -3.5x$ is the required streamlines passing through $P(-2, 7, 10)$.

Q. Find the equation of that streamline that passes through the point $P(1,4,-2)$ in the field $\vec{E} = -\frac{8x}{y}\hat{a}_x + \frac{4x^2}{y^2}\hat{a}_y$

Sol.:

Given, $\vec{E} = -\frac{8x}{y}\hat{a}_x + \frac{4x^2}{y^2}\hat{a}_y$

where, $E_x = \frac{-8x}{y}$ & $E_y = \frac{4x^2}{y^2}$

Now, we have from differential equation

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{4x^2/y^2}{-8x/y} = -\frac{x}{2y}$$

or, $2y\,dy = -x\,dx$

Integrating bothsides

$$\frac{2y^2}{2} = -\frac{x^2}{2} + \frac{c}{2}$$

or, $y^2 = \frac{-x^2}{2} + \frac{c}{2}$

Since, this streamline passes through $P(1,4,-2)$

$$16 = -\frac{1}{2} + \frac{c}{2}$$

or, $32 = -1 + c$

or, $c = 33$

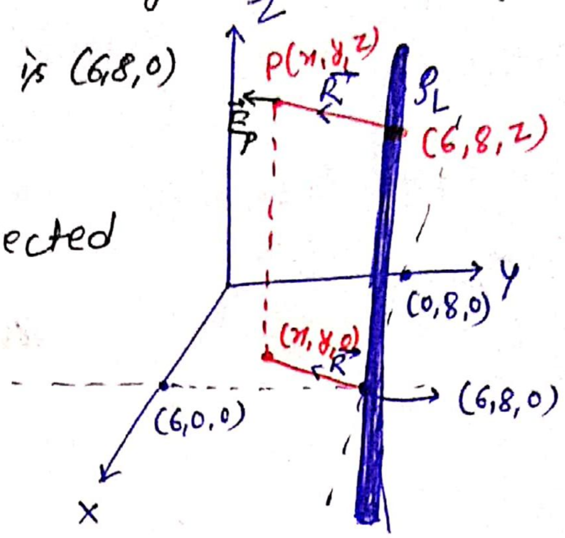
$\therefore$ The Eqn. of streamline is

$$y^2 = -\frac{x^2}{2} + \frac{33}{2}$$

or, $x^2 + 2y^2 = 33$ //

Qn: A uniform line charge having charge density ρ_L is placed at $x=6$ and $y=8$. Find the electric field intensity due to this line charge at point $P(x, y, z)$.

Soln: Here, we have ρ_L = line charge density of a line charge at $x=6$ & $y=8$. So, the point in XY-plane is $(6,8,0)$ where the line charge is passed.



Also, the point $(x, y, 0)$ is the projected point of $P(x, y, z)$ in XY-plane.

Now, the distance between the line charge and the point $P(x, y, z)$ can be calculated by calculating the distance betn $(6,8,0)$ & $(x, y, 0)$ as:

$$|\vec{R}| = R = \sqrt{(x-6)^2 + (y-8)^2}$$

$$\& \quad \vec{R} = (x-6)\hat{a}_x + (y-8)\hat{a}_y$$

Again, the electric field intensity at point P is

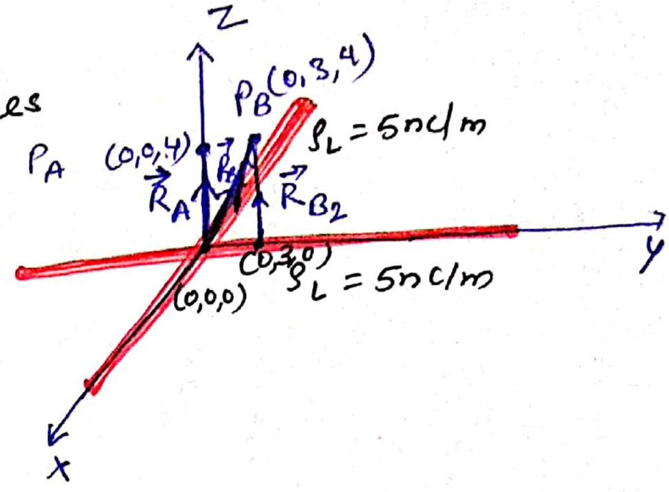
$$\vec{E}_P = \frac{\rho_L}{2\pi\varepsilon_0 R} \cdot \hat{a}_R = \frac{\rho_L}{2\pi\varepsilon_0} \cdot \frac{1}{\sqrt{(x-6)^2 + (y-8)^2}} \cdot \hat{a}_R$$

where, $$\hat{a}_R = \frac{(x-6)\hat{a}_x + (y-8)\hat{a}_y}{\sqrt{(x-6)^2 + (y-8)^2}}$$

$$\therefore \quad \vec{E}_P = \frac{\rho_L\left[(x-6)\hat{a}_x + (y-8)\hat{a}_y\right]}{2\pi\varepsilon_0\left[(x-6)^2 + (y-8)^2\right]}$$ where, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

Qn: Infinite uniform line charges of 5nC/m lie along the +ve and −ve X and Y axes in free space. Find $\vec{E}$ at ⓐ $P_A(0,0,4)$ and ⓑ $P_B(0,3,4)$

Sol.ⁿ Here, the infinite uniform line charges of charge density $\rho_L = 5nC/m$ lie along the +ve and −ve X & Y axes as shown in figure.



Now, ⓐ for the $\vec{E}$ at $P_A(0,0,4)$ we need to calculate the distance vector $\vec{R}_A$ as:

$$\vec{R}_A = (0-0)\hat{a}_x + (0-0)\hat{a}_y + (4-0)\hat{a}_z$$
$$= 4\hat{a}_z$$

& $R_A = |\vec{R}_A| = \sqrt{4^2} = 4$ also, $\hat{a}_{RA} = \hat{a}_z$

So, $\vec{E}$ at $P_A(0,0,4)$ due to line charge at X-axis is

$$\vec{E}_{A_x} = \frac{\rho_L}{2\pi\epsilon_0 R_A}\cdot\hat{a}_{RA} = \frac{5\times10^{-9}}{2\pi\times8.854\times10^{-12}\times4}\cdot\hat{a}_{RA} = \frac{5\times10^{-9}}{2\pi\times8.854\times10^{-12}}\cdot\frac{\hat{a}_z}{4}$$

$$= \frac{5\times10^{-9+12}}{8\pi\times8.854}\cdot\hat{a}_z$$

$$\therefore \vec{E}_{A_x} = \frac{5000}{70.832\pi}\hat{a}_z \text{ V/m}.$$

Again, the $\vec{E}$ at $P_A(0,0,4)$ due to line charge at Y-axis is same as due to line charge at X-axis.

$$\therefore \vec{E}_{A_y} = \vec{E}_{A_x} = \frac{5000}{70.832\pi}\hat{a}_z \text{ V/m}.$$

Hence, the total field intensity at $P_A(0,0,4)$ due to line charge at x & y axes is

$$\vec{E} = \vec{E}_{Ax} + \vec{E}_{Ay} = 2 \times \frac{5000}{70.832\pi} \cdot \hat{a}_z \text{ V/m}.$$

$$\therefore \vec{E} = 45 \hat{a}_z \text{ V/m}.$$

Again, (b) For the $\vec{E}$ at $P_B(0,3,4)$ due to the line charge at x-axis

is $$\vec{E}_{Bx} = \frac{\rho_L}{2\pi\varepsilon_0 R_{Bx}} \cdot \hat{a}_{RBx}$$

where, $$\vec{R}_{Bx} = (0-0)\hat{a}_x + (3-0)\hat{a}_y + (4-0)\hat{a}_z$$
$$= 3\hat{a}_y + 4\hat{a}_z$$

$$R_{Bx} = |\vec{R}_{Bx}| = \sqrt{3^2+4^2} = 5$$

& $$\hat{a}_{RBx} = \frac{\vec{R}_{Bx}}{|\vec{R}_{Bx}|} = \frac{3\hat{a}_y + 4\hat{a}_z}{5} = 3/5\,\hat{a}_y + 4/5\,\hat{a}_z$$

$$\therefore \vec{E}_{Bx} = \frac{5\times10^{-9}}{2\pi\times8.854\times10^{-12}} \cdot \frac{(3\hat{a}_y + 4\hat{a}_z)}{25}$$

$$= (10.78\,\hat{a}_y + 14.38\,\hat{a}_z) \text{ V/m}$$

Again, For the $\vec{E}$ at $P_B(0,3,4)$ due to the line charge at y-axis is

$$\vec{E}_{By} = \frac{\rho_L}{2\pi\varepsilon_0 R_{By}} \cdot \hat{a}_{RBy}$$

where, $$\vec{R}_{By} = (0-0)\hat{a}_x + (3-3)\hat{a}_y + (4-0)\hat{a}_z = 4\hat{a}_z$$

$$R_{By} = |\vec{R}_{By}| = 4$$

& $$\hat{a}_{RBy} = \frac{\vec{R}_{By}}{|\vec{R}_{By}|} = \frac{4\hat{a}_z}{4} = \hat{a}_z$$

$$\therefore \vec{E}_{B_y} = \frac{5\times10^{-9}}{2\pi\times8.854\times10^{-12}} \cdot \frac{\hat{a}_z}{4}$$

$$= 22.47\,\hat{a}_z \text{ V/m}.$$

$\therefore$ Total $\vec{E}$ at $P_B(0,3,4)$ due to line charges at X & Y axes

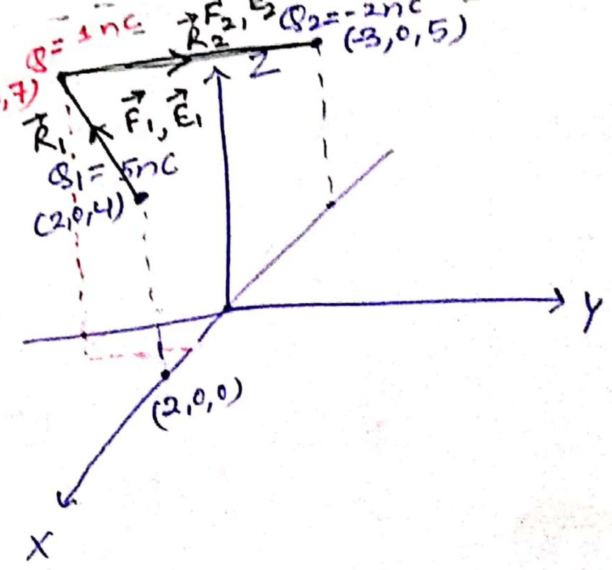
is $\vec{E} = \vec{E}_{B_x} + \vec{E}_{B_y} = (10.78\,\hat{a}_y + 14.38\,\hat{a}_z) + 22.47\hat{a}_z$

$$= 10.78\,\hat{a}_y + 36.9\,\hat{a}_z \text{ V/m}.$$

Qn: Point charges 5nC and −2nC are located at (2,0,4) and (−3,0,5) respectively.

ⓐ Determine the force on a 1 nC point charge located at (1, −3, 7) $-1.004\hat{a}_x - 1.284\hat{a}_y + 1.4\hat{a}_z$ nN

ⓑ Find the electric field $\vec{E}$ at (1, −3, 7). $-1.004\hat{a}_x - 1.284\hat{a}_y + 1.4\hat{a}_z$ V/m.



Sol.ⁿ Here, two charges $Q_1 = 5nC$ and $Q_2 = -2nC$ are located at (2,0,4) and (−3,0,5) respectively.

A unit charge $Q = 1nC$ is located at (1,−3,7).

Now, the force due to $Q_1 = 5nC$ at (2,0,4) to $Q = 1nC$ at (1,−3,7)

is $$\vec{F}_1 = \frac{Q_1 \cdot Q}{4\pi\varepsilon_0 R_1^2} \cdot \hat{a}_{R_1}$$

where, $\vec{R}_1 = (1-2)\hat{a}_x + (-3-0)\hat{a}_y + (7-4)\hat{a}_z$

$$= -1\hat{a}_x - 3\hat{a}_y + 3\hat{a}_z$$

$$R_1 = |\vec{R}_1| = \sqrt{(-1)^2 + (-3)^2 + 3^2} = \sqrt{19} = 4.36$$

$$\& \ \hat{a}_{R_1} = \frac{\vec{R}_1}{|\vec{R}_1|} = \frac{-1}{4.36}\hat{a}_x - \frac{3}{4.36}\hat{a}_y + \frac{3}{4.36}\hat{a}_z$$

$$\therefore \vec{F}_1 = \frac{5\times 1\times 10^{-9}\times 10^{-9}}{4\pi \times \left(\frac{10^{-9}}{36\pi}\right)} \cdot \frac{(-\hat{a}_x - 3\hat{a}_y + 3\hat{a}_z)}{(4.36)^3}$$

$$= \frac{45\times 10^{-9}}{82.88} \cdot (-\hat{a}_x - 3\hat{a}_y + 3\hat{a}_z)$$

$$= (-0.543\hat{a}_x - 1.63\,\hat{a}_y + 1.63\,\hat{a}_z)\ \text{nN}$$

Also, the force due to $Q_2 = -2\,\text{nC}$ at $(-3,0,5)$ to $Q = 1\,\text{nC}$ at $(1,-3,7)$ is

$$\vec{F}_2 = \frac{Q_2 \cdot Q}{4\pi\varepsilon_0 R_2^2} \cdot \hat{a}_{R_2}$$

where, $\vec{R}_2 = (1+3)\,\hat{a}_x + (-3-0)\,\hat{a}_y + (7-5)\,\hat{a}_z$

$$= 4\hat{a}_x - 3\hat{a}_y + 2\hat{a}_z$$

$$R_2 = |\vec{R}_2| = \sqrt{4^2 + (-3)^2 + 2^2} = \sqrt{29} = 5.385$$

& $\hat{a}_{R_2} = \frac{\vec{R}_2}{|\vec{R}_2|} = \frac{4}{5.385}\hat{a}_x$

$$\therefore \vec{F}_2 = \frac{-2\times 1\times 10^{-9}\times 10^{-9}}{4\pi \times \left(\frac{10^{-9}}{36\pi}\right)} \cdot \frac{(4\hat{a}_x - 3\hat{a}_y + 2\hat{a}_z)}{5.385^3}$$

$$= \frac{-18}{5.385^3} \cdot (4\hat{a}_x - 3\hat{a}_y + 2\hat{a}_z)$$

$$= -0.115\,(4\hat{a}_x - 3\hat{a}_y + 2\hat{a}_z)$$

$$= (-0.461\,\hat{a}_x + 0.346\,\hat{a}_y - 0.2305\,\hat{a}_z)\ \text{nN}$$

∴ Total force at $(1,-3,7)$ is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = -0.543\hat{a}_x - 1.63\hat{a}_y + 1.63\hat{a}_z - 0.461\hat{a}_x$$
$$+ 0.346\hat{a}_y - 0.2305\hat{a}_z$$
$$= (-1.004\hat{a}_x - 1.284\hat{a}_y + 1.4\hat{a}_z)\ \text{nN}$$

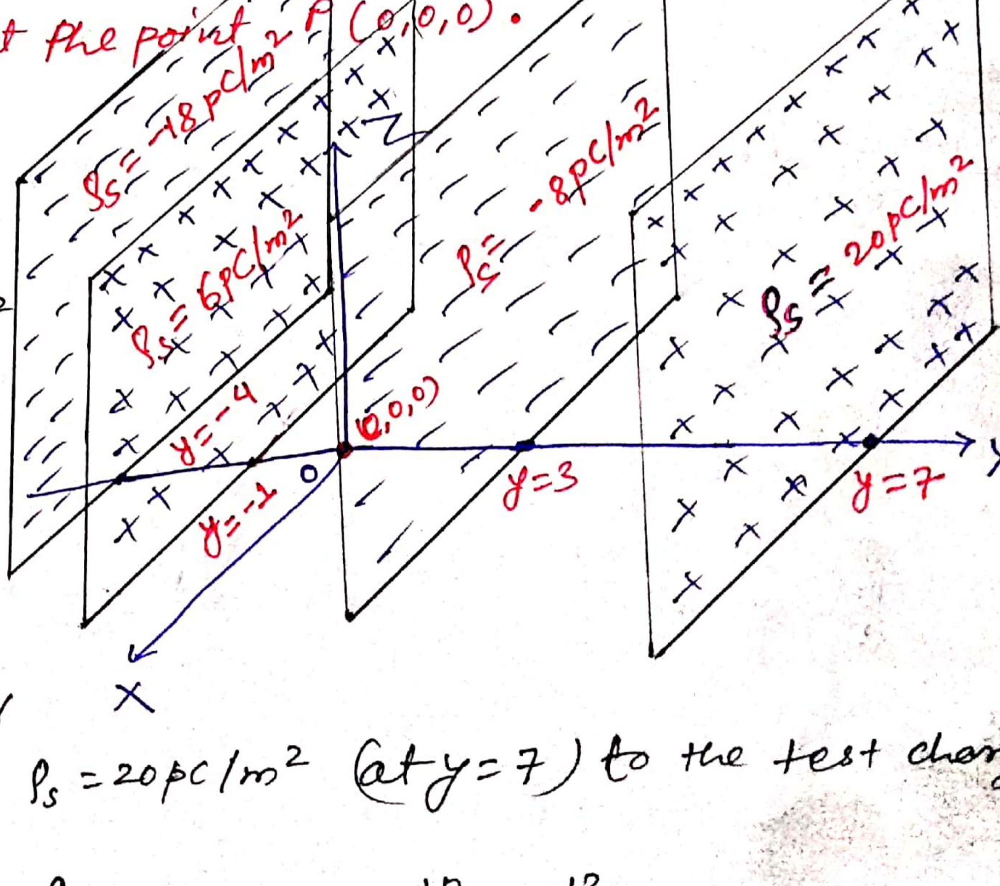
Now, (b) $\vec{E}$ at $(1,-3,7)$ is

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{(-1.004\hat{a}_x - 1.284\hat{a}_y + 1.4\hat{a}_z)\times 10^{-9}}{1\times 10^{-9}}$$

$$\therefore \vec{E} = (-1.004\hat{a}_x - 1.284\hat{a}_y + 1.4\hat{a}_z)\ \text{V/m}.$$

Qn: Four infinite uniform sheets of charge are located as follows: $20\ pC/m^2$ at $y=7$, $-8\ pC/m^2$ at $y=3$, $6\ pC/m^2$ at $y=-1$ and $-18\ pC/m^2$ at $y=-4$. Find $\vec{E}$ at the point $P(0,0,0)$.

Sol.: Here, the sheet charges of charge densities $20\ pC/m^2$, $-8\ pC/m^2$, $6\ pC/m^2$ and $-18\ pC/m^2$ are located at $y=7$, $y=3$, $y=-1$ & $y=-4$ respectively.



then, the electric field intensity due to $\rho_s = 20\ pC/m^2$ (at $y=7$) to the test charge at $P(0,0,0)$ is

$$\vec{E}_{20} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_N = \frac{20\times 10^{-12}}{2\times\left(\frac{10^{-9}}{36\pi}\right)}\cdot(-\hat{a}_y)$$

$\therefore \vec{E}_{20} = -360\pi \times 10^{-3}\, \hat{a}_y$ V/m.

Again,

Electric field intensity due to $\rho_s = -8\,pC/m^2$ (at $y=3$) to the test charge at $P(0,0,0)$ is

$$\vec{E}_{-8} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_N = \frac{\overset{4}{\cancel{-8}} \times 10^{-12}}{\frac{\cancel{2} \times 10^{-9}}{36\pi}} \cdot (-\hat{a}_y)$$

$\therefore \vec{E}_{-8} = 144\pi \times 10^{-3}\, \hat{a}_y$ V/m

Also,

Eletric field intensity due to $\rho_s = 6\,pC/m^2$ (at $y=-1$) to the test charge at $P(0,0,0)$ is

$$\vec{E}_6 = \frac{\rho_s}{2\varepsilon_0}\hat{a}_N = \frac{\overset{3}{\cancel{6}} \times 10^{-12}}{\frac{\cancel{2} \times 10^{-9}}{36\pi}}\, \hat{a}_y$$

$\therefore \vec{E}_6 = 108\pi \times 10^{-3}\, \hat{a}_y$ V/m

Similary, Electric field intensity due to $\rho_s = -18\,pC/m^2$ (at $y=-4$) to the test charge at $P(0,0,0)$ is

$$\vec{E}_{-18} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_N = \frac{-\overset{9}{\cancel{18}} \times 10^{-12}}{\frac{\cancel{2} \times 10^{-9}}{36\pi}} \cdot \hat{a}_y$$

$\therefore \vec{E}_{-18} = -324\pi \times 10^{-3}\, \hat{a}_y$ V/m

Finally,

Total electric field intensity at $P(0,0,0)$ is

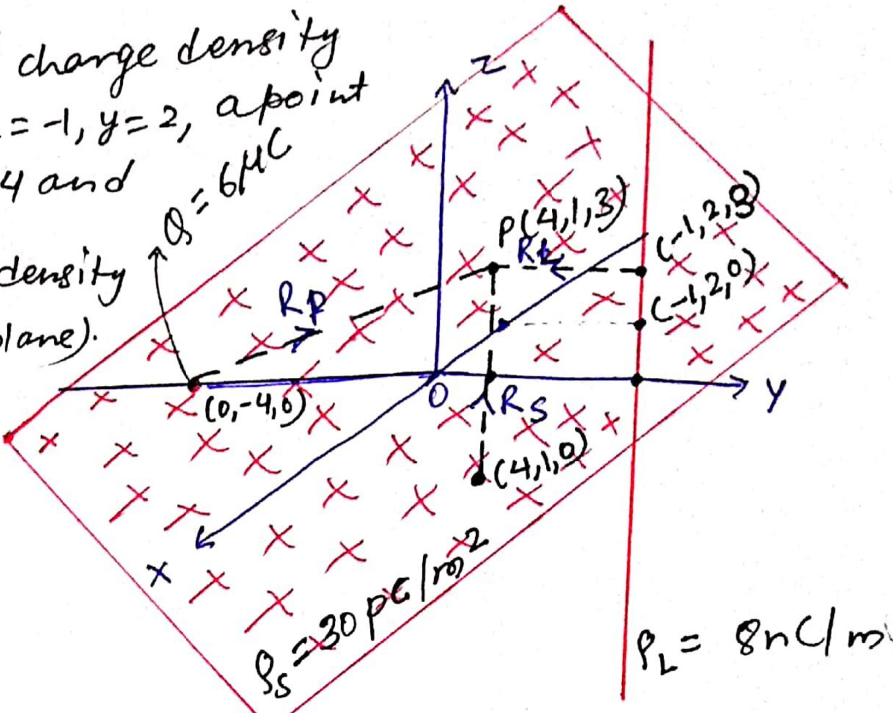
$$\vec{E} = \vec{E}_{20} + \vec{E}_{-8} + \vec{E}_6 + \vec{E}_{-18}$$

$$= (-360\pi \times 10^{-3} + 144\pi \times 10^{-3} + 108\pi \times 10^{-3} - 324\pi \times 10^{-3})\, \hat{a}_y \text{ V/m}$$

$\therefore \vec{E} = -1.187\, \hat{a}_y$ V/m. / $-1.357\, \hat{a}_y$ V/m

Qn. A line charge of 8nC/m is located at $x=-1$, $y=2$, a point charge of 6μC at $y=-4$ and a surface charge of 30pC/m² at $z=0$. Find $\vec{E}$ at $P(4,1,3)$.

Sol.ⁿ Here, a line charge of charge density $\rho_L = 8nC/m$ is located at $x=-1, y=2$, a point charge of $Q = 6\mu C$ at $y=-4$ and a surface charge of charge density $\rho_s = 30pC/m^2$ at $z=0$ (i.e. xy-plane).



then, The electric field intensity at point P(4,1,3) due to a line charge is

$$\vec{E}_L = \frac{\rho_L}{2\pi\epsilon_0 R_L} \cdot \hat{a}_{R_L}$$

where,
$$\vec{R}_L = [(4-(-1))\hat{a}_x + (1-2)\hat{a}_y + (3-3)\hat{a}_z]$$
$$= 5\hat{a}_x - \hat{a}_y.$$
& $R_L = |\vec{R}_L| = \sqrt{5^2+1^2} = \sqrt{26}$, $\hat{a}_{R_L} = \dfrac{5\hat{a}_x - \hat{a}_y}{\sqrt{26}}$

$$\therefore \vec{E}_L = \frac{8\times10^{-9}[5\hat{a}_x - \hat{a}_y]}{2\pi \cdot \frac{10^{-9}}{36\pi} \cdot \sqrt{26}\cdot\sqrt{26}}$$
$$= \frac{8\times18}{26}[5\hat{a}_x - \hat{a}_y]$$
$$= 27.69\hat{a}_x - 5.54\hat{a}_y \text{ V/m}$$

Again, the electric field intensity due to point charge is

$$\vec{E}_P = \frac{Q}{4\pi\epsilon_0 R_P^2}\hat{a}_{R_P}$$

where, $\vec{R}_P = (4-0)\hat{a}_x + (1+4)\hat{a}_y + (3-0)\hat{a}_z$
$= 4\hat{a}_x + 5\hat{a}_y + 3\hat{a}_z$

$$R_P = |\vec{R}_P| = \sqrt{16+25+9} = \sqrt{50} = 5\sqrt{2}$$

& $$\hat{a}_{R_P} = \frac{4\hat{a}_x + 5\hat{a}_y + 3\hat{a}_z}{5\sqrt{2}}$$

$$\therefore \vec{E}_P = \frac{6\times10^{-6}(4\hat{a}_x + 5\hat{a}_y + 3\hat{a}_z)}{4\pi \times \frac{10^{-9}}{36\pi} \times (5\sqrt{2})^3}$$

$$= \frac{9\times 6 \times 10^{3}}{353.55}(4\hat{a}_x + 5\hat{a}_y + 3\hat{a}_z)$$

$$= 152.74\,(4\hat{a}_x + 5\hat{a}_y + 3\hat{a}_z)$$

$$= 610.95\,\hat{a}_x + 763.68\,\hat{a}_y + 458.21\,\hat{a}_z \text{ V/m}.$$

And,

the electric field intensity due to sheet charge is

$$\vec{E}_s = \frac{\rho_s}{2\varepsilon_0}\hat{a}_{R_s}$$

where, $\vec{R}_s = (4-4)\hat{a}_x + (1-1)\hat{a}_y + (3-0)\hat{a}_z$

$$= 3\hat{a}_z$$

$$R_s = |\vec{R}_s| = 3$$

& $\hat{a}_{R_s} = \frac{3\hat{a}_z}{3} = \hat{a}_z$ [or directly $\hat{a}_{R_s} = \hat{a}_z$]

$$\therefore \vec{E}_s = \frac{30\times10^{-12}}{2\times \frac{10^{-9}}{36\pi}}(\hat{a}_z)$$

$$= 30 \times 18\pi \times 10^{-3}\,\hat{a}_z$$

$$= 1.696\,\hat{a}_z \text{ V/m}.$$

$\therefore$ Total electric field intensity at $P(4,1,3)$ is

$$\vec{E} = \vec{E}_L + \vec{E}_P + \vec{E}_s = 638.64\,\hat{a}_x + 758.14\,\hat{a}_y + 459.91\,\hat{a}_z \text{ V/m}$$

Electric Flux Density

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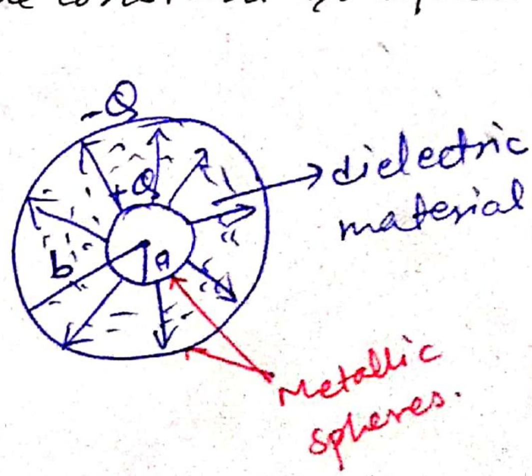
Faraday's Law

In 1837 Faraday performed an experiment about the static charge. He made the concentric metallic spheres one with small diameter and put the dielectric material in betⁿ them. He charged the inner sphere with a known positive charge & the outersphere was discharged by connecting it to ground. He then separated the outer sphere to measure the induced -ve charge on it. He found that the total charge on the outer sphere was equal in magnitude to the original charge placed on the inner sphere and that was true regardless of the dielectric material in betⁿ the two spheres. He concluded that there was some sort of "displacement" from inner sphere to the outer which was independent of the medium; which is referred to as displacement or displacement flux or simply electric flux.

Faraday's experiments also showed that larger the charge in inner sphere; larger will be the induced charge in outer sphere; leading to direct proportionality betⁿ the electric flux and the charge on the inner sphere. The constant of proportionality is different for different system of units involved. In SI unit the constant is equal to unity.

i.e ψ (flux) $\propto Q$

or, $\psi = KQ$



where, $K = 1$ for SI unit

$\therefore \boxed{\psi = Q}$

The electric flux is measured in Coulombs.

Consider an inner sphere of radius a and an outer sphere of radius b, with charges of $+Q$ and $-Q$ respectively. The paths of electric flux ψ extending from the inner sphere to the outer sphere are indicated by symmetrically distributed streamlines drawn radially from one sphere to the other.

At the surface of inner sphere, ψ coulombs of electric flux are produced by the charge $Q = (\psi)$ coulombs distributed uniformly over the surface of area $4\pi a^2$ m². The density of flux at this surface is $\psi/4\pi a^2$ or $Q/4\pi a^2$ C/m² and this quantity is called electric flux density; measured in coulombs per square meter (or lines per square meter for each line due to one coulomb). It is represented by $\vec{D}$.

The direction of $\vec{D}$ at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

∴ Electric flux density of inner and outer spheres are

$$\vec{D}_{inner} = \frac{Q}{4\pi a^2}\hat{a}_r \quad \& \quad \vec{D}_{outer} = \frac{Q}{4\pi b^2}\hat{a}_r$$

and in betⁿ the two spheres with radius r the electric flux density will be

$$\vec{D}_{bet^n} = \frac{Q}{4\pi r^2}\hat{a}_r$$

If we assume inner sphere (as very small) as point charge then the electric flux density at inbetⁿ spheres with charge Q at inner sphere will be same as:

$$\vec{D}_{bet^n} = \frac{Q}{4\pi r^2}\hat{a}_r$$

So, we have from electric field intensity of point charge

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r = \frac{\vec{D}}{\epsilon_0} \Rightarrow \boxed{\vec{D} = \epsilon_0\vec{E}}$$

Now, for a volume charge with density ρ_v the intensity is

$$\vec{E} = \int_{vol.} \frac{\rho_v \, dv}{4\pi\epsilon_0 R^2} \hat{a}_R \qquad \text{Also, } \Psi \text{ (flux)} = \int \vec{D} \cdot \vec{ds}$$

$$\therefore \boxed{\vec{D} = \int_{vol.} \frac{\rho_v \, dv}{4\pi R^2} \hat{a}_R}$$

Gauss's Law

It states that the total electric flux Ψ passing through any closed surface is equal to the total charge enclosed by that surface.

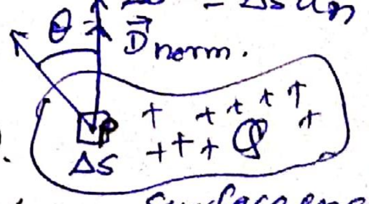
Thus, $\Psi = Q_{enclosed}$

i.e. $\Psi = \oint d\Psi = \oint_S \vec{D} \cdot \vec{ds}$ = Total charge enclosed

Also, for volume charge

$$\boxed{\Psi = \oint_S \vec{D} \cdot \vec{ds} = \int_{vol.} \rho_v \, dv}$$

We can assume the small incremental area of the surface enclosing the small charge ΔQ.



If P be any point where $\vec{D}$ makes an angle θ with $\vec{\Delta S}$ (incremental surface). Then, the electric flux passing through the incremental surface $\vec{\Delta S}$ is given by

$$\Delta\Psi = \vec{D}_{norm} \cdot \vec{\Delta S} = |\vec{D}| \cos\theta \, . \, \Delta S$$

or, $\Delta\Psi = \Delta Q = \vec{D} \cdot \vec{\Delta S}$

$\therefore$ Total flux passing through closed surface is $\Psi = Q_{enclosed} = \oint_S \vec{D} \cdot \vec{ds}$

For point charge, total charge $= \sum_{i=1}^{n} Q_i$ (n-charges)

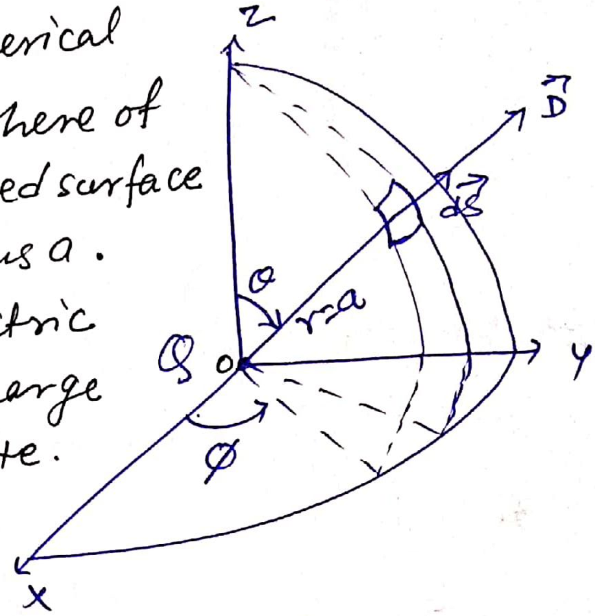
For line charge, total charge $= \int_L \rho_L \, dl$ (length - L)

For surface charge, total charge $= \int_S \rho_s \, ds$.

For volume charge, total charge $= \int_{vol.} \rho_v \, dv$

Gauss's Law Proof:

To prove Gauss's Law, let us consider the point charge, Q is placed at origin of the spherical co-ordinate system with sphere of radius a. Here, the enclosed surface will be the sphere of radius a.



From the derivation of electric field intensity of point charge at distance r we can write.

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{a}_r$$

and $\vec{D} = \epsilon_0 \vec{E}$

we have, $\vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r$

At the surface of the sphere, $\vec{D} = \frac{Q}{4\pi a^2} \hat{a}_r$

Also,
The differential element of area on a spherical surface is

$$ds = r^2 \sin\theta \, d\theta \, d\phi \quad \text{or} \quad \vec{ds} = r^2 \sin\theta \, d\theta \cdot d\phi \, \hat{a}_r$$

At the surface, $\vec{ds} = a^2 \sin\theta . d\theta \, d\phi \, \hat{a}_r$

Now,

$$\psi = \oint_S \vec{D} \cdot \vec{ds} = \oint \frac{Q}{4\pi a^2} \hat{a}_r \cdot a^2 \sin\theta . d\theta . d\phi \, \hat{a}_r$$

$$= \oint \frac{Q}{4\pi} \sin\theta . d\theta . d\phi$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \sin\theta \, d\theta . d\phi$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{\phi=2\pi} (-\cos\theta)\Big|_{\theta=0}^{\pi} d\phi$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{2\pi} 2 \, d\phi$$

$$= \frac{Q}{4\pi} \cdot 2 \cdot 2\pi$$

$\therefore \psi = Q$ = enclosed charge

i.e Q coulombs of electric flux are crossing the surface which is equal to the enclosed charge.

Application of Gauss's Law

Gauss Law is applicable only for symmetrical cases and symmetrical charge distributions. Here, Gauss's Law can only be applied if we consider the closed surface called Gaussian surface. The surface must satisfy the following conditions:

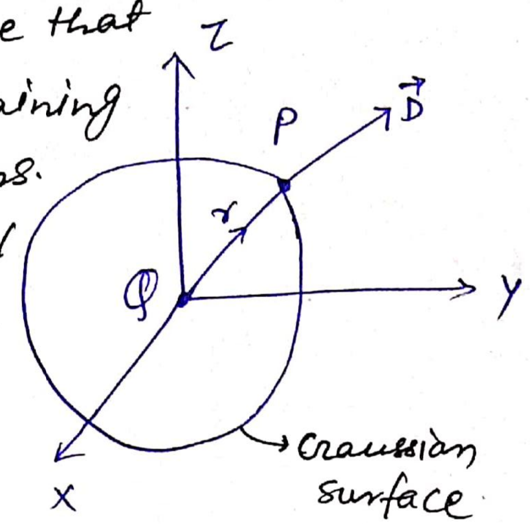
① $\vec{D}$ is everywhere either normal or tangential to the closed surface, so that $\vec{D}.\vec{ds}$ becomes either $D\,ds$ or zero, respectively.

② If $\vec{D}.\vec{ds}$ is not zero then $\vec{D}$ must be constant over the surface.

Application of Gauss's law To find the Electric field intensity due to point charge

Let us consider a point charge is located at origin. To determine $\vec{D}$ at a point p, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions.

Thus, a spherical surface centered at origin is a Gaussian Surface.



Since, $\vec{D}$ is everywhere normal to the Gaussian surface,

i.e $\vec{D} = D_r \hat{a}_r$

applying Gauss's law

$$\psi = Q_{enclosed} = Q = \oint \vec{D} \cdot \vec{ds} = D_r \oint ds = D_r 4\pi r^2$$

where $\oint ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \cdot d\theta \cdot d\phi = 4\pi r^2$ is the surface area of the Gaussian surface.

$$\therefore \boxed{\vec{D} = \frac{Q}{4\pi r^2} \hat{a}_r}$$

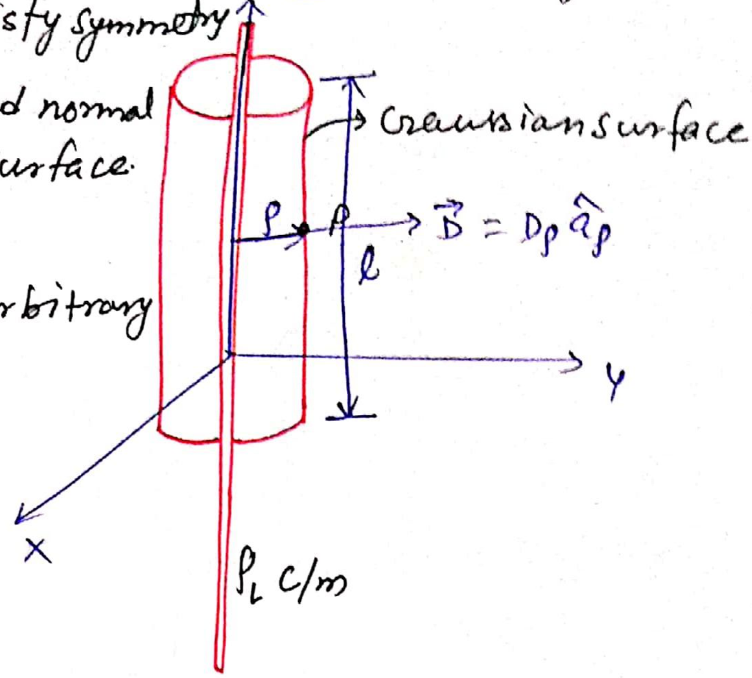
Since, $\vec{D} = \varepsilon_0 \vec{E}$

we can evaluate $\vec{E}$ as:

$$\boxed{\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{a}_r}$$

Field Due to Line charge

Let us consider the infinite line of uniform charge ρ_L C/m lies along z-axis. To determine $\vec{D}$ at point P, we choose a cylindrical surface containing P to satisfy symmetry condition. $\vec{D}$ is constant on and normal to the cylindrical Gaussian surface. i.e $\vec{D} = D_\rho \hat{a}_\rho$.



Applying Gauss's Law to an arbitrary length l of the line charge.

$$\rho_L l = Q = \oint \vec{D} \cdot \vec{dS}$$

$$= D_\rho \oint dS$$

$$= D_\rho \, 2\pi \rho l$$

where, $\oint dS = \oint \rho \, dl \cdot d\phi = \int_{l=0}^{l} \int_{\phi=0}^{2\pi} \rho \, dl \, d\phi$

$$\text{or, } \oint dS = \rho \int_{l=0}^{l} 2\pi \, dl$$

$$\text{or, } \oint dS = \rho \, 2\pi l$$ is a Gaussian surface

Since, z-component of $\vec{D}$ is zero the $\oint \vec{D} \cdot \vec{dS}$ at top and bottom surface will be zero. i.e tangential to those surfaces.

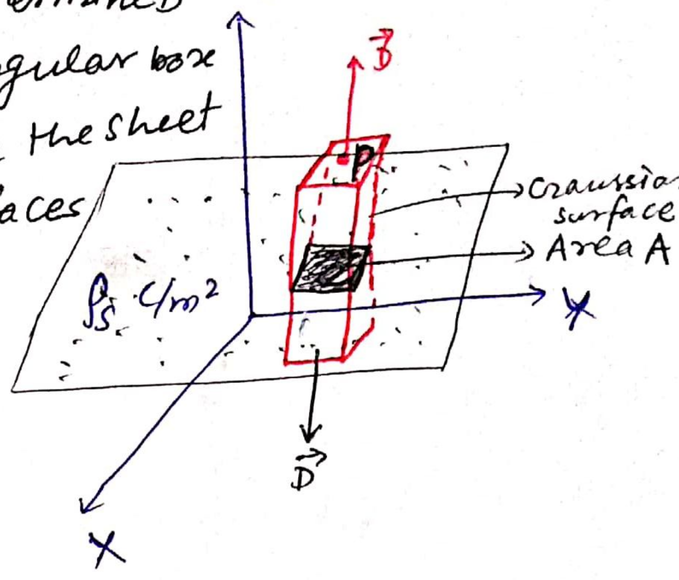
$$\therefore \boxed{\vec{D} = \frac{Q}{2\pi\rho l} \hat{a}_\rho = \frac{\rho_L l}{2\pi \rho l} \hat{a}_\rho = \frac{\rho_L}{2\pi\rho} \hat{a}_\rho}$$

Again, we have $\vec{D} = \varepsilon_0 \vec{E}$

So, $$\boxed{\vec{E} = \frac{\rho_L \hat{a}_\rho}{2\pi \rho \varepsilon_0} = \frac{\rho_L}{2\pi \varepsilon_0 \rho} \hat{a}_\rho}$$

Field due to infinite sheet of charge

Let us Consider the infinite sheet of uniform charge ρ_s C/m² lying on the z=0 plane. To determine $\vec{D}$ at point P, we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet. As $\vec{D}$ is normal to the sheet, $\vec{D} = D_z \hat{a}_z$.



Applying Gauss's law

$$\rho_s \int ds = Q = \oint \vec{D} \cdot \vec{ds}$$

$$= D_z \left[\int_{top} ds + \int_{bottom} ds \right]$$

Here, $\vec{D} \cdot \vec{ds}$ evaluated on the sides of box is zero because $\vec{D}$ has no components along $\hat{a}_x$ & $\hat{a}_y$. If the top and bottom area of the box each has area A.

$$\rho_s A = D_z [A + A]$$

$$\text{or, } D_z = \frac{\rho_s}{2}$$

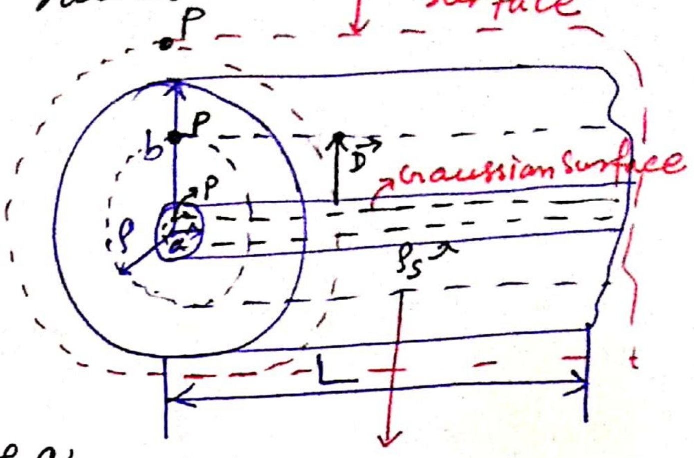
$$\therefore \boxed{\vec{D} = \frac{\rho_s}{2} \hat{a}_z = \frac{\rho_s}{2} \hat{a}_N}$$

Again, $\vec{D} = \varepsilon_0 \vec{E}$

$$\text{So, } \boxed{\vec{E} = \frac{\rho_s}{2\varepsilon_0} \hat{a}_N}$$

Field in a Co-axial Cable

Let us consider a Co-axial cable of infinite length having inner Conductor of radius a and outer conductor of radius b. A uniformly distributed surface charge density ρ_s is placed on the outer surface of inner conductor.



To determine $\vec{D}$ at point P in $\rho < a$, $\rho > b$ and $a < \rho < b$, we chose a cylindrical surfaces as shown in figure for symmetry conditions to satisfy. Since, the circular surface of the cylinder has $\vec{D}$ tangential to surface $\oint \vec{D} \cdot \vec{ds} = 0$; so only $\vec{D}$ along radius will be present; i.e $\vec{D} = D_\rho \hat{a}_\rho$.

Now, for $\rho < a$

Applying Gauss's law.

$$\oint \vec{D} \cdot \vec{ds} = \text{charge enclosed} = 0 \quad [\text{no charge enclosed}]$$

Hence, $\vec{D} = 0$ & $\vec{E} = 0$.

for $a < \rho < b$

Applying Gauss's law

$$\rho_s \int ds_I = Q = \oint \vec{D} \cdot \vec{ds} = D_\rho \oint ds_G = D_\rho \, 2\pi L \rho$$

where, $\int ds_G$ = surface area of Gaussian Surface $= 2\pi \rho L$

$\int ds_I$ = surface area of inner conductor $= 2\pi a L$

L = arbitrary length of the Co-axial cable.

$$\therefore \rho_s \cdot 2\pi a L = D_\rho \, 2\pi \rho L$$

or, $D_\rho = \dfrac{\rho_s a}{\rho}$ or, $\boxed{\vec{D} = \dfrac{\rho_s a}{\rho} \hat{a}_\rho}$

Since, $\vec{D} = \varepsilon_0 \vec{E}$

we have,

$$\boxed{\vec{E} = \frac{\rho_s a}{\varepsilon_0 \rho} \hat{a}_\rho}$$

for $\rho > b$

Since, due to induction the equal but opposite charge will be seen in outer conductor.

Applying Gauss's Law.

$$Q_{enclosed} = \rho_{s_I} \int ds_I + \rho_{s_0} \int ds_0 = 0 = \oint \vec{D} \cdot \vec{ds}$$

Thus, the field outside the co-axial cable is zero.

but $\rho_{s_I} \cdot (2\pi a L) = -\rho_{s_0} (2\pi b L)$

or, $\boxed{\rho_{s_0} = \frac{-a}{b} \rho_{s_I}}$

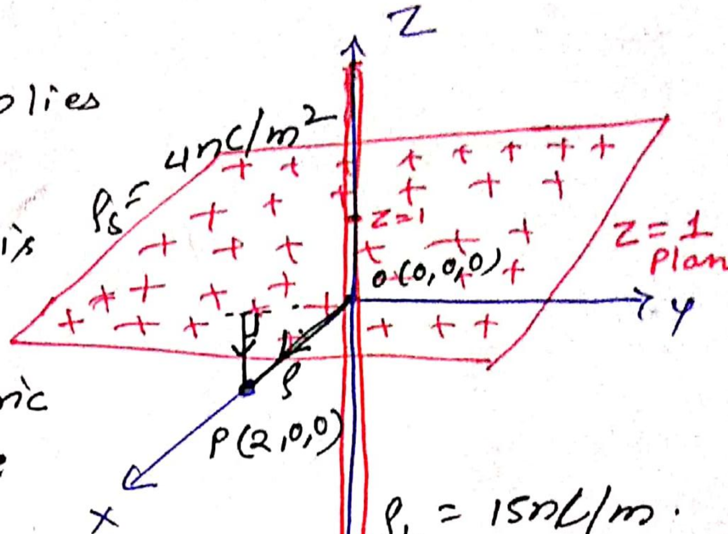
And, since total charge enclosed $= 0$, $\vec{D} = 0$ & $\vec{E} = 0$.

(27) Qn: A uniform line charge of 15 nC/m lies along the z-axis and a uniform sheet of charge ρ_s of 4 nC/m² is located at the plane z = 1. Find the electric flux density $\vec{D}$ in spherical co-ordinate system at point (2,0,0).

Sol.^{n}

Here, line charge of ρ_L = 15 nC/m lies along the z-axis.

Sheet charge of ρ_s = 4 nC/m² is located at z = 1 plane



Now, we have to find the electric flux density at point P(2,0,0);

Firstly, we calculate the electric field intensity due to both charges as :—

$\vec{E}$ due to line charge ρ_L = 15 nC/m is

$$\vec{E_l} = \frac{\rho_L}{2\pi\varepsilon_0 \rho} \cdot \hat{a}_\rho$$

where, $\vec{\rho} = (2-0)\hat{a}_x + (0-0)\hat{a}_y + (0-0)\hat{a}_z = 2\hat{a}_x$

$\rho = |\vec{\rho}| = 2$ & $\hat{a}_\rho = \hat{a}_x$

$$\therefore \vec{E_l} = \frac{15\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}\times2}\hat{a}_x = 15\times9\,\hat{a}_x = 135\,\hat{a}_x \text{ V/m}$$

Again, $\vec{E}$ due to sheet charge ρ_s = 4 nC/m² is

$$\vec{E_s} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_N$$

where, $\hat{a}_N = -\hat{a}_z$

$$\therefore \vec{E_s} = \frac{-4\times10^{-9}}{2\times\frac{10^{-9}}{36\pi}} = -226.19\,\hat{a}_z \text{ V/m}$$

ii. $\vec{E}$ total at point $P(2,0,0) = \vec{E}_L + \vec{E}_S = 135\hat{a}_x - 226.19\hat{a}_z$ V/m

Now, we have, electric flux density is

$$\vec{D} = \varepsilon_0 \vec{E} = \frac{10^{-9}}{36\pi} \times (135\hat{a}_x - 226.19\hat{a}_z)$$

$$= (1195.29\hat{a}_x - 2000\hat{a}_z) \times 10^{-12} \ C/m^2$$

$$= (1195.29\hat{a}_x - 2000\hat{a}_z) \ pC/m^2$$

Since, this value is in Cartesian co-ordinate system we need to change it to spherical co-ordinate system at point $P(2,0,0)$,
so, we need to have values of $r, \phi, \theta, D_r, D_\phi$ & D_θ.

or:

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{2^2+0^2+0^2} = 2$$

$$\phi = \tan^{-1}\left(\frac{0}{2}\right) = \tan^{-1}(0) = 0^\circ$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{2}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}.$$

$$D_r = \vec{D}\cdot\hat{a}_r = 1195.29(\hat{a}_x\cdot\hat{a}_r) - 2000(\hat{a}_z\cdot\hat{a}_r)$$

$$= 1195.29\sin\theta\cdot\cos\phi - 2000\cos\theta$$

$$= 1195.29\cdot\sin\left(\tfrac{\pi}{2}\right)\cdot\cos 0 - 2000\cos\tfrac{\pi}{2}$$

$$= 1195.29 \ pC/m^2$$

$$D_\theta = \vec{D}\cdot\hat{a}_\theta = 1195.29(\hat{a}_x\cdot\hat{a}_\theta) - 2000(\hat{a}_z\cdot\hat{a}_\theta)$$

$$= 1195.29\cdot\cos\theta\cdot\cos\phi - 2000(-\sin\theta)$$

$$= 1195.29\cdot\cos\tfrac{\pi}{2}\cdot\cos 0 + 2000\cdot\sin\tfrac{\pi}{2}$$

$$= 2000 \ pC/m^2$$

$$D_\phi = \vec{D}\cdot\hat{a}_\phi = 1195.29(\hat{a}_x\cdot\hat{a}_\phi) - 2000(\hat{a}_z\cdot\hat{a}_\phi)$$

$$= 1195.29 \ \cancel{\cos\phi}\,(-\sin\phi) - 2000\times 0$$

$$= 1195.29\times\sin 0$$

$$= 0 \ pC/m^2$$

$\therefore \vec{D}_{spherical} = D_r \hat{a}_r + D_\theta \hat{a}_\theta + D_\phi \hat{a}_\phi$

$= 1195.29\, \hat{a}_r + 2050\, \hat{a}_\theta + 0 \cdot \hat{a}_\phi$

$= 1195.29\, \hat{a}_r + 2050\, \hat{a}_\theta \quad pC/m^2$

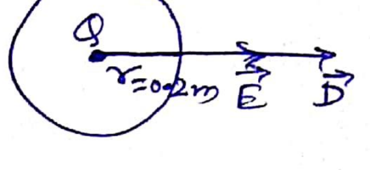
m: Given, $\vec{D} = \frac{r}{3}\hat{a}_r$ nC/m² in free space (a) Find $E\,[=|\vec{E}|]$ at $r = 0.2$ m. (b) Find the total charge within the sphere $r = 0.2$ m. (c) Find the total electric flux leaving the sphere $r = 0.3$ m.

Sol^n Here,

$\vec{D} = \frac{r}{3}\hat{a}_r$ in spherical co-ordinate system.

(a) To evaluate $E = |\vec{E}|$ we have

$\vec{D} = \epsilon_0 \vec{E}$

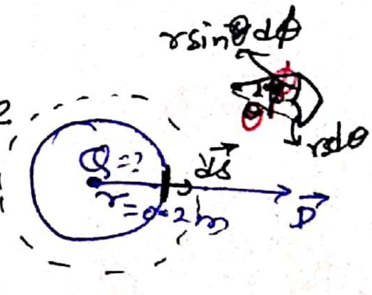


or, $\vec{E} = \frac{\vec{D}}{\epsilon_0} = \frac{r}{3\epsilon_0}\hat{a}_r$ V/m

Putting $r = 0.2$ m

$\vec{E} = \frac{0.2}{3\epsilon_0}\hat{a}_r$ V/m.

$\therefore E = |\vec{E}| = \frac{0.2}{3\epsilon_0} = \frac{0.2 \times 10^{9} \times 10^{-9}}{\frac{3}{36\pi}} = 7.53$ V/m



Again, (b) Let Q be the total charge within the sphere and assuming Gaussian surface at the surface of the sphere $r = 0.2$ m then $\vec{D}$ will be perpendicular to the surface of the sphere.

So, the differential area of the sphere, $\vec{ds} = r^2 \sin\theta \cdot d\theta \cdot d\phi\, \hat{a}_r$ $[r\sin\theta \cdot d\phi \cdot r d\theta]$

Now, from Gauss's law

$$Q = \oint_S \vec{D} \cdot \vec{ds} = \oint_S \frac{r}{3}\hat{a}_r \cdot r^2 \sin\theta \cdot d\theta \cdot d\phi\, \hat{a}_r$$

$$= \oint_S \frac{r^3}{3} \sin\theta \cdot d\theta \cdot d\phi$$

since, $r = 0.2$ m.

$$Q = \oint_S \frac{(0.2)^3}{3} \sin\theta \cdot d\theta \cdot d\phi$$

$$Q = \int_0^{\pi}\int_0^{2\pi} \frac{(0.2)^3}{3} \cdot \sin\theta \cdot d\phi \, d\theta$$

$$= \int_0^{\pi} \frac{(0.2)^3}{3} \cdot \sin\theta \cdot [2\pi - 0] \, d\theta$$

$$= \frac{(0.2)^3}{3} \times 2\pi \times [-\cos\theta]_0^{\pi}$$

$$= 33.51 \text{ pC}$$

(c) Again, since from Gauss's law, total electric flux leaving the surface is equal to the charge enclosed by the surface.

So, At $r = 0.3$ m.

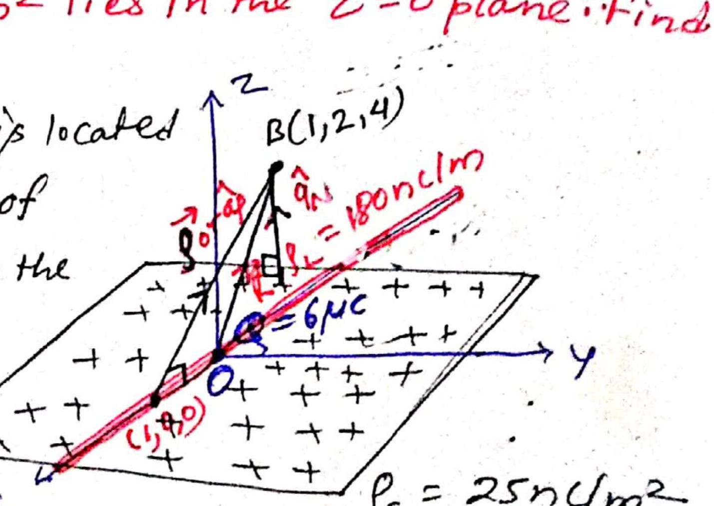
$$\Psi = Q = \oint_s \vec{D} \cdot d\vec{s} = \oint_s \frac{(0.3)^3}{3} \sin\theta \cdot d\theta \cdot d\phi$$

$$= \frac{(0.3)^3}{3} \times 2\pi \times [-\cos\theta]_0^{\pi}$$

$$= 113.097 \text{ pC}$$

Qn: A point charge of $6\mu C$ is located at the origin, a uniform line charge density of $180 nC/m$ lies along the x-axis and a uniform sheet of charge equal to $25 nC/m^2$ lies in the $z=0$ plane. Find $\vec{D}$ at $B(1,2,4)$. [2070 Ashad]

Sol.ⁿ Here, a point charge of $Q = 6\mu C$ is located at origin, a uniform line charge of density $\rho_L = 180 nC/m$ lies along the x-axis and a uniform sheet of charge density, $\rho_s = 25 nC/m^2$ lies in the $z=0$ plane as shown in figure.



Now, the electric flux density due to a point charge at origin, O (0,0,0) is

$$\vec{D}_P = \frac{Q}{4\pi R^2}\hat{a}_R$$

where, $\vec{R} = (1-0)\hat{a}_x + (2-0)\hat{a}_y + (4-0)\hat{a}_z$

$$= \hat{a}_x + 2\hat{a}_y + 4\hat{a}_z$$

$$R = |\vec{R}| = \sqrt{1^2+2^2+4^2} = \sqrt{21}$$

$$\& \ \hat{a}_R = \frac{\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z}{\sqrt{21}}$$

$$\therefore \vec{D}_P = \frac{6\times10^{-6}}{4\pi\times 21}\cdot\left(\frac{\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z}{\sqrt{21}}\right)$$

$$= 4.96\times10^{-9}(\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z)$$

$$= (4.96\,\hat{a}_x + 9.92\,\hat{a}_y + 19.84\,\hat{a}_z)\ nC/m^2$$

Again, Electric flux density due to line charge is

$$\vec{D}_L = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$$

where, $\vec{\rho} = (1-1)\hat{a}_x + (2-0)\hat{a}_y + (4-0)\hat{a}_z$

$$= 2\hat{a}_y + 4\hat{a}_z$$

$$\rho = |\vec{\rho}| = \sqrt{2^2+4^2} = \sqrt{20}$$

$$\& \ \hat{a}_\rho = \frac{2\hat{a}_y + 4\hat{a}_z}{\sqrt{20}}$$

$$\therefore \vec{D}_L = \frac{180\times10^{-9}}{2\pi\times\sqrt{20}}\cdot\frac{(2\hat{a}_y + 4\hat{a}_z)}{\sqrt{20}}$$

$$= 1.432\times10^{-9}(2\hat{a}_y + 4\hat{a}_z)$$

$$= (2.86\,\hat{a}_y + 5.73\,\hat{a}_z)\ nC/m^2$$

Again, Electric flux density due to sheet charge is

$$\vec{D}_S = \frac{\rho_S}{2}\hat{a}_N \quad \text{where, } \hat{a}_N = \hat{a}_z$$

$$\therefore \vec{D}_S = \frac{25\times10^{-9}}{2}\cdot\hat{a}_z$$
$$= 12.5\hat{a}_z \ nC/m^2$$

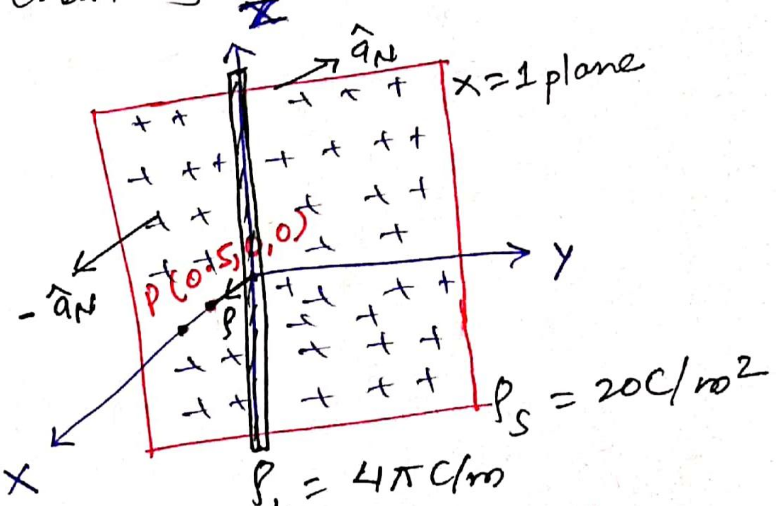
Total flux density at $B(1,2,4)$; $\vec{D} = \vec{D}_P + \vec{D}_L + \vec{D}_S$

$$= 4.96\hat{a}_x + 9.92\hat{a}_y + 19.84\hat{a}_z + 2.86\hat{a}_y + 5.73\hat{a}_z + 12.5\hat{a}_z$$
$$= (4.96\hat{a}_x + 12.78\hat{a}_y + 18.23\hat{a}_z) \ nC/m^2$$

Qn: Along z-axis there is a uniform line charge of density $\rho_L = 4\pi$ C/m and in the $x=1$ plane there is a surface charge with $\rho_s = 20$ C/m². Find the electric flux density at $(0.5, 0, 0)$. [2070 Chaitra]

Sol.ⁿ $\vec{D}_L = \frac{\rho_L}{2\pi\rho}\hat{a}_\rho$ & $\vec{D}_S = \frac{\rho_S}{2}\hat{a}_N$

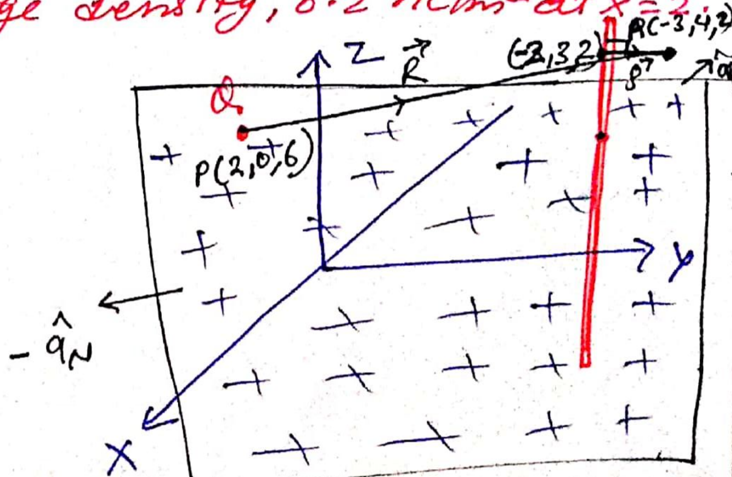
$$\therefore \vec{D} = \frac{4\cancel{\pi}}{2\cancel{\pi}\times0.5}\hat{a}_x + \frac{\overset{10}{\cancel{20}}}{2}(-\hat{a}_x)$$
$$= 4\hat{a}_x - 10\hat{a}_x$$
$$= -6\hat{a}_x \ C/m^2$$



Qn: Find $\vec{D}$ at the point $(-3,4,2)$ if the following charge distributions are present in free space: point charge, 12 nC at $P(2,0,6)$; uniform line charge density, 3 nC/m at $x=-2$, $y=3$; uniform surface charge density, 0.2 nC/m² at $x=2$.

Sol.ⁿ $\vec{D} = \vec{D}_P + \vec{D}_L + \vec{D}_S$

$$= \frac{Q}{4\pi R^2}\hat{a}_R + \frac{\rho_L}{2\pi\rho}\hat{a}_\rho + \frac{\rho_S}{2}\hat{a}_N$$



Qn: Let a uniform line charge density, 3nC/m at $y=3$; uniform surface charge density, $0.2 nC/m^2$ at $x=2$. Find $\vec{E}$ at origin. [2074 Ashwin]

Qn: A uniform line charge density of 20nC/m is located at $y=3$ and $z=5$. Find $\vec{E}$ at P(5,6,1). [2072 Chaitra]

Qn: Find Electric flux density at Point P(5,4,3) due to a uniform line Charge of 2nC/m at $x=5$, $y=3$, point charge 12nC at Q(2,0,6) and uniform surface charge density of $0.2 nC/m^2$ at $x=2$. [2073 Chaitra]

Qn: Along the z-axis there is a uniform line of charge with $\rho_L = 4\pi\ C m^{-1}$ and in the $x=1$ plane there is a surface charge with $\rho_s = 20\ C m^{-2}$. Find the Electric flux Density at (0.5,0,0). [2070 Chaitra]

Qn: Find $\vec{D}$ at the point (−3,4,2) if the following charge distributions are present in free space: point charge, 12nC at P(2,0,6); uniform line charge density, 3nC/m at $x=-2$, $y=3$; uniform surface charge density, $0.2 nC/m^2$ at $x=2$.

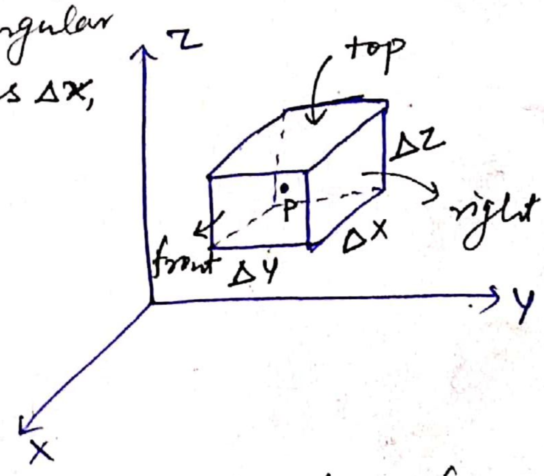
[2071 Shrawan]

(73)

Application of Gauss's Law : Differential Volume Element

Gauss's Law is only applicable for the symmetrical charge distributions and the Gaussian surface must be simple where the flux density has normal component either constant or zero everywhere on the surface. Without such surface, the integral cannot be evaluated. So, for unsymmetrical cases; there is only away to apply Gauss's Law by considering or chosing a very small closed surface that $\vec{D}$ is almost constant over the surface. The result will become more nearly correct as the volume enclosed by the Gaussian surface decreases.

Let us consider any point P in a rectangular co-ordinate system with co-ordinate $P(x_0, y_0, z_0)$. The value of $\vec{D}$ at the point P may be expressed in rectangular components, $\vec{D}_0 = D_{x0}\hat{a}_x + D_{y0}\hat{a}_y + D_{z0}\hat{a}_z$. Here, we choose a small rectangular box centered at P, having sides of lengths Δx, Δy & Δz.



Applying Gauss's Law.

$$\oint_S \vec{D}\cdot d\vec{S} = Q$$

In order to evaluate the integral over the closed surface, the integral must be broken into six for each face.

$$\oint_S \vec{D}\cdot d\vec{S} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{bottom} + \int_{top}$$

Now,

$$\int_{front} = \vec{D}_{front}\cdot \Delta\vec{S}_{front}$$

$$= \vec{D}_{front}\cdot \Delta y\,\Delta z\,\hat{a}_x$$

$$= D_{x,front}\,\Delta y\,\Delta z$$

Since the front face is at a distance of $\frac{\Delta x}{2}$ from P, and hence approximating D_x value at the front face.

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

or, $D_{x,front} = D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$

where D_{x0} is the value of D_x at P and where a partial derivative must be used to express the rate of change of D_x with x, as D_x in general vary with y & z also. This expression could ~~have~~ been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's series expansion for D_x in the neighborhood of P.

$$\therefore \int_{front} = \left(D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \cdot \Delta z$$

Similarly for back surface,

$$\int_{back} = \vec{D}_{back} \cdot \vec{\Delta S}_{back}$$

$$= \vec{D}_{back} \cdot (-\Delta y \cdot \Delta z \hat{a}_x)$$

$$= D_{x,back} \Delta y \cdot \Delta z$$

and $D_{x,back} = D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$

$$\int_{back} = \left(-D_{x0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

Combining we get

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x}\Delta x \Delta y \Delta z$$

Doing same for other surfaces, we get

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial y}\Delta x \Delta y \cdot \Delta z$$

&

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z}\Delta x \Delta y \Delta z.$$

Taylor's Series expansion.

$$D_x(x,y,z) = D_x(x_0,y_0,z_0) + (x-x_0)\left.\frac{\partial D_x}{\partial x}\right|_P + (y-y_0)\left.\frac{\partial D_x}{\partial x}\right|_P + (z-z_0)\cdot\left.\frac{\partial D_x}{\partial x}\right|_P + \text{higher order terms.}$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!} + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(75) Therefore, we have

$$\oint_S \vec{D}\cdot d\vec{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \cdot \Delta y \Delta z$$

$$\text{or, } \oint_S \vec{D}\cdot d\vec{S} = Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta V$$

Hence, $\boxed{\text{Charge enclosed in volume } \Delta V = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta V}$

Divergence and Maxwell's first Equation:

The divergence of $\vec{D}$ at a given point P is the outward flux per unit volume as the volume shrinks about P.

i.e
$$\text{divergence of } \vec{B} = \nabla\cdot\vec{D} = \lim_{\Delta V\to 0} \frac{\oint_S \vec{B}\cdot d\vec{S}}{\Delta V}$$

From above expression, if we allow the ΔV to shrink to zero, then

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta V\to 0} \frac{\oint \vec{D}\cdot d\vec{S}}{\nabla V} = \lim_{\Delta V\to 0}\frac{Q}{\Delta V} = \rho_V$$

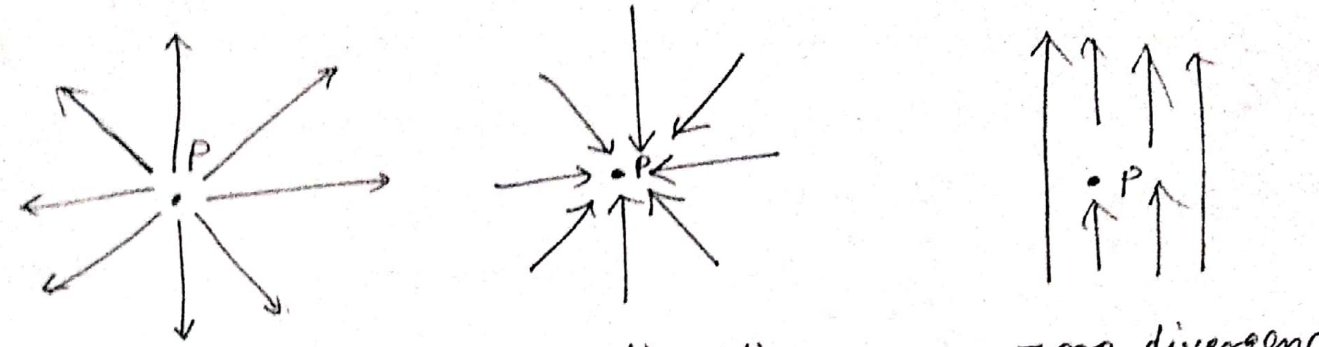
Also, if we assume any general vector, $\vec{A}$ then

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \lim_{\Delta V\to 0} \frac{\oint \vec{A}\cdot d\vec{S}}{\Delta V}$$

$$\therefore \text{divergence of } \vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta V\to 0} \frac{\oint_S \vec{A}\cdot d\vec{S}}{\Delta V}$$

where, $\boxed{\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$

The divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.



Positive divergence at Point P

Negative divergence at point P

zero divergence

Physically, we may regard the divergence of the vector field $\vec{A}$ at a given point as a measure of how much the field diverges or emanates from that point. In above figures, the vector diverges (or spreads out) in first figure, so it is +ve. The vector converges at P in second figure so, it is −ve divergence. But in 3rd figure, the vector neither diverges nor converges, so it has zero divergence.

Divergence Theorem

It states that the total outward flux of a vector field $\vec{A}$ through the closed surface S is the same as the volume integral of the divergence of $\vec{A}$.

i.e $\boxed{\oint_S \vec{A}\cdot\vec{ds} = \int_{vol.} \nabla\cdot\vec{A}\, dv}$

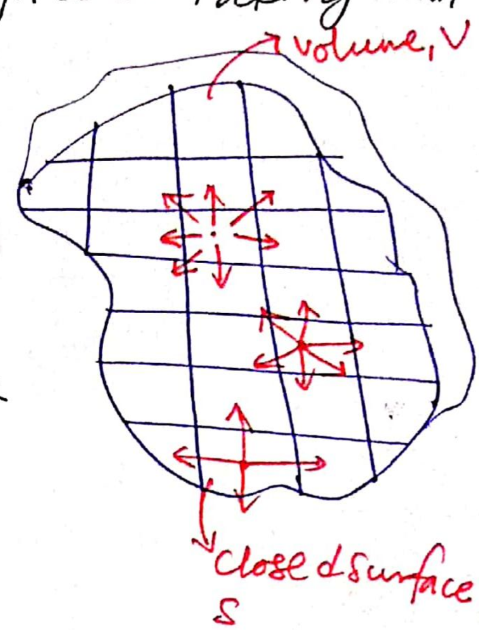
or

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

To prove the divergence theorem, let us subdivide volume v into a large number of small cells. If the kth cell has volume ΔV_K and is bounded by surface S_K

$$\oint_S \vec{A}\cdot\vec{ds} = \sum_K \oint_{S_K} \vec{A}\cdot\vec{ds} = \sum_K \frac{\oint_{S_K} \vec{A}\cdot\vec{ds}}{\Delta V_K}\cdot \Delta V_K$$

Since the outward flux to one cell is inward to some neighbouring cells, there is cancellation on every interior surface, so the sum of the surface integrals over S_k's the same as the surface integral over the surfaces S. Taking limit of right side and incorporating



$$\text{div.} \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v}$$

and

$$\oint_S \vec{A} \cdot \vec{ds} = \sum_K \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v_K} \cdot \Delta v_K$$

gives,

$$\boxed{\oint_S \vec{A} \cdot \vec{ds} = \int_{vol.} \nabla \cdot \vec{A} \, dv}$$

which is a divergence theorem. The theorem applies to any volume V bounded by the closed surface S.

Maxwell's first Equation

From divergence theorem.

$$\text{div.} \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v}$$

or, $\boxed{\nabla \cdot \vec{A} = \rho_v}$

where, $\rho_v = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \cdot \vec{ds}}{\Delta v}$, volume charge density.

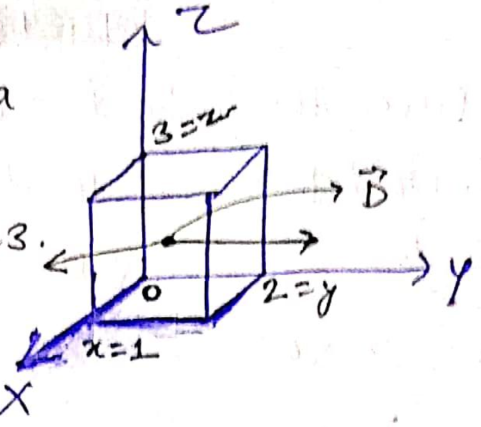
This is the Maxwell's first equation or point form of Gauss's Law.

78. Qn: Evaluate both sides of the divergence theorem for the field $\vec{D} = 2xy\,\hat{a}_x + x^2\hat{a}_y$ C/m² and the rectangular parallelopiped formed by the planes $x = 0$ & 1, $y = 0$ & 2 and $z = 0$ & 3.

Sol.ⁿ

Given,

$\vec{D} = 2xy\,\hat{a}_x + x^2\hat{a}_y$ C/m² and a rectangular parallelopiped formed by the planes $x = 0$ & 1, $y = 0$ & 2 and $z = 0$ & 3.



now, from divergence theorem.

$$\oint_S \vec{D}\cdot d\vec{S} = \int_{vol.} \nabla\cdot\vec{D}\,dv$$

For left part

$$\oint_S \vec{D}\cdot d\vec{S} = \int_0^3\int_0^2 (\vec{D})_{x=0}\cdot(-dy\,dz\,\hat{a}_x) + \int_0^3\int_0^2 (\vec{D})_{x=1}\cdot(dy\,dz\,\hat{a}_x)$$

$$+ \int_0^3\int_0^1 (\vec{D})_{y=0}\cdot(-dx\cdot dz\,\hat{a}_y) + \int_0^3\int_0^1 (\vec{D})_{y=2}\cdot(dx\cdot dz\,\hat{a}_y)$$

$$+ \int_0^2\int_0^1 (\vec{D})_{z=0}\cdot(-dx\cdot dy\,\hat{a}_z) + \int_0^2\int_0^1 (\vec{D})_{z=3}\cdot(dx\,dy\,\hat{a}_z)$$

since, the z-component of $\vec{D}$ is zero; the $\vec{D}$ seems to be parallel to $z=0$ & $z=3$ planes.

$$\therefore \oint_S \vec{D}\cdot d\vec{S} = -\int_0^3\int_0^2 (D_x)_{x=0}\,dy\,dz + \int_0^3\int_0^2 (D_x)_{x=1}\,dy\,dz$$

$$-\int_0^3\int_0^1 (D_y)_{y=0}\,dx\,dz + \int_0^3\int_0^1 (D_y)_{y=2}\,dx\,dz$$

For, $x = 0$

$\vec{D} = 0$ so, $\oint_S \vec{D}\cdot d\vec{S} = \int_0^3\int_0^2 (D_x)_{x=1}\,dy\,dz + \int_0^3\int_0^1 (D_y)_{y=2}\,dx\,dz$

$$= 0 + 0 + \int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz + 0 + 0 - \cancel{\int_{z=0}^{3}\int_{x=0}^{1} x^2\,dz\,dx}$$

at $x=1$ at $y=0$

$$+ \cancel{\int_{z=0}^{3}\int_{x=0}^{1} x^2\,dz\,dx}$$

at $y=2$

$$= \int_{y=0}^{2}\int_{z=0}^{3} 2xy\,dy\,dz$$

at $x=1$

$$= \int_{y=0}^{2}\int_{z=0}^{3} 2y\,dy\,dz$$

$$= \int_{y=0}^{2} 6y\,dy$$

$$= \frac{6y^2}{2}\Big|_0^2$$

$$= 3\,[2^2 - 0^2]$$

$$= 12\ C$$

Again taking right part only.

$$\int_{vol.} (\nabla\cdot\vec{B})\,dv = \int_{vol.} \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx\,dy\,dz$$

Here, $D_x = 2xy$, $D_y = x^2$ & $D_z = 0$.

So, $$\int_{vol.} (\nabla\cdot\vec{B})\,dv = \int_{vol.} \left[\frac{\partial(2xy)}{\partial x} + \frac{\partial(x^2)}{\partial y} + \frac{\partial(0)}{\partial z}\right] dx\,dy\,dz$$

$$\int_{vol.} (\nabla\cdot\vec{D})\, dv = \int_{vol.} [2y+0+0]\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} 2y\, dx\, dy\, dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{2} 2y\times 3\, dx\, dy$$

$$= \int_{x=0}^{1} 6 \left.\frac{y^2}{2}\right|_0^2 dx$$

$$= 12\,[1-0]$$

$$= 12C$$

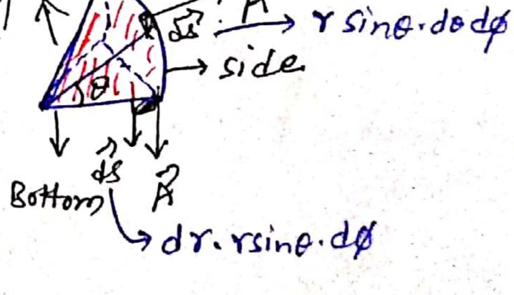
$$\therefore \oint_S \vec{D}\cdot d\vec{s} = \int_{vol.} (\nabla\cdot\vec{D})\cdot dv = 12C //$$

Qn: Verify the divergence Theorem [Calculate bothsides of the divergence theorem] for the function $\vec{A} = r^2\hat{a}_r + r\sin\theta\cdot\cos\phi\,\hat{a}_\theta$, over the surface of quarter of a hemisphere defined by: $0<r<3,\ 0<\phi<\frac{\pi}{2},\ 0<\theta<\pi/2$. [2069 chaitra]

Sol.^{n}

Here, $\vec{A} = r^2\hat{a}_r + r\sin\theta\cdot\cos\phi\,\hat{a}_\theta$

& $0<r<3,\ 0<\phi<\frac{\pi}{2},\ 0<\theta<\frac{\pi}{2}$



from divergence Theorem

$$\oint_S \vec{A}\cdot d\vec{s} = \int_{vol.} (\nabla\cdot\vec{A})\, dv$$

Taking only left part

$$\oint_S \vec{A}\cdot d\vec{s} = -\oint_{S\ top\ (\theta=0)} \vec{A}\cdot d\vec{s} + \oint_{S\ bottom\ (\theta=\pi/2)} \vec{A}\cdot d\vec{s} + \oint_{S\ side\ (r=3)} \vec{A}\cdot d\vec{s} + \oint_{S\ \phi=0} \vec{A}\cdot d\vec{s} + \oint_{S\ \phi=\pi} \vec{A}\cdot d\vec{s}$$

(81)

$$= -\oint_{s,\,\theta=0} (A_\theta \hat{a}_\theta)\cdot(r\sin\theta\, d\phi\, dr\, \hat{a}_\theta) + \oint_{s,\,\theta=\pi/2} (A_\theta \hat{a}_\theta)\cdot(r\sin\theta\, d\phi\, dr\, \hat{a}_\theta) + \oint_{s,\,r=3} (A_r \hat{a}_r)\cdot(r^2\sin\theta\, d\theta\, d\phi\, \hat{a}_r)$$

$$= -\oint_{s,\,\theta=0} r^2 \sin^2\theta \cdot \cos\phi\, d\phi\, dr + \oint_{s,\,\theta=\pi/2} r^2\sin^2\theta \cos\phi\, d\phi\, dr + \oint_{s,\,r=3} r^4 \sin\theta\, d\theta\, d\phi$$

$$= -\oint_s r^2 \sin^2(0)\cdot\cos\phi\, d\phi\, dr + \oint_{s,\,\theta=\pi/2} r^2\sin^2(\pi/2)\cos\phi\, d\phi\, dr + \oint_{s,\,r=3} (3)^4 \sin\theta\, d\theta\cdot d\phi$$

$$= \int_{r=0}^{3}\int_{\phi=0}^{\pi/2} r^2\cos\phi\, d\phi\, dr + 81\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \sin\theta\, d\theta\, d\phi$$

$$= \int_{r=0}^{3} r^2 [\sin\phi]_0^{\pi/2} dr + 81\int_{\theta=0}^{\pi/2} \sin\theta\cdot[\pi/2 - 0]\, d\theta$$

$$= \frac{(3)^3}{3}\cdot 1 + 81\times\frac{\pi}{2}\times[-\cos\theta]_0^{\pi/2}$$

$$= 9 + \frac{81\pi}{2}$$

Again, Taking only right part

$$\int_{vol.} (\nabla\cdot\vec{A})\, dv = \int_{vol.} \left[\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\cdot\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\cdot\frac{\partial A_\phi}{\partial\phi}\right].$$
$$[r^2\sin\theta\cdot dr\cdot d\theta\cdot d\phi]$$

$$= \int_{vol.}\left[\frac{1}{r^2}\times 4r^3 + \frac{1}{r\sin\theta}\frac{\partial(r\sin^2\theta\cdot\cos\phi)}{\partial\theta} + 0\right]\left[r^2\sin\theta\cdot dr\cdot d\theta\cdot d\phi\right]$$

$$= \int_{vol.}\left[4r + \frac{r\cos\phi}{r\sin\theta}\cdot 2\sin\theta\cdot\cos\theta\right][r^2\sin\theta\cdot dr\cdot d\theta\, d\phi]$$

$$= 4\int_{r=0}^{3} r^3\int_{\theta=0}^{\pi/2}\sin\theta\,[\tfrac{\pi}{2}-0]\, dr\, d\theta + \int_{r=0}^{3} r^2\int_{\theta=0}^{\pi/2}\sin 2\theta\,[\sin\phi]_0^{\pi/2}\, dr\, d\theta$$

$$= 2\pi\int_{r=0}^{3} r^3 [1]\, dr + \frac{1}{2}\int_{r=0}^{3} r^2 [-\cos 2\theta]_0^{\pi/2}\, dr$$

(82)

$$= \frac{2\pi}{4}\left[r^4\right]_0^3 + \frac{1}{2}\int_{r=0}^{3} r^2\,[-\cos\pi + \cos 0]\,dr$$

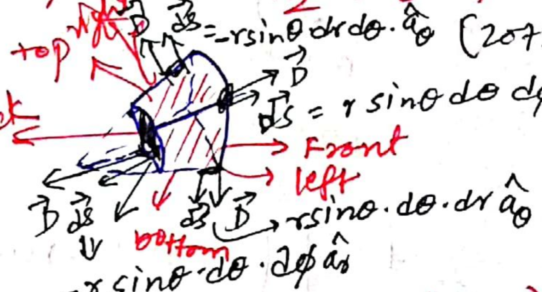
$$= \frac{81\pi}{2} + \frac{1}{2}\cdot 2\left[\frac{r^3}{3}\right]_0^3$$

$$= \frac{81\pi}{2} + 9$$

$$= 9 + \frac{81\pi}{2}$$

$$\therefore \oint_S \vec{A}\cdot d\vec{S} = \int_{vol.} (\nabla\cdot\vec{A})\cdot dv = 9 + \frac{81\pi}{2} \quad //$$

Qn: Given the flux density $\vec{D} = (2\cos\theta/r^3)\,\hat{a}_r + (\sin\theta/r^3)\,\hat{a}_\theta$ c/m²; evaluate both sides of the divergence theorem for the region defined by $1<r<2$, $0<\theta<\frac{\pi}{2}$, $0<\phi<\frac{\pi}{2}$. [2073 Shrawan]



Sol.ⁿ Given,

$$\vec{D} = \frac{2\cos\theta}{r^3}\hat{a}_r + \frac{\sin\theta}{r^3}\hat{a}_\theta$$

& $1<r<2$, $0<\theta<\frac{\pi}{2}$, $0<\phi<\frac{\pi}{2}$.

(six faces)

[Ans: 3.5 C]

From divergence Theorem.

$$\oint_S \vec{D}\cdot d\vec{S} = \int_{vol.} (\nabla\cdot\vec{D})\,dv$$

Taking only left part

$$\oint_S \vec{D}\cdot d\vec{S} = \underset{\substack{\text{front}\\(r=2)}}{\oint_S \vec{D}\cdot d\vec{S}} - \underset{\substack{\text{back}\\(r=1)}}{\oint_S \vec{D}\cdot d\vec{S}} + \underset{\substack{\text{bottom}\\(\theta=\pi/2)}}{\oint_S \vec{D}\cdot d\vec{S}} - \underset{\substack{\text{top}\\(\theta=0)}}{\oint_S \vec{D}\cdot d\vec{S}} + \underset{\substack{\text{left}\\ \phi=\pi/2}}{\oint} + \underset{\substack{\text{right}\\ \phi=0}}{\oint}$$

* Taking only right part

$$\int_{vol.} (\nabla\cdot\vec{D})\,dv = \int_{vol.}\left[\frac{1}{r^2}\frac{\partial(r^2 D_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial \sin\theta D_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}\right]\left(r^2\sin\theta\, d\theta\, d\phi\, dr\right)$$

$$\nabla \cdot \vec{D} = \frac{1}{r^2}\frac{\partial\left(r^2 \frac{2\cos\theta}{r^3}\right)}{\partial r} + \frac{1}{r\sin\theta}\cdot\frac{\partial\left(\sin\theta \cdot \frac{\sin\theta}{r^3}\right)}{\partial\theta} + 0$$

$$= -\frac{2\cos\theta}{r^2}\cdot\frac{1}{r^2} + \frac{1}{r\cancel{\sin\theta}}\cdot\frac{1}{r^3}\cdot 2\cancel{\sin\theta}\cdot\cos\theta$$

$$= -\frac{2\cos\theta}{r^4} + \frac{2\cos\theta}{r^4}$$

$$= 0$$

$$\int_{vol.} (\nabla \cdot \vec{D})\, dv = 0.$$

$$\oint_S \vec{D}\cdot d\vec{s} = -\underset{(r=1)}{\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}} \frac{2\cos\theta}{r^3}\cdot r^2\sin\theta\, d\phi\, d\theta + \underset{(r=2)}{\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}} \frac{2\cos\theta}{r^3}\cdot r^2\sin\theta\, d\phi\, d\theta$$

$$+ \underset{(\theta=\frac{\pi}{2})}{\int_{r=1}^{2}\int_{\phi=0}^{\pi/2}} \frac{\sin\theta}{r^3}\cdot r\sin\theta\, d\phi\, dr - \underset{(\theta=0)}{\int_{r=1}^{2}\int_{\phi=0}^{\pi/2}} \cancel{\frac{\sin\theta}{r^3}\cdot r\sin\theta\, d\phi\, dr}^{\,0}$$

$$= -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \frac{\sin 2\theta\, d\phi\, d\theta}{1} + \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} \frac{\sin 2\theta}{2}\, d\phi\, d\theta$$

$$+ \int_{r=1}^{2}\int_{\phi=0}^{\pi/2} \frac{1}{r^2}\, d\phi\, dr - 0$$

$$= +\frac{1}{2}\left[\frac{\cos 2\theta}{2}\right]_0^{\pi/2}\cdot\left[\frac{\pi}{2}-0\right] + (-1)\left[\frac{1}{r}\right]_1^2\cdot\left[\frac{\pi}{2}-0\right]$$

Qn: Given $\vec{D} = 8xyz^4\hat{a}_x + 4x^2z^4\hat{a}_y + 16x^2yz^3\hat{a}_z \ pC/m^2$ verify both sides of divergence Theorem for the rectangular parallelopiped $0 < x < 2, 1 < y < 3, 0 < z < 2$.

[Ans) 1774.9 pC]

Qn: Given the field $\vec{D} = 6\rho \sin\frac{1}{2}\phi\, \hat{a}_\rho + 1.5\rho \cos\frac{1}{2}\phi\, \hat{a}_\phi \ C/m^2$ evaluate both sides of the divergence theorem for the region bounded by $\rho = 2, \phi = 0, \phi = \pi, z = 0$ & $z = 5$.

[Ans :- 225 & 225C]

Qn: Let $\vec{D} = 8\rho \sin\phi\, \hat{a}_\rho + 4\rho \cos\phi\, \hat{a}_\phi \ C/m^2$, evaluate both sides of divergence theorem for the region $0 < \rho < 1.8$, $20° < \phi < 70°$, $2.4 < z < 3.1$. [8.13 C]

Soln:

$$\nabla\cdot\vec{D} = \frac{1}{\rho}\frac{\partial(\rho D_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$= \frac{6}{\rho}\frac{\partial(\rho^2 \sin\frac{\phi}{2})}{\partial\rho} + \frac{1}{\rho}\frac{\partial(1.5\rho\cos\frac{\phi}{2})}{\partial\phi} + 0$$

$$= \frac{6}{\rho}\times 2\rho \sin\frac{\phi}{2} + \frac{1}{\rho}\times\frac{1.5\rho}{2}\left[-\sin\frac{\phi}{2}\right]$$

$$= 12\sin\frac{\phi}{2} + 0.75\sin\frac{\phi}{2}$$

$$= 11.25\sin\frac{\phi}{2}$$

$$\int_{vol.}(\nabla\cdot\vec{D})\,dv = \int_{z=0}^{5}\int_{\phi=0}^{\pi}\int_{\rho=0}^{2} 11.25\sin\frac{\phi}{2}\,\rho\, d\phi\, dz\, d\rho$$

$$= 11.25\left[\frac{\rho^2}{2}\right]_0^2\left[-\cos\frac{\phi}{2}\right]_0^\pi \times .[5-0]\times 2$$

$$= 11.25\times\frac{4}{2}\times 1\times 5\times 2$$

$$= 225\ C.$$

Qn: If $\vec{D} = 15\rho^2 \sin 2\phi \,\hat{a}_\rho + 10\rho^2 \cos 2\phi \,\hat{a}_\phi$ C/m², evaluate both sides of divergence theorem for the region:

$1 < \rho < 2\,m$, $1 < \phi < 2\,rad$, $1 < z < 2\,m$.

[Ans - 6.93C]

~~Qn: If [illegible] C/m², evaluate both sides of~~

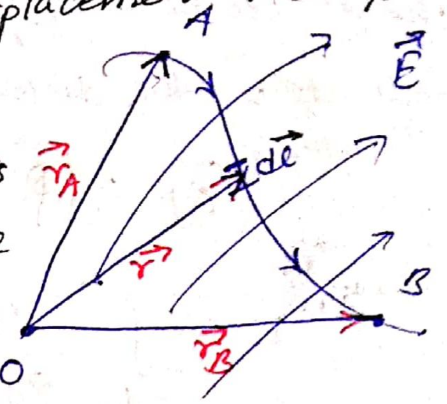
Qn: Given the flux density, $\frac{16}{r} \cos 2\theta \,\hat{a}_\theta$ C/m², evaluate both sides of divergence theorem for the region: $1 < r < 2\,m$, $1 < \theta < 2\,rad$, $1 < \phi < 2\,rad$.

[Ans: - - 3.91C]

(82) Electric Potential & Work (Potential field of a point charge)

If a charge Q is placed in an electric field $\vec{E}$, then the force due to electric field is given by $\vec{F} = Q\vec{E}$ [Coulomb's Law]. Suppose we wish to move a point charge from point A to point B in an electric field $\vec{E}$ as in figure. Then, the workdone in moving the charge Q in electric field $\vec{E}$ for a displacement of $\vec{dl}$ is

$$dW = -\vec{F}\cdot\vec{dl} = -Q\vec{E}\cdot\vec{dl}$$



The -ve sign indicates that the work is being done by an external agent. Thus the total workdone, or the potential energy required, in moving Q from A to B is

$$\boxed{W = -Q\int_A^B \vec{E}\cdot\vec{dl}}$$

$\vec{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$

$\vec{dl} = d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$

$\vec{dl} = dr\,\hat{a}_r + r\,d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi$

Again,

$$\boxed{\frac{W}{Q} = V_{BA} = -\int_A^B \vec{E}\cdot\vec{dl}}$$

This is the potential energy per unit charge, denoted by V_{BA} and also known as potential difference betn points A & B.

Note:

1.) In determining V_{BA}, A is the initial point while B is final point.

2.) If V_{BA} is -ve, there is a loss in potential energy in moving Q from A to B; this implies that the work done is by the field. However, if V_{BA} is +ve, there is a gain in potential energy in the movement; an external agent performs the work.

3.) V_{BA} is independent of the path taken.

4.) V_{BA} is measured in joules per coulomb, commonly as volts (V).

(84):

→ Work done by external source against field → +ve

→ Work done by field → −ve

We have, from Coulomb's Law, the field $\vec{E}$ due to point charge Q located at origin is

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r$$

Then,

$$V_{BA} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r \cdot dr\,\hat{a}_r$$

or, $$V_{BA} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right].$$

Potential difference is defined as the workdone (by an external source) in moving a unit positive charge from one point to another in an electric field.

i.e $V_{AB} = -\int_A^B \vec{E}\cdot\vec{dl}$

or, $$V_{BA} = V_B - V_A$$

where, V_B and V_A are the potentials (or absolute potentials) at B and A, respectively. Thus the potential difference V_{BA} may be regarded as the potential at B with reference to A. For point charge, it is customary to choose infinity as reference; that is, we assume the potential at infinity is zero. Thus, if $V_A = 0$ as $r_A \to \infty$, then the potential at any point $(r_B \to r)$ due to a point charge Q located at the origin is.

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}}$$

If reference is not at infinity then $V = \frac{Q}{4\pi\epsilon_0 r} + C_1$, C_1 is potential at the reference point.

Since, $\vec{E}$ points in the radial direction ($\hat{a}_r$), any contribution from a displacement in θ or ϕ direction is wiped out by the dot product $\vec{E}\cdot\vec{dl} = E\cos\theta\, dl = E dr$. Hence, potential difference is independent of the path.

The potential at any point is the potential difference betn that point and a choosen point at which the potential is zero.

85. In other words, by assuming zero potential at ∞, the potential at a distance r from the point charge is the workdone per unit charge by an external agent in moving a test charge from infinity to that point.

i.e $V = -\int_{\infty}^{r} \vec{E} \cdot \vec{dl}$

If a point charge Q is not located at origin but at a point whose position vector is $\vec{r}'$, then, the potential $V(x, y, z)$ or $V(\vec{r})$ at $\vec{r}$ becomes:

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|}$$

Then, for n point charges, $Q_1, Q_2, \cdots Q_n$ located at points with position vectors $\vec{r}_1, \vec{r}_2, \cdots \vec{r}_n$; the potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{Q_1}{4\pi\epsilon_0 |\vec{r} - \vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0 |\vec{r} - \vec{r}_2|} + \cdots + \frac{Q_n}{4\pi\epsilon_0 |\vec{r} - \vec{r}_n|}$$

$$\text{or, } V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|\vec{r} - \vec{r}_k|}$$

Now, for line, surface & volume charges

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_L(\vec{r}')\, dl'}{|\vec{r} - \vec{r}'|}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_S(\vec{r}')\, ds'}{|\vec{r} - \vec{r}'|}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{Vol.} \frac{\rho_V(\vec{r}')\, dv'}{|\vec{r} - \vec{r}'|}.$$

Finite length line charge

Consider the uniform distribution of line charge ρ_L of finite length $2L$ centered on the z-axis. Distinguishing betn position of charge element $dq = \rho_L dz'$ at z' and the field point at co-ordinate z, the distance betn source and field is

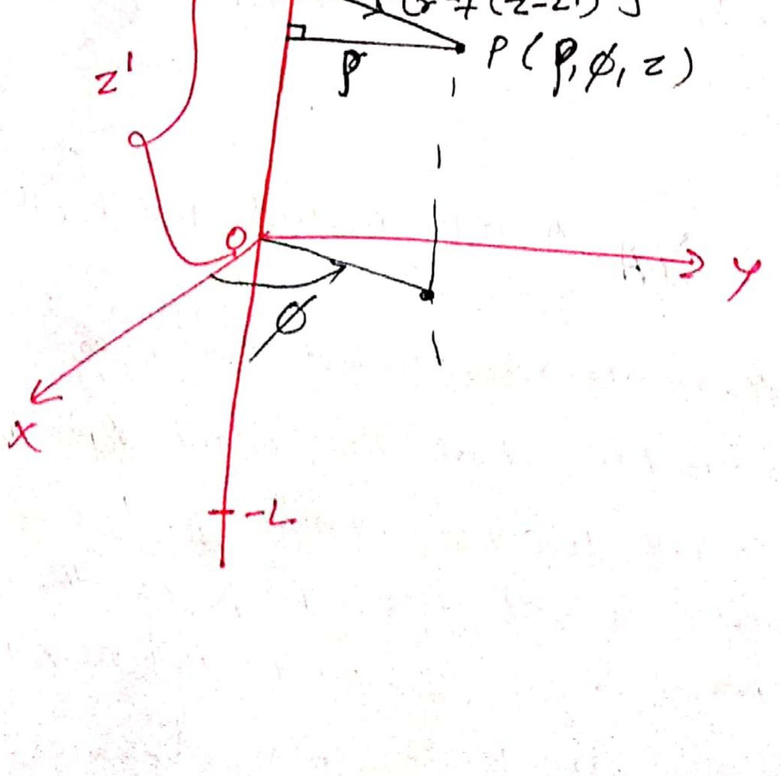
$$r_{QP} = [\rho^2 + (z-z')^2]^{1/2}$$

Then,

$$V = \int_{-L}^{L} \frac{\rho_L dz'}{4\pi\varepsilon_0 [\rho^2 + (z-z')^2]^{1/2}}$$

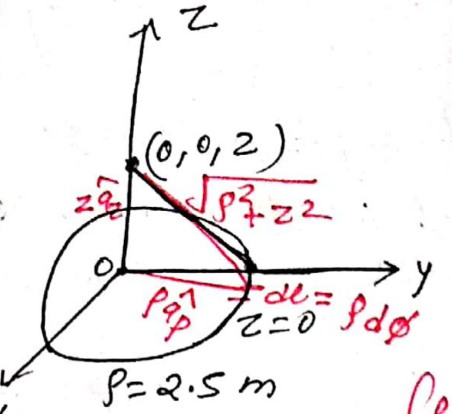
$$= -\frac{\rho_L}{4\pi\varepsilon_0} \ln \left(\frac{z-L + [\rho^2 + (z-L)^2]^{1/2}}{z+L + [\rho^2 + (z+L)^2]^{1/2}} \right)$$

$$= -\frac{\rho_L}{4\pi\varepsilon_0} \left(\sinh^{-1} \frac{z-L}{\rho} - \sinh^{-1} \frac{z+L}{\rho} \right)$$



Qn: If we take a reference ... for potential at infinity find the potential at (0, 0, 2) caused by this charge configuration in free space: (a) 12 nC/m on the line $\rho = 2.5\,m$, $z = 0$ (b) point charge of 18 nC at (1, 2, −1) (c) 12 nC/m on the line $y = 2.5$, $z = 0$, $-1.0 < x < 1.0$. [Ans: 529 V; 43.2 V, 66.3 V]

Sol.n (a) For line charge



$$V \text{ at } (0,0,2) = \int_0^{2\pi} \frac{\rho_L \cdot \rho \cdot d\phi}{4\pi\epsilon_0 \sqrt{\rho^2+z^2}}$$

$$= \int_0^{2\pi} \frac{12\times10^{-9} \times 2.5 \times d\phi}{4\pi\epsilon_0 \sqrt{2.5^2+2^2}}$$

$$\therefore V = \int \frac{\rho_L\, dl}{4\pi\epsilon_0 \sqrt{\rho^2+z^2}}$$

$$dl = \rho\, d\phi$$

$$= \frac{108 \times 2.5 \times 2\pi}{\sqrt{2.5^2+2^2}}$$

$$= 529.885\,V$$

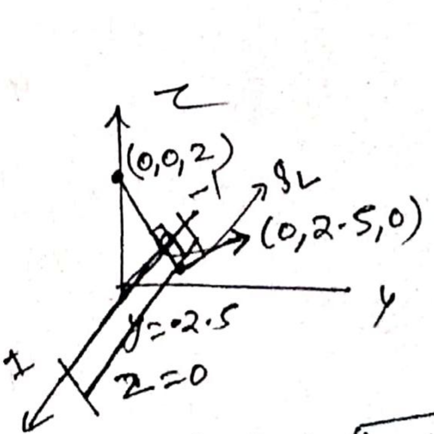
(b) For point charge

$$V \text{ at } (0,0,2) = \frac{Q}{4\pi\epsilon_0 r} = \frac{18\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi} \times \sqrt{(0-1)^2+(0-2)^2+(2+1)^2}}$$

$$= \frac{18\times9}{\sqrt{14}}$$

$$= 43.29\,V$$

(c) For line charge



$$V \text{ at } (0,0,2) = \int_{-1}^{1} \frac{\rho_L\, dx}{4\pi\epsilon_0 \sqrt{\rho^2+x^2}}$$

$$= \frac{12\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}} \cdot \ln\left(x + \sqrt{\rho^2+x^2}\right)\Big|_{-1}^{1}$$

where, $\rho = \sqrt{(0-0)^2 + (0-2.5)^2 + (2-0)^2}$
$= 3.20156$

$$= 108 \left[\ln\left(1+\sqrt{1+10.25}\right) - \ln\left(-1+\sqrt{1+10.25}\right)\right] = 66.41\,V$$

Qn: Assume a zero reference at infinity. Find the potential at $P(0,0,10)$ that is caused by this charge configuration in free space: (a) 20 nC at the origin; (b) 10 nC/m along the line $x=0$, $z=0$, $-1<y<1$; (c) 10 nC/m along the line $x=0, y=0, -1<z<1$.

Ans: (a) 17.98 V (b) 17.95 V (c) 18.04 V

Sol.^{n} (a) For point charge

$$V_P = \frac{Q}{4\pi\epsilon_0 r_P} \quad \text{where, } r_P = \sqrt{(0-0)^2+(0-0)^2+(10-0)^2} = 10$$

$$\therefore V_P = \frac{20\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}\times10} = 18\text{ V or } 17.98 \text{ if } \epsilon_0 = 8.85\times10^{-12}\text{ F/m}$$

(b) $$V_P = \int_{-1}^{1} \frac{\rho_L\, dy}{4\pi\epsilon_0\sqrt{\rho^2+y^2}}$$

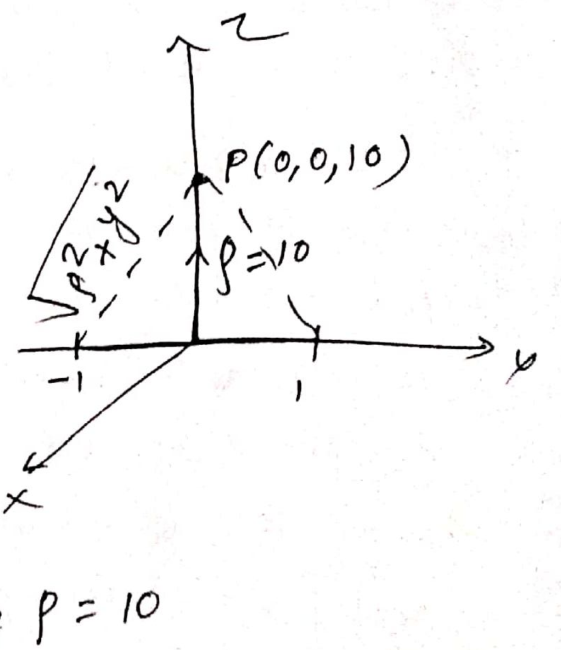
$$= \int_{-1}^{1} \frac{10\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}} \frac{dy}{\sqrt{\rho^2+y^2}}$$

$$= 90\times\int_{-1}^{1}\frac{dy}{\sqrt{\rho^2+y^2}}$$

$$= 90\times \ln\left(y+\sqrt{\rho^2+y^2}\right)\Big|_{-1}^{1} \quad \text{where } \rho = 10$$

$$= 90\left[\ln\left(1+\sqrt{100+1}\right) - \ln\left(-1+\sqrt{100+1}\right)\right]$$

$$= 17.97\text{ V}$$



Workdone is independent of Path taken (or Physical interpretation of line integral)

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इंजिनियर

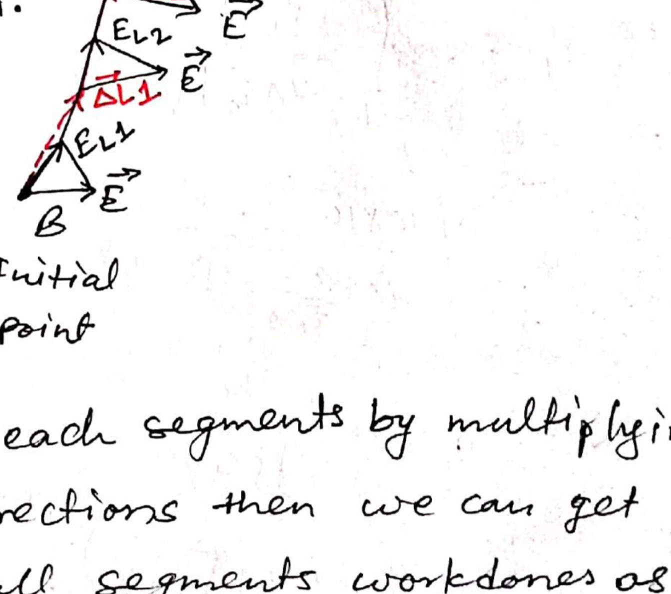
The workdone in moving a point charge Q from one point to another is given by

$$W = -Q \int_{Initial}^{Final} \vec{E} \cdot \vec{dl}$$ where, $\vec{dl}$ = differential vector path length.

$$= -Q \int_{Initial}^{Final} E_L \, dl$$

where, E_L is the component of $\vec{E}$ along $\vec{dl}$.

Consider a region having uniform electric field intensity as shown in figure. We have to calculate the workdone in carrying a charge Q from B to A.



From above workdone value in integral form, we can calculate the total workdone but if the path B to A is divided into very small segments and calculate the workdone for each segments by multiplying $\vec{E}$ by $\vec{\Delta L}$ along the path directions then we can get total workdone by adding all segments workdones as:

$$W = -Q\left(E_{L_1}\Delta L_1 + E_{L_2}\Delta L_2 + E_{L_3}\Delta L_3 + \cdots E_{L_N}\Delta L_N\right)$$

Using vector notation

$$W = -Q\left(\vec{E}_1 \cdot \vec{\Delta L}_1 + \vec{E}_2 \cdot \vec{\Delta L}_2 + \cdots + \vec{E}_N \cdot \vec{\Delta L}_N\right)$$

Since, $\vec{E}_1 = \vec{E}_2 = \vec{E}_3 = \cdots = \vec{E}_N = \vec{E}$ for Uniform field

Then,

$$W = -Q\vec{E}\cdot(\vec{\Delta L_1} + \vec{\Delta L_2} + \vec{\Delta L_3} + \cdots + \vec{\Delta L_N}) = \vec{L}_{BA}$$

= a vector directed from initial point o final point.

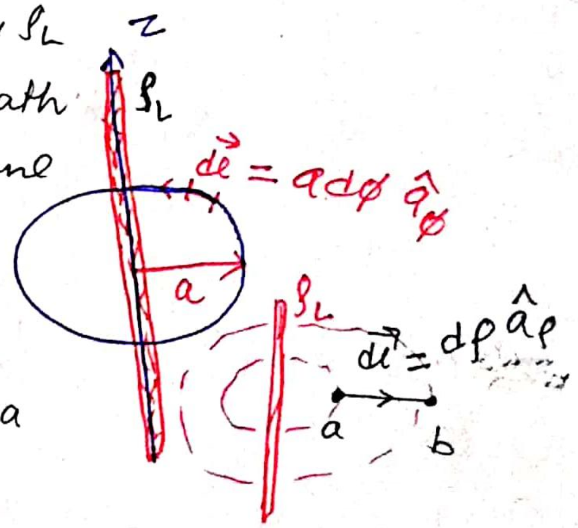
Since, the vector from initial point to final point will be same whether the path is different, the work-done in moving a point charge from initial point to final point is independent of the path taken.

i.e $\boxed{W = -Q\vec{E}\cdot\vec{L}_{BA}}$

This shows that workdone is independent of the path taken.

Workdone in carrying a charge Q around a circular path of radius, a in presence of the field due to a uniform line charge at center of the circular Path.

Here, a uniform line charge of density ρ_L is placed in the center of circular path where a charge, Q is moved from one point to another.



From previous derivations

$\vec{E}$ due to uniform line charge at a point is

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$

Let us take a differential displacement $\vec{dl} = a\,d\phi\,\hat{a}_\phi$ in cylindrical Co-ordinate systems.

Then, the workdone is given by

$$W = -Q \int_{initial}^{Final} \vec{E} \cdot \vec{dl}$$

$$= -Q \int_{\phi=0}^{2\pi} \frac{\rho_L}{2\pi \epsilon_0 \rho} \hat{a}_\rho \cdot a \, d\phi \, \hat{a}_\phi$$

$$= 0 \quad [\hat{a}_\rho \cdot \hat{a}_\phi = 0]$$

i.e the workdone in moving the point charge Q, in circular path is zero.

Again, if we choose the differential displacement in the path $\rho = a$ to $\rho = b$ in radial path then

$\vec{dl} = d\rho \, \hat{a}_\rho$ and $W = -Q \int_{initial}^{Final} \frac{\rho_L}{2\pi \epsilon_0 \rho} \hat{a}_\rho \cdot d\rho \, \hat{a}_\rho$

or $W = -Q \int_a^b \frac{\rho_L}{2\pi \epsilon_0 \rho} d\rho$

$$\therefore \boxed{W = \frac{-Q \rho_L}{2\pi \epsilon_0} \ln\left(\frac{b}{a}\right)}$$

Since, b is greater than a so the value $\ln(b/a)$ becomes +ve & workdone, W will be −ve i.e by field.

Conservative field: Any field that satisfies $\oint \vec{E} \cdot \vec{dl} = 0$ (no work is done in carrying the unit charge around any closed path) is said to be a conservative field. i.e no work is done (or the energy is conserved) around a closed path.

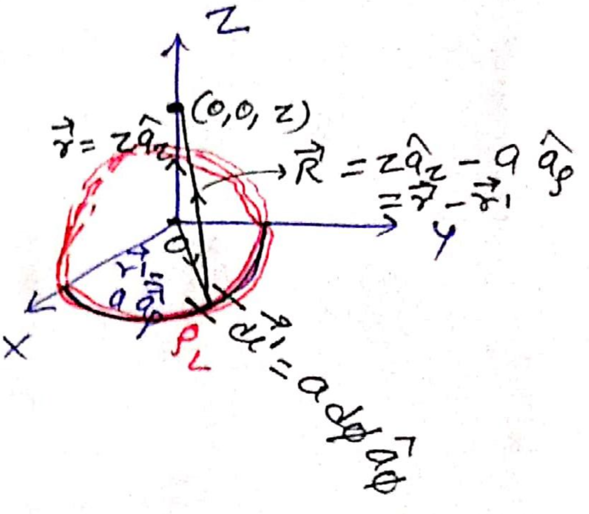
(9) The gravitational field is also conservative for any energy expended in moving (raising) an object against the field is recovered exactly when the [illegible] is returned (lowered) to its original position.

Qn: Find the electric potential at a point on the z-axis due to a line charge ρ_L in the form of a circular ring of radius 'a' placed on the xy-plane (z=0 plane).

Soln

we have,

$$V = \int \frac{\rho_L' \, dl'}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|}$$



$$= \int_{\phi=0}^{2\pi} \frac{\rho_L a \, d\phi}{4\pi\varepsilon_0 \sqrt{a^2+z^2}}$$

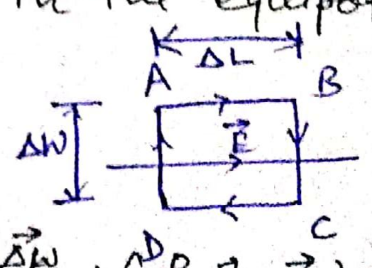
$$= \frac{2\pi a \rho_L}{4\pi\varepsilon_0 \sqrt{a^2+z^2}}$$

$$= \frac{a \rho_L}{2\varepsilon_0 \sqrt{a^2+z^2}} \quad //$$

Equipotential Surface

The surface over which the potential is same at all the points is known as equipotential surface.

If A and B are two points in the equipotential surface then



$$V_A - V_B = 0$$

i.e From figure

$$\oint_{ABCDA} \vec{E} \cdot \vec{dl} = \int_A^B \vec{E} \cdot \vec{\Delta L} + \int_B^C \vec{E} \cdot \vec{\Delta W} + \int_C^D (-\vec{E} \cdot \vec{\Delta L}) + \int_D^A (-\vec{E} \cdot \vec{\Delta W})$$

$= \cancel{E\Delta L} + 0 - \cancel{E\Delta L} + 0$

$= 0$

i.e $\oint \vec{E} \cdot \vec{dl} = 0$ [It is only applicable for static fields]

These surfaces for which $\oint \vec{E} \cdot \vec{dl} = 0$ is known as equipotential surface. $\vec{E}$ is known as conservative field.

Qn: A uniform sheet of charge, $\rho_s = 40\epsilon_0\ C/m^2$ is located in the plane $x = 0$ in free space. A uniform line charge ρ_L of 0.6 nC/m lies along the z-axis in free space. Find the potential at point P(3,4,2) if V = 5V at A (2,−9,3).

Sol.

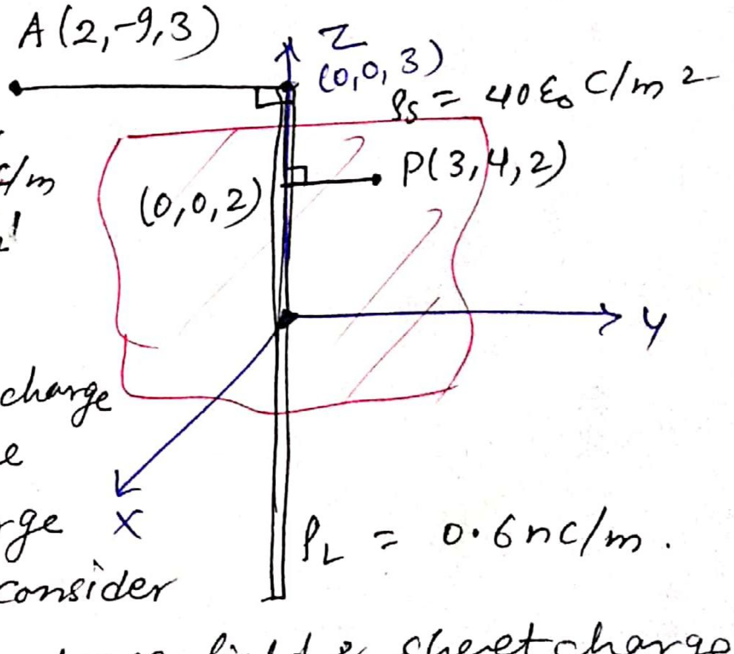
Given,

uniform sheet charge,

$\rho_s = 40\,\epsilon_0\ C/m^2$ at $x=0$ plane,

uniform line charge, $\rho_L = 0.6\,nC/m$

along z-axis and V = 5V at A (2,−9,3).



Since, the field due to line charge & sheet charge affects the movement of the unit charge in the field, we need to consider the potential due to line charge field & sheet charge fields.

Now,

Potential at P due to line charge is

$$V_{P_\ell} = \frac{-\rho_L}{2\pi\epsilon_0} \ln(\rho_p) + C_1$$

where, $\rho_p = \sqrt{(3-0)^2 + (4-0)^2 + (2-2)^2}$

$= 5$

(b). Again, potential at p due to sheet charge is

$$V_{P_s} = -\frac{\rho_s}{2\varepsilon_0} x_p + C_2$$

where, $x_p = 3$

Then,

total potential at P, $V_P = V_{P_\ell} + V_{P_s}$

$$= -\frac{\rho_L}{2\pi\varepsilon_0} \ln(5) + \left(\frac{-\rho_s}{2\varepsilon_0} \cdot 3\right) + C_1 + C_2$$

$$= \frac{-0.6\times10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \cdot \ln(5) - \frac{40\varepsilon_0}{2\varepsilon_0} \times 3 + C$$

or, $V_P = -0.6 \times 18 \times 1.61 - 20\times3 + C$

or, $V_P = -77.39 + C$

where, $C = C_1 + C_2$

since, $V_A = 5V$

$$\text{or, } -\frac{\rho_L}{2\pi\varepsilon_0} \ln(\rho_A) - \frac{\rho_s}{2\varepsilon_0} x_A + C = 5$$

$$\text{or, } -\frac{0.6\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}} \ln(\rho_A) - \frac{40\varepsilon_0}{2\varepsilon_0} \cdot x_A + C = 5$$

where, $\rho_A = \sqrt{(2-0)^2 + (-9-0)^2 + (3-3)^2}$

$= \sqrt{4+81}$

$= \sqrt{85}$

& $x_A = 2$

(92) $or, -0.6 \times 18 \ln(\sqrt{85}) - 20 \times 2 + C = 5$

$or, -29.99 - 40 + C = 5$

$or, \ C = 5 + 59.99 = 64.99$

Now,

$V_p = -77.39 + C = -77.39 + 64.99$

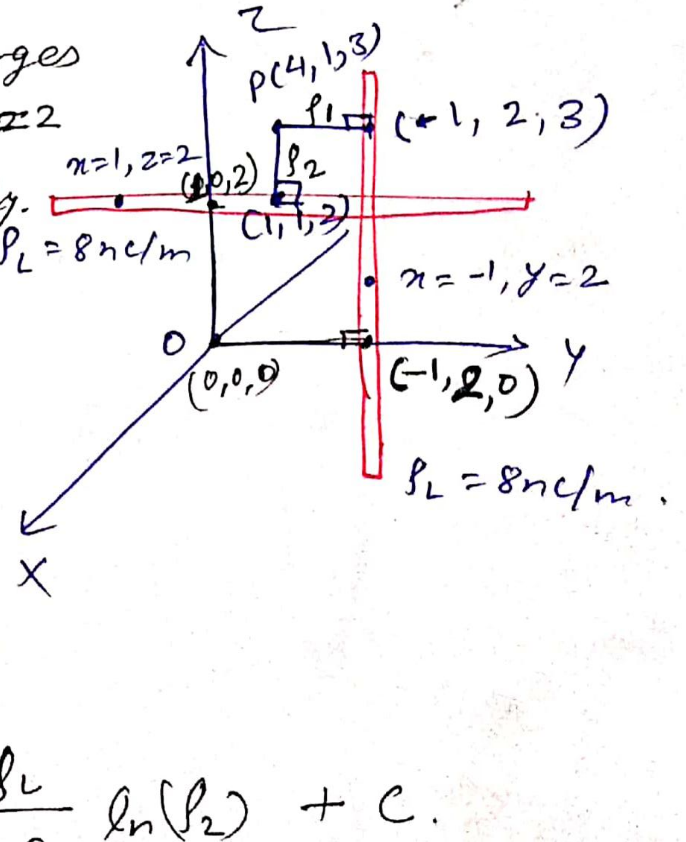
$\therefore V_p = -8.4V$

Qn: Two uniform line charges 8nC/m each are located at $x=1, z=2$ and at $x=-1, y=2$ in free space. If the potential at the origin is 100V, find V at P(4,1,3).
[2071 shrawan]

Sol^n

Given, uniform line charges of $\rho_L = 8nC/m$ lies at $x=1, z=2$ and $x=-1, y=2$ respectively.

Potential at origin, $V_0 = 100V$.



Now,

we have, the potential at Point P (4,1,3) is due to two line charges. so,

$$V_p = \frac{-\rho_L}{2\pi\varepsilon_0} \ln(\rho_1) - \frac{\rho_L}{2\pi\varepsilon_0} \ln(\rho_2) + C.$$

$$= -\frac{8\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}} \cdot \ln(\rho_1) - \frac{8\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}} \cdot \ln(\rho_2) + C$$

$$= -8\times18 \ln(\rho_1) - 8\times18 \ln(\rho_2) + C$$

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where, $\rho_1 = \sqrt{(4+1)^2 + (1-2)^2 + (3-3)^2}$

$= \sqrt{25+1}$

$= \sqrt{26}$

& $\rho_2 = \sqrt{(4-1)^2 + (1-1)^2 + (3-2)^2}$

$= \sqrt{9+1}$

$= \sqrt{10}$

$\therefore V_P = -144 \ln(\sqrt{26}) - 144 \ln(\sqrt{10}) + C$

or, $V_P = -234.58 - 165.786 + C$

or, $V_P = -400.37 + C$

Again,

$V_0 = 100\,V$

or, $\frac{(-\rho_L)}{2\pi\varepsilon_0} \ln(\rho_{10}) + \frac{(-\rho_L)}{2\pi\varepsilon_0} \ln(\rho_{20}) + C = 100$

or, $-8\times18 \ln(\rho_{10}) - 8\times18 \ln(\rho_{20}) + C = 100$

where, $\rho_{10} = \sqrt{(0-1)^2 + (0-0)^2 + (0-2)^2} = \sqrt{5}$

& $\rho_{20} = \sqrt{(0+1)^2 + (0-2)^2 + (0-0)^2} = \sqrt{5}$

$\therefore -144 \ln(\sqrt{5}) - 144 \ln(\sqrt{5}) + C = 100$

or, $-288 \ln(\sqrt{5}) + C = 100$

or, $-231.76 + C = 100$

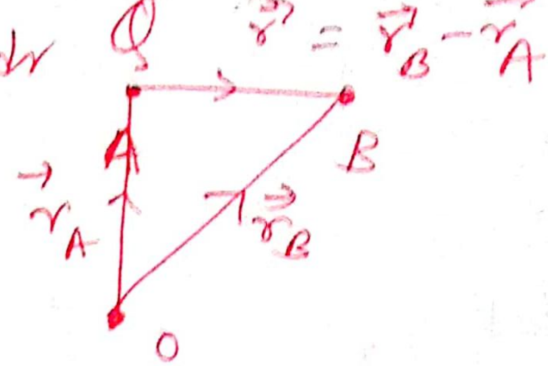
or, $C = 331.76 \Rightarrow V_p = -400.37 + 331.76 = -68.61\,V$

For point charge

$$W = -q \int_A^B \vec{E} \cdot \vec{dl}$$

$$= -q \int_A^B \frac{Q'}{4\pi\varepsilon_0 r^2} \hat{a}_r \cdot dr\, \hat{a}_r$$

$$= -\frac{q \cdot Q'}{4\pi\varepsilon_0} \int_A^B \frac{1}{r^2} dr$$



$$= -\frac{qQ'}{4\pi\varepsilon_0} \left(\frac{-1}{r}\right)\Bigg|_A^B$$

$$= \frac{qQ'}{4\pi\varepsilon_0} \left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$= q\left[\frac{Q'}{4\pi\varepsilon_0 r_B} - \frac{Q'}{4\pi\varepsilon_0 r_A}\right]$$

or, $\dfrac{W}{q} = V_{BA} = V_B - V_A$ $\Rightarrow$

where,

$$\boxed{V_B = \frac{Q'}{4\pi\varepsilon_0 r_B} + C}$$ → reference point

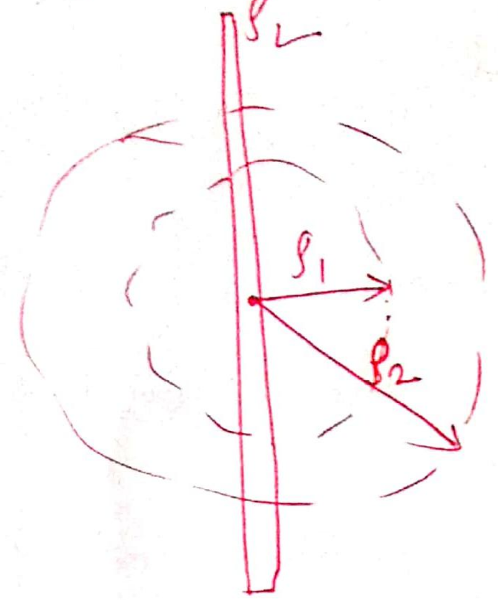
& $$\boxed{V_A = \frac{Q'}{4\pi\varepsilon_0 r_A} + C}$$ → reference point

* If $C = 0$ then reference is at ∞.

& $V_{\infty B} = \dfrac{Q'}{4\pi\varepsilon_0 r_B}$.

line charge

$$W = -Q\int_{\rho_1}^{\rho_2} \vec{E}\cdot\vec{dl} = -Q\int_{\rho_1}^{\rho_2} \frac{\rho_L}{2\pi\epsilon_0\rho}\hat{a}_\rho \cdot d\rho\, \hat{a}_\rho$$



$$= -Q\frac{\rho_L}{2\pi\epsilon_0}\left[\ln(\rho_2) - \ln(\rho_1)\right]$$

$$= -Q\frac{\rho_L}{2\pi\epsilon_0}\ln\left(\frac{\rho_2}{\rho_1}\right)$$

or, $$\frac{W}{Q} = \frac{-\rho_L}{2\pi\epsilon_0}\ln\left(\frac{\rho_2}{\rho_1}\right)$$

or, $$V_{\rho_2\rho_1} = \frac{-\rho_L}{2\pi\epsilon_0}\ln\left(\frac{\rho_2}{\rho_1}\right)$$

$$= \frac{-\rho_L}{2\pi\epsilon_0}\ln(\rho_2) + \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_1)$$

$$= V_{\rho_2} - V_{\rho_1}$$

i.e, $$\boxed{V_{\rho_1} = \frac{-\rho_L}{2\pi\epsilon_0}\ln(\rho_1) + C_1}$$ reference point

if ref. is at ∞ then $C_1 = 0 = C_2$

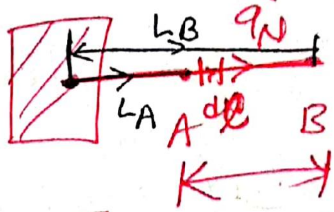
& $$\boxed{V_{\rho_2} = \frac{-\rho_L}{2\pi\epsilon_0}\ln(\rho_2) + C_2}$$

For sheet charge

$$W = -Q \int_A^B \vec{E} \cdot \vec{dl}$$

$$= -Q \int_A^B \frac{\rho_s}{2\varepsilon_0} \cdot \hat{a}_N \cdot dl\, \hat{a}_N$$

$$= -Q \frac{\rho_s}{2\varepsilon_0} \cdot [L_B - L_A]$$



or, $$\frac{W}{Q} = V_{BA} = \frac{-\rho_s}{2\varepsilon_0} [L_B - L_A]$$ L_A L_B

or, $V_{BA} = V_B - V_A$

$$\therefore \boxed{V_B = \frac{-\rho_s}{2\varepsilon_0} L_B + C_1}$$

→ reference point
↳ if reference is at ∞ then $C_1 = 0 = C_2$

$$\& \boxed{V_A = \frac{-\rho_s}{2\varepsilon_0} L_A + C_2}$$

(94) Qn: An electric field is expressed in Cartesian Co-ordinate system by $\vec{E} = 6x^2\hat{a}_x + 6y\hat{a}_y + 4\hat{a}_z$ V/m. Find (a) V_{MN} if points M & N are specified by M(2, 6, −1) and N(−3, −3, 2) (b) V_M if $V_Q = 0$ at Q(4, −2, −35) (c) V_N if $V = 2$ at P(1, 2, −4).

Sol.ⁿ Given, $\vec{E} = 6x^2\hat{a}_x + 6y\hat{a}_y + 4\hat{a}_z$ V/m.

(a)

$$V_{MN} = -\int_N^M (6x^2\hat{a}_x + 6y\hat{a}_y + 4\hat{a}_z)\,(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z)$$

$$= -\int_{-3}^{2} 6x^2\,dx - 6\int_{-3}^{6} y\,dy - 4\int_{2}^{-1} dz$$

$$= -\frac{6}{3}\left[(2)^3 - (-3)^3\right] - \frac{6}{2}\left[6^2 - (-3)^2\right] - 4(-1-2)$$

$$= -139\text{ V}.$$

Again,

(b)

$V_Q = 0$ at Q(4, −2, −35) then

$$V_{MQ} = -\int_Q^M \vec{E}\cdot\vec{dl}$$

$$= \int_Q^M (6x^2\hat{a}_x + 6y\hat{a}_y + 4\hat{a}_z)\cdot(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z)$$

$$= -\int_4^2 6x^2\,dx - 6\int_{-2}^{6} y\,dy - 4\int_{-35}^{-1} dz$$

$$= -\frac{6}{3}\left[2^3 - 4^3\right] - \frac{6}{2}\left[6^2 - (-2)^2\right] \mp 4\left[-1+35\right]$$

$$= -120\text{ V}$$

Now,

$$V_{MQ} = V_M - V_Q$$

or, $-120 = V_M - 0 \Rightarrow V_M = -120\text{ V}$

(95) (c) $V_P = 2V$ at $P(1,2,-4)$

then,

$$V_{NP} = -\int_P^N \vec{E}\cdot\vec{dl}$$

$$= -\int_P^N (6x^2\hat{a}_x + 6y\hat{a}_y + 4\hat{a}_z)\cdot(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z)$$

$$= -6\int_1^{-3} x^2dx - 6\int_2^{-3} y\,dy - 4\int_{-4}^{2} dz$$

$$= \frac{-6}{3}\left[(-3)^3 - 1^3\right] - \frac{6}{2}\left[(-3)^2 - 2^2\right] - 4\left[2+4\right]$$

$$= 17V.$$

Now,

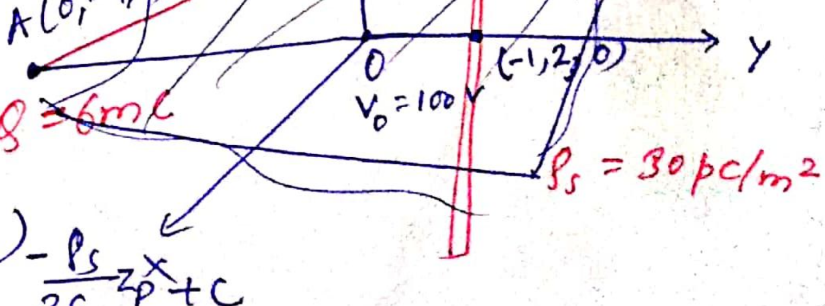
$$V_{NP} = V_N - V_P$$

or, $17 = V_N - 2$

or, $V_N = 19V$ //

Qn: A line charge of 8nc/m is located at $x=-1, y=2$, a point charge of 6mc at $y=-4$ and a surface charge of 30pc/m² at $z=0$. If the potential at origin is 100V, find the potential at $P(4,1,3)$. [2072 Kartik]

Sol.ⁿ



$$V_P = \frac{Q}{4\pi\epsilon_0 r_P} + \frac{(-\rho_L)}{2\pi\epsilon_0}\ln(\rho_P) + \frac{(-\rho_s)}{2\epsilon_0} z_P + C$$

$$= \frac{6\times10^{-3}}{4\pi\times\frac{10^{-9}}{36\pi} r_P} - \frac{8\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}}\ln(\rho_P) - \frac{\rho_s}{2\epsilon_0} z_P + C$$

where, $r_p = \sqrt{(4-0)^2 + (1+4)^2 + (3-0)^2}$

$= \sqrt{16+25+9}$

$= \sqrt{50}$

$\rho_p = \sqrt{(4+1)^2 + (1-2)^2 + (3-3)^2}$

$= \sqrt{25+1}$

$= \sqrt{26}$

$z_p = 3$

so,

$$V_p = \frac{54\times10^6}{\sqrt{50}} - 144\ln(\sqrt{26}) - \frac{30\times10^{-12}}{\frac{2\times10^{-9}}{36\pi}}\times 3 + C$$

or, $V_p = 7.64\times10^6 - 234.58 - 5.089 + C$

or, $V_p = 7639760.331 + C$

Again,

$V_0 = 100\,V$

$$\text{or, } \frac{6\times10^{-3}}{\frac{4\pi\times10^{-9}}{36\pi} r_0} - \frac{8\times10^{-9}}{\frac{2\pi\times10^{-9}}{36\pi}}\ln(\rho_0) - \frac{30\times10^{-12}}{\frac{2\times10^{-9}}{36\pi}}\cdot z_0 + C = 100$$

$$\text{or, } \frac{54\times10^6}{r_0} - 144\ln(\rho_0) - 15\times36\pi\times10^{-3} z_0 + C = 100$$

where, $r_0 = 4$, $\rho_0 = \sqrt{(0+1)^2 + (0-2)^2 + (0-0)^2} = \sqrt{5}$

$z_0 = 0$

So,

$$\frac{54\times10^6}{4} - 144\ln(\sqrt{5}) - 0 + C = 100$$

or $13.5\times10^6 - 115.88 + C = 100$

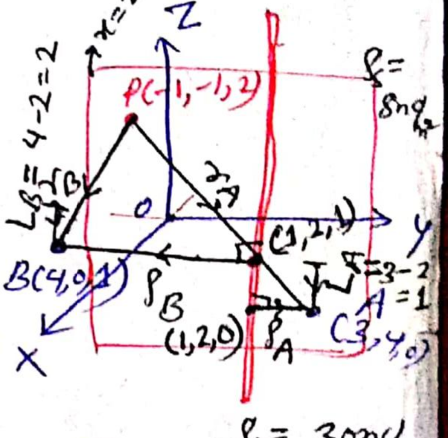
$\therefore C = -13499784.12$

Now,

$$V_P = 7639760.331 - 13499784.12$$
$$= -5860023.739$$
$$= -5.86 \times 10^6 \text{ V}$$
$$= -5860 \text{ kV}.$$

Qn: Given a surface charge density of $8\,nC/m^2$ on the plane $x = 2$, a line charge density of $30\,nC/m$ on the line $x = 1, y = 2$ and a $1\,\mu C$ point charge at $P(-1,-1,2)$, find V_{AB} for points $A(3,4,0)$ and $B(4,0,1)$. [Ans:- $V_{AB} = 193V$]

Soln:



$$V_A = \frac{Q}{4\pi\epsilon_0 r_A} - \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_A) - \frac{\rho_s}{2\epsilon_0}L_A + C$$

$$V_B = \frac{Q}{4\pi\epsilon_0 r_B} - \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_B) - \frac{\rho_s}{2\epsilon_0}L_B + C$$

$$\therefore V_A - V_B = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] + \frac{\rho_L}{2\pi\epsilon_0}\left[\ln\left(\frac{\rho_B}{\rho_A}\right)\right] + \frac{\rho_s}{2\epsilon_0}[L_B - L_A]$$
$$= 193V$$

Qn: Let a uniform surface charge density of $5\,nC/m^2$ be present at the $z = 0$ plane, a uniform line charge density of $8\,nC/m$ be located at $x = 0, z = 4$ and a point charge of $2\,\mu C$ be present at $P(2,0,0)$. If $V = 0$ at $M(0,0,5)$ then find V at $N(1,2,3)$.

Soln:

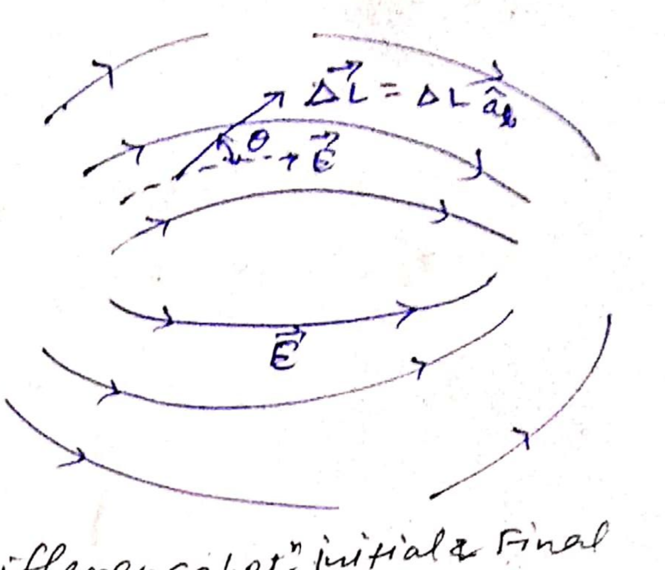
$$V_M = 0 = \frac{Q}{4\pi\epsilon_0 r_M} - \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_M) - \frac{\rho_s}{2\epsilon_0}L_M + C$$

$$V_N = \frac{Q}{4\pi\epsilon_0 r_N} - \frac{\rho_L}{2\pi\epsilon_0}\ln(\rho_N) - \frac{\rho_s}{2\epsilon_0}L_N + C = 1.98\text{ kV}$$

Potential Gradient

Consider a region as shown in figure, when both $\vec{E}$ and V change from point to point, we have

$$V = - \int_{\text{Initial}}^{\text{Final}} \vec{E} \cdot \vec{dl}$$



Let us assume the small incremental length $\vec{\Delta L}$ where $\vec{E}$ is almost uniform & Constant.

Then,

$$\Delta V = - \vec{E} \cdot \vec{\Delta L}$$

which is the incremental potential difference bet.ⁿ initial & Final points of $\vec{\Delta L}$.

If $\vec{\Delta L}$ makes an angle θ with $\vec{E}$ as shown in figure, then

$$\Delta V = - E \cos\theta \, \Delta L$$

$$\text{or,} \quad \frac{\Delta V}{\Delta L} = - E \cos\theta$$

Here, V may be interpreted as a function $V(x, y, z)$. If we assume a specified starting point or zero reference and then let our end point be (x, y, z), then the result of the integration is a unique function of the end point (x, y, z) because $\vec{E}$ is a conservative field. Using limit we get

$$\lim_{\Delta L \to 0} \frac{\Delta V}{\Delta L} = - E \cos\theta$$

$$\text{or,} \quad \frac{dV}{dL} = - E \cos\theta$$

Here, the value of $\vec{E}$ is a definite value at the point at which we need to calculate & is independent of the direction of $\vec{\Delta L}$. The magnitude ΔL is also constant, and the variable is $\hat{a}_L$ only the unit vector showing the direction of $\vec{\Delta L}$. From above expression, maximum +ve increment of potential, ΔV_{max} will occur when $\cos\theta$ is -1, or $\vec{\Delta L}$ points in the direction opposite to $\vec{E}$.

(99) For this,

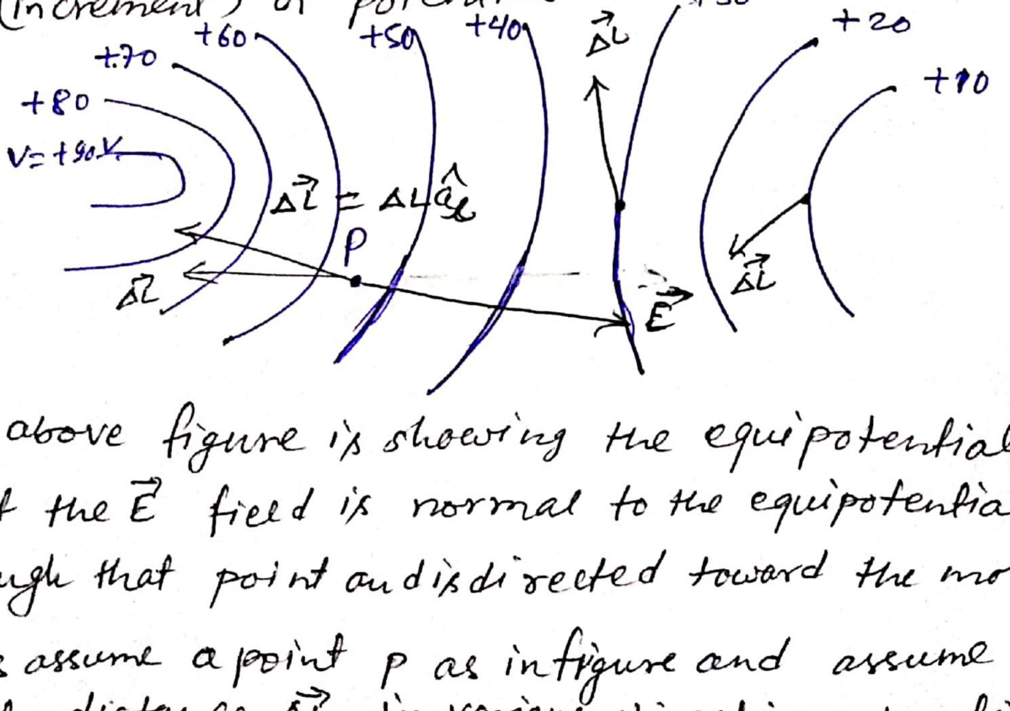
$$\left.\frac{dv}{dl}\right|_{max} = +E$$

This shows that,

① The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.

② The direction of $\vec{\Delta L}$ is exactly opposite to $\vec{E}$ or the dirⁿ of $\vec{E}$ is opposite to the dirⁿ in which the potential is increasing the most rapidly.

Hence, the dirⁿ of $\vec{E}$ is opposite of maximum positive change (increment) of potential.



The above figure is showing the equipotential surfaces. At any point the $\vec{E}$ field is normal to the equipotential surface passing through that point and is directed toward the more negative surfaces.

Let us assume a point P as in figure and assume a small incremental distance $\vec{\Delta L}$ in various directions, hunting for that dirⁿ in which the potential is increasing the most rapidly.

From the sketch, the dirⁿ of maxᵐ increasing of potential appears to be left and slightly upward. It seems that the dirⁿ of $\vec{E}$ is opposite to the maxᵐ increment of potential.

Also, the dirⁿ in which the potential is increasing the most rapidly is perpendicular to the equipotential surfaces. So, if $\vec{\Delta L}$ has dirⁿ along equipotential surface then $\Delta V = 0$.

i.e $\Delta V = -\vec{E} \cdot \vec{\Delta l} = 0$

Since, neither $\vec{E}$ nor $\Delta \vec{l}$ is zero, $\vec{E}$ must be perpendicular to $\vec{\Delta l}$ or perpendicular to the equipotential surfaces. Then, assuming $\hat{a}_N$ as the unit vector along normal to the equipotential surface and directed toward the higher potentials.

$$\therefore \vec{E} = -\left.\frac{dv}{dl}\right|_{max} \hat{a}_N$$

which shows that the magnitude of $\vec{E}$ is given by the max.^{m} rate of change of V and the direction of $\vec{E}$ is normal to the equipotential surface (in the dir.^{n} of decreasing potential).

Since, $\left.\frac{dv}{dl}\right|_{max}$ occurs in $\hat{a}_N$ dir.^{n}, we may write as:

$$\left.\frac{dv}{dl}\right|_{max} = \frac{dv}{dN} \quad \& \quad \therefore \vec{E} = -\frac{dv}{dN}\hat{a}_N$$

The operation on V by which $-\vec{E}$ is obtained is known as gradient and the gradient of a scalar field T is defined as

$$\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\hat{a}_N$$

where, $\hat{a}_N$ is a unit vector normal to the equipotential surfaces, and that normal is chosen which points in the dir.^{n} of increasing values of T.

Now,

$$\boxed{\vec{E} = -\text{grad } V}$$

Since, V is a unique function of x, y & z, we may take its total differential

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

Also, we have,

$$dv = -\vec{E}\cdot\vec{dl} = -E_x\,dx - E_y\,dy - E_z\,dz$$

(124) Since both expressions are true for any dx, dy & dz

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y}$$

$$E_z = -\frac{\partial V}{\partial z}$$

$$\therefore \boxed{\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)}$$

Since, $\text{grad}\cdot V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z = \nabla V$

The gradient of any scalar gives the maximum rate of change of that scalar w.r.t. distance and also indicates where that max^m rate of change w.r.t. distance is taking place. This is physical interpretation of gradient.

Hence, $\boxed{\vec{E} = -\nabla V}$

example

Assuming reference at infinity the potential at any point due to the field of point charge Q is

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

Now,

$$\vec{E} = -\nabla V$$

$$= -\frac{\partial V}{\partial r}\hat{a}_r \quad [\because \vec{E} \text{ is only function of } \vec{r}]$$

$$= -\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0 r}\right)\hat{a}_r$$

$$= \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r$$

$$\therefore \boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{a}_r}$$

Qn: A potential field is given as $V = 100 e^{-5x} \sin 3y \cdot \cos 4z$ V. Let point $P(0.1, \frac{\pi}{12}, \frac{\pi}{24})$ be located at a conductor-free space boundary. At point P, find the magnitude of (a) V. (b) $\vec{E}$.

Sol.ⁿ

(a) V at $P(0.1, \frac{\pi}{12}, \frac{\pi}{24}) = V_P = 100 e^{-0.05} \sin(3\times\frac{\pi}{12}) \cdot \cos(4\times\frac{\pi}{24})$

$= 37.142$ V

(b) $\vec{E} = -\nabla V = 500 e^{-5x} \sin 3y \cos 4z \, \hat{a}_x - 300 e^{-5x} \cos 3y \cos 4z \, \hat{a}_y + 400 e^{-5x} \sin 3y \cdot \sin 4z \, \hat{a}_z$

Substituting co-ordinates of P in $\vec{E}$, we get

$\vec{E}$ at $P = \vec{E}_P = 185.711 \hat{a}_x - 111.427 \hat{a}_y + 85.7769 \hat{a}_z$ V/m

Qn: Within the cylinder $\rho = 2$, $0 < z < 1$ the potential given by: $V = 100 + 50\rho + 150\rho \sin\phi$ V find:

(a) Electric field Intensity $(\vec{E})$ at $P(1, 60°, 0.5)$ in freespace

(b) Potential gradient $\left(\frac{dV}{dN}\right)$

(c) Volume charge Density at $P(1, 60°, 0.5)$ in free space

(d) How much charge lies within the cylinder?

[2071 Chaitra]

Sol.ⁿ

Given, potential within the cylinder is

$V = 100 + 50\rho + 150\rho \sin\phi$ V

(a) $\vec{E}$ at $P(1, 60°, 0.5)$ is

$$\vec{E}_P = -\nabla V = -\frac{\partial V}{\partial \rho} \hat{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi - \frac{\partial V}{\partial z} \hat{a}_z$$

$$= [-50 - 150 \sin\phi] \hat{a}_\rho - \frac{1}{\rho} [150\rho \cos\phi] \hat{a}_\phi - 0 \hat{a}_z$$

$$= -[50 + 150 \sin\phi] \hat{a}_\rho - 150 \cos\phi \, \hat{a}_\phi$$

(103) Substituting value of $P(1, 60^\circ, 0.5)$

$$\vec{E}_P = -[50 + 150\sin 60^\circ]\,\hat{a}_\rho - 150\cos 60^\circ\,\hat{a}_\phi$$
$$= -179.9\,\hat{a}_\rho - 75\,\hat{a}_\phi \text{ V/m}$$

Again,

(b) Potential Gradient $\left(\frac{dV}{dN}\right) = +|\vec{E}| = +\sqrt{(-179.9)^2 + (-75)^2}$

$$= +194.90 \text{ V/m}$$

Also,

(c) $\rho_v = \nabla \cdot \vec{D}_P$ [From Maxwell's first Equation]

$$= \epsilon_0 \nabla \cdot \vec{E}_P$$

$$= \epsilon_0 \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\{\rho(-50 - 150\sin\phi)\} + \frac{1}{\rho}\frac{\partial}{\partial \phi}(-150\cos\phi) \right]$$

$$= \epsilon_0 \cdot \left[-50 - \cancel{\frac{1}{\rho}150\sin\phi} + \cancel{\frac{1}{\rho}150\sin\phi}\right]$$

$$= -\frac{50}{\rho}\,\epsilon_0$$

Substituting $P(1, 60^\circ, 0.5)$

$$\rho_v = -\frac{50}{1} \times 8.85\times 10^{-12}$$

$$= -442.5 \text{ pC/m}^3.$$

(d) For the cylinder, $\rho = 2$, $0 < z < 1$, the charge withing the cylinder will be

$$Q = \int_{vol.} \rho_v \, dv = \int_{\rho=0}^{2}\int_{\phi=0}^{2\pi}\int_{z=0}^{1} -\frac{50}{\rho}\epsilon_0\, \rho \, d\rho\, d\phi\, dz$$

(104)

$$Q = \int_{\rho=0}^{2}\int_{\phi=0}^{2\pi} (-50\,\varepsilon_0)\, d\rho\, d\phi$$

$$= \int_{\rho=0}^{2} -50\,\varepsilon_0 \cdot 2\pi\, d\rho$$

$$= -50 \times 2\pi\, \varepsilon_0 \times 2$$

$$= -2.78\ nC \times 2$$

$$= -5.56\ nC$$

Qn: Given the potential field $V = \dfrac{100xz}{x^2+4}$ volts in free space:

(a) Find $\vec{D}$ at the surface $z=0$

(b) Show that the $z=0$ surface is an equipotential surface

(c) Assume that ~~the~~ $z=0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0<x<2$, $-3<y<0$. [2069 Chaitra]

Sol.ⁿ Given the potential field

$$V = \frac{100xz}{x^2+4} \text{ volts.}$$

(a) $\vec{E} = -\nabla V$

$$= -\left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right)\left(\frac{100xz}{x^2+4}\right) = \frac{-100x}{x^2+4}\hat{a}_z \text{ V/m}$$

& $\vec{D} = \varepsilon_0\vec{E}$. (Put $z=0$) $= -\dfrac{100\varepsilon_0 x}{x^2+4}\hat{a}_z$ C/m².

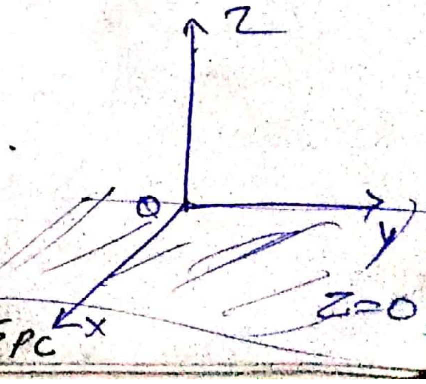
(b) To show $z=0$ surface is an equipotential surface

$$V = \frac{100 \times x \times 0}{x^2+4} = 0\ V$$

Since, V is constant for any value of x & y.

(c) Total charge within that conductor is

$$Q = \oint_S \vec{D}\cdot d\vec{S} = \int_S \vec{D}\cdot(dx\,dy)\,\hat{a}_z = -920.15\ pC$$



$$\frac{d(u/v)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2) + C$$

$$\vec{E} = -\left[\frac{\partial}{\partial x}\left(\frac{100xz}{x^2+4}\right)\hat{a}_x + \frac{\partial}{\partial y}\left(\frac{100xz}{x^2+4}\right)\hat{a}_y + \frac{\partial}{\partial z}\left(\frac{100xz}{x^2+4}\right)\hat{a}_z\right]$$

$$= -\left[100z\left\{\frac{(x^2+4) - x\cdot 2x}{(x^2+4)^2}\right\}\hat{a}_x + 0\hat{a}_y + \frac{100x}{x^2+4}\hat{a}_z\right]$$

Put $z = 0$

$$\therefore \vec{E} = -\frac{100x}{x^2+4}\hat{a}_z \text{ V/m}.$$

$$\& \ \vec{D} = \frac{-100\varepsilon_0 x}{x^2+4}\hat{a}_z \text{ C/m}^2$$

$$Q = \int_S \vec{D}\cdot d\vec{S} = \int_S \left(-\frac{100\varepsilon_0 x}{x^2+4}\cdot\hat{a}_z\right)\cdot(dx\,dy\,\hat{a}_z)$$

$$= \int_{x=0}^{2}\int_{y=-3}^{0} -\frac{100\varepsilon_0 x}{x^2+4}\,dx\,dy$$

$$= -\int_{x=0}^{2} \frac{100\varepsilon_0 x}{x^2+4}(0+3)\,dx$$

$$= -\frac{300\varepsilon_0}{2}\ln(x^2+4)\Big|_0^2 = -150\varepsilon_0 \times 0.693$$

$$= -920.15 \text{ pC}$$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

Integrals Using Beta Gamma function.

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2} \beta\left(\underset{\substack{\Downarrow \\ m}}{\frac{p+1}{2}}, \underset{\substack{\Downarrow \\ n}}{\frac{q+1}{2}}\right)$$

$$\beta(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

Gamma

$$\Gamma(m) = \Gamma(m-1+1) = (m-1)\Gamma(m-1) = (m-1)!$$

$$\Gamma(m+n) = (m+n-1)!$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(\tfrac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\tfrac{3}{2}\right)$$

$$= \frac{3}{2}\Gamma\left(\tfrac{1}{2}+1\right) = \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\tfrac{1}{2}\right)$$

$$\exists \sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$$

$$\Gamma 4 = \Gamma(3+1) = 3$$

Qn: For a potential field $V = r^2 z^2 \sin\phi$ ($V = \rho^2 z^2 \sin\phi$) at point $P(1, 45°, 1)$ in cylindrical co-ordinate system. Determine (a) Potential V (b) Electric field $\vec{E}$. (c) Electric flux density, $\vec{D}$ (d) Volume charge density ρ_v and (e) unit vector in direction of $\vec{E}$.

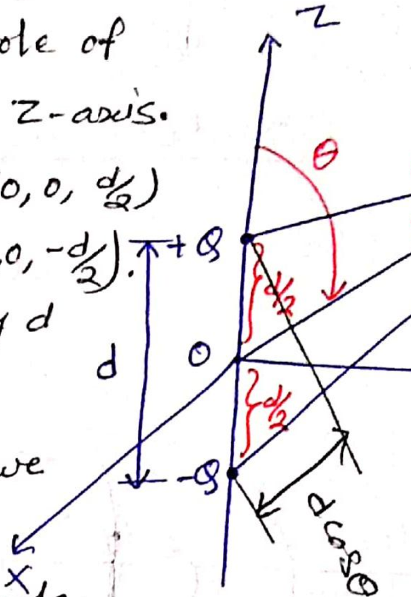
[2068 Shrawan]

Qn: Given the potential $V = \frac{10}{r^2} \sin\theta \cdot \cos\phi$; find the electric flux density $\vec{D}$ at $(2, \frac{\pi}{2}, 0)$. [2073 Chaitra]

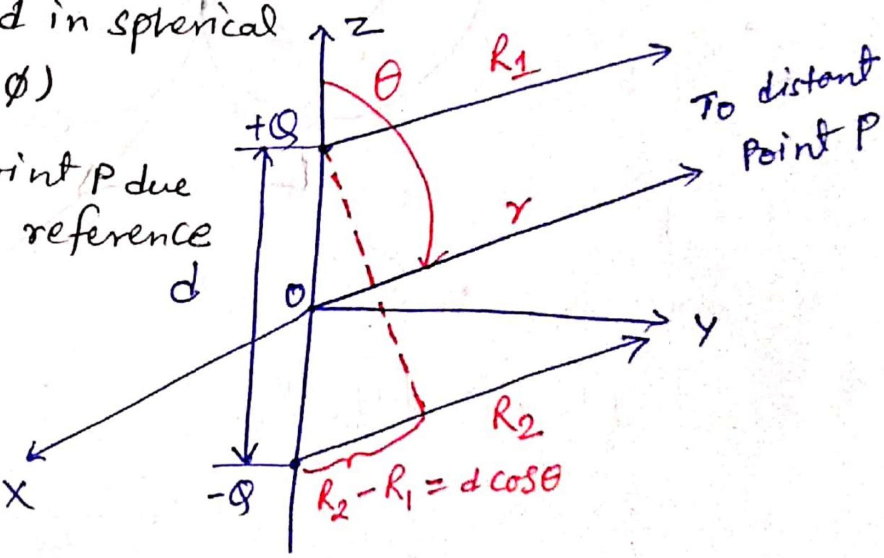
Electric Dipole

An electric dipole is formed when two point charges of equal magnitude but opposite sign are separated by a small distance. The distance betn two charges is very small compared to the distance to the point at which we want to know the electric and potential fields.

Consider an electric dipole of magnitude Q is placed on the Z-axis. The +ve charge $+Q$ is placed at $(0, 0, \frac{d}{2})$ and -ve charge $-Q$ is placed at $(0, 0, -\frac{d}{2})$. i.e two charges are separated by d distance as shown in figure.



Let us take the point P where we need to find the electric field intensity due to the electric dipole. Let the point P be described in spherical co-ordinate system as $P(r, \theta, \phi)$



Now, the potential at point P due to field of dipole, assuming reference at infinity is given by

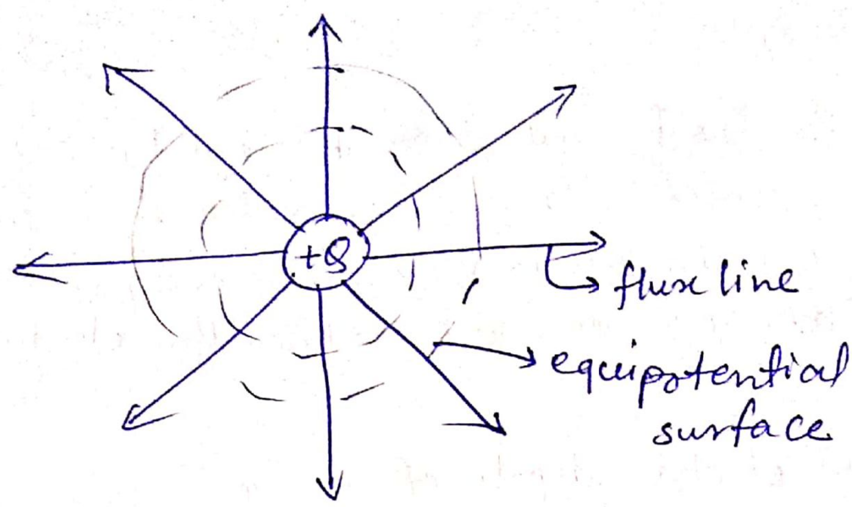
$$V = \frac{Q}{4\pi\epsilon_0 R_1} + \frac{(-Q)}{4\pi\epsilon_0 R_2}$$

$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

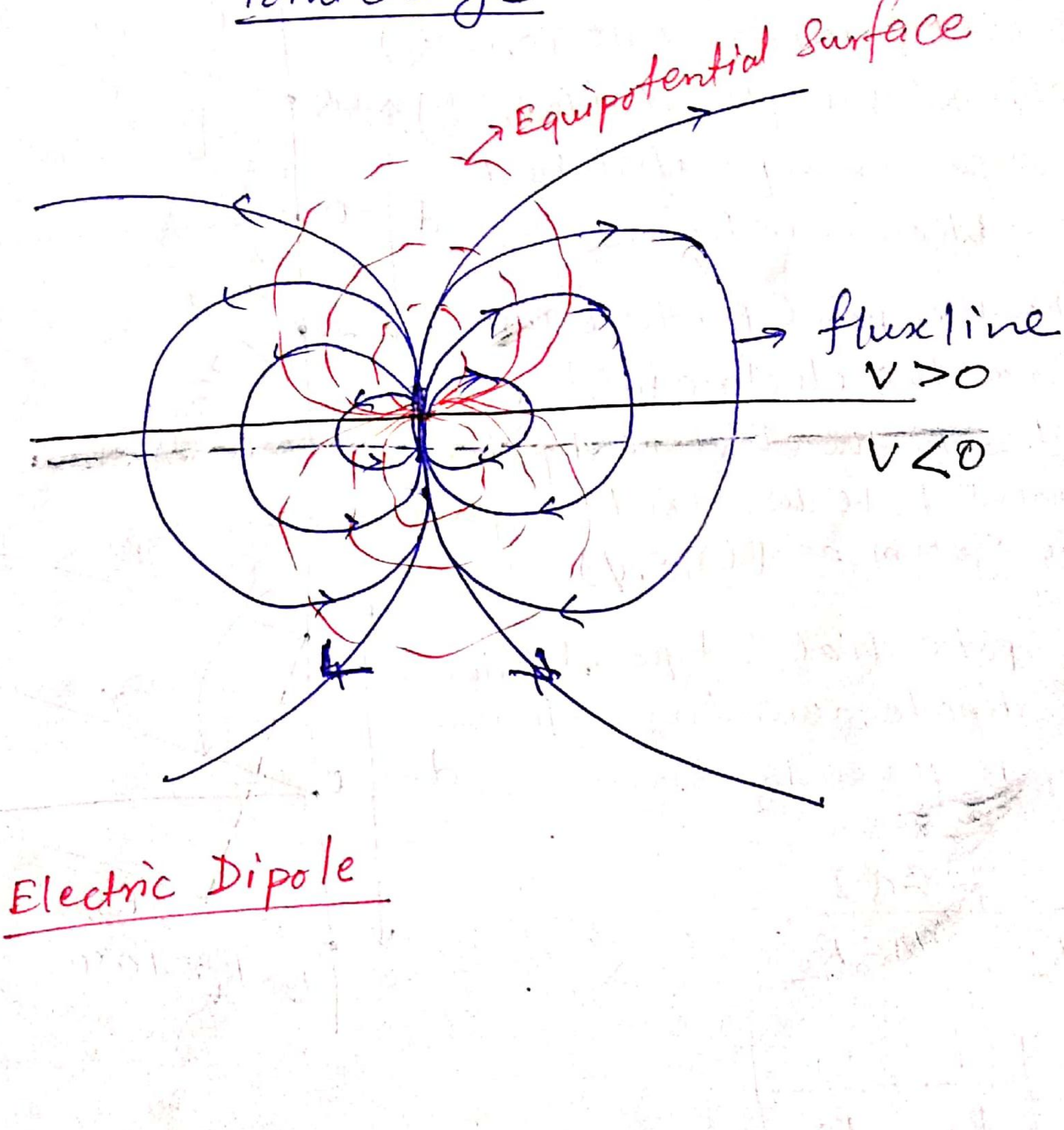
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{R_2 - R_1}{R_1 R_2}\right]$$

where, R_1 = ~~distance~~ distance betn P & $+Q$.

& R_2 = distance betn P & $-Q$



Pointcharge



Electric Dipole

If point P is very far and $r \gg d$ then

$R_2 = R_1 = r$, $R_2 - R_1 \approx d\cos\theta$ and $R_2 \cdot R_1 \approx r^2$

So,

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{R_2 - R_1}{r^2}\right] = \frac{Q}{4\pi\epsilon_0} \cdot \frac{d\cos\theta}{r^2}$$

Now, since, $d\cos\theta = \vec{d} \cdot \hat{a}_r$, where $\vec{d} = d\hat{a}_z$, we define

$$\vec{p} = Q\vec{d}$$

where $\vec{p}$ = the dipole moment, directed from $-Q$ to $+Q$.

Then,

$$\boxed{V = \frac{Q}{4\pi\epsilon_0} \cdot \frac{d\cos\theta}{r^2} = \frac{Q \cdot \vec{d} \cdot \hat{a}_r}{4\pi\epsilon_0 r^2} = \frac{\vec{p} \cdot \hat{a}_r}{4\pi\epsilon_0 r^2}}$$

If the dipole center is not at the origin but at $\vec{r}'$

Then,

$$V(\vec{r}) = \frac{\vec{p} \cdot (\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}$$

where, $\vec{r}$ locates the fieldpoint & $\vec{r}'$ locates the center of dipole.

Again, The electric field due to the dipole with center at the origin is given by

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$$

$$= \frac{Qd\cos\theta}{2\pi\epsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}\hat{a}_\theta$$

$$\boxed{\therefore \vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right] \quad V/m.}$$

This is the electric field intensity due to the dipole.

where, $p = |\vec{p}| = Qd$

(107)

Qn: A dipole for which $\vec{p} = 3\hat{a}_x - 5\hat{a}_y + 10\hat{a}_z$ nC.m is located at origin ~~the~~ point P(1, 2, −4). Find $\vec{E}$ at P. [2072 Chaitra]

Sol.ⁿ

Given, $\vec{p} = 3\hat{a}_x - 5\hat{a}_y + 10\hat{a}_z$ nC/m.

The electric field intensity due to the dipole is

$$\vec{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right]$$

Since, $\vec{E}$ is in spherical co-ordinate system, we need to chang P(1, 2, −4) to spherical co-ordinate system.

$$r = \sqrt{1^2+2^2+(-4)^2} = \sqrt{21}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\sqrt{5}}{-4}\right) = 180^\circ - 29.2^\circ = 150.79^\circ$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{1}\right) = 63.43^\circ$$

Now, $|\vec{p}| = p = \sqrt{3^2+(5)^2+10^2} = 11.575$ nCm.

$$\vec{E} = \frac{11.575\times10^{-9}}{\frac{4\pi\times10^{-9}}{36\pi}\times(21)^{3/2}}\left[2\cos(150.79^\circ)\hat{a}_r + \sin(150.79^\circ)\hat{a}_\theta\right]$$

$$= 1.0825\left[-1.7457\,\hat{a}_r + (+0.488)\,\hat{a}_\theta\right]$$

$$= \left[-1.889\,\hat{a}_r + 0.528\,\hat{a}_\theta\right] \text{ V/m}$$

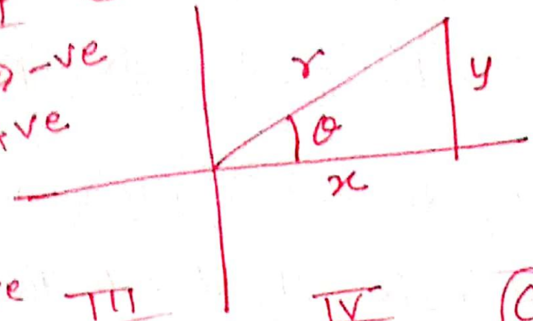
$\theta = \rho^2 ? \sin x$

$\Rightarrow y/x,\ x/r\ \&\ y/r \Rightarrow +ve$

S $+y$ A

II I

$180° - \theta$ { $y/x\ \&\ x/r \Rightarrow -ve$, $y/r \Rightarrow +ve$ }



$-x$ $+x$ (r is always $+ve$)

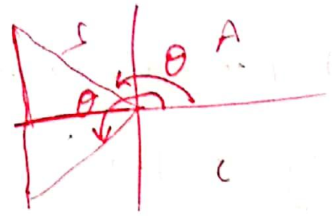
III IV C

$180° + \theta$ { $y/x \Rightarrow +ve$, $x/r\ \&\ y/r \Rightarrow -ve$ }

$-y$ T

$\frac{x}{r} \Rightarrow +ve$, $y/r\ \&\ y/x \Rightarrow -ve$ } $360° - \theta$

eg:- $\cos\theta = \frac{-5}{13} \Rightarrow$ (x = -5, r = 13)



$67.38°$

$\Rightarrow 112.619°$

247.38

-0.38

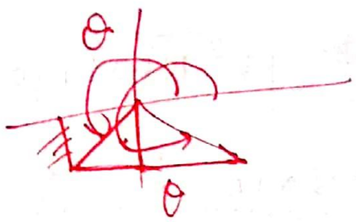
$\theta = \cos^{-1}\left(\frac{-5}{13}\right)$

$= 180° - 67.38$ or $180° + 67.38°$

336.42

203.578

$\sin\theta = \frac{-2}{5}$ (y = -2, r = 5)



$\theta = \sin^{-1}\left(\frac{-2}{5}\right) = 180° + 23.578°$ or $360° - 23.578°$

(108) Qn: A dipole of moment $\vec{p} = 6\hat{a}_z$ nC·m is located at the origin in freespace. (a) Find V at $P(r=4, \theta=20°, \phi=0°)$ (b) Find $\vec{E}$ at P.

Sol^n

Given, $\vec{p} = 6\hat{a}_z$

Converting it to spherical co-ordinate

$$\vec{p}_{sphe.} = 6\cos\theta\,\hat{a}_r + 6(-\sin\theta)\,\hat{a}_\theta$$
$$= [6\cos\theta\,\hat{a}_r - 6\sin\theta\,\hat{a}_\theta] \text{ nC·m}$$

(a) so, $$V = \frac{\vec{p}\cdot\hat{a}_r}{4\pi\epsilon_0 r^2}$$

$$= \frac{[6\cos\theta\,\hat{a}_r - 6\sin\theta\,\hat{a}_\theta]\cdot\hat{a}_r}{4\pi\epsilon_0 r^2}$$

$$= \frac{6\cos\theta}{4\pi\epsilon_0 r^2}$$

Now,

at $P(r=4, \theta=20°, \phi=0)$

$$V_p = \frac{6\cos 20° \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi} \times 4^2} = 3.17\text{ V}$$

Again,

(b) $$\vec{E} \text{ at } P = \vec{E}_p = \frac{p}{4\pi\epsilon_0 r^3}[2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta]$$

$$= \frac{6\times 10^{-9}}{4\pi \times \frac{10^{-9}}{9} \times 4^3}[2\cos 20°\,\hat{a}_r + \sin 20°\,\hat{a}_\theta]$$

$$= [1.585\,\hat{a}_r + 0.288\,\hat{a}_\theta] \text{ V/m}.$$

(129) Qn: A dipole having a moment $\vec{P} = 3\hat{a}_x - 5\hat{a}_y + 10\hat{a}_z$ nC·m is located at Q (1, 2, -4) in free space. Find V at P(2, 3, 4) ~~& $\vec{E}$ at P~~.

Sol.ⁿ Given, $\vec{p} = 3\hat{a}_x - 5\hat{a}_y + 10\hat{a}_z$ nC·m at point Q(1,2,-4)

we have,

$$V = \frac{\vec{p}\cdot(\vec{r}-\vec{r'})}{4\pi\epsilon_0 |\vec{r}-\vec{r'}|^3}$$

where, $\vec{r'} = \hat{a}_x + 2\hat{a}_y - 4\hat{a}_z$

$\vec{r} = 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$

so, $\vec{r} - \vec{r'} = -\hat{a}_x + \hat{a}_y + 8\hat{a}_z$

& $|\vec{r} - \vec{r'}| = \sqrt{1+1+64} = \sqrt{66}$

Then, $$V_p = \frac{(3\hat{a}_x - 5\hat{a}_y + 10\hat{a}_z)\cdot(\hat{a}_x + \hat{a}_y + 8\hat{a}_z)}{4\pi \times \frac{10^{-9}}{36\pi}\cdot(66)^{3/2}} \times 10^{-9}$$

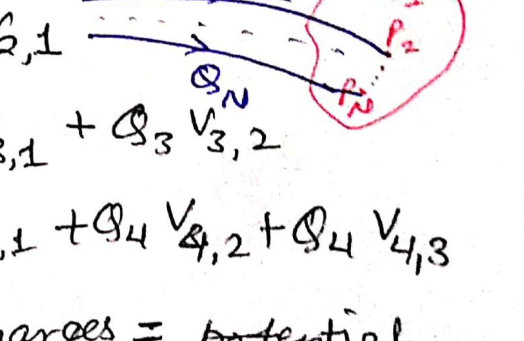
$$= \frac{(3 - 5 + 80)\times 9}{(66)^{3/2}}$$

$$= 1.31\,V$$

Energy Density in Electrostatic field.

To determine the energy present in a system of charges, we must find the work done by an external source in assemblying the charges.

Let us consider an empty universe and position the point charges brought from infinity. For initial condition, there is no charge in universe so, the workdone in bringing pointcharge Q_1 from infinity to that universe is zero. Then, the workdone in bringing & positioning charge Q_2 from outside into that universe requires an amount of work against the field due to charge Q_1; which is given by $V_{2,1} \cdot Q_2$ where, $V_{2,1}$ is the potential at point P_2 due to the charge Q_1 at point P_1.



i.e workdone in positioning $Q_2 = Q_2 V_{2,1}$

workdone in positioning $Q_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$

workdone in positioning $Q_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$

So, total work done to position the N charges = potential energy of field

$$= W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \cdots + Q_N V_{N,1} + Q_N V_{N,2} + \cdots + Q_N V_{N,N-1}$$

$$\therefore W_E = Q_2 V_{2,1} + Q_3 (V_{3,1} + V_{3,2}) + \cdots + Q_N (V_{N,1} + \cdots + V_{N,N-1})$$

(iii) Also, we have,

$$Q_3 V_{3,1} = Q_3 \cdot \frac{Q_1}{4\pi\epsilon_0 R_{13}} = \frac{Q_1 Q_3}{4\pi\epsilon_0 R_{31}}$$

where, R_{13} & R_{31} each represent the scalar distance betn Q_1 & Q_3.

Then, the total workdone in positioning different charges in reverse order is

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + \cdots + V_{1,N}$$

Adding the two energy expressions.

$$2W_E = Q_1 (V_{1,2} + V_{1,3} + V_{1,4} + \cdots) + Q_2 (V_{2,1} + V_{2,3} + V_{2,4} + \cdots)$$

$$\cdots + Q_{N-1} (\cdots V_{N-1,N})$$

Each sum of potentials in parenthesis is the combined potential due to all the charges except for the charge at the point where this combined potential is being found. i.e $V_{1,2} + V_{1,3} + V_{1,4} + \cdots + V_{1,N} = V_1$

$$V_{2,1} + V_{2,3} + V_{2,4} + \cdots + V_{2,N} = V_2$$

Therefore,

$$W_E = \frac{1}{2} (Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \cdots)$$

$$\boxed{\therefore W_E = \frac{1}{2} \sum_{m=1}^{m=N} Q_m V_m} \text{ - For discrete charges.}$$

In order to obtain an expression for the energy stored in a region of continuous charge distribution, each charge is replaced by $\rho_v\, dv$ and the summation becomes an integral.

$$W_E = \frac{1}{2} \int_{vol.} \rho_v V \, dv$$

From Maxwell's first Equation

$$\rho_v = \nabla \cdot \vec{D}$$

So,

$$W_E = \frac{1}{2} \int_{vol.} (\nabla \cdot \vec{D}) V \, dv$$

Again, from vector identity

$$(\nabla \cdot \vec{D}) V = \nabla (V \vec{D}) - \vec{D} \cdot (\nabla V)$$

vector identity

$$\nabla \cdot (V\vec{D}) = V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$$

or, $V(\nabla \cdot \vec{D}) = \nabla \cdot (V\vec{D}) - \vec{D} \cdot (\nabla V)$

so, $$W_E = \frac{1}{2} \int_{vol.} [\nabla \cdot (V\vec{D}) - \vec{D} \cdot (\nabla V)] \, dv$$

from divergence theorem.

$$\int_{vol.} (\nabla \cdot \vec{D}) \, dv = \oint_s \vec{D} \cdot \vec{ds}$$

so, $$\oint_s (V\vec{D}) \cdot \vec{ds} = \int_{vol.} \nabla \cdot (V\vec{D}) \, dv.$$

$$\therefore W_E = \frac{1}{2} \oint_s (V\vec{D}) \cdot \vec{ds} - \frac{1}{2} \int_{vol.} \vec{D} \cdot (\nabla V) \, dv$$

Here, in $\frac{1}{2} \oint_s V\vec{D} \cdot \vec{ds}$, $\vec{D}$ varies with $\frac{1}{r^2}$, and $\vec{ds}$ varies with r^2 & V varies with $\frac{1}{r}$. So, the effect of variation with $\frac{1}{r^2}$ is cancelled but effect due to $\frac{1}{r}$ is remained &

if $r \to \infty$ then $\frac{1}{r} \to 0$

So, $\frac{1}{2} \oint_s V\vec{D} \cdot \vec{ds} = 0$ as we are bringing the charges from infinity to an empty Universe

$$\therefore W_E = \frac{-1}{2}\int_{vol.} \vec{D}\cdot(\nabla V)\, dv$$

Again, from potential gradient, we have

$$\vec{E} = -\nabla V$$

The W_E becomes,

$$W_E = \frac{1}{2}\int_{vol.} \vec{D}\cdot(\vec{E})\, dv$$

$$= \frac{1}{2}\int_{vol.} (\epsilon_0\vec{E})\cdot\vec{E}\, dv$$

$$\boxed{\therefore W_E = \frac{1}{2}\int_{vol.} \epsilon_0 E^2\, dv}$$

Again, from, $W_E = \frac{1}{2}\int_{vol.} (\vec{D}\cdot\vec{E})\, dv$

we can use, the differential form as:

$$dW_E = \frac{1}{2}\vec{D}\cdot\vec{E}\, dv$$

or, $\boxed{\frac{dW_E}{dv} = \frac{1}{2}\vec{D}\cdot\vec{E}}$ — This is called Energy density in electrostatic field.

It has the dimension of Energy density or Joules per cubic meter. If we integrate this energy over the entire field containing volume, the result is the total energy present in that volume.

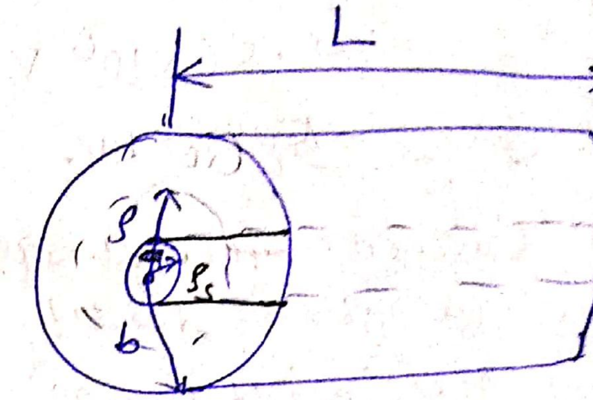
Also, $w_E = \frac{dW_E}{dv} = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\epsilon_0 E^2 = \frac{D^2}{2\epsilon_0}$

$W_E = \int w_E\, dv$

Energy density

Energy stored in the electrostatic field of section of a coaxial cable or capacitor of length L.

We have, $\vec{E} = \dfrac{a \rho_s}{\varepsilon_0 \rho} \hat{a}_\rho$



& $\vec{D} = \dfrac{a \rho_s}{\rho} \hat{a}_\rho$

where, ρ_s = Surface charge density on the inner conductor.

Thus,

$$W_E = \frac{1}{2} \int_{vol.} \vec{D} \cdot \vec{E} \, dv = \frac{1}{2} \int_{vol} \varepsilon_0 E^2 \, dv$$

$$= \frac{1}{2} \int_0^L \int_0^{2\pi} \int_0^{\pi} \varepsilon_0 \frac{a^2 \rho_s^2}{\varepsilon_0^2 \rho^2} \rho \, d\rho \, d\phi \, dz$$

$$= \frac{\pi L a^2 \rho_s^2}{\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

If we choose the outer conductor as zero potential reference then the potential of the inner conductor is

$$V_a = -\int_b^a E_\rho \, d\rho = -\int_b^a \frac{a \rho_s}{\varepsilon_0 \rho} d\rho = \frac{a \rho_s}{\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

Qn: An electric dipole located at the origin in free space has a moment $\vec{p} = 3\hat{a}_x - 2\hat{a}_y + \hat{a}_z$ nC·m (a) Find V at $P_A(2,3,4)$ (b) Find V at $r = 2.5, \theta = 30°, \phi = 40°$.

Ans: 0.230 V, 1.973 V

(11.4) Qn: Find the energy stored in free space for the region $0<\rho<a, 0<\phi<\pi, 0<z<2$, given the potential field $V=$: (a) $V_0(\rho/a)\,V$, (b) $V_0(\rho/a)\cos^2\phi\,V$.

Sol.^{n} (a) We have, $V = V_0(\rho/a)$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial \rho}\hat{a}_\rho\right] = \frac{-\partial}{\partial\rho}\left(\frac{V_0\rho}{a}\right)\hat{a}_\rho = \frac{-V_0}{a}\hat{a}_\rho$$

$$\therefore |\vec{E}| = E = \frac{V_0}{a}$$

Now, we have, the ~~e~~energy stored is given by

$$W_E = \frac{1}{2}\int_{vol.}\epsilon_0 E^2 dv$$

$$= \frac{1}{2}\int_{vol.}\epsilon_0\left(\frac{V_0}{a}\right)^2 dv$$

$$= \frac{1}{2}\frac{\epsilon_0 V_0^2}{a^2}\int_{vol.} d\rho\cdot\rho\, d\phi\cdot dz$$

$$= \frac{\epsilon_0 V_0^2}{2a^2}\int_{\rho=0}^{a}\int_{\phi=0}^{\pi}\int_{z=0}^{2}\rho\, d\rho\, d\phi\, dz$$

$$= \frac{\epsilon_0 V_0^2}{2a^2}\cdot\int_{\rho=0}^{a}\rho\,(\pi-0)\cdot(2-0)\,d\rho$$

$$= \frac{\cancel{2}\pi\epsilon_0 V_0^2}{\cancel{2}a^2}\cdot\left.\frac{\rho^2}{2}\right|_{\rho=0}^{a}$$

$$= \frac{\epsilon_0 V_0^2\pi}{a^2}\cdot\frac{a^2}{2}$$

$$= \frac{\pi}{2}\epsilon_0 V_0^2 = 1.571\,\epsilon_0 V_0^2\ J. //$$

(115) Again, $V = V_0 \left(\frac{\rho}{a}\right) \cdot \cos^2\phi$

(b) $\vec{E} = -\nabla V = -\left[\frac{\partial}{\partial \rho}\left(\frac{V_0 \rho \cos^2\phi}{a}\right)\hat{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{V_0 \rho}{a}\cos^2\phi\right)\hat{a}_\phi\right]$

$$= -\frac{V_0}{a}\left(\cos^2\phi\, \hat{a}_\rho - 2\sin\phi \cdot \cos\phi\, \hat{a}_\phi\right)$$

$\therefore$ $E = |\vec{E}| = \frac{V_0}{a}\sqrt{(\cos^2\phi)^2 + (-2\sin\phi \cdot \cos\phi)^2}$

$$= \frac{V_0}{a}\cos\phi \sqrt{\cos^2\phi + 4\sin^2\phi}$$

$$= \frac{V_0}{a}\cos\phi \sqrt{1 + 3\sin^2\phi}$$

Now,

$$W_E = \frac{\varepsilon_0}{2}\int_{vol.} E^2 dv$$

$$= \frac{\varepsilon_0}{2}\int_{\rho=0}^{a}\int_{\phi=0}^{\pi}\int_{z=0}^{2}\frac{V_0^2}{a^2}\cos^2\phi\,(1+3\sin^2\phi)\,\rho\, d\rho\, d\phi\, dz$$

$$= \frac{\varepsilon_0 V_0^2}{2a^2}\left[\frac{\rho^2}{2}\right]_0^a \left[\frac{1}{2}\left(\phi + \frac{\sin 2\phi}{2}\right) + \frac{3}{8}\left(\phi - \frac{\sin 4\phi}{4}\right)\right]_{\phi=0}^{\pi} \cdot [z]_0^2$$

$$= \frac{\varepsilon_0 V_0^2}{2a^2} \times \frac{a^2}{2} \times 2.749 \times 2$$

$$= 1.374\, \varepsilon_0 V_0^2\ J.$$

Qn: Find the energy stored in free space for the region $2\,mm < r < 3\,mm$, $0 < \theta < 90^\circ$; $0 < \phi < 90^\circ$; given the potential field V =: (a) $\frac{200}{r}$ V; (b) $\frac{300 \cos\theta}{r^2}$ V.

Ans: $\cancel{1.391\,pJ}$, 36.7 J.
46.4 μJ
↳ corrected in 8th edition.

Current and Current Density

रविन कासुला
इन्जिनियर

Current (in amperes) through a given area is the electric charge passing through the area per unit time. Also, the rate of movement of charge passing a given reference point (or crossing a given reference plane) of one coulomb per second. Current is symbolized by I and given by

$$\boxed{I = \frac{dQ}{dt}}$$

current is thus defined by the motion of positive charges, even though conduction in metals takes place through the motion of electrons.

Thus, in a current of one ampere, charge is being transferred at a rate of one coulomb per second.

Current density is the current per unit area measured in Ampere per meter square and represented by $\vec{J}$.
If ΔI current flows through a surface ΔS, the current density is

$$\boxed{J_N = \frac{\Delta I}{\Delta S}}$$

or, $\Delta I = J_N \Delta S$

assuming that the current density is perpendicular to the surface. If current density is not normal to the surface then,

$$\Delta I = \vec{J} \cdot \vec{\Delta S}$$

Then, total current is obtained by integrating

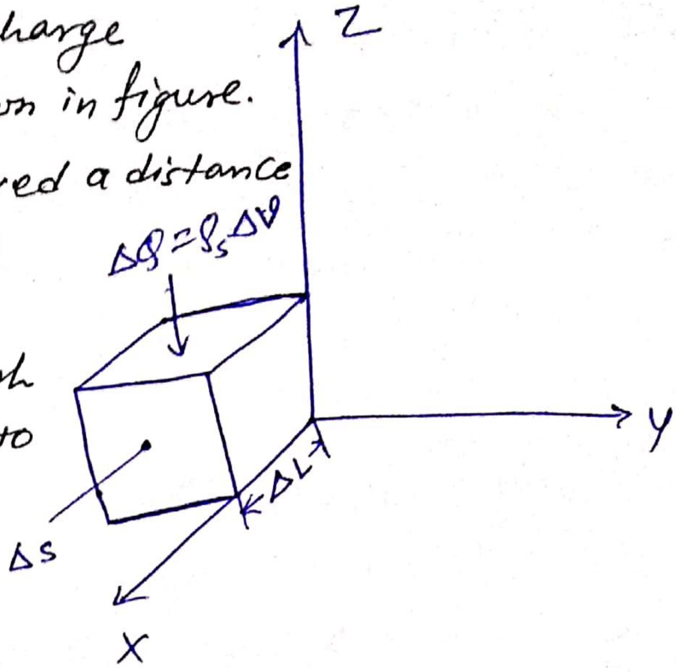
$$\boxed{I = \int_S \vec{J} \cdot \vec{dS}}$$

(117) Current density may be related to the velocity of volume charge density at a point.

Let us consider the element of charge $\Delta Q = \rho_v \Delta v = \rho_v \Delta S \Delta L$ as shown in figure.

If the element of charge is moved a distance Δx in Δt time interval the we will find that $\Delta Q = \rho_v \Delta S \Delta x$

Charge has been moved through a reference plane perpendicular to the direction of motion.

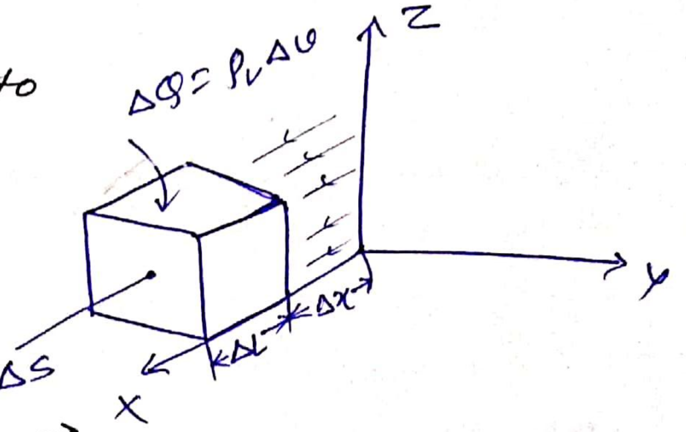


So, resultant current is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta x}{\Delta t}$$

Taking the limit with respect to time

$$\Delta I = \rho_v \Delta S \, v_x$$



where v_x represents the x-component of the velocity $\vec{v}$. In terms of current density,

so, $J_x = \rho_v v_x$

and in general

$$\boxed{\vec{J} = \rho_v \vec{v}}$$

This shows that the charge in motion constitutes the current, and this type of current is called convection current and $\vec{J}$ or $\rho_v \vec{v}$ is the convection current density.

Depending on how I is produced, there are different kinds of current densities: convection current density, conduction current density and displacement current density.

Continuity of Current (Continuity Equation)

Conservation of charge – The principle of conservation of charge states that charges can be neither created nor destroyed, although equal amounts of positive and negative charge may be simultaneously created, obtained by separation, destroyed or lost by recombination.

The continuity equation follows the conservation of charge principle when any region bounded by a closed surface is considered. The current through closed surface is given by

$$I = \oint_S \vec{J} \cdot \vec{ds}$$

This is the outflow of positive charge from the closed surface and must be balanced by the decrease of positive charge (or increase of negative charge) within the closed surface. If the charge inside the closed surface is denoted by Q_i, then the rate of decrease of charge is $-\frac{dQ_i}{dt}$ and the principle of conservation of charge requires,

$$\boxed{I = \oint_S \vec{J} \cdot \vec{ds} = -\frac{dQ_i}{dt}}$$

The sign of $-\frac{dQ_i}{dt}$ depends on the type of charge; for +ve it is −ve & for −ve charge it is +ve. The above current is the outward flowing current. The above equation is integral form of continuity equation and the point or differential form is obtained by using the divergence theorem.

$$\oint_S \vec{J} \cdot \vec{ds} = \int_{vol.} (\nabla \cdot \vec{J})\, dv$$

Also, we can use the volume charge density for Q_i as:

$$Q_i = \int_{vol.} \rho_v \, dv$$

So,

$$\int_{vol.} (\nabla \cdot \vec{J}) \, dv = \frac{-d}{dt} \int_{vol.} \rho_v \, dv$$

If the surface is kept constant the derivative becomes a partial derivative and

$$\int_{vol.} (\nabla \cdot \vec{J}) \, dv = \int_{vol.} \frac{-\partial \rho_v}{\partial t} \, dv$$

Since, the expression is true for any volume, however small it is also true for an incremental or differential volume.

So,

$$(\nabla \cdot \vec{J}) \, \Delta v = \frac{-\partial \rho_v}{\partial t} \, \Delta v$$

&

$$\boxed{(\nabla \cdot \vec{J}) = -\frac{\partial \rho_v}{\partial t}}$$

This is the point form of continuity Equation. This indicates that the current or charge persecond, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

Let us consider a current density that is directed radially outward and decreases exponentially with time,

$$\vec{J} = \frac{1}{r} e^{-t} \hat{a}_r \ A/m^2$$

selecting an instant of time $t = 1s$, we may calculate the total outward current at $r = 5m$;

$$I = J_r \cdot S = \left(\frac{1}{5} e^{-1}\right)(4\pi 5^2) = 23.1 \ A$$

For $r = 6m$ & $t = 1s$,

$$I = J_r \cdot S = \left(\frac{1}{6} e^{-1}\right) \cdot (4\pi 6^2) = 27.7 \ A$$

Then, from Continuity Eq^n:

$$-\frac{\partial \rho_v}{\partial t} = \nabla \cdot \vec{J} = \nabla \cdot \left(\frac{1}{r} e^{-t} \hat{a}_r\right) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{1}{r} e^{-t}\right) = \frac{1}{r^2} e^{-t}$$

Integrating w.r.to t:

$$\rho_v = -\int \frac{1}{r^2} e^{-t} dt + K(r) = \frac{1}{r^2} e^{-t} + K(r)$$

If we assume that $\rho_v \to 0$ as $t \to \infty$ then $K(r) = 0$

& $\boxed{\rho_v = \frac{1}{r^2} e^{-t} \ C/m^3}$

Now, we may use $\vec{J} = \rho_v \vec{v}$ to find velocity

$$v_r = \frac{J_r}{\rho_v} = \frac{\frac{1}{r} e^{-t}}{\frac{1}{r^2} e^{-t}} \ m/s = r \ m/s.$$

Point Form of Ohm's Law

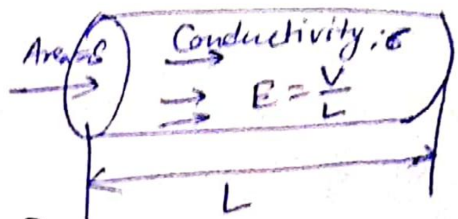
For any metallic Conductors, the relation betn $\vec{J}$ and $\vec{E}$ is given by $\boxed{\vec{J} = \sigma \vec{E}}$ or $\vec{J} = -\rho_e \mu_e \vec{E}$ – This is the point form of Ohm's law

where, σ = Conductivity of that metallic conductor
$= -\rho_e \mu_e$

& ρ_e is free electron charge density & μ_e is mobility of the electron.

Let us consider for any cylindrical metallic conductor, $\vec{J}$ & $\vec{E}$ are uniform. Then the current through the metallic conductor is given by

$$I = \int_s \vec{J} \cdot \vec{ds} = JS$$



and $$V_{ab} = -\int_b^a \vec{E} \cdot \vec{dl} = -\vec{E} \cdot \int_b^a \vec{dl} = -\vec{E} \cdot \vec{L}_{ba} = \vec{E} \cdot \vec{L}_{ab}$$

or, $V = EL$

So, $$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L}$$

& $$V = \frac{L}{\sigma S} I$$

The ratio of the potential difference betn two ends of the cylinder to the current entering the more positive end is the resistance of the cylinder and hence

$\boxed{V = IR}$

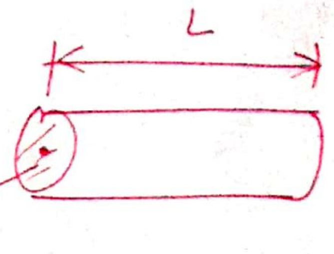
For nonuniform fields

where, $\boxed{R = \frac{L}{\sigma S}}$

$$\boxed{R = \frac{V_{ab}}{I} = \frac{-\int_b^a \vec{E} \cdot \vec{dl}}{\int_s \sigma \vec{E} \cdot \vec{ds}}}$$

Point form of ohm's law (Derivation)

Resistivity, $s = \frac{1}{\sigma} = \frac{R \cdot S}{L}$



where, σ = conductivity of the conductor

so, $\sigma = \frac{L}{RS}$ ——— (a)

Again, Assuming that the field $\vec{E}$ & current density $\vec{J}$ for metallic conductors are uniform, then,

Potential difference betⁿ two ends of the conductor is given by

$V = EL \quad \left[\because V_{ab} = -\int_b^a \vec{E} \cdot \vec{dl} = \vec{E} \cdot \vec{l}_{ab} \right]$ ——— (b)

& the current passing through the conductor is

$I = JS \quad \left[\because I = \int_S \vec{J} \cdot \vec{ds} = JS \right]$ ——— (c)

Again, from ohm's Law.

$V = IR$

or, $EL = JS \cdot R$

or, $J = \frac{L}{R \cdot S} E$

or, $J = \sigma E$

so, in vector form $\boxed{\vec{J} = \sigma \vec{E}}$ This is point form of ohm's Law.

12.1 Relaxation Time

शाजानगर

Let us consider any conductor containing charge. When we see the charge on any conductor they will apper at the surface as surface charge. Initially, when we introduce the charge into the conductor the charge will be seen at every point of the conductor. After some time it will move towards surface and finally all charge will be seen at the surface. So, the time taken to move the charge from the point in conductor to the surface of conductor is termed as the relaxation constant for that conductor.

Now, from ohm's Law (point form)

$$\vec{J} = \sigma \vec{E}$$

and from Continuity Equation (point form)

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

in which $\vec{J}$ & ρ_v both involve only free charges.

$$\nabla \cdot \sigma \vec{E} = -\frac{\partial \rho_v}{\partial t}$$

or, $\nabla \cdot \frac{\sigma}{\varepsilon} \vec{D} = -\frac{\partial \rho_v}{\partial t}$ $\quad$ $[\because \varepsilon = \varepsilon_0 \varepsilon_r]$ $\hookrightarrow$ permitivity of conductor.

If we assume that the medium is homogeneous, so that σ & ε are not functions of position.

$$\nabla \cdot \vec{D} = -\frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$$

or, $\rho_v = -\frac{\varepsilon}{\sigma} \frac{\partial \rho_v}{\partial t}$ $\quad$ using Maxwell's first Equation $\nabla \cdot \vec{D} = \rho_v$

Relaxation time or rearrangement time:

It is the time it takes a charge placed in the interior of a material to drop to $e^{-1} = 36.8$ percent of its initial value.

If we assume that σ is not a function of ρ_v, then we will get the solution as:

$$\boxed{\rho_v = \rho_0 e^{-(\sigma/\varepsilon)t}}$$

where, ρ_0 = Charge density at $t=0$. This shows an exponential decay of charge density at every point. with a time constant of ε/σ; which is called Relaxation time.

i.e $\boxed{\text{Relaxation time} = \frac{\varepsilon}{\sigma}}$

For distilled water,

$$\frac{\varepsilon}{\sigma} = \frac{80 \times 8.854 \times 10^{-12}}{2 \times 10^{-4}} = 3.54\,\mu s.$$

Qn: Let $\vec{J} = \frac{400 \sin\theta}{r^2+4} \hat{a}_r$ A/m² (a) Find the total current flowing through that portion of the spherical surface $r=0.8$ bounded by $\theta = 0.1\pi, \theta = 0.3\pi, 0 < \phi < 2\pi$ (b) Find the average value of J over the defined area.

Solⁿ Given,

$$\vec{J} = \frac{400 \sin\theta}{r^2+4} \hat{a}_r \text{ A/m}^2$$

(a) Spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$. Then,

the current flowing through that portion, $I = \int_s \vec{J} \cdot \vec{ds}$.

$$= \int_s \vec{J} \cdot (r^2 \sin\theta \, d\theta \, d\phi) \hat{a}_r = \int_{\phi=0}^{2\pi} \int_{\theta=0.1\pi}^{0.3\pi} \frac{400 \sin\theta}{(0.8)^2+4} \cdot (0.8)^2 \sin\theta \cdot d\theta \cdot d\phi$$

$$= \frac{400(0.8)^2 2\pi}{4.84} \int_{\theta=0.1\pi}^{0.3\pi} \sin^2\theta \, d\theta$$

$$= 346.5 \int_{\theta=0.1\pi}^{0.3\pi} \frac{1}{2}[1-\cos 2\theta]\, d\theta$$

$$= 77.4\ A$$

(b) The area is given by

$$\text{Area} = \int_{\phi=0}^{2\pi} \int_{\theta=0.1\pi}^{0.3\pi} (0.8)^2 \sin\theta \, d\theta \, d\phi$$

$$= 1.46\ m^2$$

The average current density is thus

$$\vec{J}_{avg} = \frac{I}{\text{Area}} \hat{a}_r = \frac{77.4}{1.46} \hat{a}_r = 53.0\, \hat{a}_r\ A/m^2$$

Qn: The current density in certain region is approximated by $\vec{J} = \left(\frac{0.1}{r}\right) e^{-10^6 t}\, \hat{a}_r\ A/m^2$ in spherical co-ordinates.

(a) How much current is crossing the surface $r = 50$ cm, at $t = 1\ \mu s$? (b) Find $\rho_v(r,t)$ assuming that $\rho_v \to 0$ as $t \to \infty$.

[2072 Kartik]

Sol.ⁿ Given, $\vec{J} = \frac{0.1}{r} e^{-10^6 t}\, \hat{a}_r\ A/m^2$

(a) The current crossing the surface $r = 50$ cm at $t = 1\ \mu s$ is

$$I = J_r \cdot S = \left(\frac{0.1}{0.5} e^{-10^6 \times 10^{-6}}\right) \cdot 4\pi \times (0.5)^2$$

$$= \frac{1}{5} e^{-1} \cdot 4\pi \times 0.25 = 0.2 \times 0.3678 \times 4\pi \times 0.25$$

$$= 0.231\ A$$

Again, (b) we have, From continuity Eq^n: $-\frac{\partial \rho_v}{\partial t} = (\nabla \cdot \vec{J})$

$\rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r)$

$$\rho_v(r,t) = -\int \frac{0.1}{r^2} e^{-10^6 t}\, dt$$

$$= -\frac{0.1}{r^2} \cdot \frac{1}{(-10^6)} \cdot e^{-10^6 t} + K(r)$$

$$= \frac{10^{5}}{r^2} e^{-10^6 t} + K(r)$$

If $\rho_v \rightarrow 0$ as $t \rightarrow \infty$

then, $K(r) = 0$

so, $\boxed{\rho_v(r,t) = \frac{10^{5}}{r^2} e^{-10^6 t}}$

For, $r = 50\,cm$ & $t = 1\,\mu s$.

$$\rho_v = \frac{10^5}{0.5} e^{-10^6 \times 10^{-6}}$$

$$= \frac{10^5}{0.5} \times 0.3678$$

$$= 73560$$

$$= 73.56 \times 10^3\ C/m^3$$

Qn: A current density in certain region is given as:

$\vec{J} = 20\sin\theta\cos\phi\,\hat{a}_r + \frac{1}{r}\hat{a}_\phi$ A/m², Find:

(i) The average value of J_r over the surface $r=1$, $0<\theta<\pi/2$, $0<\phi<\pi/2$ [2073 Shrawan]

(ii) $\frac{\partial \rho_v}{\partial t} = ?$

Sol.ⁿ Given, $\vec{J} = 20\sin\theta\cos\phi\,\hat{a}_r + \frac{1}{r}\hat{a}_\phi$ A/m²

(i)

$$\text{Area} = \int_S r^2\sin\theta\, d\theta\, d\phi$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} (1)^2 \sin\theta \cdot d\theta \cdot d\phi$$

$$= \frac{\pi}{2}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}\cdot 1$$

$$= \frac{\pi}{2}\ m^2$$

&

$$I = \int_S \vec{J}\cdot d\vec{s} = \int_S \left(20\sin\theta\cos\phi\,\hat{a}_r + \frac{1}{r}\hat{a}_\phi\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\,\hat{a}_r\right)$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} 20\sin^2\theta\cdot\cos\phi\cdot d\theta\, d\phi$$

So, $\vec{J}_{r(avg)} = \frac{I}{\text{Area}}\hat{a}_r = \ldots\hat{a}_r$ A/m²

(ii)

$$\frac{\partial \rho_v}{\partial t} = -\nabla\cdot\vec{J} = -\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 J_r) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta J_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial J_\phi}{\partial\phi}\right)$$

$$= \ldots$$

Electric Properties of Material Medium

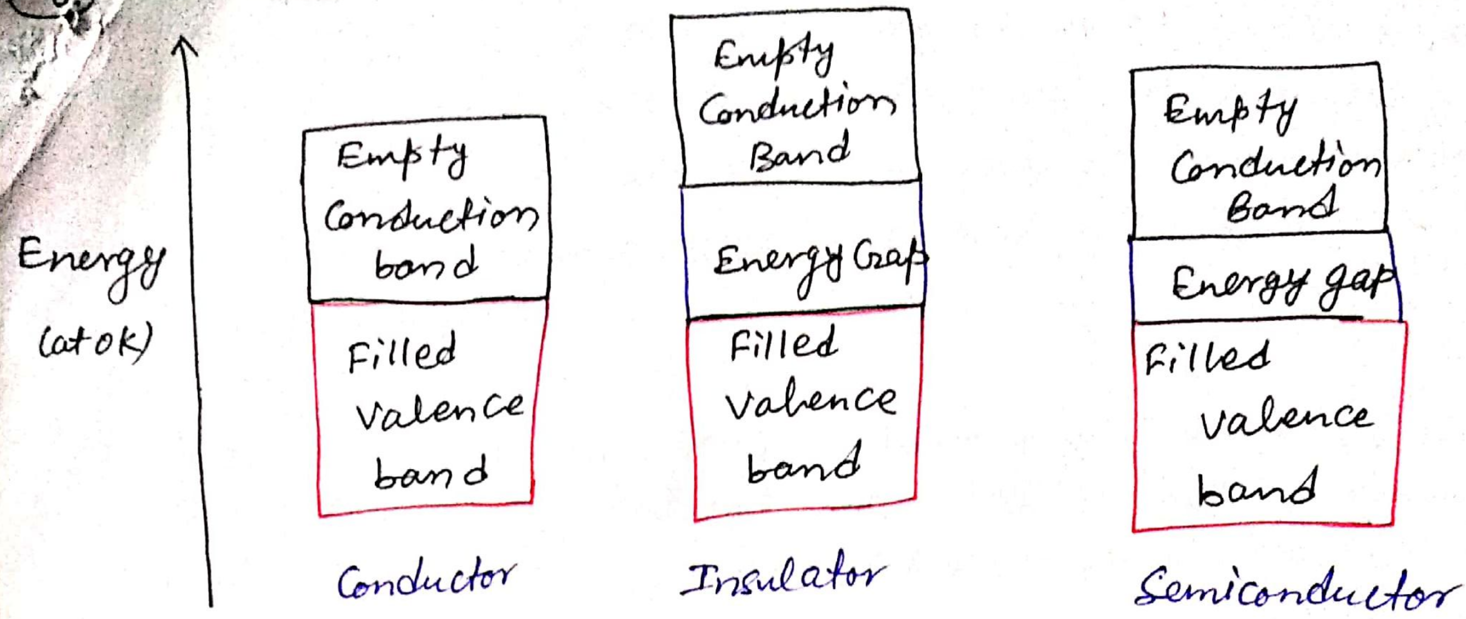
As electric fields can exist in freespace, they can exist in material media. Materials are broadly classified in terms of their electrical properties as conductors and non conductors. Non conducting materials are usually referred to as insulators or dielectrics.

In a broad sense materials may be classified in terms of their conductivity σ, in mhos per meter ($\mho/m$) or siemens per meter (S/m), as conductors and non conductors, or technically as metals and insulators (or dielectrics). The conductivity of a material usually depends on temperature and frequency. A material with high conductivity ($\sigma >> 1$) is referred to as a metal whereas one with low conductivity ($\sigma << 1$) is referred to as an insulator. A material whose conductivity lies somewhere bet.^{n} those of metals and insulators is called semiconductor. Copper and aluminium are metals, silicon and germanium as semiconductors and glass and rubber are insulators.

The conductivity of metals generally increases with decrease in temperature. At temperatures near absolute zero ($T=0°K$), some conductors exhibit infinite conductivity and are called superconductors. Lead and aluminium are typical examples of such metals. The major difference bet.^{n} a metal and an insulator lies in the amount of electrons available for conduction of current in contrast to metals, which have an abundance of free electrons.

For conductors, current is due to movement of electrons. and with field $\vec{E}$ due to charge $Q = -e$

$\therefore$ Force, $\vec{F} = -e\vec{E}$.



In free space, the electron would accelerate and continuously increase its velocity (and energy); in the crystalline material the progress of the electron is impeded by continual collisions with the thermally excited crystalline lattice structure, and a constant average velocity is soon attained. The velocity $\vec{v}_d$ is termed as the drift velocity and linearly related to the electric field intensity by the mobility of the electron in the given material.

$$\therefore \vec{v}_d = -\mu_e \vec{E}$$

where, μ_e = mobility of an electron and is positive by definition.

Also, $\vec{J} = -\rho_e \mu_e \vec{E}$

where, ρ_e = free electron charge density, a -ve value.

Free and Bound Charges

Free Charges: It is the one that moves freely in space i.e without an external electromagnetic field. Any electric charge that can be placed on a conductor or on or within a dielectric.

eg:- Electrons in a conductor, which are free to move throughout the material. Electrons or ions in a vaccum are also free charges.

(127) **Bound charges** :- It is one that cannot move freely or will move only in response to the external electromagnetic force. Charges on the surface of an insulator are an examples. Also, the electric charge that is bound to an atom or molecule is bound charges.

Polar Molecules :- The molecules that have a permanent displacement existing betn the centers of gravity of the positive and negative charges, and each pair of charges acts as dipole.

Non polar Molecules: Does not have the dipole arrangement until after a field is applied. i.e sufficiently strong field is required to produce an additional displacement betn the positive and negative charges.

Polarization

When the atoms of a material are under the influence of the electric field, the electrons shift their positions and the atoms are said to be polarized; the phenomenon being is called polarization. The direction of polarization is from -ve to +ve charges or in the dirn of $\vec{E}$.

Also, polarization is the dipole moment per unit volume.

$$\text{i.e } \vec{P} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i=1}^{n\Delta V} \vec{p}_i$$

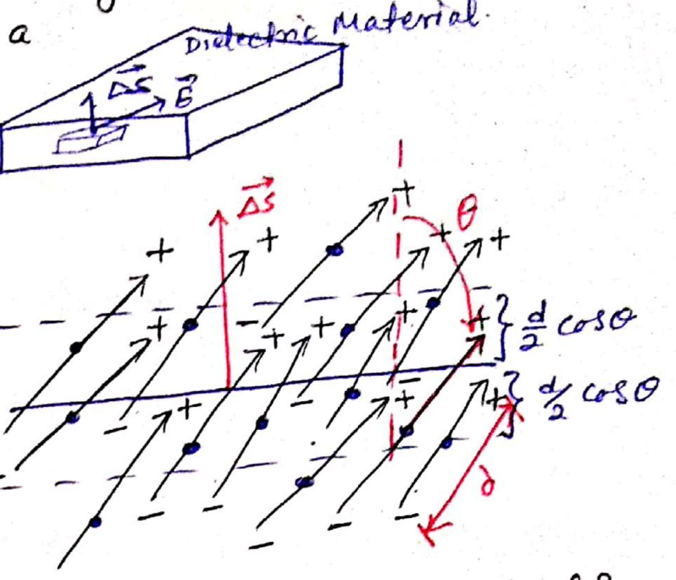
where, $\vec{p}_i$ = dipole moment

& given by $\vec{p} = Q\vec{d}$; $\vec{d}$ = vector from -ve to +ve charge

$$\vec{p}_{total} = \sum_{i=1}^{n\Delta V} \vec{p}_i = \text{total dipole moment}$$

If there is n dipoles per unit volume, or ΔV then there will be $n\Delta V$ dipoles.

(128): Let us consider the dielectric containing non polar molecules. No molecule has a dipole moment, and $\vec{P}=0$ throughout the material. Somewhere in the interior of dielectric let us select an incremental surface element $\vec{\Delta S}$ as shown in figure; and apply the electric field $\vec{E}$. The field $\vec{E}$ produces a dipole moment $\vec{p}=Q\vec{d}$ in each molecule such that $\vec{p}$ & $\vec{d}$ make an angle θ with $\vec{\Delta S}$.



Let us inspect the movement of bound charges across $\vec{\Delta S}$. Each of the charges associated with the creation of a dipole must have moved a distance $\frac{d}{2}\cos\theta$ in the direction perpendicular to $\vec{\Delta S}$. Thus, any positive charges initially lying below the surface $\vec{\Delta S}$ and within the distance $\frac{d}{2}\cos\theta$ of the surface must have crossed $\vec{\Delta S}$ going upward. Also, any negative charges initially lying above the surface and within that distance $(\frac{d}{2}\cos\theta)$ from $\vec{\Delta S}$ must have crossed $\vec{\Delta S}$ going downward. Therefore since there are n molecules/m3, the net total charge which crosses the elemental surface in an upward direction is equal to $nQd\cos\theta\,\Delta S$ or $\Delta Q_b = nQ\vec{d}\cdot\vec{\Delta S}$

where, ΔQ_b = bound charge.

Now, in terms of polarization

$$\Delta Q_b = \vec{P}\cdot\vec{\Delta S}$$

If we interpret $\vec{\Delta S}$ as an element of a closed surface inside the dielectric material, then the dir^n of $\vec{\Delta S}$ is outward, and the net increase in the bound charge within the closed surface is obtained through the integral

$$Q_b = -\oint_s \vec{P}\cdot\vec{ds}$$

Similar, expression from Gauss's Law

$$Q_T = \oint_s \varepsilon_0 \vec{E}\cdot\vec{ds} \qquad \text{where, } Q_T = Q_b + Q$$

and Q is the total free charge enclosed by the surface S,

Now,

$$Q = Q_T - Q_b = \oint_S (\epsilon_0 \vec{E} + \vec{P}) \cdot \vec{ds}$$

we may define $\vec{D}$ in more general terms as:

$$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$$

Thus, for polarizable material the added $\vec{P}$ (polarization) is present in the relation

$$Q = \oint_S \vec{D} \cdot \vec{ds}.$$

where, Q = free charge enclosed.

Isotropic Materials – The materials in which $\vec{E}$ & $\vec{P}$ are linearly related. The $\vec{E}$ & $\vec{P}$ are always parallel, regardless of the orientation of the field.

Nonisotropic or Anisotropic Materials – The materials in which $\vec{E}$ and $\vec{P}$ are non linearly related and $\vec{E}$ & $\vec{P}$ does not have same direction.

Relative Permittivity (ϵ_R)

For Ferroelectric materials the relationship betn $\vec{P}$ & $\vec{E}$ is not only nonlinear, but also shows hysteresis effects; that is the polarization produced by a given electric field intensity depends on the past history of the sample. eg:- barium titanate used in ceramic capacitors.

The linear relationship betn $\vec{P}$ and $\vec{E}$ is

$$\boxed{\vec{P} = \chi_e \epsilon_0 \vec{E}}$$

where, χ_e(chi) is the dimensionless quantity called the electric susceptibility of the material.

Dielectric Strength - It is the maximum electric field that a dielectric can tolerate or withstand without breakdown.

Now,

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E} = (\chi_e + 1)\varepsilon_0 \vec{E} = \varepsilon_R \varepsilon_0 \vec{E}}$$

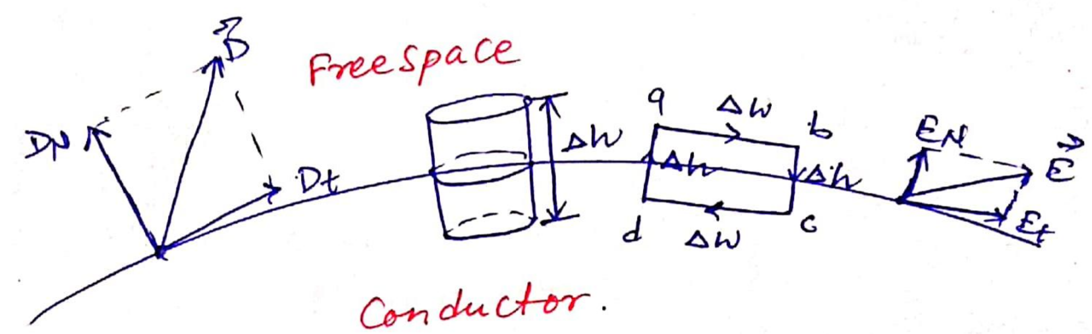
where, $\boxed{\varepsilon_R = \chi_e + 1}$ is the dimensionless quantity known as relative permitivity or dielectric constant of the material.

$$\boxed{\therefore \vec{D} = \varepsilon_0 \varepsilon_R \vec{E} = \varepsilon \vec{E}}$$

where, $\boxed{\varepsilon = \varepsilon_0 \varepsilon_R}$ = is the permitivity of dielectric material.

Electric Boundary Conditions

① Boundary Conditions between freespace and conductor



Let us consider the boundary of free surface and the conductor. Since, all the charges resides on the exterior surface of the conductor the charge density within the conductor is zero. For static conditions in which no current may flow, the electric field intensity within the conductor is zero; from Ohm's Law.

Let us consider $\vec{E}$ be the external fields to the charge on the surface of the conductor then it can be decomposed to the tangential component and normal component to the conductor surface. The tangential component E_t is seen to be zero. If it is not zero then it would be applied to the elements of the surface charge, resulting in their motion and nonstatic conditions. Since, static condition is assumed, the tangential electric field intensity and electric flux density are zero.

For the region interior ~~of~~ conductor the both fields are zero in the conductor.

For tangential components,

$$\oint \vec{E} \cdot \vec{dl} = 0$$ around the small closed path abcda.

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Since, $\vec{E} = 0$ within the conductor, $ab = cd = \Delta W$ & $bc = da = \Delta h$

then,

$$E_t \Delta W - E_{N,\text{at}\,b} \cdot \frac{1}{2} \Delta h + E_{N,\text{at}\,a} \cdot \frac{1}{2} \Delta h = 0$$

As, Δh to approach zero, keeping ΔW small but finite. we get

$$E_t \Delta W = 0$$

& $\boxed{E_t = 0}$

The Condition on the normal field is found most readily by considering D_N rather than E_N and choosing a small cylinder as the Gaussian surface. Let the height be Δh & the area of the top and bottom faces be ΔS. Again, if Δh tends to zero,

$$\oint_S \vec{D} \cdot \vec{dS} = Q$$

or, $$\int_{top} + \int_{bottom} + \int_{sides} = Q.$$

Since, no fields within conductor & tangential components of fields are ~~zero~~ $\int_{bottom} \vec{D} \cdot \vec{dS} = \int_{sides} \vec{D} \cdot \vec{dS} = 0$.

$$\therefore \int_{top} \vec{D} \cdot \vec{dS} = Q$$

or, $D_N \Delta S = Q = \rho_s \Delta S$

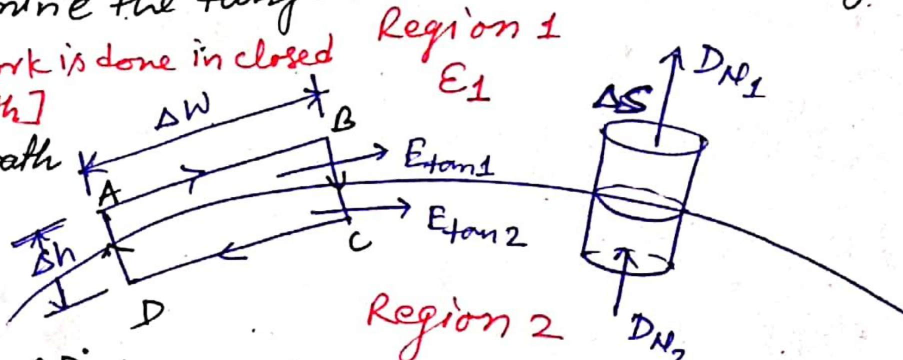
or, $\boxed{D_N = \rho_s}$

So, the desired boundary conditions for the conductor - free space boundary in electrostatics are:

$$\boxed{\begin{aligned} D_t &= E_t = 0 \\ \& \; D_N &= \epsilon_0 E_N = \rho_s \end{aligned}}$$

(2) Boundary Conditions for Perfect Dielectric Materials

Let us consider the interface betn two dielectrics having permitivities ϵ_1 & ϵ_2 and occupying regions 1 & 2 as shown in figure. We first examine the tangential components by using $\oint \vec{E} \cdot \vec{dl} = 0$ [No work is done in closed path] around the small closed path on the left as:



$$\int_A^B \vec{E} \cdot \vec{dl} + \int_B^C \vec{E} \cdot \vec{dl} + \int_C^D \vec{E} \cdot \vec{dl} + \int_D^A \vec{E} \cdot \vec{dl} = 0$$

or, $E_{tan1} \Delta W - E_{tan2} \Delta W = 0$

or, $\boxed{E_{tan1} = E_{tan2}}$

Since, the normal component of $\vec{E}$ along the sections of length Δh becomes negligible as Δh decreases and the closed path crowds the surface.

If the tangential electric field intensity is continuous across the boundary then tangential $\vec{D}$ is discontinuous

(33)

i.e

$$\frac{D_{tan1}}{\varepsilon_1} = \frac{D_{tan2}}{\varepsilon_2}$$

or, $$\boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}}$$

Now, the boundary conditions on the normal components are found by applying Gauss's Law to the small "pill box". The sides are again very short, and the flux leaving the top & bottom surfaces is the difference

$$D_{N_1}\Delta S - D_{N_2}\Delta S = \Delta Q = \rho_S \Delta S$$

or, $$\boxed{D_{N_1} - D_{N_2} = \rho_S}$$

Assuming $\rho_S = 0$ on the interface betn two perfect dielectrics

Then,

$$\boxed{D_{N_1} = D_{N_2}}$$

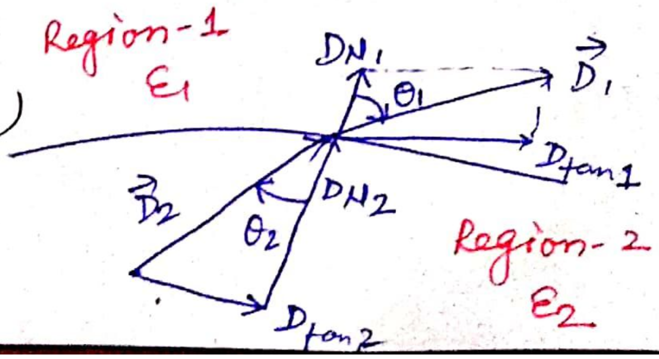
or The normal component of $\vec{D}$ is continuous &

$$\varepsilon_1 E_{N_1} = \varepsilon_2 E_{N_2}$$

or, $$\boxed{\frac{E_{N_1}}{E_{N_2}} = \frac{\varepsilon_2}{\varepsilon_1}}$$

normal component of $\vec{E}$ is discontinuous.

These conditions may be combined to show the change in the vectors $\vec{D}$ and $\vec{E}$ at the surface. Let $\vec{D}_1$ (an $\vec{E}_1$) make an angle θ_1 with a normal to the surface.



Since the normal components of $\vec{D}$ are continuous,

$$D_{N_1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N_2} \quad \text{--- (a)}$$

& the ratio of tangential component is given by

$$\frac{D_{tan1}}{D_{tan2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

or, $\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2$. --- (b)

Now, Dividing (b) by (a) we get

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

Since, we have assumed $\epsilon_1 > \epsilon_2$ and therefore $\theta_1 > \theta_2$. Direction of $\vec{E}$ on each side of the boundary is identical with the $\vec{D}$, because $\vec{D} = \epsilon\vec{E}$.

The magnitude of $\vec{D}$ in region 2 may be

$$\boxed{D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right)^2 \sin^2\theta_1}} \quad (*)$$

and the magnitude of $\vec{E_2}$ is

$$\boxed{E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1}} \quad (*)(*)$$

(*) Since, $D_{N_2} = D_1 \cos\theta_1$ & $D_{tan2} = \frac{\epsilon_2}{\epsilon_1} D_1 \sin\theta_1$

$$D_2 = \sqrt{D_{N_2}^2 + D_{tan2}^2} = \sqrt{D_1^2 \cos^2\theta_1 + \left(\frac{\epsilon_2}{\epsilon_1}\right) D_1^2 \sin^2\theta_1}$$

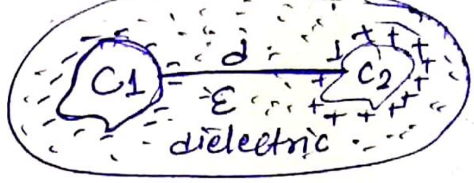
(**) Also, $E_{N_2} = \frac{\epsilon_1 E_1 \cos\theta_1}{\epsilon_2}$ & $E_{tan2} = E_1 \sin\theta_1$

$$\therefore E_2 = \sqrt{E_{N_2}^2 + E_{tan2}^2} = \sqrt{\left(\frac{\epsilon_1}{\epsilon_2}\right)^2 \cos^2\theta_1 E_1^2 + E_1^2 \sin^2\theta_1}$$

continue.

Capacitance – It is the ratio of the magnitude of total charge on either conductor to the magnitude of the potential difference betn conductors; where, two conductors with equal charges but in opposite signs are separated by small distance and a dielectric is placed in betn the two conductors.

i.e $\boxed{C = \frac{Q}{V_0}}$



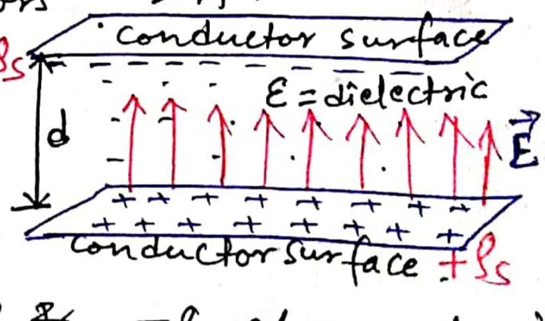
Also, we define Q by a surface integral over the positive conductors and find V_0 by carrying a unit positive charge from the –ve to +ve surface,

$$\boxed{C = \frac{\oint_S \epsilon \vec{E} \cdot d\vec{S}}{-\int_{-ve}^{+ve} \vec{E} \cdot d\vec{l}}}$$

Capacitance is measured in Farads (F) and is a function of only the physical dimentions of the system of conductors and of the permitivity of the homogeneous dielectric.

① Capacitance of the parallel plate capacitor

Let us consider two metallic conductors are separated by the distance, d. Let the space is filled with dielectric material of permitivity, ε and a uniform surface charge of ρ_s charge density is placed on conductor at z=0 & $-\rho_s$ charge density is placed on conductor at z=d. Then, the electric field intensity will be generated from +ve to –ve charges as shown in figure.



As we know, the electric field intensity in betn two plates is

$$\vec{E} = \frac{\rho_s}{\epsilon} \hat{a}_N = \frac{\rho_s}{\epsilon} \hat{a}_z \quad [\because \text{ since z-axis is considered in figure.}$$

Now, from defination of Capacitance,

$$C = \frac{\oint_S \varepsilon \vec{E}\cdot d\vec{S}}{-\int_{-ve}^{+ve} \vec{E}\cdot d\vec{l}} = \frac{Q}{V_0} = \frac{\rho_s \cdot S}{-\int_{-ve}^{+ve} \vec{E}\cdot d\vec{l}}$$

$$= \frac{\rho_s \cdot S}{-\int_{z=d}^{0} \frac{\rho_s}{\varepsilon}\cdot dz}$$

$$= \frac{\rho_s \cdot S}{-\frac{\rho_s}{\varepsilon}[0-d]}$$

$$= \frac{\cancel{\rho_s} S}{\frac{\cancel{\rho_s} d}{\varepsilon}}$$

$$= \frac{\varepsilon S}{d}$$

$$\therefore \boxed{C = \frac{\varepsilon S}{d}}$$

where, S = Area of the Conductors
& d = separation betn Conductors

Energy of capacitors:
Energy stored is

$$W_E = \frac{1}{2}\int_{Vol.} \varepsilon E^2 dv$$

$$= \frac{1}{2}\int_0^S \int_0^d \frac{\varepsilon \rho_s^2}{\varepsilon^2}\, dz\, ds$$

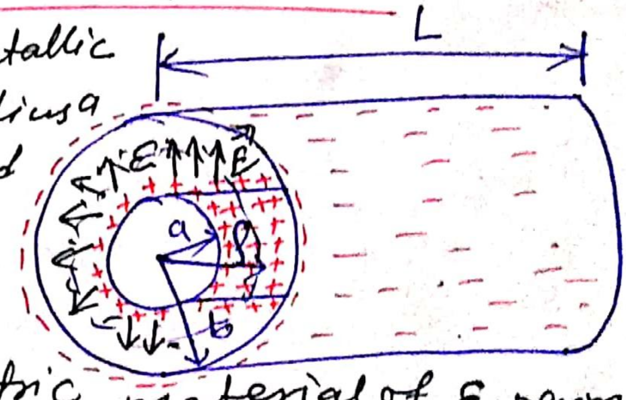
$$= \frac{1}{2}\frac{\rho_s^2}{\varepsilon} Sd$$

$$= \frac{1}{2}\frac{\varepsilon S}{d}\cdot \frac{\rho_s^2 d^2}{\varepsilon^2}$$

$$= \frac{1}{2} C V_0^2$$

② Capacitance of the Co-axial Capacitor

Let us consider two cylindrical metallic conductors of length L, inner radius a and outer radius b are separated by certain distance as shown in figure. Let the space betn two conductors is filled with dielectric material of ε permitivity. Then, the electric field intensity is produced from +ve charge to -ve charge & given by.



$$\vec{E} = \frac{\rho_s a}{\varepsilon \rho}\hat{a}_\rho \quad \text{where, } \rho = \text{radius in betn } a \,\&\, b.$$

Then,

$$C = \frac{Q}{V} = \frac{\rho_s \cdot 2\pi a L}{-\int_{-ve}^{+ve} \vec{E}\cdot d\vec{l}}$$

or,
$$C = \frac{\rho_s 2\pi a L}{-\int_b^a \frac{\rho_s a}{\epsilon \rho} \hat{a}_\rho \cdot d\rho \hat{a}_\rho}$$

$$= \frac{\cancel{\rho_s} \cdot 2\pi \cancel{a} L}{-\frac{\cancel{\rho_s} \cancel{a}}{\epsilon} \ln(\rho) \Big|_b^a}$$

$$= \frac{2\pi L \epsilon}{-\ln(a) + \ln(b)}$$

$$= \frac{2\pi L \epsilon}{\ln(b/a)}$$

$$\therefore \boxed{C = \frac{2\pi \epsilon L}{\ln(b/a)}}$$

Graphical Field Plotting (Curvilinear squares)

It is the basic and first mapping method. The method to be described is applicable only to fields in which no variation exists in the direction normal to the plane of the sketch. The procedure is based on several facts given as:

① A conductor boundary is an equipotential surface.

② The electric field intensity and electric flux density are both perpendicular to the equipotential surfaces.

③ $\vec{E}$ & $\vec{D}$ are therefore perpendicular to the conductor boundaries and possess zero tangential values.

④ The lines of electric flux, or streamlines, begin and terminate on charge and hence, in a charge free, homogeneous dielectric, begin and terminate only on the conductor boundaries.

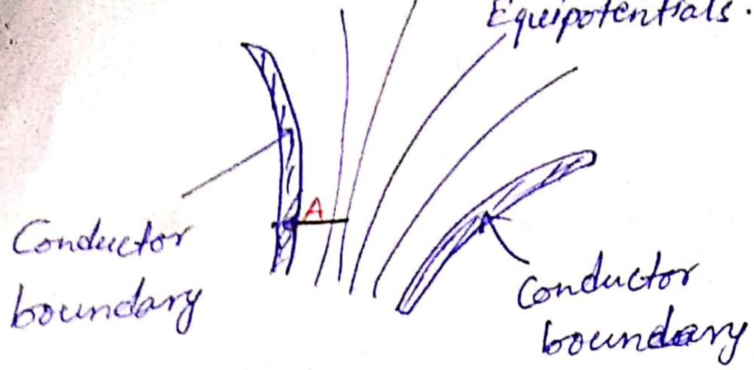
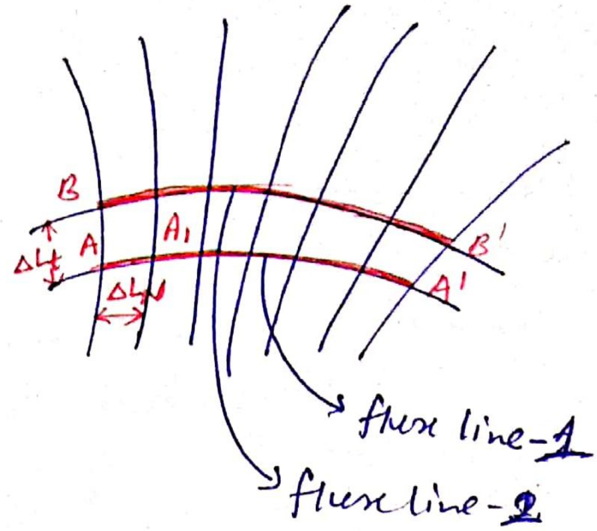


fig:- sketch of equipotential surfaces betn two conductors.



Let us construct the equipotential surfaces betn two conductors as shown in figure. Let the fluxline-1 is drawn from A to A' & another flux line-2 drawn from B to B' as shown in figure. During the construction of AA' & BB' flux lines the perpendicularity is maintened betn equipotential surfaces & the flux lines.

Here, the electric field intensity at the midpoint of the line joining A to B may be found approximately by assuming a value for the flux in the tube AB, say $\Delta\psi$, so electric flux density is $\frac{\Delta\psi}{\Delta L_t}$, where the depth of the tube into the paper is 1m and ΔL_t is the length of the line joining A to B

Magnitude of E is

$$E = \frac{1}{\epsilon} \cdot \frac{\Delta\psi}{\Delta L_t}$$

However, we may find the magnitude of the electric field intensity by dividing the potential difference betn points A & A_1, lying on two adjacent equipotential surfaces, by the distance from A to A_1. If A to A_1 is ΔL_N and ΔV is incremental potential betn equipotentials. then,

$$E = \frac{\Delta V}{\Delta L_N}$$

The value applies most accurately to the point at the middle of the line segment from A to A_1, while previous was most accurate at the midpoint of the line segment from A to B. If (ΔV small) equipotential surfaces are close together and the two streamlines are close together ($\Delta\psi$ small), the two values for E must be equal approximately.

i.e $\frac{1}{\varepsilon}\frac{\Delta\psi}{\Delta L_t} = \frac{\Delta V}{\Delta L_{N}}$

Throughout the sketch a homogeneous medium is considered (ε - constant), a constant increment of potential betn equipotentials (ΔV constant) and a constant amount of flux per tube ($\Delta\psi$ constant). To satisfy all these conditions,

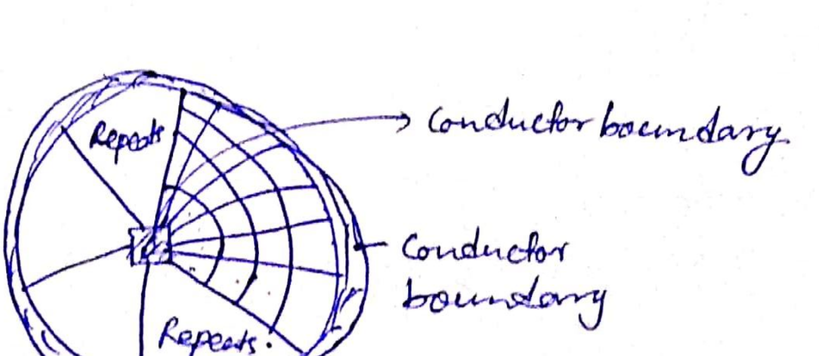
$$\boxed{\frac{\Delta L_t}{\Delta L_N} = \text{Constant} = \frac{1}{\varepsilon}\cdot\frac{\Delta\psi}{\Delta V}}$$

A similar argument might be made at any point of the sketch, and we are therefore led to a conclusion that a constant ratio must be maintained betn the distance betn streamlines as measured along an equipotential and the distance betn equipotentials as measured along the streamline. It is the ratio which must have the same value at every point.

Again, the capacitance is found from $C = \frac{Q}{V_0}$ by replacing Q by $N_Q \Delta Q = N_Q \Delta\psi$, where N_Q is number of flux tubes joining the two conductors and letting $V_0 = N_V \Delta V$, where, N_V is the number of potential increments betn conductors.

$$C = \frac{N_Q \Delta Q}{N_V \Delta V}$$

$$\text{or, } C = \frac{N_Q}{N_V}\varepsilon\frac{\Delta L_t}{\Delta L_N} = \varepsilon\frac{N_Q}{N_V} \quad \text{if } \frac{\Delta L_t}{\Delta L_N} = 1$$



$C = \varepsilon_0 \frac{6 \times 3.25}{4}$ pF/m. $N_V = 4$, $N_Q = 6 \times 3.25$

(140) Numerical Integration (The Iteration Method)

In potential problems where the potential is completely specified on the boundaries of a given region, particularly problems in which the potential does not varry in one direction (i.e two dimensional potential distributions) there exist a repetitive method yielding desired accuracy. It can not be found with Mapping method. So, digital computers should be used. The iterative method is well suited for digital computers.

Let us assume a two dimensional problem in which the potential doesnot vary with the z-co-ordinate and divide the interior of a cross section of the region where the ~~region~~ potential is desired into squares of length h on a side. The unknown values of the potential at five adjacent points are indicated as V_0, V_1, V_2, V_3 and V_4 as shown in figure. If the region is charge free and contains a homogeneous dielectric then

$\nabla \cdot \vec{D} = 0$ & $\nabla \cdot \vec{E} = 0$, from which we have

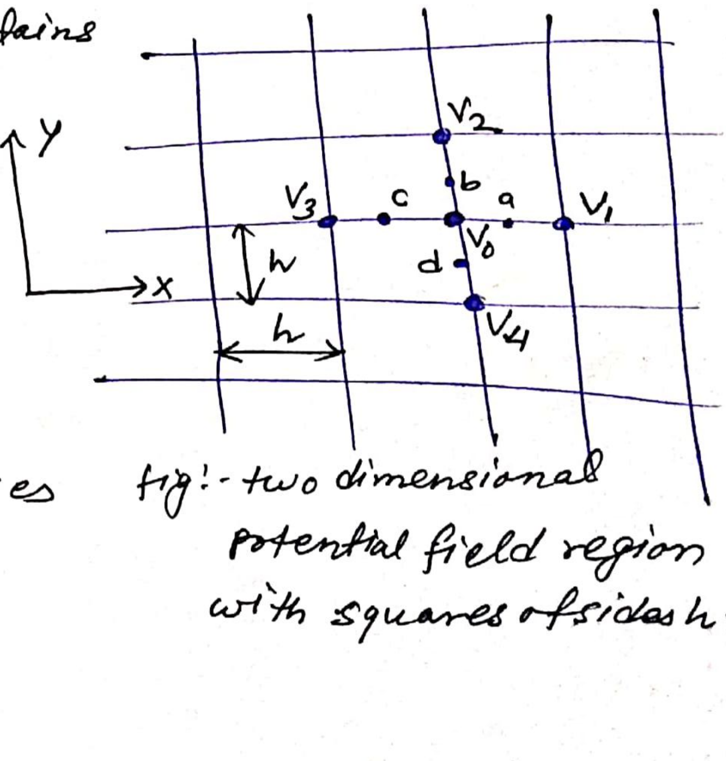


fig:- two dimensional potential field region with squares of sides h.

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

But the gradient operation gives

$$E_x = -\frac{\partial V}{\partial x} \quad \& \quad E_y = -\frac{\partial V}{\partial y}$$

so, $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Approximate values for these partial derivatives may be obtained in terms of the assumed potentials as:

$$\left.\frac{\partial V}{\partial x}\right|_a = \frac{V_1 - V_0}{h}$$

& $$\left.\frac{\partial V}{\partial x}\right|_c = \frac{V_0 - V_3}{h}$$

(141)

From which .

$$\left.\frac{\partial^2 v}{\partial x^2}\right|_0 = \frac{\left.\frac{\partial v}{\partial x}\right|_a - \left.\frac{\partial v}{\partial x}\right|_c}{h} = \frac{V_1 - V_0 - V_0 + V_3}{h}$$

& similarly,

$$\left.\frac{\partial^2 v}{\partial y^2}\right|_0 = \frac{V_2 - V_0 - V_0 + V_4}{h}$$

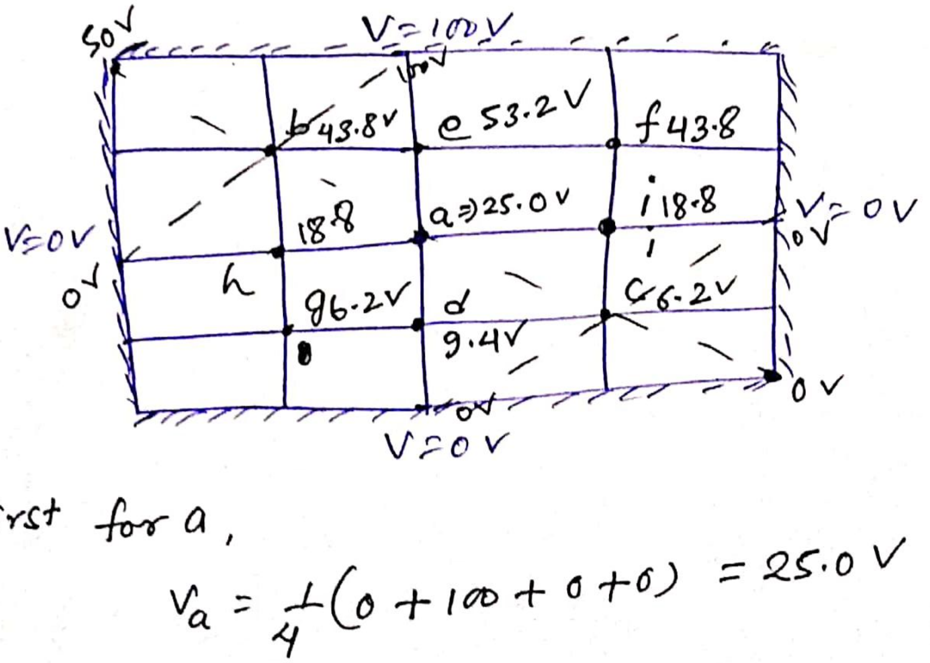
Combining .

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{h} = 0$$

or, $\boxed{\frac{(V_1 + V_2 + V_3 + V_4)}{4} = V_0}$

The expression becomes exact as h tends to zero.

eg:-



At first for a,

$$V_a = \frac{1}{4}(0 + 100 + 0 + 0) = 25.0\,V$$

Again, for b,

$$V_b = \frac{1}{4}(25 + 100 + 50 + 0) = 43.8\,V$$

for c,

$$V_c = \frac{1}{4}(25 + 0 + 0 + 0) = 6.2\,V$$

for d,

$$V_d = \frac{1}{4}(25 + 6.2 + 6.2 + 0)$$
$$= 9.4\,V$$

for e,

$$V_e = \frac{1}{4}(43.8 + 43.8 + 100 + 25)$$
$$= 53.2\,V$$

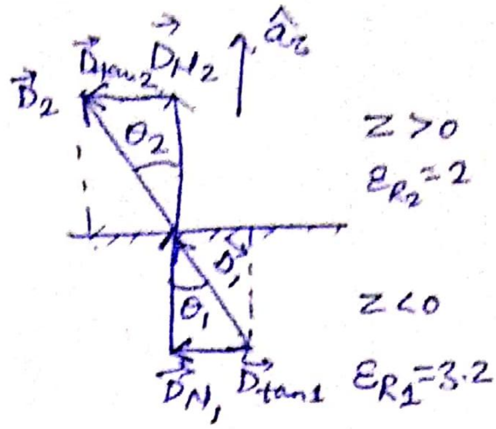
for h,

$$V_h = \frac{1}{4}(0 + 25 + 6.2 + 43.8)$$
$$= 18.8\,V$$

(142) Qn: Let the region $z < 0$ be composed of a uniform dielectric material for which $\epsilon_R = 3.2$, while the region $z > 0$ is characterised by $\epsilon_R = 2$. Let $\vec{D_1} = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z \ nC/m^2$ and find (a) D_{N1} (b) $\vec{D}_{t1}$ (c) D_{t1} (d) D_1 (e) θ_1 (f) $\vec{P}_1$ (g) $\vec{D}_{N2}$ (h) $\vec{D}_{t2}$ (i) $\vec{D}_2$ (j) $\vec{P}_2$ (k) θ_2.

Sol:^n Given,

$$\vec{D}_1 = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z \ nC/m^2$$



since, the two dielectric boundary lies in $z = 0$

we have,

(a) Normal component of $\vec{D}_1$ is

$$\vec{D}_{N1} = 70\hat{a}_z \ nC/m^2$$

$$D_{N_1} = |\vec{D}_{N_1}| = 70 \ nC/m^2$$

& (b) $\vec{D}_{tan1} = \vec{D}_{t1} = \vec{D}_1 - \vec{D}_{N1} = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z - 70\hat{a}_z$

$$= -30\hat{a}_x + 50\hat{a}_y \ nC/m^2$$

(c) $D_{t1} = |\vec{D}_{t1}| = \sqrt{(-30)^2 + (50)^2} = 58.3 \ nC/m^2$

(d) $D_1 = |\vec{D}_1| = \sqrt{(-30)^2 + (50)^2 + (70)^2} = 91.1 \ nC/m^2$

(e) We have,

$$\vec{D}_{N_1} \cdot \vec{D}_1 = |\vec{D}_{N_1}||\vec{D}_1| \cos\theta_1$$

$$\Rightarrow D_{N_1} = D_1 \cos\theta_1$$

or, $\theta_1 = \cos^{-1}\left(\frac{D_{N_1}}{D_1}\right) = \cos^{-1}\left(\frac{70}{91.1}\right) = 39.8°$

(f) We have,

$$\vec{P}_1 = \chi_{e_1} \epsilon_0 \vec{E}_1 \quad \& \quad \vec{D}_1 = \epsilon_{R_1} \epsilon_0 \vec{E}_1$$

or, $\vec{P}_1 = \frac{\chi_{e_1} \cdot (\epsilon_0 \epsilon_{R_1} \vec{E}_1)}{\epsilon_{R1}} = \frac{\chi_{e_1}}{\epsilon_{R1}} \cdot \vec{D}_1$

and $\epsilon_{R1} = \chi_{e_1} + 1$

or, $\chi_{e1} = \epsilon_{R1} - 1 = 3.2 - 1 = 2.2$

so, $\vec{P}_1 = \frac{2.2}{3.2}(-30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z)$

$= -20.6\hat{a}_x + 34.4\hat{a}_y + 48.1\hat{a}_z \ nC/m^2$

(g) From boundary conditions

$\vec{D}_{N_1} = \vec{D}_{N_2} = 70\,\hat{a}_z \ nC/m^2$

(h) Also,

from boundary conditions.

$\vec{E}_{t1} = \vec{E}_{t2}$

or, $\frac{\varepsilon_{R_1}\varepsilon_0\vec{E}_{t1}}{\varepsilon_{R_1}} = \frac{\varepsilon_{R_2}\varepsilon_0\vec{E}_{t2}}{\varepsilon_{R_2}}$

or, $\frac{\vec{D}_{t1}}{\varepsilon_{R_1}} = \frac{\vec{D}_{t2}}{\varepsilon_{R_2}}$

or, $\vec{D}_{t2} = \frac{\varepsilon_{R_2}}{\varepsilon_{R_1}}\vec{D}_{t1} = \frac{2}{3.2}(-30\hat{a}_x + 50\hat{a}_y)$

$= -18.75\hat{a}_x + 31.25\hat{a}_y \ nC/m^2$

(i) $\vec{D}_2 = \vec{D}_{N_2} + \vec{D}_{t2} = 70\hat{a}_z - 18.75\hat{a}_x + 31.25\hat{a}_y$

$= -18.75\hat{a}_x + 31.25\hat{a}_y + 70\hat{a}_z \ nC/m^2$

(j) $\vec{P}_2 = \chi_{e_2}\varepsilon_0\vec{E}_2$

& $\vec{D}_2 = \varepsilon_{R_2}\varepsilon_0\vec{E}_2$

i.e $\vec{P}_2 = \frac{\chi_{e_2}}{\varepsilon_{R_2}}\vec{D}_2$ where, $\chi_{e_2} = \varepsilon_{R_2} - 1 = 2 - 1 = 1$

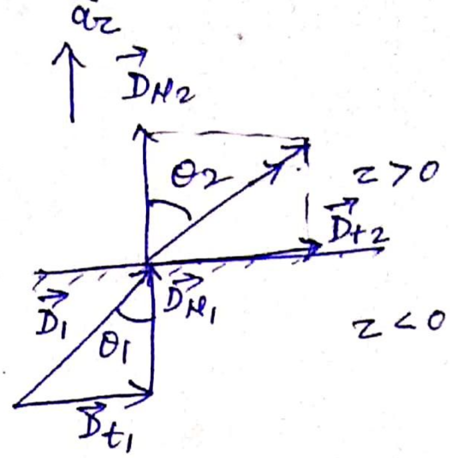
or, $\vec{P}_2 = \frac{1}{2}(-18.75\hat{a}_x + 31.25\hat{a}_y + 70\hat{a}_z)$

$= -9.38\hat{a}_x + 15.63\hat{a}_y + 35\hat{a}_z \ nC/m^2$

(k) Again, $\theta_2 = \cos^{-1}\left(\frac{D_{N2}}{D_2}\right) = \cos^{-1}\left(\frac{70}{\sqrt{(-18.75)^2 + (31.25)^2 + (70)^2}}\right) = 27.5°$

(144) Qn: The region $z<0$ contains a dielectric material for which $\epsilon_{r1} = 2.5$ while the region $z>0$ is characterized by $\epsilon_{r2} = 4$. Let $\vec{E_1} = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z$ V/m. Find: ⓐ $\vec{E_2}$ ⓑ $\vec{D_2}$ ⓒ Polarization in Region 2 ($\vec{P_2}$).

[2073 Shrawan]



Sol.ⁿ Given,

$$\vec{E_1} = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z \text{ V/m.}$$

Since, the boundary of two dielectric medium is at $z=0$

$$\vec{E}_{N1} = 70\hat{a}_z \text{ V/m.}$$

& $\vec{D}_{N1} = \epsilon_{r1}\epsilon_0 \cdot \vec{E}_{N1} = \epsilon_0 175\hat{a}_z$ C/m²

Also, $\vec{E}_{t1} = \vec{E_1} - \vec{E}_{N1} = -30\hat{a}_x + 50\hat{a}_y$ V/m.

& $\vec{D}_{t1} = \epsilon_0 \epsilon_{r1} \vec{E}_{t1} = \epsilon_0(-75\hat{a}_x + 125\hat{a}_y)$ C/m².

From boundary Conditions.

$$\vec{E}_{t2} = \vec{E}_{t1} = -30\hat{a}_x + 50\hat{a}_y \text{ V/m.}$$

Also,

$$\vec{D}_{N2} = \vec{D}_{N1} = 175\epsilon_0\hat{a}_z \text{ C/m}^2$$

or, $\cancel{\epsilon_0}\epsilon_{r2}\vec{E}_{N2} = 175\cancel{\epsilon_0}\hat{a}_z$

or, $\vec{E}_{N2} = \frac{175}{4}\hat{a}_z = 43.75\hat{a}_z$ V/m

∴ ⓐ $\vec{E_2} = \vec{E}_{t2} + \vec{E}_{N2} = -30\hat{a}_x + 50\hat{a}_y + 43.75\hat{a}_z$ V/m.

Again,

ⓑ $\vec{D_2} = \epsilon_0 \cdot \epsilon_{r2}\vec{E_2} = 8.85\times10^{-12} \times 4(-30\hat{a}_x + 50\hat{a}_y + 43.75\hat{a}_z)$

$$= -1.062\hat{a}_x + 1.77\hat{a}_y + 1.55\hat{a}_z \text{ nC/m}^2$$

Now, for polarization, $\vec{P_2}$

we have, $\vec{P_2} = \chi_{e_2} \varepsilon_0 \vec{E_2}$ and $\vec{D_2} = \varepsilon_{r_2} \varepsilon_0 \vec{E_2}$, $\chi_{e_2} = \varepsilon_{r_2} - 1$ $\Rightarrow \vec{P_2} = \vec{D_2}\left(\frac{\varepsilon_{r_2}-1}{\varepsilon_{r_2}}\right)$

$= 4-1 = 3$

So,

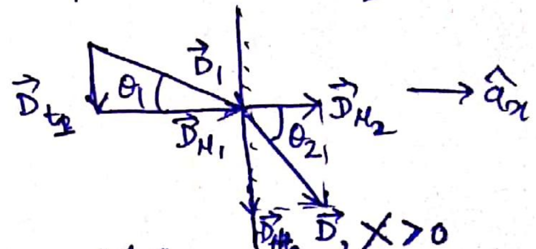
$$\vec{P_2} = 3 \times 8.85 \times 10^{-12} (-30\hat{a}_x + 50\hat{a}_y + 43.75\hat{a}_z)$$

$$= -0.796\hat{a}_x + 1.327\hat{a}_y + 1.161\hat{a}_z \ nC/m^2$$

Qn: The region $x<0$ is composed of a uniform dielectric material for which $\varepsilon_{r_1} = 3.2$, while the region $x>0$ is characterized by $\varepsilon_{r_2} = 2$. The electric flux density at region $x<0$ is $\vec{D_1} = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z \ nC/m^2$. Then find polarization ($\vec{P}$) and electric field intensity ($\vec{E}$) in both regions. [$\vec{P_1}, \vec{P_2}, \vec{E_1}$ & $\vec{E_2} = ?$] [2068 Chaitra]

Sol.ⁿ Given, $\vec{D_1} = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z \ nC/m^2$ $\varepsilon_{r_1} = 3.2$ $\varepsilon_{r_2} = 2$

Since, the boundary of two dielectrics is in $x=0$ plane



$$\vec{D}_{N1} = -30\hat{a}_x \ nC/m^2 = \vec{D}_{N2}$$

So, $\vec{E}_{N1} = \frac{\vec{D}_{N1}}{\varepsilon_{r_1}\varepsilon_0} = \frac{-30 \times 10^{-9}\hat{a}_x}{3.2 \times 8.85 \times 10^{-12}} = -1059\hat{a}_x \ V/m$.

Also, $\vec{E_1} = \frac{\vec{D_1}}{\varepsilon_{r_1}\varepsilon_0} = \frac{(-30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z) \times 10^{-9}}{3.2 \times 8.85 \times 10^{-12}}$

$$= -1059\hat{a}_x + 1765\hat{a}_y + 2471\hat{a}_z \ V/m$$

Again,

$$\chi_{e_1} = \varepsilon_{r_1} - 1 = 3.2 - 1 = 2.2$$

& $\vec{P_1} = \frac{\chi_{e_1}}{\varepsilon_{r_1}} \vec{D_1} = \frac{2.2}{3.2}(-30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z) \ nC/m^2$

$$= -20.625\hat{a}_x + 34.4\hat{a}_y + 48.1\hat{a}_z \ nC/m^2$$

Now,

$$\vec{E}_{t1} = \vec{E_1} - \vec{E}_{N1} = 1765\hat{a}_y + 2471\hat{a}_z \ V/m.$$

From boundary Conditions,

$$\vec{E}_{t_2} = \vec{E}_{t_1} = 1765\,\hat{a}_y + 2471\,\hat{a}_z \text{ V/m}.$$

and, $$\vec{E}_{N_2} = \frac{\vec{D}_{N_2}}{\varepsilon_{r_2}\varepsilon_0} = \frac{-30\,\hat{a}_x \times 10^{-9}}{2\times 8.85\times 10^{-12}} = -1694.92\,\hat{a}_x \text{ V/m}.$$

$$\therefore \vec{E}_2 = \vec{E}_{t_2} + \vec{E}_{N_2} = -1695\,\hat{a}_x + 1765\,\hat{a}_y + 2471\,\hat{a}_z \text{ V/m}.$$

Now,

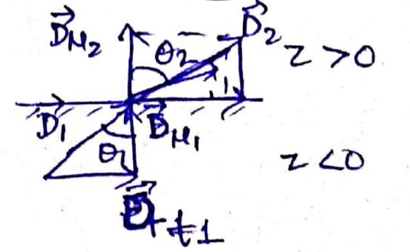
$$\vec{P}_2 = \chi_{e_2}\varepsilon_0\vec{E}_2 \quad \text{where, } \chi_{e_2} = \varepsilon_{r_2} - 1 = 2-1 = 1$$

$$\therefore \vec{P}_2 = 1\times 8.85\times 10^{-12}\left(-1695\,\hat{a}_x + 1765\,\hat{a}_y + 2471\,\hat{a}_z\right)$$

$$= -15\,\hat{a}_x + 15.62\,\hat{a}_y + 21.87\,\hat{a}_z \text{ nC/m}^2$$

Qn: Use boundary Condition to find $\vec{E}_2$ in the medium 2 with boundary located at plane $z=0$. Medium 1 is perfect dielectric characterized by $\varepsilon_{r_1} = 2.5$, medium2 is perfect dielectric characterized by $\varepsilon_{r_2} = 5$, electric field in medium 1 is $\vec{E}_1 = \hat{a}_x + 3\hat{a}_y + 3\hat{a}_z$ V/m. [2069 Ashadh]

Sol.



$$\vec{D}_{N_2} = \vec{D}_{N_1}, \quad \vec{E}_{t2} = \vec{E}_{t1}$$

$$\therefore \vec{E}_{N_2} = \frac{\vec{D}_{N_2}}{\varepsilon_{r_2}\varepsilon_0}.$$

$$\therefore \vec{E}_2 = \vec{E}_{N_2} + \vec{E}_{t_2}$$

Qn: Consider the region $y<0$ be composed of a uniform dielectric material for which the relative permitivity (ε_r) is 3.2 while the region $y>0$ is characterized by $\varepsilon_r = 2$. Let the flux density in region 1 be $\vec{D}_1 = -30\hat{a}_x + 50\hat{a}_y + 70\hat{a}_z$ nC/m². Find: ⓐ Magnitude of flux density (D_2) & electric field intensity at region 2 (E_2) ⓑ Polarization in region 1 & 2 ($\vec{P}_1$ & $\vec{P}_2$) [2070 chaitra]

Sol. $\chi_{e_1} = \varepsilon_{r_1} - 1$, $\chi_{e_2} = \varepsilon_{r_2} - 1$, $\vec{P}_1 = \frac{\chi_{e_1}}{\varepsilon_{r_1}}\vec{D}_1$, $\vec{P}_2 = \frac{\chi_{e_2}}{\varepsilon_{r_2}}\vec{D}_2$

Boundary Value Problems

The procedure for determining the electric field $\vec{E}$ are defined in many ways; using graphical plotting, coulomb's law, Gauss's Law when charge distribution is known; or using $\vec{E} = -\nabla V$ when the potential V is known throughout the region. In most practical situations, however, neither the charge distribution nor the potential distribution is known. So, for practical situations there will be only electrostatic conditions (charge & potential) at some boundaries are known and it is desired to find $\vec{E}$ & V throughout the region. Such problems are usually solved using Poisson's or Laplace's equation or the image method, and they are usually referred to as boundary value problems.

Poisson's and Laplace's Equation

Poisson's and Laplace's equations are easily derived from Gauss's Law (for a linear material medium). or maxwell's first Equation.
From point form of Gauss's Law

$$\nabla \cdot \vec{D} = \rho_v \quad \text{—— (a)}$$

Again, we have from defination of $\vec{D}$

$$\vec{D} = \varepsilon \vec{E}$$

and gradient relationship is.

$$\vec{E} = -\nabla V$$

Substituting in (a)

$$\nabla \cdot (\varepsilon \vec{E}) = \rho_v$$

or, $\nabla \cdot [\varepsilon(-\nabla V)] = \rho_v$

or $$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon}$$

This is a Poisson's Equation; where ε is constant for homogeneous region.

Again, from definition of gradient & divergence, we get

$$\left.\begin{aligned} \nabla \cdot \vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ \& \quad \nabla V &= \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z \end{aligned}\right\} \text{Cartesian Co-ordinates.}$$

Therefore,

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

So, $$\boxed{\nabla \cdot \nabla V = \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}}$$

If $\rho_v = 0$, indicating zero volume charge density, but allowing point charges, line charges, and surface chargedensity to exist at singular locations as sources of the field, then,

$$\boxed{\nabla^2 V = 0}$$

This is the Laplace's Equation. The ∇^2 (del squared) operation is called Laplacian of V. In Cartesian co-ordinates Laplace's Equation is

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0}$$

Similarly, in cylindrical and spherical Co-ordinates the Laplace's Eq^n is given as:

Cylindrical – $$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

Spherical – $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Laplace's equation is applicable for the region where volume charge density is zero; it states that every conceivable configuration of electrodes or conductors produces the field for which $\nabla^2 V = 0$. All these fields are different, with different potential values and different spatial rates of change, yet for each of them $\nabla^2 V = 0$. To certain boundary conditions we must solve the Laplace's equation.

Every physical problem must contain at least one conducting boundary and usually contains two or more. The potentials on these boundaries are assigned values; perhaps $V_0, V_1, \ldots,$ or perhaps numerical values. These definite equipotential surfaces will provide the boundary conditions for the type of problem to be solved.

Example of solution of Laplace's Equation

Let us assume that V is a function of x only and we have ~~a~~ need for boundary conditions to get exact value. So, Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2} = 0$$

and the partial derivative may be replaced by ordinary derivative, since, V is not a function of y & z,

$$\frac{d^2V}{dx^2} = 0$$

Integrating twice, we get

$$\frac{dV}{dx} = C_1$$

and $V = C_1 x + C_2$

where, C_1 & C_2 are constants of integration and are determined from boundary conditions.

Let $V = V_0$ at $x = x_0$ and $V = V_1$ at $x = x_1$ then

$$V_0 = C_1 x_0 + C_2 \quad V_1 = C_1 x_1 + C_2$$

so, $C_1 = \dfrac{V_0 - V_1}{x_0 - x_1}$. $C_2 = \dfrac{V_1 x_0 - V_0 x_1}{x_0 - x_1}$

and $V = \dfrac{V_0(x - x_1) - V_1(x - x_0)}{x_0 - x_1}$

If we assume a simple boundary conditions as:

$V = 0$ at $x = 0$ & $V = V_0$ at $x = d$ then

$$C_1 = \frac{V_0}{d} \qquad C_2 = 0$$

and $V = \dfrac{V_0 x}{d}$

(25.1)

Uniqueness Theorem - If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

Any solution of Laplace's Equation which satisfies the same boundary conditions must be the only solution regardless of the method used. This is known as the uniqueness theorem. The theorem applies to any solution of Poisson's or Laplace's equation in a given region or closed surface.

The theorem is proved by contradiction.

proof:

Let us assume that we have two solutions of Laplace's equation, V_1 & V_2, both are general functions of the co-ordinates used. Therefore.

$$\nabla^2 V_1 = 0$$

and $$\nabla^2 V_2 = 0$$

from which, we get

$$\nabla^2 (V_1 - V_2) = 0$$

Each solution must also satisfy the boundary conditions and if we assume the given potential values on the boundaries by V_b then the value of V_1 on the boundary V_{1b} and the value of V_2 on the boundary V_{2b} must both be identical to V_b,

$$V_{1b} = V_{2b} = V_b$$

or, $$V_{1b} - V_{2b} = 0$$

From vector identity

$$\nabla \cdot (V\vec{D}) \equiv V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$$

which holds for any scalar V and any vector $\vec{D}$.

152. For present condition we assume $V_1 - V_2$ as the scalar & $\nabla(V_1 - V_2)$ as the vector, resulting

$$\nabla\cdot\left[(V_1-V_2)\,\nabla(V_1-V_2)\right] \equiv (V_1-V_2)\left[\nabla\cdot\nabla(V_1-V_2)\right] + \nabla(V_1-V_2)\cdot\nabla(V_1-V_2)$$

Integrating throughout the volume enclosed by the boundary surfaces specified:

closed surface

$$\int_{vol.} \nabla\cdot\left[(V_1-V_2)\,\nabla(V_1-V_2)\right] dv \qquad \nabla^2(V_1-V_2)=0$$

$$\equiv \int_{vol.} (V_1-V_2)\left[\nabla\cdot\nabla(V_1-V_2)\right] dv + \int_{vol.} \left[\nabla(V_1-V_2)\right]^2 dv$$

From divergence theorem we can replace the volume integral on the left side of the equation by the closed surface integral over the surface surrounding the volume. This surface consists of the boundaries already specified on which $V_{1b} = V_{2b}$ and therefore.

$$\int_{vol.} \nabla\cdot\left[(V_1-V_2)\,\nabla(V_1-V_2)\right] dv = \oint_s \left[(V_{1b}-V_{2b})\,\nabla(V_{1b}-V_{2b})\right] d\vec{s} = 0$$

One of the factors of the first integral on the right side of above equation is $\nabla\cdot\nabla(V_1-V_2)$ or $\nabla^2(V_1-V_2)$, which is zero by hypothesis, and therefore that integral is zero. Hence, the remaining integral must be zero.

$$\int_{vol.} \left[\nabla(V_1-V_2)\right]^2 dv = 0$$

There are two reasons for an integral to be zero; either the integrand (the quantity under the integral sign) is everywhere zero, or the integrand is positive in some regions and negative in others, and the contributions cancel algebrically. In this case, the first reason must hold because $[\nabla(V_1-V_2)]^2$ can-not be negative. Therefore

$$[\nabla(V_1-V_2)]^2=0$$

and $\nabla(V_1-V_2)=0$

Finally, if the gradient of V_1-V_2 is everywhere zero, then V_1-V_2 cannot change with any Co-ordinates and

$$V_1=V_2=\text{constant}.$$

The constant is evaluated by considering a point on the boundary. Here, $V_1-V_2=V_{1b}-V_{2b}=0$ & we see that the constant is zero and therefore

$$V_1=V_2$$

giving two identical solutions. It is applied to poisson's equation also, where $\nabla^2V_1=-\frac{\rho_v}{\varepsilon}$ & $\nabla^2V_2=-\frac{\rho_v}{\varepsilon}$ then $\nabla^2(V_1-V_2)=0$ as before (Laplace's equation).

Capacitance of parallel plate capacitor

Capacitance is given by the ratio of charge to potential difference. The potential difference is fixed as V_0 by choosing the potential of one plate zero and the other V_0; the location of these plates are made as simple as possible by letting $V=0$ at $x=0$.

Steps to find capacitance after choosing $V=0$ at $x=0$.

1. Given V, use $\vec{E} = -\nabla V$ to find $\vec{E}$
2. Use $\vec{D} = \epsilon\vec{E}$ to find $\vec{D}$
3. Evaluate $\vec{D}$ at either capacitor plate, $\vec{D} = \vec{D_S} = D_N \hat{a}_N$.
4. Recognize that $\rho_S = D_N$
5. Find Q by a surface integration over the capacitor plate; $Q = \int_S \rho_S \, dS$.

Then, we will have

$$V = \frac{V_0 x}{d} \quad \text{[solution of Laplace's Eq}^n\text{]}$$

$$\vec{E} = -\frac{V_0}{d}\hat{a}_x$$

$$\vec{D} = -\epsilon\frac{V_0}{d}\hat{a}_x$$

$$\vec{D_S} = \vec{D}\Big|_{x=0} = -\epsilon\frac{V_0}{d}\hat{a}_x \text{ and } \hat{a}_N = \hat{a}_x$$

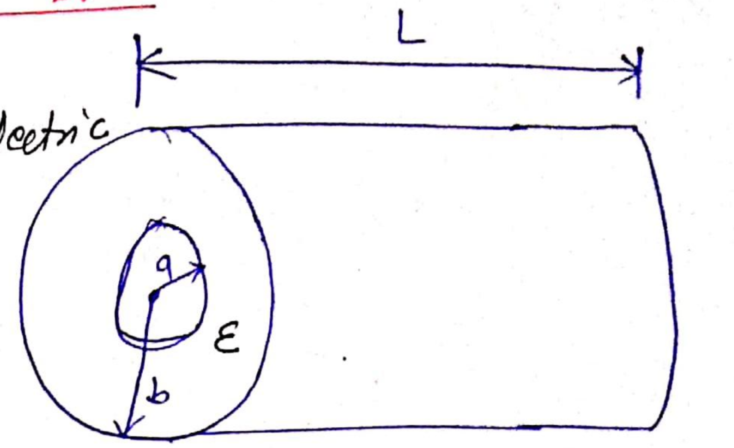
$$D_N = -\epsilon\frac{V_0}{d} = \rho_S$$

$$Q = \int_S -\frac{\epsilon V_0}{d}\, dS = -\epsilon\frac{V_0 S}{d}$$

and the capacitance is

$$\boxed{C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}}$$

(155) Capacitance of Co-axial Cable



Let us consider a cylindrical Co-axial cable filled with dielectric of permittivity ε inbet^n two cylindrical conductor plates. Let 'a' be the radius of inner conductor where the potential $V = V_0$ at $\rho = a$ and 'b' be the radius of the outer conductor where the potential $V = 0$ at $\rho = b$ $(b > a)$.

From the figure, we can see that the variation of potential is seen only in ρ direction. So, from laplace Equation.

$$\nabla^2 V = 0$$

$$\text{or, } \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

As $\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$ & $\frac{\partial^2 V}{\partial z^2} = 0$

we have,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0$$

$$\text{or, } \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0 \quad \text{[from ordinary derivative]}$$

$$\text{or, } \frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

Integrating both sides w.r.t. ρ

$$\rho \frac{dV}{d\rho} = C_1$$

or, $\frac{dV}{d\rho} = \frac{C_1}{\rho}$

Again, Integrating both sides w.r.t. ρ, we get

$$V = C_1 \ln(\rho) + C_2$$

Now, using boundary conditions $V = V_0$ at $\rho = a$ & $V = 0$ at $\rho = b$ for $b > a$. then

$$V_0 = C_1 \ln(a) + C_2 \qquad 0 = C_1 \ln(b) + C_2$$

$$V_0 = C_1 \ln(a/b)$$

$$\text{or, } C_1 = \frac{V_0}{\ln(a/b)}$$

Again,

$$C_2 = \frac{-\ln(b) \cdot V_0}{\ln(a/b)}$$

Therefore,

$$V = \frac{V_0}{\ln(a/b)} \cdot \ln(\rho) - \frac{\ln(b) V_0}{\ln(a/b)}$$

$$= \frac{V_0 \ln(\rho/b)}{\ln(a/b)}$$

$$\therefore \boxed{V = \frac{V_0 \ln(b/\rho)}{\ln(b/a)}}$$

Now, $\vec{E} = -\nabla V = \frac{V_0}{\rho} \cdot \frac{1}{\ln(b/a)} \hat{a}_\rho$

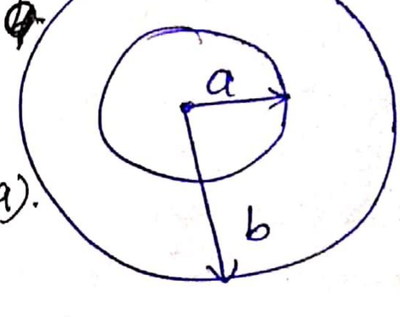
$$D_N(\rho=a) = \frac{\varepsilon V_0}{a \ln(b/a)}$$

$$\& \; Q = \frac{\varepsilon V_0 \, 2\pi a L}{a \ln(b/a)}$$

So, $$\boxed{C = \frac{2\pi \varepsilon L}{\ln(b/a)}}$$

Capacitance of Spherical Capacitor

Let us assume a spherical capacitor with inner plate of radius a & potential difference $V = V_0$ at $r = a$ and outer plate of radius b & potential difference $V = 0$ at $r = b$. $(b > a)$.



Then, from Laplace's Eq^n

$$\nabla^2 V = 0$$

or, $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

As, V only varies with r.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

or, $$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

or, $$\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0$$

Integrating twice we get

$$\frac{dV}{dr} = \frac{c_1}{r^2}$$

or, $V = -\frac{c_1}{r} + c_2$

Now, using boundary conditions, we get

$$V = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

& capacitance, $C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$

Example of solution of Poisson's Equation

Let us assume the pn-junction betn two halves of a semi-conductor bar extending in the x-dirn. Assuming the region for $x < 0$ is doped p-type and that region for $x \geq 0$ is n-type and the degree of doping is identical on each side of the junction. Then, the charge distribution is given as

$$\rho_v = 2\rho_{v_0} \operatorname{sech}\frac{x}{a} \tanh\frac{x}{a}$$

which has a maxm charge density $\rho_{v,max} = \rho_{v_0}$ that occurs at $x = 0.881a$. As the maxm chargedensity is related to the acceptor and donor concentrations N_a & N_d as:

$$\rho_{v_0} = eN_a = eN_d$$

we get from poisson's Eqn.

$$\nabla^2 V = \frac{-\rho_v}{\varepsilon}$$

Then,

$$\frac{d^2V}{dx^2} = -\frac{2\rho_{v_0}}{\varepsilon} \operatorname{sech}\frac{x}{a} \tanh\frac{x}{a}$$

(159) Integrating both sides

$$\frac{dv}{dx} = \frac{2\rho_{v0}a}{\varepsilon} \operatorname{sech}\frac{x}{a} + C_1$$

and obtain the electric field intensity.

$$E_x = \frac{-2\rho_{v0}a}{\varepsilon} \operatorname{sech}\frac{x}{a} - C_1$$

To evaluate the constant of integration C_1, we note that no net charge density and no fields can exist far from the junction. Thus, as $x \to \pm\infty$, E_x must approach zero. Therefore, $C_1 = 0$, and

$$E_x = -\frac{2\rho_{v0}a}{\varepsilon} \operatorname{sech}\frac{x}{a}$$

Integrating again

$$V = \frac{4\rho_{v0}a^2}{\varepsilon} \tan^{-1} e^{x/a} + C_2$$

Assuming arbitrarily select our zero reference of potential at the center of the junction, $x = 0$

$$0 = \frac{4\rho_{v0}a^2}{\varepsilon} \cdot \frac{\pi}{4} + C_2$$

and finally

$$V = \frac{4\rho_{v0}a^2}{\varepsilon}\left(\tan^{-1} e^{x/a} - \frac{\pi}{4}\right)$$

The potential difference V_0 across the junc^n is obtained from

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = \frac{2\pi\rho_{v0}a^2}{\varepsilon}$$

Total positive charge is

$$Q = S\int_0^\infty 2\rho_{v0} \operatorname{sech}\frac{x}{a} \tanh\frac{x}{a}\, dx = 2\rho_{v0}aS.$$

where, S = area of the junction cross section.

Then, $Q = S\sqrt{\frac{2\rho_{v_0}\varepsilon V_0}{\pi}}$ $\left[\because a = \sqrt{\frac{V_0\varepsilon}{2\pi\rho_{v_0}}}\right]$

Since, total charge is a function of potential difference, we get

$$I = \frac{dQ}{dt} = C\frac{dV_0}{dt}$$

& $C = \frac{dQ}{dV_0}$

Differentiating we get

$$C = \sqrt{\frac{\rho_{v_0}\varepsilon}{2\pi V_0}} \cdot S = \frac{\varepsilon S}{2\pi a}$$