Chapter-4 Wave Equation and Wave Propagation

Faraday's Law

After Oersted demonstrated in 1820 that an electric current affected a compass needle, Faraday professed his belief that if a current could produce a magnetic field then a magnetic field should be able to produce a current. After his experiment; that the changing magnetic field produced the deflection in the galvanometer showing that the current is produced; he concluded that the time-varying (changing) magnetic field produces an electromotive force (emf) which may establish a current in a suitable closed circuit. An electromotive force is merely the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields.

Statement: The induced emf, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of magnetic flux linkage by the circuit.

$$\boxed{\text{i.e } V_{emf} = -\frac{d\phi}{dt} \text{ V}} \quad \text{—— (a)}$$

Here, non zero value of $\frac{d\phi}{dt}$ may result from any of the followings:

1. A time-changing flux linking a stationary closed path.
   i.e $\vec{B}$ = varrying.
2. Relative motion betn a steady flux and a closed path.
   i.e $\vec{B}$ = uniform but the path is moving in field $\vec{B}$.
3. A combination of two.
   i.e $\vec{B}$ = varrying & the closed path is moving in field $\vec{B}$.

Here, the -ve sign indicates that the emf is in such a direction as to produce a current whose flux; if added to the original flux, would reduce the magnitude of emf. This statement that the induced voltage acts to produce an opposing flux is known as Lenz's Law.

If the closed path is that taken by an N-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and then

$$\boxed{V_{emf} = -N\frac{d\phi}{dt}} \quad \text{— (b)}$$

where, ϕ is the flux passing through any one of N coincident paths.

The emf is a scalar quantity and has dimension of voltage and unit is volts. So, we can write the emf as:

$$\boxed{V_{emf} = \oint_L \vec{E}\cdot\vec{dl}} \quad \text{— (c)}$$

i.e voltage about a specific closed path and if any part of the closed path is changed, the emf changes.

From definition of flux density, we have

$$\text{flux}, \ \phi = \int_S \vec{B}\cdot\vec{ds}$$

Replacing in (a) and equating with (c) we get

$$\boxed{V_{emf} = \oint_L \vec{E}\cdot\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{ds}}$$

S - area of the circuit bounded by the closed path L

where, fingers of the right hand indicate the direction of the closed path and the thumb indicates the dirn of $\vec{ds}$.

Transformer and Motional (generator) emfs

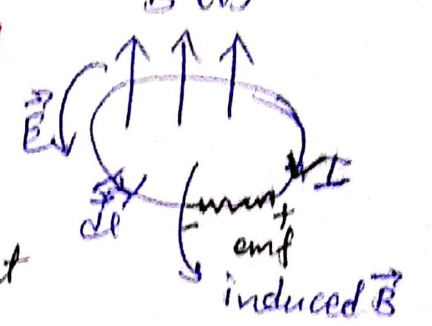
Here, we have from Faraday's Law and voltage or potential relations,

$$V_{emf} = \oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$$

Transformer emf

(i) Let us first assume the first condition, that the closed path considered is stationary and the magnetic flux only is time varying then, we have

$$\boxed{V_{emf} = \oint_L \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}}$$



This emf induced by the time varying current (producing the time-varying $\vec{B}$ field) in a stationary loop/path is often referred to as transformer emf. in power analysis since it is due to transformer action.

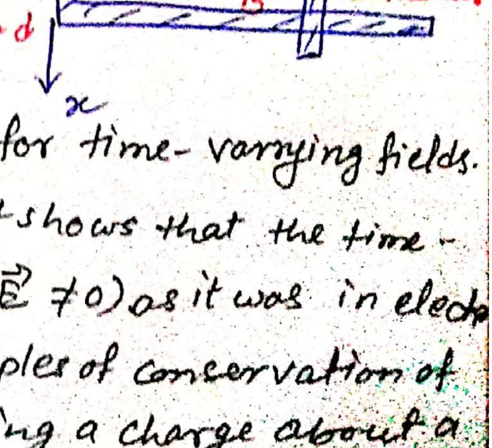
fig:- induced emf due to stationary path & time varying field $\vec{B}$.

Applying Stoke's Theorem, we get

$$\int_S (\nabla\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

or,

$$\boxed{\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$



This is one of Maxwell's equations for time-varying fields written in differential or point form. It shows that the time-varying $\vec{E}$ field is not conservative ($\nabla\times\vec{E} \neq 0$) as it was in electro-static. This does not imply that the principles of conservation of energy is violated. The work done in taking a charge about a closed path in a time varying electric field; for eg: is due to the energy from the time-varying magnetic field.

From above figure, it obeys Lenz's Law; the induced current I flows such as to produce a magnetic field that opposes $\vec{B}(t)$.

रविन कासुला
ईन्जिनियर

(b) Motional (or generator) emf.

Again, let us consider the second condition that the magnetic field is uniform or steady and the conducting closed path or loop is moving then, we have from the force produced in magnetic field $\vec{B}$ to the moving charge with uniform velocity $\vec{v}$ is:

$$\vec{F}_m = Q \cdot \vec{v} \times \vec{B}$$

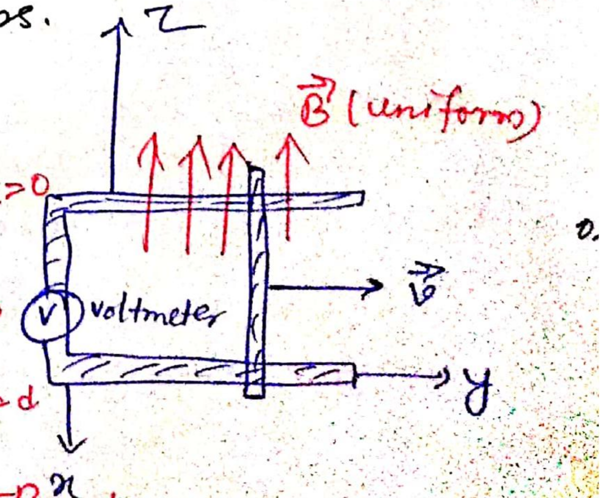
If we define the motional electric field $\vec{E}_m$ then.

$$\vec{E}_m = \frac{\vec{F}_m}{Q} = \vec{v} \times \vec{B}$$

If we consider a conducting closed path, moving with uniform velocity $\vec{v}$ as consisting of a large number of free electrons, then the emf induced in the closed path is:

$$\boxed{V_{emf} = \oint_L \vec{E}_m \cdot \vec{dl} = \oint_L (\vec{v} \times \vec{B}) \cdot \vec{dl}}$$

This type of emf is called motional or generator emf or flux cutting emf because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators.



eg: From figure, the flux passing through the surface within the closed path at any time t is then

$$\phi = Byd$$

& then, $V_{emf} = -\frac{d\phi}{dt} = -B\frac{dy}{dt}d = -B\overset{x}{v}d.$

Moving Loop/closed path in time varying field

We have, both conditions, where the field is time varying and the closed path is also moving then the transformer emf and motional emf both will be present; so the total emf is:

$$\boxed{V_{emf} = \oint_L \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_L (\vec{v} \times \vec{B}) \cdot d\vec{l}}$$

Displacement Current

Faraday's experimental Law has been used to obtain one of Maxwell's equations in differential form as:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

which shows that the time changing magnetic field produces an electric field.

Again, for static EM fields or the point form of Ampere's circuital Law as it applies to steady magnetic fields,

$$\nabla \times \vec{H} = \vec{J}$$

Taking divergence of the curl, we see that the divergence of the curl of any vector field is identically zero.

Hence, $\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J}$

But, however from equation of continuity,

$$\nabla \cdot \vec{J} = \frac{-\partial \rho_v}{\partial t}$$

Here, $\nabla \times \vec{H} = \vec{J}$ can only be true only if $\frac{\partial \rho_v}{\partial t} = 0$. This is an unrealistic limitation and hence $\nabla \times \vec{H} = \vec{J}$ must be amended before we can accept it for time varying fields. Suppose we add an unknown term $\vec{G}$ to $\nabla \times \vec{H} = \vec{J}$ as:

$$\nabla \times \vec{H} = \vec{J} + \vec{G} = \vec{J} + \vec{J}_d$$

Again, taking the divergence, we have $[\vec{G} = \vec{J}_d]$

$$0 = \nabla \cdot \vec{J} + \nabla \cdot \vec{G}$$

Thus, $\nabla \cdot \vec{G} = \frac{\partial \rho_v}{\partial t}$

Replacing ρ_v by $\nabla \cdot \vec{D}$ as $\nabla \cdot \vec{D} = \rho_v$

$$\nabla \cdot \vec{G} = \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

Hence,

$$\boxed{\vec{G} = \frac{\partial \vec{D}}{\partial t}}$$

Substituting in $\nabla \times \vec{H} = \vec{J} + \vec{G}$, we have

$\hookrightarrow \nabla \times \vec{H} = \vec{J} + \vec{J}_d$

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

which is the point form of Ampere's circuital law for time varying fields or Maxwell's equation for time varying field. The term $\vec{G} = \frac{\partial \vec{D}}{\partial t}$ is known as the displacement current density and the $\vec{J}$ is the conduction current density. $(\vec{J} = \sigma \vec{E})$. $\vec{G}$ is sometimes written as $\vec{J}_d$.

For a non-conducting medium in which no volume charge density is present as: $\vec{J} = \rho_v \vec{v} = 0$ then,

$$\boxed{\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad (\text{if } \vec{J} = 0)}$$

This is similar to

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Now, the total displacement current crossing given surface is expressed by the surface integral as:

$$\boxed{I_d = \int_s \vec{J}_d \cdot \vec{ds} \text{ or } \int_s \vec{J} \cdot \vec{ds} = \int_s \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}}$$

and we may obtain the time varying version of Ampere's circuital law by integrating above expression over the surfaces,

$$\left[\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \right]$$

$$\int_s (\nabla \times \vec{H}) \cdot \vec{ds} = \int_s \vec{J} \cdot \vec{ds} + \int_s \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}$$

Now, applying Stoke's theorem

$$\boxed{\oint_L \vec{H} \cdot \vec{dl} = I + I_d = I + \int_s \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}}$$

Maxwell's Equations

Point form or differential form		Integral form	
Time invariant or Steady field	Time variant field	Time invariant or Steady field	Time variant field
$\nabla \times \vec{H} = \vec{J}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H} \cdot \vec{dl} = \int_s \vec{J} \cdot \vec{ds}$	$\oint_L \vec{H} \cdot \vec{dl} = \int_s (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot \vec{ds}$
$\nabla \times \vec{E} = 0$	$\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$	$\oint_L \vec{E} \cdot \vec{dl} = 0$	$\oint_L \vec{E} \cdot \vec{dl} = \int_s \frac{-\partial \vec{B}}{\partial t} \cdot \vec{ds}$
$\nabla \cdot \vec{B} = 0$	$\nabla \cdot \vec{B} = 0$	$\oint_s \vec{B} \cdot \vec{ds} = 0$	$\oint_s \vec{B} \cdot \vec{ds} = 0$
$\nabla \cdot \vec{D} = \rho_v$	$\nabla \cdot \vec{B} = \rho_v$	$\oint_s \vec{D} \cdot \vec{ds} = \int_{vol.} \rho_v dv$	$\oint_s \vec{D} \cdot \vec{ds} = \int_{vol.} \rho_v dv$

Maxwell's eq^n in phasor form (only time variant)

point form

① $\nabla \times \vec{H_s} = \vec{J_s} + j\omega \vec{D_s}$

② $\nabla \times \vec{E_s} = -j\omega \vec{B_s} = -j\omega\mu \vec{H_s}$

③ $\nabla \cdot \vec{B_s} = 0$

④ $\nabla \cdot \vec{D_s} = \rho_V$

Integral form.

① $\oint_L \vec{H_s} \cdot \vec{dl} = \int_S (\vec{J_s} + j\omega \vec{D_s}) \cdot \vec{ds}$

② $\oint_L \vec{E_s} \cdot \vec{dl} = \int_S -j\omega \vec{B_s} \cdot \vec{ds}$

③ $\oint_S \vec{B_s} \cdot \vec{ds} = 0$

④ $\oint_S \vec{D_s} \cdot \vec{ds} = \int_{vol.} \rho_V \, dv$

4) **Retarded Potentials**

The time variant potentials, usually called retarded potentials for a reason which find their greatest application in radiation problems in which the distribution of the source is known approximately. The scalar electric potential V may be expressed in terms of a static charge distribution as:

$$V = \int_{vol.} \frac{\rho_v \, dv}{4\pi \varepsilon R} \quad (\text{static}) \quad \text{—— (1)}$$

and the vector magnetic potential may be found from a current distribution which is constant with time,

$$\vec{A} = \int_{vol.} \frac{\mu \vec{J} \, dv}{4\pi R} \quad (dc) \quad \text{—— (2)}$$

The differential equations satisfied by V,

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \quad (\text{static})$$

and $\vec{A}$,

$$\nabla^2 \vec{A} = -\mu \vec{J} \quad (dc)$$

may be regarded as point form of (1) & (2) respectively.

With V & $\vec{A}$ we can write as:

$$\vec{E} = -\nabla V \quad (\text{static})$$

$$\vec{B} = \nabla \times \vec{A} \quad (dc)$$

For combined form.

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad \nabla \cdot \vec{A} = -\mu \varepsilon \frac{\partial V}{\partial t}$$

$$\nabla^2 V - \mu \varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \qquad \vec{B} = \nabla \times \vec{A}.$$

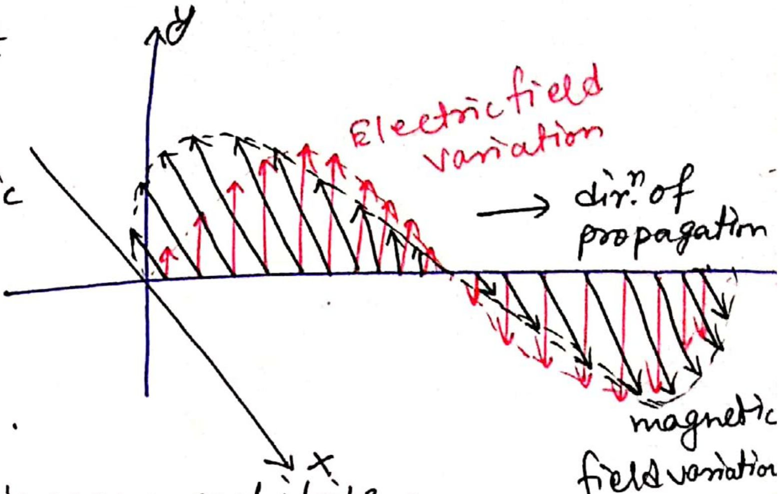
$$\nabla^2 \vec{A} - \mu \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

Polarization of wave vector

रविन कासुला
इन्जिनियर

Polarization is the phenomenon in which electromagnetic waves are restricted in the direction of vibration. Also, polarization is a property of waves that describes the orientation of their oscillators.

→ Electromagnetic waves transport energy through empty space, stored in the propagating electric and magnetic fields.



→ Magnetic field variation is perpendicular to electric field.

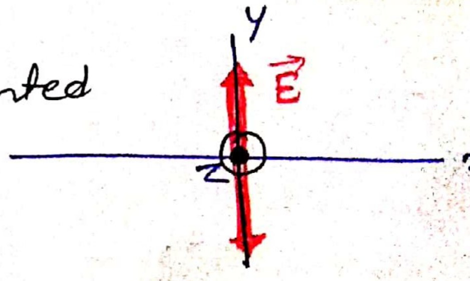
→ A single-freqⁿ electromagnetic wave exhibits a sinusoidal variation of electric and magnetic fields in space.

Types of polarization

① Linear (also called plane polarized).

- Trace of electric field vector is linear
- plane of the polarization is orthogonal to dirⁿ of propagation

Field of any polarization can be represented by two orthogonal linearly polarized components.



linear

$$E_x(t) = m_x \cos(\omega t) \text{ and } E_y(t) = m_y \cos(\omega t + \delta)$$

$$\therefore \vec{E}(t) = \hat{a}_x m_x \cos(\omega t) + \hat{a}_y m_y \cos(\omega t + \delta)$$

ⓐ For vertical polarization

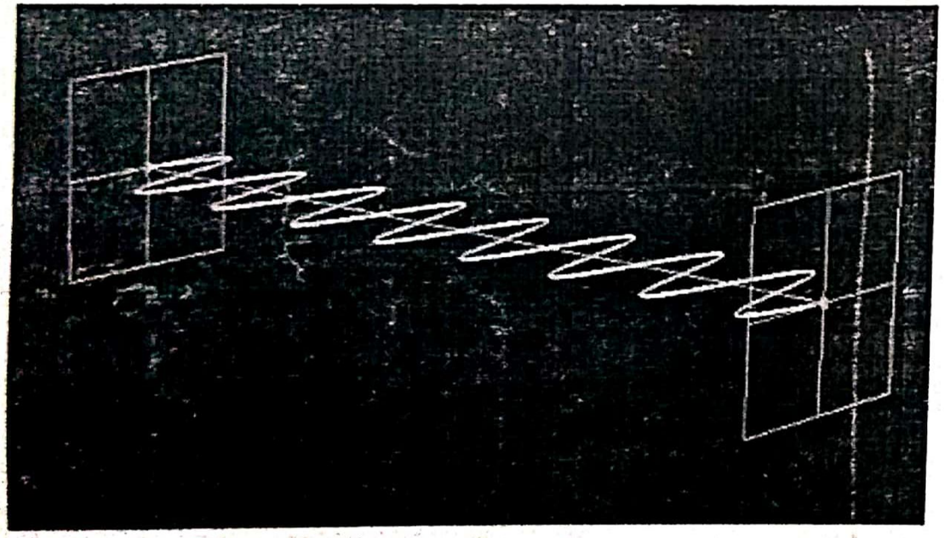
The vertical vector only passed. i.e $E_y(t) = m_y \cos(\omega t + \delta)$.

ⓑ For Horizontal Polarization

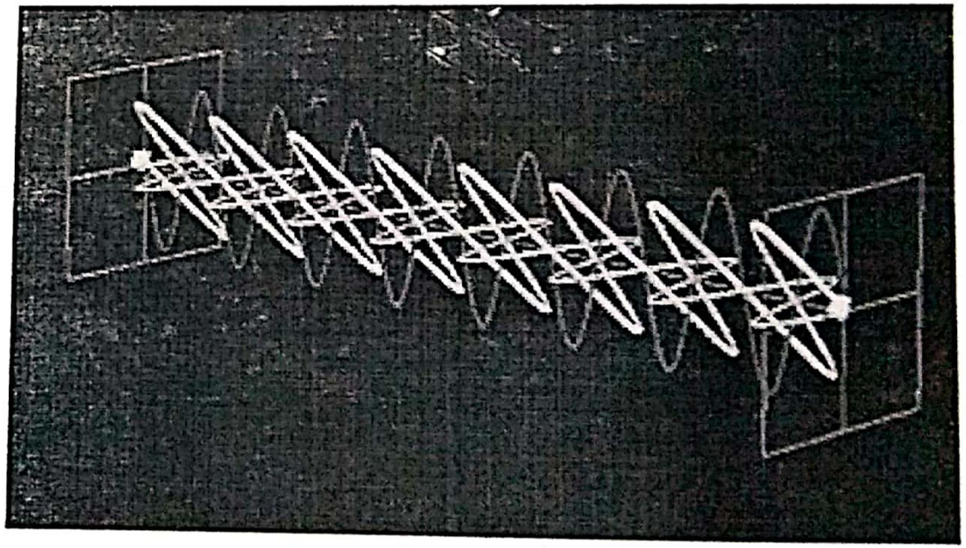
The Horizontal vector only passed. i.e $E_x(t) = m_x \cos(\omega t)$

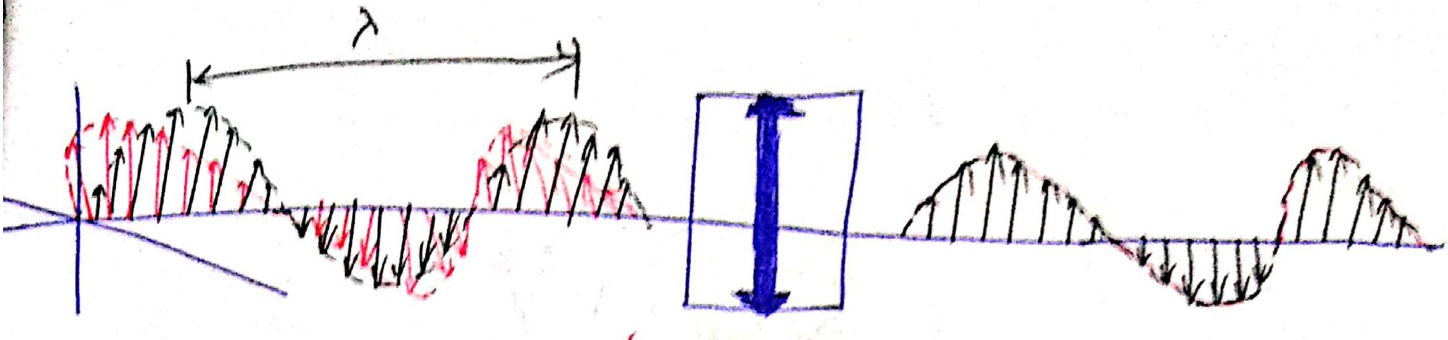
Linear Polarization (Animation)

Linear Polarization (Horizontal)

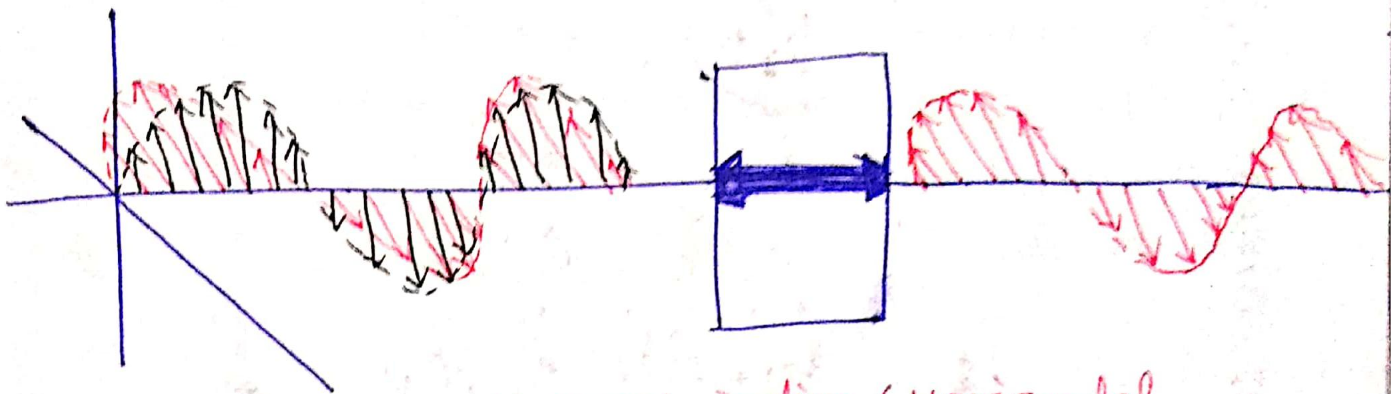


Linear Polarization (45 degrees)





Linear polarization (vertical)



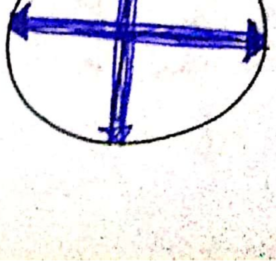
Linear polarization (Horizontal

② Circular Polarization.

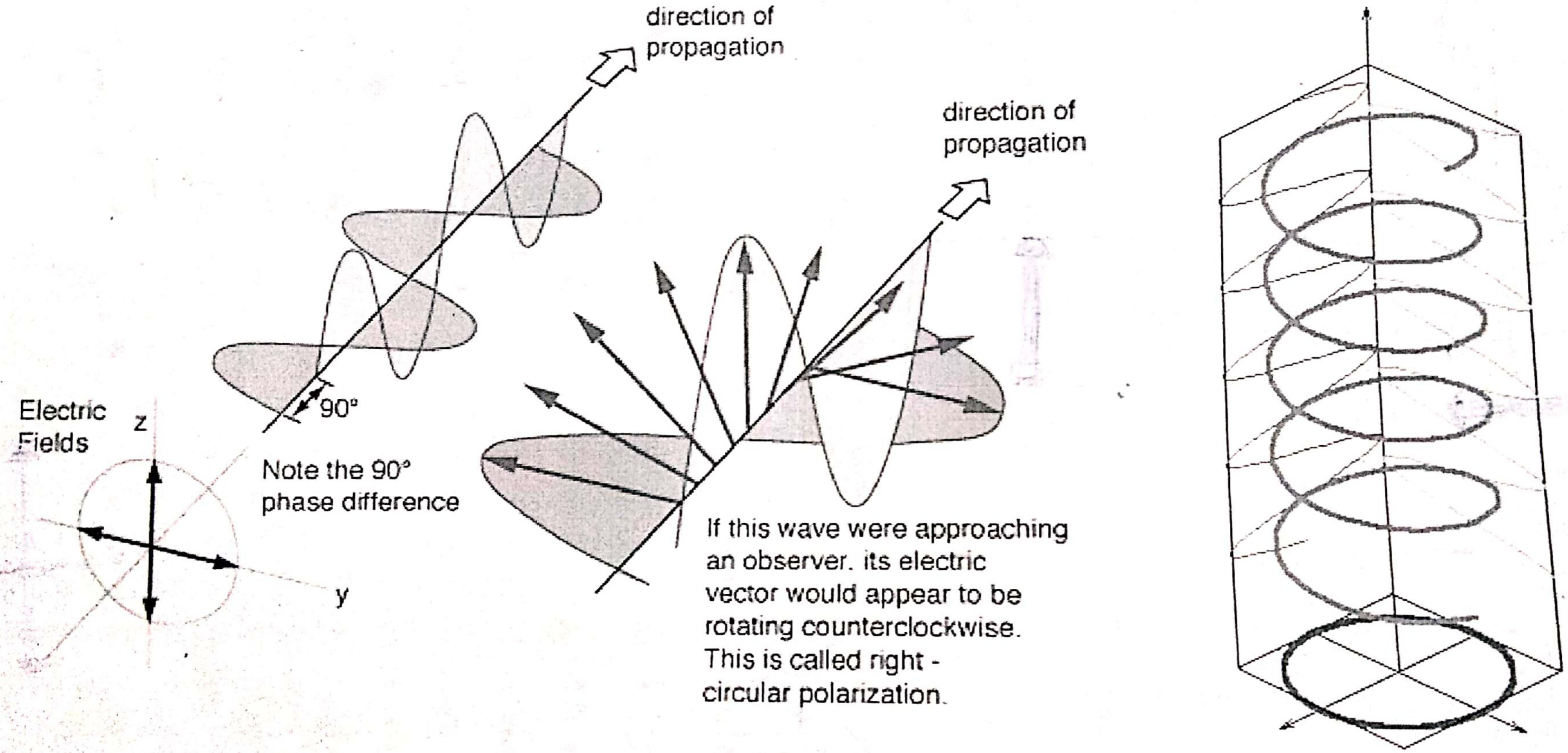
→ In $\vec{E} = E\cos(\omega t)\,\hat{a}_x + E\cos(\omega t+\theta)\,\hat{a}_y$, the tip of wave vector varies along a circle and the wave vector is circularly polarized. The component vectors along $\hat{a}_x$ & $\hat{a}_y$ dirⁿ have some amplitude but are some (α) value out of phase in time.

→ Electric vector rotates counterclockwise → right hand circular polarization

→ Electric vector rotates clockwise → left hand circular polarization



Circular Polarization



- Two perpendicular electric field components of equal amplitude with 90° difference in phase
- Electric vector rotates counterclockwise ➔ right-hand circular polarization
- Electric vector rotates clockwise ➔ left-hand circular polarization

Elliptical Polarization

→ In $\vec{E} = E_1 \cos(\omega t + \theta_1)\hat{a}_x + E_2 \cos(\omega t + \theta_2)\,\hat{a}_y$. E_1; E_2; θ_1, θ_2 are arbitrary and the wave vector is said to be elliptically polarized

Terms Used in wave propagation

Phasor $e^{i\theta} = \cos\theta + i\sin\theta$ (Phasor z is complex number that can be written as: $z = x + jy = r\angle\theta$)

It is a complex number representing a sinusoidal function whose amplitude (A), angular freq.ⁿ (ω) and initial phase (θ) are time invariant. In complex form, phasor can be represented as:

$$Ae^{jx} = A\cos x + jA\sin x$$

$$Re\{Ae^{jx}\} = A\cos x \text{ and } Im\{Ae^{jx}\} = A\sin x$$

For vector, $\vec{E} = E_0 \cos(\omega t - \beta z)\,\hat{a}_x$

$$Re\{E_0 e^{j\omega t} e^{-j\beta z}\hat{a}_x\} = Re\{E_0 e^{-j\beta z}\hat{a}_x e^{j\omega t}\}$$

$\therefore \vec{E} = Re\{\vec{E_s} e^{j\omega t}\}$ where, $\vec{E_s} = E_0 e^{-j\beta z}\hat{a}_x$ is the phasor corresponding to $\vec{E}$.

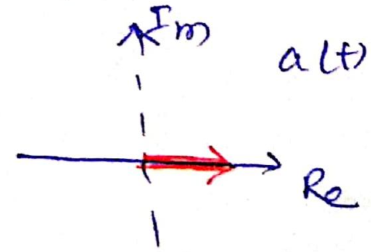
eg:

$$A\cdot\cos(\omega t + \theta) = \frac{A\left(e^{i(\omega t+\theta)} + e^{-i(\omega t+\theta)}\right)}{2}$$

or As real part only

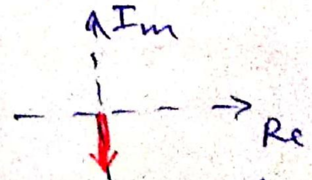
$$A\cdot\cos(\omega t + \theta) = Re\{A\cdot e^{i(\omega t+\theta)}\} = Re\{Ae^{i\theta}\cdot e^{i\omega t}\}$$

$$a(t) = Re\{\tilde{A}e^{i\omega t}\}$$



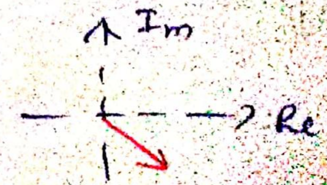
$a(t) = \cos\omega t$

$\tilde{A} = 1$



$a(t) = \sin\omega t$

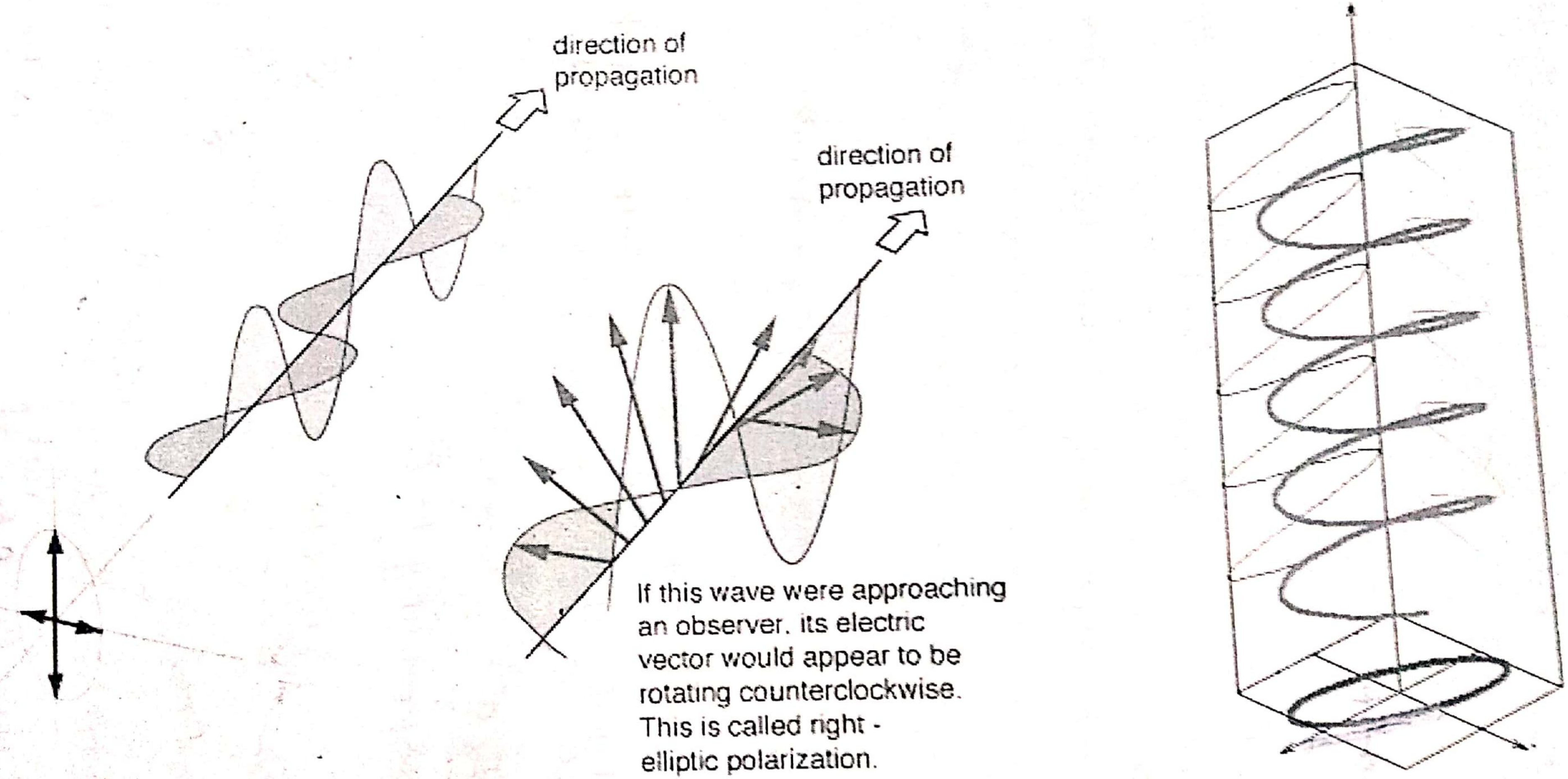
$\tilde{A} = -i$



$a(t) = \cos(\omega t) + \sin(\omega t)$

$\tilde{A} = \sqrt{2}\,e^{-i\pi/4}$

Elliptical Polarization



- Two perpendicular eletric field components not in phase, either with different amplitudes and/or not 90º out of phase
- Electric vector rotates counterclockwise ➔ right-hand elliptical polarization
- Electric vector rotates clockwise ➔ left-hand elliptical polarization
- The most general state of complete polarization is elliptical

Electromagnetic Wave Propagation

Waves are means of transporting energy or information. Typical examples of EM waves include radio waves, TV signals, radar beams and light rays. All forms of EM energy share three fundamental characteristics: they all travel at high velocity, in travelling, they assume the properties of waves and they radiate outward from a source, without benefit of any discernible physical vehicles.

Different Medias

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_0, \mu = \mu_0$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r \varepsilon_0, \mu = \mu_r \mu_0$, or $\sigma \leq \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r \varepsilon_0, \mu = \mu_r \mu_0$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_0, \mu = \mu_r \mu_0$ or $\sigma \gg \omega\varepsilon$)

where, ω is the angular frequency of the wave.

Waves in general – It is a function of both space & time.

To consider wave motion in free space, first we need to see Maxwell's Equations in terms of $\vec{E}$ & $\vec{H}$ as:

$$\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{H} = 0$$

Let us first write Maxwell's above mentioned equations for the special case of sinusoidal (more strictly, cosinusoidal) variation with time. This is accomplished by complex notation and phasors. The procedure is identical to the one we used in the sinusoidal steady state representation for current & voltage.

Then, the fector field is given by

$$\vec{E} = E_x \hat{a}_x$$

we assume E_x is given as:

$$E_x = E(x, y, z) \cos(\omega t + \psi)$$

where, $E(x,y,z)$ is real function of x, y, z, and perhaps ω, but not of time and ψ is a phase angle which may also be a function of x, y & z and ω. Using Euler's Identity,

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

we let

$$E_x = Re\{E(x,y,z) e^{j(\omega t + \psi)}\} = Re\{\underbrace{E(x,y,z)\, e^{j\psi}}_{E_{xs}} \cdot e^{j\omega t}\}$$

where, Re signifies the real part of the following quantity is to be taken. If we then simplify the nomenclature by dropping Re and suppressing $e^{j\omega t}$, the field quantity E_x becomes a phasor, or a complex quantity, which we identify using an 's' subscript, E_{xs}. Thus,

$$E_{xs} = E(x, y, z)\, e^{j\psi}$$

and $\vec{E}_s = E_{xs}\, \hat{a}_x$

The 's' can be thought of as indicating a freq^n domain quantity expressed as a function of the complex freq^n, 's', even though we shall we shall consider only those cases in which 's' is a pure imaginary, $s = j\omega$.

Again, If $\vec{H_s} = 20\, e^{-(0.1+j20)z}\, \hat{a}_x$ A/m, then

$$\vec{H}(t) = Re\{20\, e^{-0.1z} \cdot e^{-j20z}\, \hat{a}_x = 20\, e^{-0.1z} \cos(\omega t * 20z)\, \hat{a}_x \text{ A/m}$$

Now, since
$$\frac{\partial E_x}{\partial t} = \frac{\partial}{\partial t}\left[E(x,y,z)\cos(\omega t+\psi)\right] = -\omega E(x,y,z)\sin(\omega t+\psi)$$
$$= Re\{j\omega E_{xs}\, e^{j\omega t}\}$$

it is seen that taking the partial derivative of any field quantity with respect to time is equivalent to multiplying the corresponding phasor by $j\omega$.

ex: if
$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0}\frac{\partial H_y}{\partial z}$$

the corresponding phasor expression is

$$j\omega E_{xs} = -\frac{1}{\varepsilon_0}\frac{\partial H_{ys}}{\partial z}$$

where E_{xs} and H_{ys} are complex quantities

Now, applying phasor notations to Maxwell's Equations, as:

$$\boxed{\begin{aligned} \nabla \times \vec{H} &= \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \times \vec{H_s} &= j\varepsilon_0 \omega\, \vec{E_s} \end{aligned}}$$

Also,

$$\boxed{\begin{aligned} \nabla \times \vec{E_s} &= -j\omega\mu_0 \vec{H_s} \\ \nabla \cdot \vec{E_s} &= 0 \\ \nabla \cdot \vec{B_s} &= 0 \end{aligned}}$$

Wave Propagation in free Space [Hayt's method]

Method by which the wave equation is obtained could be accomplished in one line (using four equals signs on a wider sheet of paper).

$$\nabla \times \nabla \times \vec{E_s} = \nabla(\nabla\cdot\vec{E_s}) - \nabla^2\vec{E_s} = -j\omega\mu_0\, \nabla\times\vec{H_s}$$

or, $\omega^2 \mu_0 \varepsilon_0 \vec{E_s} = -\nabla^2 \vec{E_s}$

Since, $\nabla\cdot\vec{E_s} = 0$. Thus

$$\boxed{\nabla^2 \vec{E_s} = -k_0^2 \vec{E_s}}$$ – Vector Helmholtz equation

where k_0, the free space wave number, is identified as:

$$\boxed{k_0 = \omega\sqrt{\mu_0 \epsilon_0}}$$

From $\nabla^2 \vec{E_s} = -k_0^2 \vec{E_s}$, we can write x component as:

$$\nabla^2 \vec{E}_{xs} = -k_0^2 \vec{E}_{xs}$$

and the expression of ∇ operator leads to the second order partial differential equations

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} = -k_0^2 \cdot E_{xs}$$

Let us attempt to find the solution assuming that a simple solution is possible in which E_{xs} does not vary with x & y, so that the two corresponding derivatives are zero, leading to the ordinary differential equation

$$\frac{d^2 E_{xs}}{dz^2} = -k_0^2 E_{xs}$$

By inspection we may write down one solution as:

$$E_{xs} = E_{x0}\, e^{-jk_0 z}$$

Again, we reinsert the $e^{j\omega t}$ factor and take a real part,

$$E_x(t) = Re\{E_{x0}\, e^{-jk_0 z} \cdot e^{j\omega t}\}$$

$$\therefore \boxed{E_x(z,t) = E_{x0} \cos(\omega t - k_0 z)} \quad \text{(a)}$$

where, Amplitude factor E_{x0}, is the value of E_x at $z=0, t=0$.

Also, we can find an alternative solution of vector Helmholtz equation as:

$$\boxed{E_x'(z,t) = E_{x0}' \cos(\omega t + k_0 z)} \quad \text{(b)}$$

The solutions in Ⓐ & Ⓑ are the real instantaneous forms of electric field. Here, ωt and $k_0 z$ are usually expressed in radians & ω is radian times frequency i.e. rad/sec. And k_0 is termed as the spatial freq^n, which in the present case measures the phase shift per unit distance along the z-dir^n. Its units are rad/m; Also, k_0 is named as phase constant for a uniform wave in free space.

Also,

$$\frac{dz}{dt} = c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times10^8 \text{ m/s} \text{ [velocity of light]}$$

we can thus write as:

$$\boxed{k_0 = \omega/c}$$

And, above solution can be written as:

$$E_x(z,t) = E_{x0} \cos[\omega t - z/c)]$$

Let $t=0$ then

$$E_x(z,0) = E_{x0} \cos\left(\frac{\omega z}{c}\right) = E_{x0} \cos(k_0 z)$$

which is a simple periodic function that repeats every incremental distance λ, known as wavelength. The requirement is that $k_0\lambda = 2\pi$ and so,

$$\boxed{\lambda = \frac{2\pi}{k_0} = \frac{c}{f} = \frac{3\times10^8}{f}} \quad \text{(freespace)}$$

For multiple wave crests we can write; for mth crest

$$k_0 z = 2m\pi \quad (t=0)$$

if $t \neq 0$ then

$$\omega t - k_0 z = \omega(t - z/c) = 2m\pi$$

↳ this says that the wave moves in positive z-dir^n and $(\omega t + k_0 z)$ describes a wave that moves in the -ve z-direction.

Again, we can determine wave eq^n in the form of $\vec{H}$ field.

Given, $\vec{E}_s$, $\vec{H}_s$ is most easily obtained as:

$$\nabla \times \vec{E}_s = -j\omega\mu_0 \vec{H}_s$$

which is greatly simplified for a single $\vec{E}_{xs}$ component varying only with z,

$$\frac{dE_{xs}}{dz} = -j\omega\mu_0 H_{ys}$$

using $E_{xs} = E_{x0} e^{-jk_0 z}$

$$H_{ys} = -\frac{1}{j\omega\mu_0}(-jk_0) E_{x0} e^{-jk_0 z} = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}} e^{-jk_0 z}$$

So, we can write the real instantaneous form as:

$$\boxed{H_y(z,t) = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}} \cos(\omega t - k_0 z)}$$

where, E_{x0} is assumed real.

We therefore find the x-directed $\vec{E}$ field that propagates in the positive z-direction is accompanied by a y-directed $\vec{H}$ field. The ratio of Electric and magnetic field intensities is given by

$$\frac{E_x}{H_x} = \sqrt{\frac{\mu_0}{\epsilon_0}}$$ is constant. i.e E_x and H_y are "in phase".

Intrinsic Impedance (η - eta)

The square root of the ratio of the permeability to the permittivity is called the intrinsic impedance, η,

$$\boxed{\eta = \sqrt{\frac{\mu}{\epsilon}}}$$

where, η has the dimension of ohms. The intrinsic impedance of free space is

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \doteq 120\pi\ \Omega}$$

This wave is called a uniform plane wave because its value is uniform throughout any plane $z =$ constant.

Eg: Express $E_y = 100\cos(10^8 t - 0.5z + 30^\circ)$ V/m as a phasor.

Sol.ⁿ First using Exponential notation

$$E_y = Re\{100 e^{j(10^8 t - 0.5z + 30^\circ)}\} = Re\{100 e^{j10^8 t} \cdot e^{-j0.5z + j30^\circ}\}$$

and then drop Re and suppress $e^{j10^8 t}$ to obtain the phasor as:

$$E_{ys} = 100 e^{-j0.5z + j30^\circ}$$

Here, E_y is real, but E_{ys} is in general complex. A mixed nomenclature is used for the angle; As 0.5z is in radian & 30° is in degree.

Q: Given, the field intensity vector $\vec{E}_s = 100\angle 30^\circ \hat{a}_x + 20\angle -50^\circ \hat{a}_y + 40\angle 210^\circ \hat{a}_z$ V/m, identified as a phasor by its subscript s, the we need to calculate the vector as a real function of time.

Sol.ⁿ Vector in phasor is

$$\vec{E}_s = 100\angle 30^\circ \hat{a}_x + 20\angle -50^\circ \hat{a}_y + 40\angle 210^\circ \hat{a}_z \text{ V/m.}$$

Let us assume that freqⁿ is specified as 1 MHz. then we first select the exponential notation.

$$\vec{E}_s = 100 e^{j30^\circ} \hat{a}_x + 20 e^{-j50^\circ} \hat{a}_y + 40 e^{j210^\circ} \hat{a}_z \text{ V/m.}$$

Now, reinserting $e^{j\omega t}$ factor.

$$\vec{E}_s(t) = (100 e^{j30^\circ} \hat{a}_x + 20 e^{-j50^\circ} \hat{a}_y + 40 e^{j210^\circ} \hat{a}_z) e^{j2\pi f t} \Rightarrow 10^6$$

$$= 100 e^{j(2\pi 10^6 t + 30^\circ)} \hat{a}_x + 20 e^{j(2\pi 10^6 t - 50^\circ)} \hat{a}_y + 40 e^{j(2\pi 10^6 t + 210^\circ)} \hat{a}_z \text{ V/m}$$

Now, taking real part only.

$$\vec{E}(t) = 100\cos(2\pi 10^6 t + 30^\circ) \hat{a}_x + 20\cos(2\pi 10^6 t - 50^\circ) \hat{a}_y + 40\cos(2\pi 10^6 t + 210^\circ) \hat{a}_z \text{ V/m.}$$

Wave eq^n for Lossless Medium ($\sigma \ll \omega\varepsilon, \mu, \varepsilon$)

$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$ - Maxwell's Eq^n.

Using Vector Tripple Product Formula

$$\nabla \times \nabla \times \vec{E}_s = \nabla \cdot (\nabla \cdot \vec{E}_s) - \vec{E}_s (\nabla \cdot \nabla)$$

or, $\nabla \times (-j\omega\mu \vec{H}_s) = \nabla(\cdot 0) - \nabla^2 \vec{E}_s$

or, $-j\omega\mu (\nabla \times \vec{H}_s) = -\nabla^2 \vec{E}_s$

or, $j\omega\mu (\vec{J}_s + j\omega\varepsilon \vec{E}_s) = \nabla^2 \vec{E}_s$

or, $j\omega\mu (\sigma \vec{E}_s + j\omega\varepsilon \vec{E}_s) = \nabla^2 \vec{E}_s$

or, $j\omega\mu (\sigma + j\omega\varepsilon) \vec{E}_s = \nabla^2 \vec{E}_s$

Let $j\omega\mu(\sigma + j\omega\varepsilon) = \gamma^2$

Since, $\sigma \ll \omega\varepsilon$

$\gamma^2 = j\omega\mu \times j\omega\varepsilon$

or, $\gamma = j\omega\sqrt{\mu\varepsilon}$

So, $\gamma^2 \vec{E}_s = \nabla^2 \vec{E}_s$

Let the variation of Electric field be along x-axis and propagation of wave be along z-axis then we can write

$$\gamma^2 E_{xs} = \frac{d^2 E_{xs}}{dz^2}$$

Let $E_{xs} = e^{\psi z}$ then $\gamma^2 e^{\psi z} = \frac{d^2}{dz^2} e^{\psi z}$

or, $\gamma^2 e^{\psi z} = \psi^2 e^{\psi z}$

$\Rightarrow$ ~~γz~~ $\psi = \pm \gamma$

Hence, the solution will be

$$\vec{E}_{xs} = (C_1 e^{\gamma z} + C_2 e^{-\gamma z})\, \hat{a}_x$$

Let the +ve z-direction propagation only and $C_2 = E_{xo}$

then $\vec{E}_{xs} = E_{xo} e^{-\gamma z}\, \hat{a}_x$

Again, $\gamma = \alpha + j\beta \rightarrow$ phase constant
$\hookrightarrow$ attenuation constant

or, $j\omega\sqrt{\mu\varepsilon} = \alpha + j\beta$

$\Rightarrow \alpha = 0$ & $\beta = \omega\sqrt{\mu\varepsilon}$

Then,

$$\vec{E}_{xs} = E_{xo} e^{-j\beta z} \hat{a}_x$$

changing to time domain

$$\vec{E}(z,t) = Re\{E_{xo} e^{-j\beta z} \cdot e^{j\omega t}\} \hat{a}_x$$

$$= E_{xo} \cos(\omega t - \beta z) \hat{a}_x$$

Again.

$$\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$$

$$\text{or, } \vec{H}_s = \frac{\nabla \times \vec{E}_s}{-j\omega\mu} = \frac{\nabla \times E_{xo} e^{-j\beta z} \hat{a}_x}{-j\omega\mu}$$

$$= \frac{E_{xo}}{-j\omega\mu} \frac{d(e^{-j\beta z})}{dz} \hat{a}_y$$

$$= E_{xo} \cdot \frac{-j\beta}{-j\omega\mu} e^{-j\beta z} \hat{a}_y$$

$$= E_{xo} \cdot \frac{\cancel{j\omega}\sqrt{\mu\varepsilon}}{\cancel{j\omega}\mu} e^{-j\beta z} \hat{a}_y$$

$$= \frac{E_{xo}}{\sqrt{\frac{\mu}{\varepsilon}}} e^{-j\beta z} \hat{a}_y$$

where, $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

$$\text{or, } \vec{H}_s = \frac{E_{xo}}{\eta} e^{-j\beta z} \hat{a}_y$$

changing to time domain

$$\vec{H}(z,t) = \frac{E_{xo}}{\eta} \cos(\omega t - \beta z) \hat{a}_y$$

$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{xo} e^{-j\beta z} & 0 & 0 \end{vmatrix} = \frac{\partial E_{xo} e^{-j\beta z}}{\partial z} \hat{a}_y$$

रविन कासुला
इन्जिनियर

Wave Propagation in Lossy dielectric

Let the medium be linear, homogeneous and isotropic and charge density, ρ_v be zero in the medium. A lossy dielectric is a medium in which an EM wave loses power as it propagates due to poor conduction. In other words, the lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect or good dielectric) in which $\sigma = 0$.

Thus, from Maxwell's equation, $\nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s$

$$\nabla \times \nabla \times \vec{E}_s = \nabla(\nabla \cdot \vec{E}_s) - \nabla^2 \vec{E}_s$$

since, $\nabla \cdot \vec{E}_s = 0$, $\nabla \times \nabla \times \vec{E}_s = -\nabla^2 \vec{E}_s$

$$\text{or, } \nabla \times (-j\omega\mu \vec{H}_s) = -\nabla^2 \vec{E}_s \quad [\because \nabla \times \vec{E}_s = -j\omega\mu \vec{H}_s]$$

$$\text{or, } -j\omega\mu (\nabla \times \vec{H}_s) = -\nabla^2 \vec{E}_s$$

$$\text{or, } -j\omega\mu (\vec{J}_s + j\omega\varepsilon \vec{E}_s) = -\nabla^2 \vec{E}_s \quad [\because \nabla \times \vec{H}_s = \vec{J}_s + j\omega\varepsilon \vec{E}_s]$$

$$\text{or, } -j\omega\mu (\sigma \vec{E}_s + j\omega\varepsilon \vec{E}_s) = -\nabla^2 \vec{E}_s \quad [\because \vec{J}_s = \sigma \vec{E}_s]$$

$$\text{or, } j\omega\mu (\sigma + j\omega\varepsilon) \vec{E}_s = \nabla^2 \vec{E}_s$$

Let $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$ then

$$\nabla^2 \vec{E}_s = \gamma^2 \vec{E}_s$$ where, $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$ is propagation constant.

Assuming $\vec{E}_s$ has only x-component that varries only with z then,

$$\frac{d^2 E_{xs}}{dz^2} = \gamma^2 E_{xs}$$

Let $E_{xs} = e^{\gamma z}$ then, from $\frac{d^2 e^{\gamma z}}{dz^2} = \gamma^2 e^{\gamma z}$

$\psi^2 e^{\psi z} = \gamma^2 e^{\psi z}$.

or, $\psi^2 = \gamma^2$

or, $\psi = \pm\gamma$

Then, $E_{xs} = C_1 e^{\gamma z} + C_2 e^{-\gamma z}$ [since $C_1 e^{\gamma z}$ denotes wave travelling in $-\hat{a}_z$ axis]

$E_{xs} = E_{x0} e^{-\gamma z}$ $[C_2 = E_{x0}]$

$\therefore \vec{E}_x(z,t) = Re\{E_{x0} e^{-\gamma z} \cdot e^{j\omega t} \hat{a}_x\}$

$= Re\{E_{x0} e^{-\alpha z} \cdot e^{-j\beta z} \cdot e^{j\omega t} \hat{a}_x\}$

$= Re\{E_{x0} e^{-\alpha z} \cdot e^{j(\omega t - \beta z)} \hat{a}_x\}$

or, $\vec{E}_x(z,t) = E_{x0} e^{-\alpha z} \cdot \cos(\omega t - \beta z)\, \hat{a}_x$

where, $\gamma = \alpha + j\beta$ → phase constant or ~~phase~~ wavenumber

↳ attenuation factor/constant

Again,

$\nabla \times \vec{E}_s = \left(\frac{\partial E_{zs}}{\partial y} - \frac{\partial E_{ys}}{\partial z}\right)\hat{a}_x + \left(\frac{\partial}{\partial z} E_{xs} - \frac{\partial E_{zs}}{\partial x}\right)\hat{a}_y +$

$\left(\frac{\partial}{\partial x} E_{ys} - \frac{\partial}{\partial y} E_{xs}\right)\hat{a}_z$

or, $-j\omega\mu \vec{H}_s = \frac{\partial}{\partial z} E_{xs}\, \hat{a}_y = \frac{\partial}{\partial z} E_{x0} e^{-\gamma z} \hat{a}_y = -\gamma E_{x0} e^{-\gamma z} \hat{a}_y$

or, $\vec{H}_s = \frac{-\gamma E_{x0} e^{-\gamma z} \hat{a}_y}{-j\omega\mu}$

or, $\vec{H}_s = \frac{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}}{j\omega\mu} E_{x0} e^{-\gamma z} \hat{a}_y$

or, $\vec{H}_s = \sqrt{\frac{\sigma + j\omega\epsilon}{j\omega\mu}} E_{x0} e^{-\gamma z} \hat{a}_y$

or, $\vec{H}_s = \frac{E_{x0}}{\sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}} e^{-\gamma z} \hat{a}_y = \frac{E_{x0}}{\eta} e^{-\gamma z} \hat{a}_y$

Binomial Expansion $(x+y)^n = x^n + \frac{n}{1!} x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \cdots$

where, $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$ is a complex quantity known as the intrinsic impedance (in ohms) of the medium.

Also, $$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta| \angle \theta_\eta = |\eta| e^{j\theta_\eta}$$

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}}$$

Then, $$\vec{H}_y(z,t) = Re\left\{ \vec{H}_s e^{j\omega t} \right\}$$

$$= Re\left\{ \frac{E_{x0}}{\eta} e^{-\gamma z} \hat{a}_y \cdot e^{j\omega t} \right\}$$

$$= Re\left\{ \frac{E_{x0}}{|\eta| e^{j\theta_\eta}} e^{-\gamma z} \hat{a}_y \cdot e^{j\omega t} \right\}$$

$$= Re\left\{ \frac{E_{x0}}{|\eta|} e^{-\alpha z} \cdot e^{j(\omega t - \beta z - \theta_\eta)} \hat{a}_y \right\}$$

$$= \frac{E_{x0}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \hat{a}_y$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$= \sqrt{\mu/\epsilon} \left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{-1/2}$$

Using Binomial Expansion

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \left\{ (1)^{-1/2} + \frac{(-\frac{1}{2})}{1!} * \frac{\sigma}{j\omega\epsilon} + \cdots \right\}$$

$$= \sqrt{\frac{\mu}{\epsilon}} \left\{ 1 - \frac{\sigma}{j 2\omega\epsilon} \right\}$$

$$= \sqrt{\frac{\mu}{\epsilon}} \left(1 + j\frac{\sigma}{2\omega\epsilon}\right) \text{ as } j^2 = -1$$

So, the wave equations are

$$\vec{E}_x(z,t) = E_{x0} e^{-\alpha z} \cos(\omega t - \beta z) \hat{a}_x$$

$$\vec{H}_y(z,t) = \frac{E_{x0}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \hat{a}_y$$

For Lossy dielectric medium, the electric and magnetic field vectors are not in Same phase. (not in same phase)

we can also have,

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

$$\text{or, } \gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = j\omega\mu\sigma - \omega^2\mu\epsilon$$

$$\text{or, } (\alpha + j\beta)^2 = j\omega\mu\sigma - \omega^2\mu\epsilon$$

D: $\therefore \alpha^2 - \beta^2 + j\alpha \cdot \beta \times 2$

$Re\{\gamma^2\} = \alpha^2 - \beta^2 = -\omega^2 \mu \varepsilon$ —— (1)

Also, $|\gamma^2| = \sqrt{\omega^2 \mu^2 \sigma^2 + \omega^4 \mu^2 \varepsilon^2}$

$= \sqrt{(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2}$

$= (\alpha^2 + \beta^2)^2.$

$= \alpha^2 + \beta^2$

then,

$$\alpha^2 + \beta^2 = \sqrt{\omega^2 \mu^2 \sigma^2 + \omega^4 \mu^2 \varepsilon^2} = \omega^2 \mu \varepsilon \sqrt{1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}} \quad \text{—— (2)}$$

Adding (1) & (2)

$$2\alpha^2 = -\omega^2 \mu \varepsilon + \omega^2 \mu \varepsilon \sqrt{1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}} = \omega^2 \mu \varepsilon \left(\sqrt{1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}} - 1 \right)$$

or, $\boxed{\alpha = \omega \sqrt{\frac{\mu \varepsilon}{2}} \sqrt{\sqrt{1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}} - 1}}$

Subtracting (1) from (2)

$$2\beta^2 = \omega^2 \mu \varepsilon \sqrt{1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}} + \omega^2 \mu \varepsilon$$

or, $\boxed{\beta = \omega \sqrt{\frac{\mu \varepsilon}{2}} \cdot \sqrt{\sqrt{1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}} + 1}}$

(1) when the medium is free space

$\sigma = 0$, $\varepsilon = \varepsilon_0$ & $\mu = \mu_0$

so, $\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}}$, $\gamma = j\omega \sqrt{\mu_0 \varepsilon_0}$, $\beta = \omega \sqrt{\mu_0 \varepsilon_0}$, $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$\Rightarrow \beta = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda}$

② when the medium is perfect dielectric / Lossless

$\sigma = 0,\ \varepsilon = \varepsilon_r \varepsilon_0,\ \mu = \mu_r \mu_0.$

then,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}},\quad \gamma = j\omega\sqrt{\mu\varepsilon},\quad \beta = \omega\sqrt{\mu\varepsilon}.$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \qquad \Rightarrow \beta = \frac{\omega}{v} = \frac{2\pi f}{v} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \alpha = 0$$

③ When the medium is Conductor ($\sigma \gg \omega\varepsilon$)

$\sigma \cong \infty,\ \varepsilon = \varepsilon_0;\ \mu = \mu_r \mu_0.$

then,

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}} \cdot \sqrt{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - 1}$$

$$\simeq \omega\sqrt{\frac{\mu\varepsilon}{2}} \cdot \sqrt{\frac{\sigma}{\omega\varepsilon}}$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$= \sqrt{\pi f \mu\sigma}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}} \sqrt{\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1}$$

$$\simeq \omega\sqrt{\frac{\mu\varepsilon}{2}} \cdot \sqrt{\frac{\sigma}{\omega\varepsilon}}$$

$$= \sqrt{\frac{\omega^2 \mu\varepsilon\sigma}{2\omega\varepsilon}}$$

$$= \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$= \sqrt{\pi f \mu\sigma}$$

& $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \cong \sqrt{\frac{j\omega\mu}{\sigma}}$

Wave propagation in Free Space

Let us consider the wave is propagating in free space where $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ is seen. Then, from Maxwell's eq^n in phasor form.

$$\nabla \times \vec{E_s} = -j\omega\mu_0 \vec{H_s}$$

$$\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

or, $\nabla \times \nabla \times \vec{E_s} = \nabla(\nabla \cdot \vec{E_s}) - \nabla^2 \vec{E_s}$ [bac-cab rule]

Since, $\nabla \cdot \vec{E_s} = 0$,

$$\nabla \times (-j\omega\mu_0 \vec{H_s}) = -\nabla^2 \vec{E_s}$$

or, $-j\omega\mu_0 (\nabla \times \vec{H_s}) = -\nabla^2 \vec{E_s}$

or, $-j\omega\mu_0 (j\omega\varepsilon_0 \vec{E_s}) = -\nabla^2 \vec{E_s}$

or, $-j^2\omega^2\mu_0\varepsilon_0 \vec{E_s} = -\nabla^2 \vec{E_s}$

or, $j^2\omega^2\mu_0\varepsilon_0 \vec{E_s} = \nabla^2 \vec{E_s}$

Let $\gamma^2 = j^2\omega^2\mu_0\varepsilon_0 \Rightarrow \gamma = j\omega\sqrt{\mu_0\varepsilon_0}$ is propagation constant

Then, $\gamma^2 \vec{E_s} = \nabla^2 \vec{E_s}$

Assuming $\vec{E_s}$ has only x-component that varries with z-di

then, $$\frac{d^2 E_{xs}}{dz^2} = \gamma^2 E_{xs}$$

Let $E_{xs} = e^{\psi z}$ then from $\frac{d^2 e^{\psi z}}{dz^2} = \gamma^2 e^{\psi z}$

$\psi^2 e^{\psi z} = \gamma^2 e^{\psi z}$

or, $\psi^2 = \gamma^2 \Rightarrow \psi = \pm\gamma$

then $E_{xs} = C_1 e^{\gamma z} + C_2 e^{-\gamma z}$

Taking the portion travelling in $\hat{a}_z$ dir^n. only

$$E_{xs} = E_{x_0} e^{-\gamma z} \quad [C_2 = E_{x_0}]$$

$$\therefore \vec{E}_x(z,t) = Re\left\{ E_{x_0} e^{-\gamma z} \cdot e^{j\omega t} \hat{a}_x \right\}$$

$$= Re\left\{ E_{x_0} e^{-\alpha z} \cdot e^{-j\beta z} \cdot e^{j\omega t} \hat{a}_x \right\}$$

$$= Re\left\{ E_{x_0} \cdot e^{-\alpha z} \cdot e^{j(\omega t - \beta z)} \hat{a}_x \right\}$$

or, $\vec{E}_x(z,t) = E_{x_0} e^{-\alpha z} \cdot \cos(\omega t - \beta z)\, \hat{a}_x$

where, $\gamma = \alpha + j\beta$.

Again,

$$\nabla \times \vec{E}_s = -j\omega\mu_0 \vec{H}_s$$

or, $\dfrac{\partial E_{xs}}{\partial z} \hat{a}_y = -j\omega\mu_0 \vec{H}_s$

or, $-j\omega\mu_0 \vec{H}_s = -\gamma E_{x_0} \cdot e^{-\gamma z} \hat{a}_y$

or, $\vec{H}_s = \dfrac{\gamma E_{x_0} e^{-\gamma z}}{j\omega\mu_0}$

or, $\vec{H}_s = \dfrac{j\omega\sqrt{\mu_0 \epsilon_0}}{j\omega\mu_0} E_{x_0} e^{-\gamma z}$

or, $\vec{H}_s = \dfrac{E_{x_0}}{\sqrt{\frac{\mu_0}{\epsilon_0}}} e^{-\gamma z}$

or, $\vec{H}_s = \dfrac{E_{x_0}}{\eta} e^{-\gamma z}$ where, $\eta = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = |\eta| \angle \theta_\eta = 0$

Similar process can be done for

(a) lossless dielectric ($\sigma = 0, \epsilon, \mu$ where $\sigma \ll \omega\epsilon$)

(b) Good conductors ($\sigma \simeq \infty$ or $\sigma \gg \omega\epsilon, \mu, \epsilon$)

Plane Wave Propagation in Lossy Dielectrics

$$\nabla \times \nabla \times \vec{E}_s = -j\omega\mu \, \nabla \times \vec{H}_s$$

Using vector identity $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

we get,

$$\nabla^2 \vec{E}_s = \gamma^2 \vec{E}_s \rightarrow \text{vector Helmholtz eq}^n\text{s}$$

where, $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$; γ = propagation constant.

Also, for $\vec{H}_s$, $\nabla^2 \vec{H}_s = \gamma^2 \vec{H}_s \Rightarrow$ Vector Helmholtz's eq^{n}s.

Let $\gamma = \underbrace{\alpha}_{\text{attenuation constant}} + j\underbrace{\beta}_{\text{phase constant}}$.

Then, $-\mathrm{Re}\{\gamma^2\} = \beta^2 - \alpha^2 = \omega^2\mu\varepsilon$

& $|\gamma^2| = \beta^2 + \alpha^2 = \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2}$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$$

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{v}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \cdot \frac{1}{\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}}$$

Without loss we assume, $\vec{E}_s = E_{xs}(z)\,\hat{a}_x$

Then, $(\nabla^2 - \gamma^2)\, E_{xs}(z) = 0$

or, $\left(\frac{d^2}{dz^2} - \gamma^2\right) E_{xs}(z) = 0$

So, sol.^{n} is

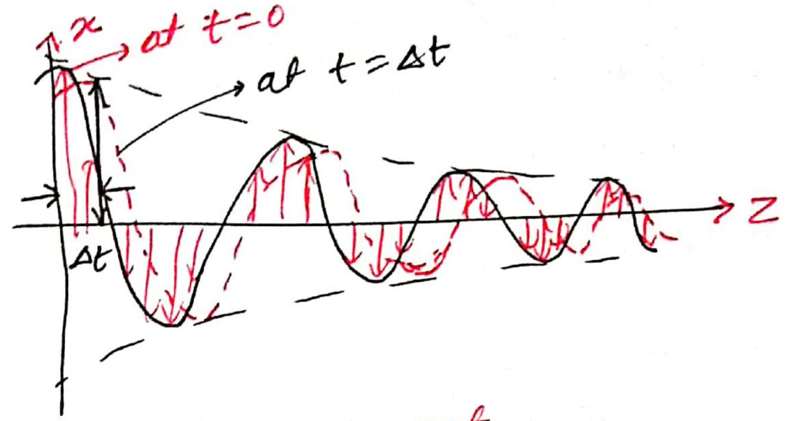
$$E_{xs}(z) = E_0 e^{-\gamma z} + E_0' e^{\gamma z} \quad [\because E_0 \,\&\, E_0' \Rightarrow \text{constants}]$$

$e^{\gamma z}$ denotes a wave travelling in $-\hat{a}_z$ dir^n. & $E_0' = 0$ for finite field at ∞. $\vec{E}(z,t) = \mathrm{Re}\left[E_{xs}(z)\, e^{j\omega t}\,\hat{a}_x\right]$

or, $\vec{E}(z,t) = Re[E_o e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a}_x]$

$= E_o e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$

Sketch of $|\vec{E}|$ at times $t=0$ & $t=\Delta t$ is



Then, doing similar steps we get

$\vec{H}(z,t) = Re[H_o e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a}_y]$

where, $H_o = \frac{E_o}{\eta}$; η - complex quantity known as the intrinsic impedance (in ohms) of the medium.

where, $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| \angle \theta_\eta = |\eta| e^{j\theta_\eta}$

with $\boxed{|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}}$; $\boxed{\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}}$

where $0^\circ < \theta_\eta \le 45^\circ$

then, $\vec{H} = Re\left[\frac{E_o}{|\eta| e^{j\theta_\eta}} \cdot e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a}_y\right]$

or, $\vec{H} = \frac{E_o}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \hat{a}_y$

Here, wave propagates in $\hat{a}_z$ dirⁿ, it decreases or attenuates in amplitude by factor of $e^{-\alpha z}$ and hence α is known as the attenuation constant or factor of the medium.

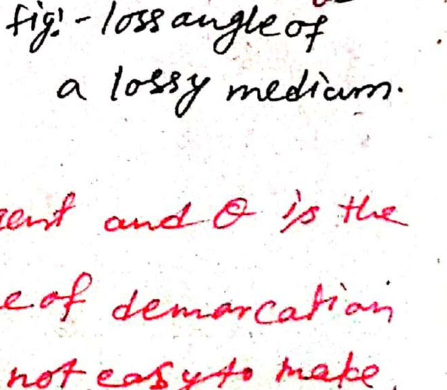
For free space, $\alpha = 0$ & for lossless medium $\sigma = 0$.

β is the measure of phase shift per length and is called the phase constant or wave number.

$$u = \frac{\omega}{\beta}, \quad \lambda = \frac{2\pi}{\beta}; \quad u = \text{wave velocity } \& \quad \lambda = \text{wavelength}$$

Loss Tangent

Since, $\vec{E}$ & $\vec{H}$ are out of phase by θ_η at any instant of time due to the complex intrinsic impedance of the medium, $\vec{E}$ leads $\vec{H}$ (or $\vec{H}$ lags $\vec{E}$) by θ_η. Then, the ratio of the magnitude of conduction current density, $\vec{J}_{\sigma s}$ to that of the displacement current density, $\vec{J}_{ds}$ in a lossy medium is



$$\frac{|\vec{J}_{\sigma s}|}{|\vec{J}_{ds}|} = \frac{|\sigma\vec{E}_s|}{|j\omega\varepsilon\vec{E}_s|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta$$

fig:- loss angle of a lossy medium.

or, $$\boxed{\frac{\sigma}{\omega\varepsilon} = \tan\theta}$$

where $\tan\theta$ is known as the loss tangent and θ is the loss angle of the medium. Although the line of demarcation betⁿ good conductors and lossy dielectric is not easy to make, $\tan\theta$ or θ may be used to determine how lossy a medium is. A medium is said to be a good (lossless or perfect) dielectric if $\tan\theta$ is very small ($\sigma \ll \omega\varepsilon$) or a good conductor if $\tan\theta$ is very large ($\sigma \gg \omega\varepsilon$). So, the characteristic behaviour of a medium depends on the frequency of operation also.

from above equations $\theta = 2\theta_\eta$

Power and Poynting vector

रविन कासुला
इन्जिनियर

In order to find the power in a uniform plane wave, it is necessary to develop a theorem for the electromagnetic field known as the poynting theorem. It was originally postullated by an English physicist, John. H. poynting in 1884.

From Maxwell's Equations

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \quad \text{— (a)}$$

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \quad \text{— (b)}$$

Taking Dot each side of the equation (b) with $\vec{E}$.

$$\vec{E}\cdot(\nabla\times\vec{H}) = \sigma E^2 + \vec{E}\cdot\varepsilon\frac{\partial\vec{E}}{\partial t}$$

But for any vector fields $\vec{A}$ & $\vec{B}$.

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

Applying this vector identity letting $\vec{A}=\vec{H}$ & $\vec{B}=\vec{E}$, gives

$$\vec{H}\cdot(\nabla\times\vec{E}) + \nabla\cdot(\vec{H}\times\vec{E}) = \sigma E^2 + \vec{E}\cdot\varepsilon\frac{\partial\vec{E}}{\partial t}$$

From (a)

$$\vec{H}\cdot(\nabla\times\vec{E}) = \vec{H}\cdot\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) = -\frac{\mu}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H})$$

so, $$-\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \nabla\cdot(\vec{E}\times\vec{H}) = \sigma E^2 + \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t}$$

Rearranging the terms and taking the volume integral of both sides,

$$\int_{Vol.}\nabla\cdot(\vec{E}\times\vec{H})\,dv = \frac{-\partial}{\partial t}\int_{Vol.}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right)dv - \int_{Vol}\sigma E^2\,dv$$

Applying divergence theorem to left hand side,

$$\underbrace{\oint_S(\vec{E}\times\vec{H})\cdot d\vec{s}}_{\text{Total Power Leaving the volume}} = \underbrace{\frac{-\partial}{\partial t}\int_{Vol.}\left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right]dv}_{\text{Rate of decrease in energy stored in E and H fields}} - \underbrace{\int_{Vol}\sigma E^2\,dv}_{\text{Ohmic power dissipated}} \quad \text{— (c)}$$

i.e, Total power leaving the volume = Rate of decrease in energy stored in electric and magnetic fields − Ohmic power dissipated.

Equation (C) is referred to as Poynting's Theorem. The varius various terms in the equation are identified using energy conservation arguments for EM fields. The first term in right hand side is interpreted as the rate of decrease in energy stored in the electric and magnetic fields. The second term is power dissipated due to the fact that the medium is conducting ($\sigma \neq 0$). The quantity $\vec{E} \times \vec{H}$ on the left handside is known as the Poynting vector, $\mathcal{P}$ in watts per square meter (W/m^2);

i.e $\boxed{\mathcal{P} = \vec{E} \times \vec{H}}$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface. Here, $\mathcal{P}$ is normal to both $\vec{E}$ & $\vec{H}$ and so is therefore along the direction of wave propagation $\hat{a}_k$ for uniform plane waves. Thus, $\hat{a}_k = \hat{a}_E \times \hat{a}_H$

Again, if we assume that

$$\vec{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_n$$

& $$\vec{H}(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \hat{a}_y$$

and $$\mathcal{P}(z,t) = \vec{E} \times \vec{H}$$

$$= \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cdot \cos(\omega t - \beta z - \theta_\eta)\, \hat{a}_z$$

$$= \frac{E_0^2}{2|\eta|} e^{-2\alpha z} [\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)]\, \hat{a}_z$$

Since, $\cos A \cdot \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

To determine time average Poynting vector $\mathcal{P}_{avg}(z)$ in W/m², which is of more practical value than the instantaneous Poynting vector $\mathcal{P}(z,t)$, we integrate above relation over the period of $T = 2\pi/\omega$; i.e

$$\boxed{\mathcal{P}_{avg}(z) = \frac{1}{T}\int_0^T \mathcal{P}(z,t)\,dt}$$

Also,

$$\boxed{\mathcal{P}_{avg}(z) = \frac{1}{2} Re\,(\vec{E_s} \times \vec{H_s}^{*})}$$

* - denotes complex conjugate.
eg: 3+4i has complex conjugate 3−4i.

& $\mathcal{P}_{avg}(z) = \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta \, \hat{a}_z$

∴ The total time average power crossing a given surfaces is given by $\boxed{P_{avg} = \int_S \mathcal{P}_{avg} \cdot \vec{ds}}$

Here, $\mathcal{P}(x,y,z,t)$ is the poynting vector in watts/meter and is time varrying, $\mathcal{P}_{avg}(x,y,z)$ also in watts/meter is the time average of the poynting vector $\mathcal{P}$; it is a vector but time invariant. P_{avg} is a total time average power through a surface in watts; it is a scalar.

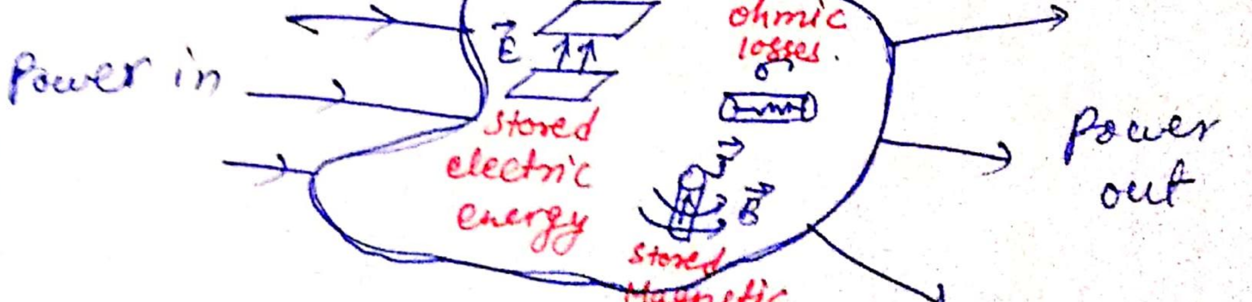


fig:- Power balance of EM fields.

Plane waves in Good Conductors: $[\sigma \simeq \infty, \varepsilon = \varepsilon_0, \mu = \mu_0\mu_r]$

Hence, $\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} = \frac{1}{\delta}$

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}} \; ; \; \lambda = \frac{2\pi}{\beta}$$

Also, $\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$

& $\vec{E}$ leads $\vec{H}$ by 45°.

$$\vec{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$$

then, $\vec{H} = \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ)\, \hat{a}_y$

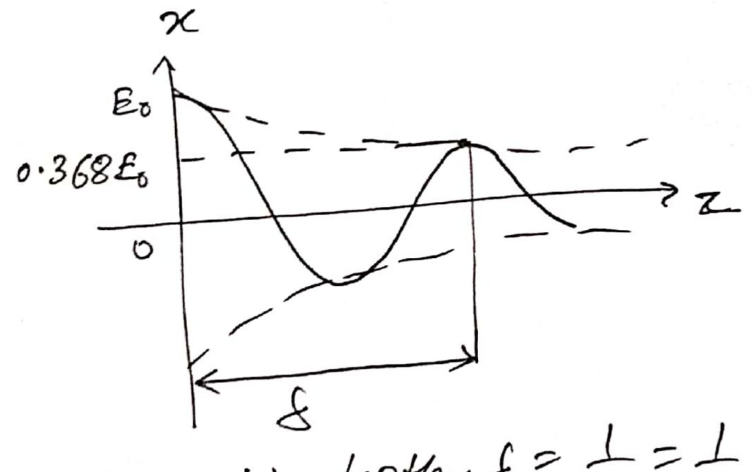


fig:. skin depth. $\delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu\sigma}}$

& η for good conductor is

$$\eta = \frac{1}{\sigma\delta} \sqrt{2}\, e^{j\pi/4} = \frac{1+j}{\sigma\delta}$$

Skin depth (δ) or penetration depth.

The skin depth is a measure of the depth to which an EM wave can penetrate the medium.

Also, skin depth in the medium is the distance in medium at which the travelling wave in the medium is attenuated to $e^{-1}E_0$, where E_0 is the amplitude before entering the medium. Also, skin depth (δ) is the distance where the wave amplitude decreases by a factor e^{-1} (about 37%). It is also called penetration depth of the medium.

When the ($\vec{E}$ or $\vec{H}$) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$.

So, when $e^{-\alpha z}$ becomes e^{-1} the depth is skin depth.

i.e $E_0 e^{-\alpha \delta} = E_0 e^{-1}$ then, $\boxed{\delta = \frac{1}{\alpha}} = \frac{1}{\beta}$ [$\because \alpha = \beta$]

It is valid for any material medium. For good Conductors,

$$\boxed{\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}} \quad \& \quad \eta = \frac{1}{\sigma\delta}\sqrt{2}\, e^{j\pi/4} = \frac{1+j}{\sigma\delta}$$

For partially conducting medium, the skin depth can be considerably large

For good conductors at skin depth we can write as:

$$\vec{E} = E_0 e^{-z/\delta} \cos(\omega t - \frac{z}{\delta})\hat{a}_x$$

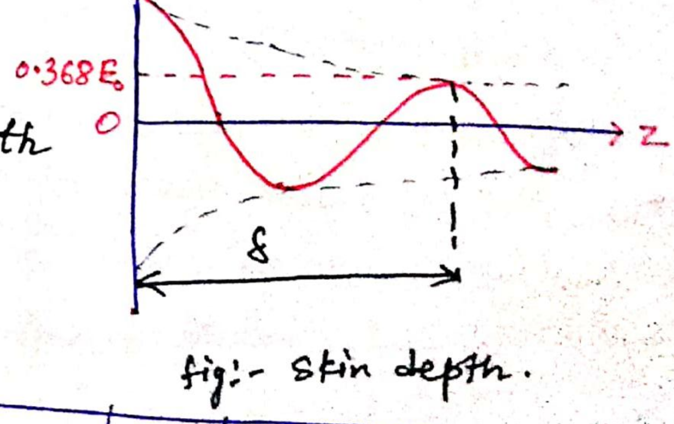


fig:- skin depth.

Skin depth in Copper

Freqⁿ (Hz)	10	60	100	500	10^4	10^8	10^{10}
skin depth (mm)	20.8	8.6	6.6	2.99	0.66	6.6×10^{-3}	6.6×10^{-4}

for Copper, $\sigma = 5.8\times10^7$ mho/m, $\mu = \mu_0$, $\delta = 66.1/\sqrt{f}$ (in mm)

(263). Skin depth decreases with increase in freq^n. So, $\vec{E}$ & $\vec{H}$ can hardly propagate through good conductors.

Skin effect The phenomenon whereby field intensity in a conductor rapidly decreases is known as skin effect. The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface. ∴ Skin effect appears in different guises in such problems as attenuation in waveguides, effective or ac resistance of transmission lines, an electromagnetic sheilding.

This effect is used as advantage in many applications. Such as silver plating is used to reduce material cost in waveguide components as pure silver has very small skin depth. Also, the hollow tubular conductors are used instead of solid conductors in outdoor television antennas.

Skin depth is useful in calculating the ac resistance due to the skin effect.

The dc resistance is $\boxed{R_{dc} = \dfrac{l}{\sigma S}}$ — (a)

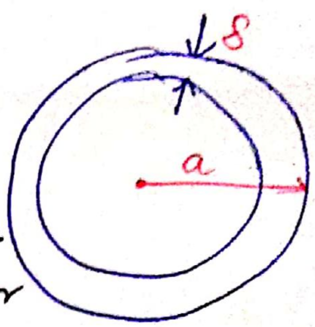


fig:- skin depth at high freq^n, $\delta << a$

Again, the surface or skin resistance R_s in Ω/m^2 as the real part of the η for a good conductor is

$$\boxed{R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}}$$

This is a resistance of a unit width & unit length of the conductor. It is equivalent to the dc resistance for a unit length of the conductor having cross sectional area $1 \times \delta$. Thus, for a given width w and length l; the ac resistance is calculated using the familiar dc resistance relation (a) and assuming a uniform current flow in the conductor of thickness δ;

i.e $\boxed{R_{ac} = \dfrac{l}{\sigma \delta w} = \dfrac{R_s l}{w}}$ where, $S \simeq \delta w$.

For a conductor of radius a ; $w = 2\pi a$, so

$$\frac{R_{ac}}{R_{dc}} = \frac{\frac{l}{\sigma 2\pi a \delta}}{\frac{l}{\sigma \pi a^2}} = \frac{a}{2\delta} \Rightarrow \boxed{R_{ac} = \frac{l}{\sigma 2\pi a \delta}}$$

Since $\delta << a$ at high frequencies, this shows that R_{ac} is far greater than R_{dc}.

Example

For a conductor of radius $a = 1\times10^{-3}$ m, length $l = 10\times10^{-3}$ m and conductivity 5×10^{7} mhos/m, if source freq^n is 1 GHz then find the DC and AC resistances of the conductor.

S^n

$$R_{dc} = \frac{\rho L}{A} = \frac{l}{\sigma A} = \frac{10\times10^{-3}}{5\times10^{7}\times\pi(1\times10^{-3})^{2}} = 6.36\times10^{-5}\,\Omega$$

$$\& \; R_{ac} = \frac{l}{\sigma A} = \frac{l}{\sigma 2\pi a \delta}$$

$$\text{where, } \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi\times1\times10^{9}\times4\pi\times10^{-7}\times10^{7}\times5}} = 2.25\times10^{-6}\,m$$

$$\text{Then, } R_{ac} = \frac{10\times10^{-3}}{5\times10^{7}\times2\pi\times10^{-3}\times2.25\times10^{-6}} = 14.147\times10^{-3}\,\Omega$$

Reflection of a Plane wave at Normal Incidence

Previously, we have considered the uniform plane waves travelling in unbounded, homogeneous media. When a plane wave from one medium meets a different medium, it is partly reflected and partly transmitted. The portion of the incident wave that is reflected or transmitted depends on the constitutive parameters (ϵ, μ, σ) of the two media involved.

In this case, we assume that the incidence of plane waves is normal to the boundary betn the media.

Suppose that a plane wave propagating in the +ve z-dirn is incident normally on the boundary $z=0$ betn medium 1 $(z<0)$ characterized by $\sigma_1, \epsilon_1, \mu_1$ and medium 2 $(z>0)$ characterized by $\sigma_2, \epsilon_2, \mu_2$ as shown in figure. Let subscripts i, r, t denotes incident, reflected and transmitted waves, respectively.

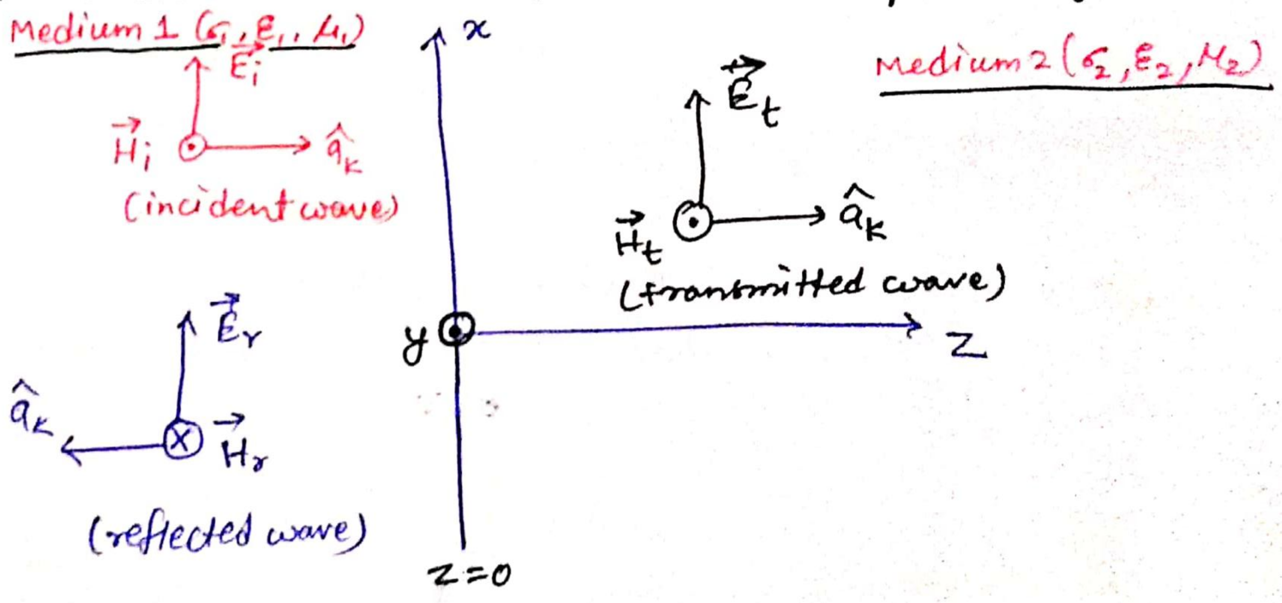


fig:- plane wave incident normally on an interface betn two different medias.

Incident wave

$(\vec{E}_i, \vec{H}_i)$ is travelling along $+\hat{a}_z$ in medium 1. If we suppress the time factor $e^{j\omega t}$ and assume that

$$\vec{E}_{is}(z) = E_{i0} \, e^{-\gamma_1 z} \, \hat{a}_x$$

then,

$$\vec{H}_{is}(z) = H_{io}\, e^{-\gamma_1 z}\, \hat{a}_y = \frac{E_{io}}{\eta_1} e^{-\gamma_1 z}\, \hat{a}_y$$

Reflected waves

$(\vec{E}_r, \vec{H}_r)$ is traveling along $-\hat{a}_z$ dirⁿ in medium 1. If

$$\vec{E}_{rs}(z) = E_{ro}\, e^{\gamma_1 z}\, \hat{a}_x$$

then,

$$\vec{H}_{rs}(z) = H_{ro}\, e^{\gamma_1 z}(-\hat{a}_y) = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z}\, \hat{a}_y$$

where, $\vec{E}_{rs}$ is assumed to be along $\hat{a}_x$ and also $\vec{E}_i$, $\vec{E}_r$ & $\vec{E}_t$ have same polarization.

Transmitted wave

$(\vec{E}_t, \vec{H}_t)$ is travelling along $+\hat{a}_z$ dirⁿ in medium 2. If

$$\vec{E}_{ts}(z) = E_{to}\, e^{-\gamma_2 z}\, \hat{a}_x$$

then,

$$\vec{H}_{ts}(z) = H_{to}\, e^{-\gamma_2 z}\, \hat{a}_y = \frac{E_{to}}{\eta_2} e^{-\gamma_2 z}\, \hat{a}_y$$

where, E_{io}, E_{ro}, E_{to} are, respectively, the magnitudes of the incident, reflected and transmitted electric fields at $z=0$.

Here, we see that

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r,\ \vec{H}_1 = \vec{H}_i + \vec{H}_r$$
$$\vec{E}_2 = \vec{E}_t,\ \vec{H}_2 = \vec{H}_t$$

Using boundary conditions at boundary $z=0$, the tangential components of $\vec{E}$ and $\vec{H}$ fields must be continuous at the boundary. Since, the waves are transverse, $\vec{E}$ & $\vec{H}$ fields are entirely tangential to the interface. Hence, at $z=0$,
$\vec{E}_{1tan} = \vec{E}_{2tan}$ and $\vec{H}_{1tan} = \vec{H}_{2tan}$.

this imply that

$$\left.\begin{aligned} &\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_t(0) \\ &\Rightarrow E_{io} + E_{ro} = E_{to} \\ &\& \ \vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0) \\ &\Rightarrow \frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{E_{to}}{\eta_2} \end{aligned}\right\}$$

Put $z=0$ in above wave equations.

So, arranging these two equations, we obtain

$$E_{ro} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{io}$$

and

$$E_{to} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{io}$$

Now, we define the reflection Co-efficient Γ (Tau) and the transmission Co-efficient (γ) from above eq.^{n}s.

$$\boxed{\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}$$

or, $E_{ro} = \Gamma E_{io}$

and

$$\boxed{\gamma = \frac{2\eta_2}{\eta_2 + \eta_1}}$$

or, $E_{to} = \gamma E_{io}$

$$\gamma = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\eta_2 + \eta_2 - \eta_1 + \eta_1}{\eta_2 + \eta_1} = 1 + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

or $\boxed{\gamma = 1 + \Gamma}$

Tx. Co-efficient — reflection Co-efficient

Here,

1) $1 + \Gamma = \gamma$

2) Both Γ and γ are dimensionless and may be complex

3) $0 \leq |\Gamma| \leq 1$

(3) Let us consider a special case, when the medium 1 is a perfect dielectric (lossless) ($\sigma_1 = 0$) & medium 2 is a perfect conductor ($\sigma_2 \simeq \infty$). For this case, $\eta_2 = 0$ (as $\sigma_2 = \infty$), hence $\Gamma = -1$ and $\tau = 0$ showing that the wave is totally reflected.

So, there will be no transmitted wave ($\vec{E}_2 = \vec{E}_t = 0$).

The totally reflected wave combines with the incident wave to form a standing wave. i.e the wave "stands" but doesnot travel; it consists of two travelling waves ($\vec{E}_i$ & $\vec{E}_r$) of equal amplitude but in opposite directions, given by.

$$\vec{E}_{1s} = \vec{E}_{is} + \vec{E}_{rs} = \left(E_{io}\, e^{-\gamma_1 z} + E_{ro}\, e^{\gamma_1 z}\right)\hat{a}_x \quad [\because \Gamma = -1]$$

But $\Gamma = \dfrac{E_{ro}}{E_{io}} = -1,\ \sigma_1 = 0,\ \alpha_1 = 0,\ \gamma_1 = j\beta_1$

Hence,

$$\vec{E}_{1s} = -E_{io}\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\hat{a}_x$$

$$\text{or, } \vec{E}_{1s} = -2j\, E_{io} \sin\beta_1 z\ \hat{a}_x \quad \left[\sin\beta_1 z = \frac{e^{j\beta_1 z} - e^{-j\beta z}}{2}\right]$$

$$\left[\cos\beta_1 z = \frac{e^{j\beta_1 z} + e^{-j\beta_1 z}}{2}\right]$$

Thus, $\vec{E}_1 = Re\{\vec{E}_{1s}\, e^{j\omega t}\}$

$$\therefore \boxed{\vec{E}_1 = 2E_{io} \sin\beta_1 z \cdot \sin\omega t\ \hat{a}_x} \quad [\because j^2 = -1]$$

By taking similar steps, we can derive

$$\boxed{\vec{H}_1 = \frac{2E_{io}}{\eta_1} \cos\beta_1 z \cdot \cos\omega t\ \hat{a}_y}$$

we have, Here, time period of standing wave will be

$$T = \frac{2\pi}{\omega}$$

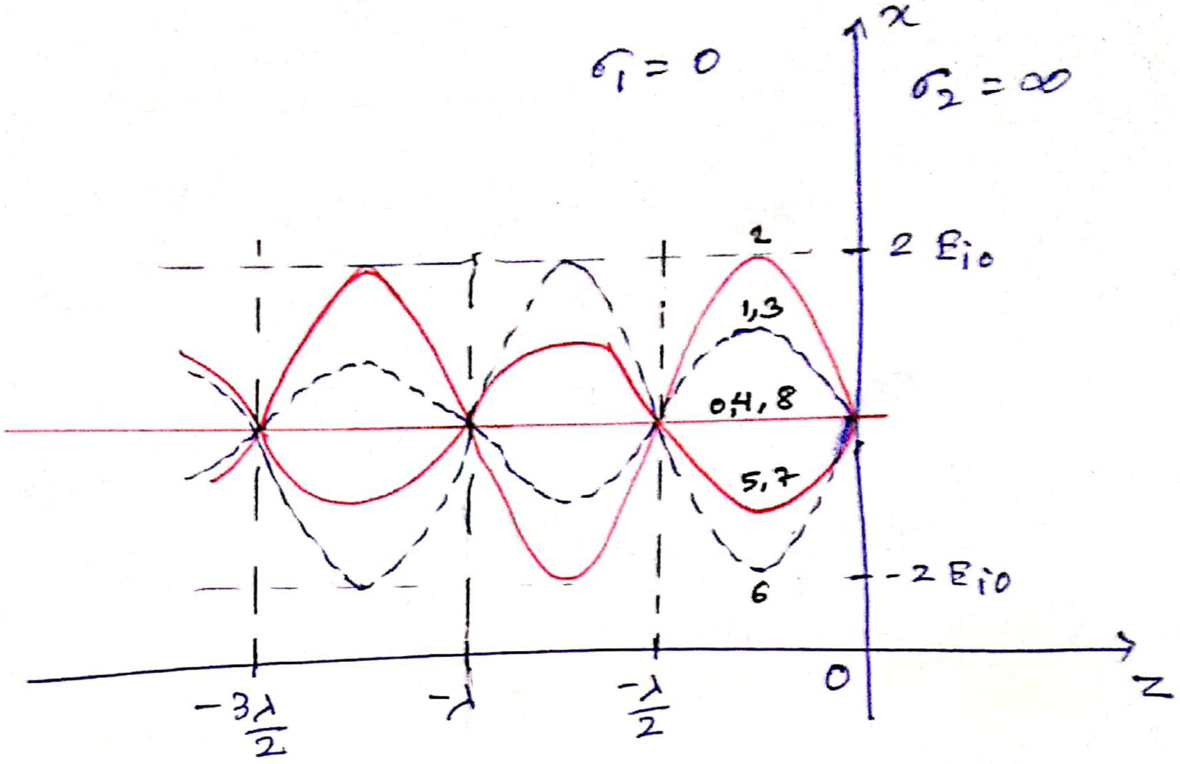


fig:- Standing waves, $E = 2E_{io}\sin\beta_1 z \sin\omega t\, \hat{a}_x$; curves 0, 1, 2, 3, 4, ... are, respectively at times $t = 0, T/8, T/4, 3T/8, T/2, \ldots, \lambda = \frac{2\pi}{\beta_1}$

Again, if media 1 and media 2 are both lossless we have another case ($\sigma_1 = 0 = \sigma_2$). In this case, η_1 & η_2 are real and so are Γ and τ. Let us consider following cases.

<u>Case A</u>

If $\eta_2 > \eta_1$, $\Gamma > 0$. Again, there is a standing wave in medium 1 but there is also a transmitted wave in medium 2. However, the amplitudes of incident and reflected waves are not equal in magnitudes. It can be shown that the max^n values of $|\vec{E}_1|$ occur at

$$-\beta_1 z_{max} = n\pi$$

[✱✱]

$$\text{or, } z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\lambda_1}{2}, \quad n = 0, 1, 2, \ldots$$

and the minimum value of $|\vec{E}_1|$ occur at

$$-\beta_1 z_{min} = (2n+1)\frac{\pi}{2}$$

$$\text{or, } z_{min} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)}{4}\lambda_1, \quad n = 0, 1, 2, \ldots$$

[**]

In medium 1

$$\vec{E}_{1s} = E_{io}\, e^{-j\beta_1 z}\, \hat{a}_x + E_{ro}\, e^{j\beta_1 z}\, \hat{a}_x$$

$$= E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right) \hat{a}_x$$

$$= E_{io}\left(e^{-j\beta_1 z} + |\Gamma| e^{j\theta_r} . e^{j\beta_1 z}\right) \hat{a}_x$$

$$\vec{E}_1 = Re\left\{ E_{io}\left(e^{-j\beta_1 z} + |\Gamma| e^{j\theta_r} . e^{j\beta_1 z}\right) \hat{a}_x \right\}$$

For Maximum value of $|\vec{E}_1|$

$$-\beta_1 z = \theta_r + \beta_1 z + 2n\pi$$

$$\text{or, } -2\beta_1 z = \theta_r + 2n\pi$$

$$\text{or, } z = \frac{-1}{2\beta_1}(\theta_r + 2n\pi) = \frac{-\lambda}{4\pi}(\theta_r + 2n\pi)$$

$$\left[\because \lambda = 2\pi/\beta_1\right]$$

For minimum value of $|\vec{E}_1|$

$$-\beta_1 z = \theta_r + \beta_1 z + (2n+1)\pi$$

$$\text{or, } z = -\frac{1}{2\beta_1}\left[\theta_r + (2n+1)\pi\right]$$

$$\text{or, } z = \frac{-\lambda_1}{4\pi}\left[\theta_r + (2n+1)\pi\right]$$

(270)

Case B

If $\eta_2 < \eta_1$, $\Gamma < 0$. For this case, the locations of $|\vec{E}_1|$ maximum are given by

$$z_{max} = -\frac{(2n+1)}{4}\lambda_1, \quad n = 0, 1, 2, \ldots$$

& minimum of $|\vec{E}_1|$ are given by

$$z_{min} = -\frac{n\lambda_1}{2}, \quad n = 0, 1, 2, \ldots$$

Also,

(1) $|\vec{H}_1|$ minimum occurs wherever there is $|\vec{E}_1|$ minimum and vice versa.

(2) The transmitted wave is purely traveling wave in medium 2, so no maxima & minima in this region.

The ratio of $|\vec{E}_1|_{max}$ to $|\vec{E}_1|_{min}$ (or $|\vec{H}_1|_{max}$ to $|\vec{H}_1|_{min}$) is called the standing wave ratio, s, i.e.

$|\vec{E}_1|_{max}$ at $z_{max} = \frac{-n\pi}{\beta_1}$

$|\vec{E}_1|_{min}$ at $z_{min} = \frac{(2n+1)\pi}{2\beta_1}$

$$\boxed{s = \frac{|\vec{E}_1|_{max}}{|\vec{E}_1|_{min}} = \frac{|\vec{H}_1|_{max}}{|\vec{H}_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

or, $\boxed{|\Gamma| = \frac{s-1}{s+1}}$

Since $|\Gamma| \le 1$, it follows that $1 \le s \le \infty$. The standing wave ratio (SWR) is dimensionless and it is customarily expressed in decibels (dB) as

$$\boxed{s \text{ in dB} = 20\log_{10} s}$$

Intrinsic Impedance – It is the ratio of electric field component to the magnetic field component.

$$\eta_{in} = \frac{E_{io} e^{-j\beta_1 z} + E_{ro} e^{j\beta_1 z}}{H_{io} e^{-j\beta_1 z} - H_{ro} e^{j\beta_1 z}}$$

$$= \frac{E_{io} e^{-j\beta_1 z} + \Gamma E_{io} e^{j\beta_1 z}}{\frac{E_{io}}{\eta_1} e^{-j\beta_1 z} - \Gamma \frac{E_{io} e^{j\beta_1 z}}{\eta_1}}$$

$$= \frac{E_{io}\left(e^{-j\beta_1 z} + \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} e^{j\beta_1 z}\right)}{\frac{E_{io}}{\eta_1}\left(e^{-j\beta_1 z} - \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} e^{j\beta_1 z}\right)}$$

$$= \eta_1 \frac{\left[(\eta_2 + \eta_1) e^{-j\beta_1 z} + (\eta_2 - \eta_1) e^{j\beta_1 z}\right]}{\left[(\eta_2 + \eta_1) e^{-j\beta_1 z} - (\eta_2 - \eta_1) e^{j\beta_1 z}\right]}$$

$$= \eta_1 \frac{\left[\eta_2 (e^{-j\beta_1 z} + e^{j\beta_1 z}) + \eta_1 (e^{-j\beta_1 z} - e^{j\beta_1 z})\right]}{\left[\eta_2 (e^{-j\beta_1 z} - e^{j\beta_1 z}) + \eta_1 (e^{-j\beta_1 z} + e^{j\beta_1 z})\right]}$$

$$= \eta_1 \left[\frac{2\eta_2 \cos\beta_1 z - 2\eta_1 \sin\beta_1 z}{-2\eta_2 \sin\beta_1 z + 2\eta_1 \cos\beta_1 z}\right]$$

$$\therefore \boxed{\eta_{in} = \eta_1 \left[\frac{\eta_2 - \eta_1 \tan\beta_1 z}{\eta_1 - \eta_2 \tan\beta_1 z}\right]}$$

Reflection of a plane wave at Oblique Incidence

Here, we consider the incident wave is incident obliquely in the boundary of two media. To simplify the analysis, let us assume that the media is lossless. The general form of a plane wave is

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$$
$$= Re\{\vec{E}_0 e^{j(\vec{k}\cdot\vec{r} - \omega t)}\}$$

where, $\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$ is the radius or position vector and $\vec{k} = k_x\hat{a}_x + k_y\hat{a}_y + k_z\hat{a}_z$ is the wave number vector or the propagation vector; $\vec{k}$ is always in the dirⁿ of wave propagation. The magnitude of k is related to ω according to the dispersion relation.

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\epsilon$$

Thus, for lossless media, k is essentially the same as β. With the general form of $\vec{E}$, Maxwell's equations reduce to

$$\vec{k}\times\vec{E} = \omega\mu\vec{H}$$
$$\vec{k}\times\vec{H} = -\omega\epsilon\vec{E}$$
$$\vec{k}\cdot\vec{H} = 0$$
$$\vec{k}\cdot\vec{E} = 0$$

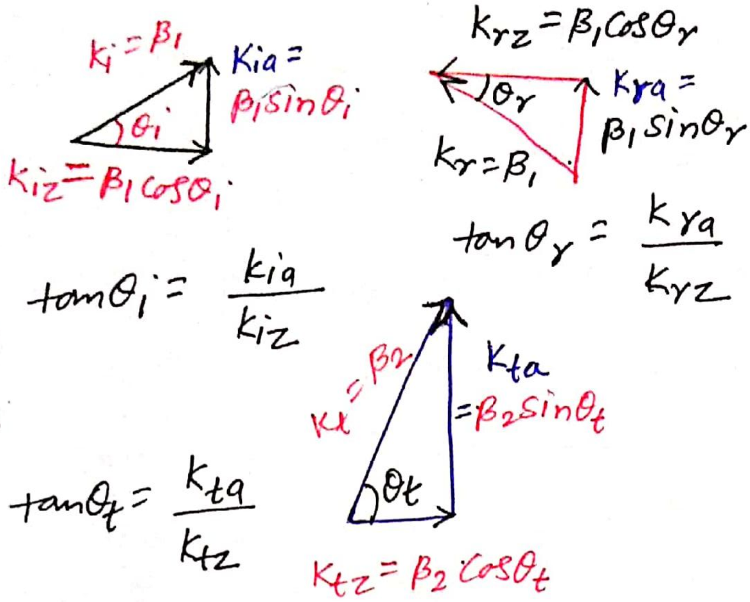
showing that $\vec{k}$, $\vec{E}$ & $\vec{H}$ are mutually orthogonal and $\vec{E}$ & $\vec{H}$ lie on the plane

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z = \text{constant}$$

Then, from $\vec{k}\times\vec{E} = \omega\mu\vec{H}$ we have

$$\vec{H} = \frac{1}{\omega\mu}\vec{k}\times\vec{E} = \frac{\hat{a}_k\times\vec{E}}{\eta}$$

(273) Having Expressed $\vec{E}$ & $\vec{H}$ in the general form, we can now consider the oblique incidence of a uniform plane wave at a plane boundary as shown in figure:



$\tan\theta_i = \dfrac{k_{ia}}{k_{iz}}$

$\tan\theta_r = \dfrac{k_{ra}}{k_{rz}}$

$\tan\theta_t = \dfrac{k_{ta}}{k_{tz}}$

fig:- Normal and tangential components of $\vec{k}$.

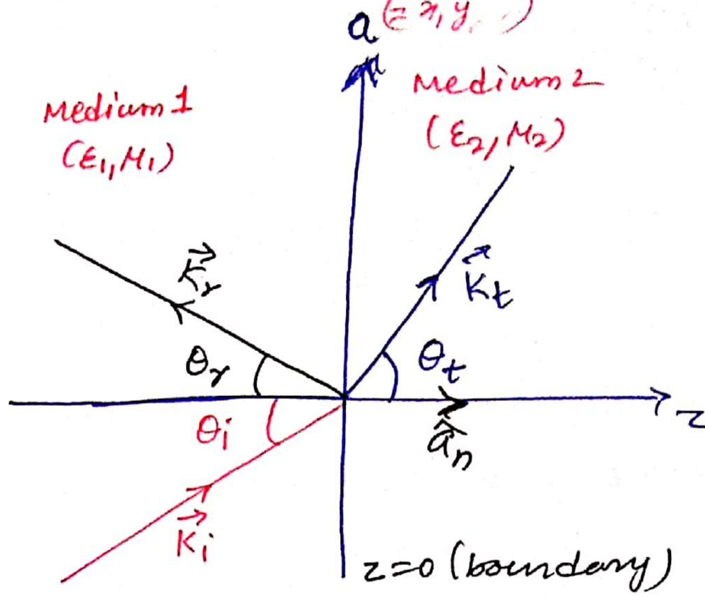


fig:- Oblique incidence of plane wave.

The plane defined by the propagation vector $\vec{k}$ and a unit normal vector $\hat{a}_n$ to the boundary is called a plane of incidence. The angle θ_i betn $\vec{k}$ and $\hat{a}_n$ is the angle of incidence.

Again, both the incident and reflected waves are in medium 1 while the transmitted (or refracted wave) is in medium 2. Let

$$\vec{E}_i = \vec{E}_{io} \cos(k_{ix} x + k_{iy} y + k_{iz} z - \omega_i t)$$

$$\vec{E}_r = \vec{E}_{ro} \cos(k_{rx} x + k_{ry} y + k_{rz} z - \omega_r t)$$

$$\vec{E}_t = \vec{E}_{to} \cos(k_{tx} x + k_{ty} y + k_{tz} z - \omega_t t)$$

where, k_i, k_r and k_t with their normal and tangential components shown in above figures. Since the tangential components of $\vec{E}$ must be continuous at the boundary $z=0$

$$\vec{E}_i (z=0) + \vec{E}_r (z=0) = \vec{E}_t (z=0)$$

(294)

the only way this boundary condition will be satisfied by the above waves for all x and y is that

(a) $\omega_i = \omega_r = \omega_t = \omega$

(b) $k_{ix} = k_{rx} = k_{tx} = k_x$

(c) $k_{iy} = k_{ry} = k_{ty} = k_y$

Condition (a) implies that freq^n is unchanged and conditions (b) & (c) require that the tangential components of the propagation vectors be continuous (called phase matching conditions). This means that the propagation vectors $\vec{k_i}$, $\vec{k_t}$ and $\vec{k_r}$ must all lie in the plane of incidence. Thus by conditions (b) & (c)

$$k_i \sin\theta_i = k_r \sin\theta_r$$

$$k_i \sin\theta_i = k_t \sin\theta_t$$

where θ_r is the angle of reflection and θ_t is the angle of transmission. But for lossless media,

$$k_i = k_r = \beta_1 = \omega\sqrt{\mu_1\epsilon_1}$$

$$k_t = \beta_2 = \omega\sqrt{\mu_2\epsilon_2}$$

So, from above relations it is seen that

$$\boxed{\theta_r = \theta_i}$$

& $$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \frac{u_2}{u_1} = \sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}$$

where, $u = \omega/k$ is the phase velocity. Then, it can be written as

$$n_1 \sin\theta_i = n_2 \sin\theta_t \quad \text{— Snell's law.}$$

where $n_1 = c\sqrt{\mu_1\epsilon_1} = c/u_1$ and $n_2 = c\sqrt{\mu_2\epsilon_2} = c/u_2$ are refractive indices of the media. Based on these general preliminaries on oblique incidence, we have specifically two special cases: one with the $\vec{E}$ field perpendicular to the plane of incidence, the other with the $\vec{E}$ field parallel to it. Any other polarization may be considered as a linear combination of these two cases:

(A) Parallel Polarization

In this case, the $\vec{E}$ field lies in the xz plane, the plane of incidence as shown in figure. In medium 1, we have both incident and reflected fields given by

$$\vec{E}_{is} = E_{io}(\cos\theta_i\,\hat{a}_x - \sin\theta_i\,\hat{a}_z)\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_{is} = \frac{E_{io}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\,\hat{a}_y$$

$$\vec{E}_{rs} = E_{ro}(\cos\theta_r\,\hat{a}_x + \sin\theta_r\,\hat{a}_z)\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_{rs} = -\frac{E_{ro}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\,\hat{a}_y$$

where, $\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$

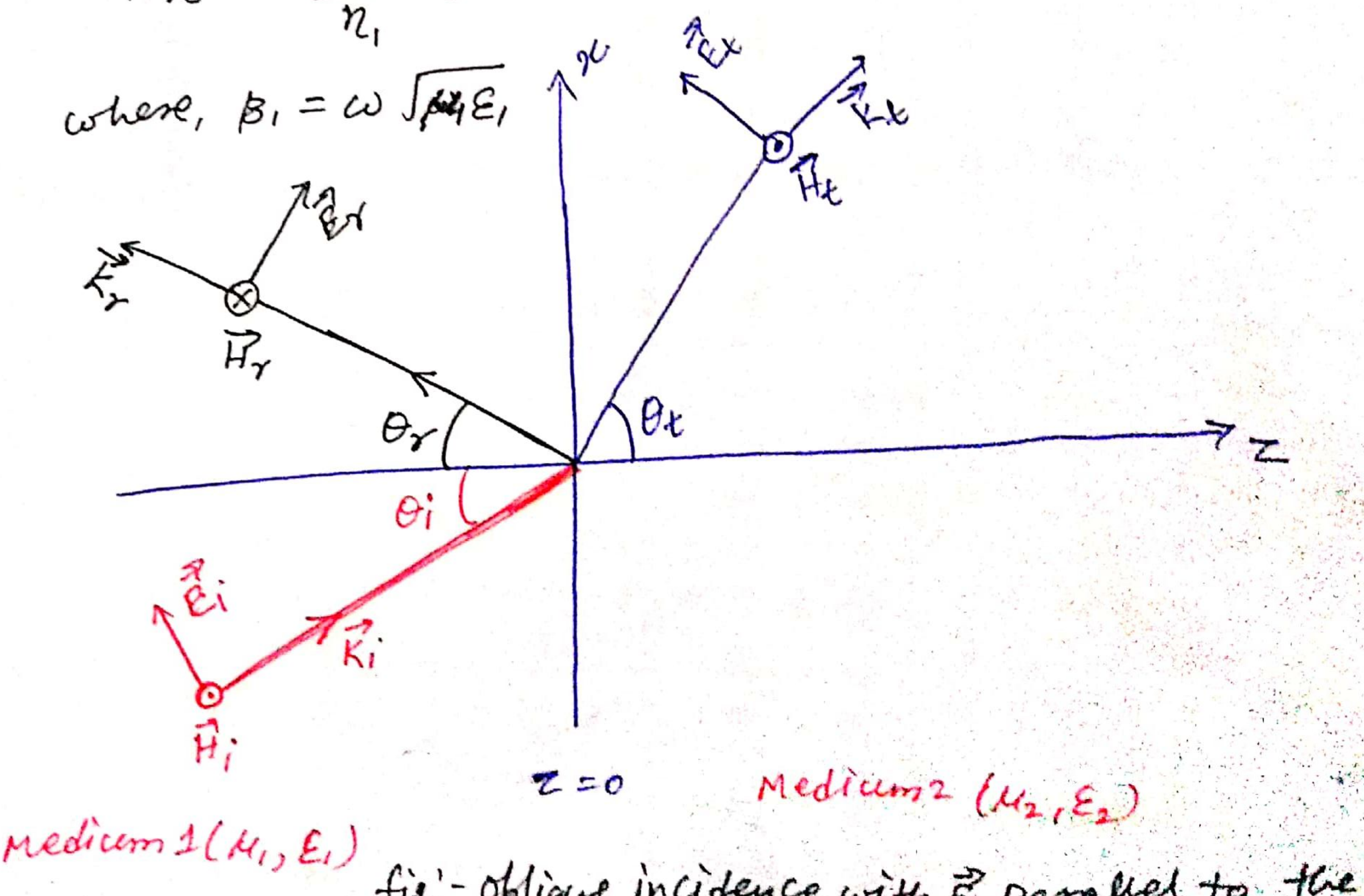


fig:- oblique incidence with $\vec{E}$ parallel to the plane of incidence.

Here, we first derive the values of incident, reflected and transmitted waves $\vec{k}$. Once $\vec{k}$ is known we define $\vec{E}_s$ such that $\nabla \cdot \vec{E}_s = 0$ or $\vec{k} \cdot \vec{E}_s = 0$ and then $\vec{H}_s$ is obtained from $\dfrac{\vec{k} \times \vec{E}_s}{\omega \mu} = \hat{a}_k \times \dfrac{\vec{E}}{\eta}$.

The transmitted fields exist in medium 2 and are given by

$$\vec{E}_{ts} = E_{to}(\cos\theta_t \hat{a}_x - \sin\theta_t \hat{a}_z) e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_{ts} = \frac{E_{to}}{\eta_2} e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)} \hat{a}_y$$

where, $\beta_2 = \omega\sqrt{\mu_2 \varepsilon_2}$.

Requiring that $\theta_r = \theta_i$ and the tangential components of $\vec{E}$ & $\vec{H}$ be continuous at the boundary $z=0$, we get

$$\left.\begin{aligned} (E_{io} + E_{ro})\cos\theta_i &= E_{to}\cos\theta_t \\ \frac{1}{\eta_1}(E_{io} - E_{ro}) &= \frac{1}{\eta_2} E_{to} \end{aligned}\right\} \text{Boundary Conditions}$$

Expressing E_{ro} & E_{to} in terms of E_{io}, we obtain

$$\boxed{\Gamma_1 = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}}$$

or, $E_{ro} = \Gamma_1 E_{io}$

and $$\boxed{\tau_1 = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}}$$

Fresnel's equations.

or, $E_{to} = \tau_1 E_{io}$

(477)

When $\theta_i = \theta_t = 0$, the equations reduce to

$$\Gamma_1 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad \& \quad \tau_1 = \frac{2\eta_2}{\eta_2 + \eta_1}$$

Since, θ_i & θ_t are related according to Snell's law.

$$n_1 \sin\theta_i = n_2 \sin\theta_t \quad \left[\because n_1 = c\sqrt{\mu_1 \varepsilon_1} = c/u_1 \ \& \ n_2 = c\sqrt{\mu_2 \varepsilon_2} = c/u_2 \right]$$

We can write,

$$\frac{c}{u_1} \sin\theta_i = \frac{c}{u_2} \sin\theta_t$$

$$\text{or, } \sin\theta_t = \frac{u_2}{u_1} \sin\theta_i$$

$$\text{or, } \cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \left(\frac{u_2}{u_1}\right)^2 \sin^2\theta_i}$$

From above equations of Γ_1 & τ_1 we can see that

$$\boxed{1 + \Gamma_1 = \tau_1 \left(\frac{\cos\theta_t}{\cos\theta_i} \right)}$$

Here, from equation of Γ_1 we can have $\Gamma_1 = 0$ as the numerator is the difference of two terms. which means no reflection (i.e $E_{ro} = 0$) and the incident angle at which this takes place is called the Brewster angle θ_{B1}. The Brewster angle is also known as Polarizing angle because an arbitrarily polarized incident wave will be reflected with only the component of $\vec{E}$ perpendicular to the plane of incidence. The Brewster angle is obtained by setting $\theta_i = \theta_{B1}$ when $\Gamma_1 = 0$.

(B) Perpendicular Polarization

In this case the $\vec{E}$ field is perpendicular to the plane of incidence (the xz plane) as shown in figure.

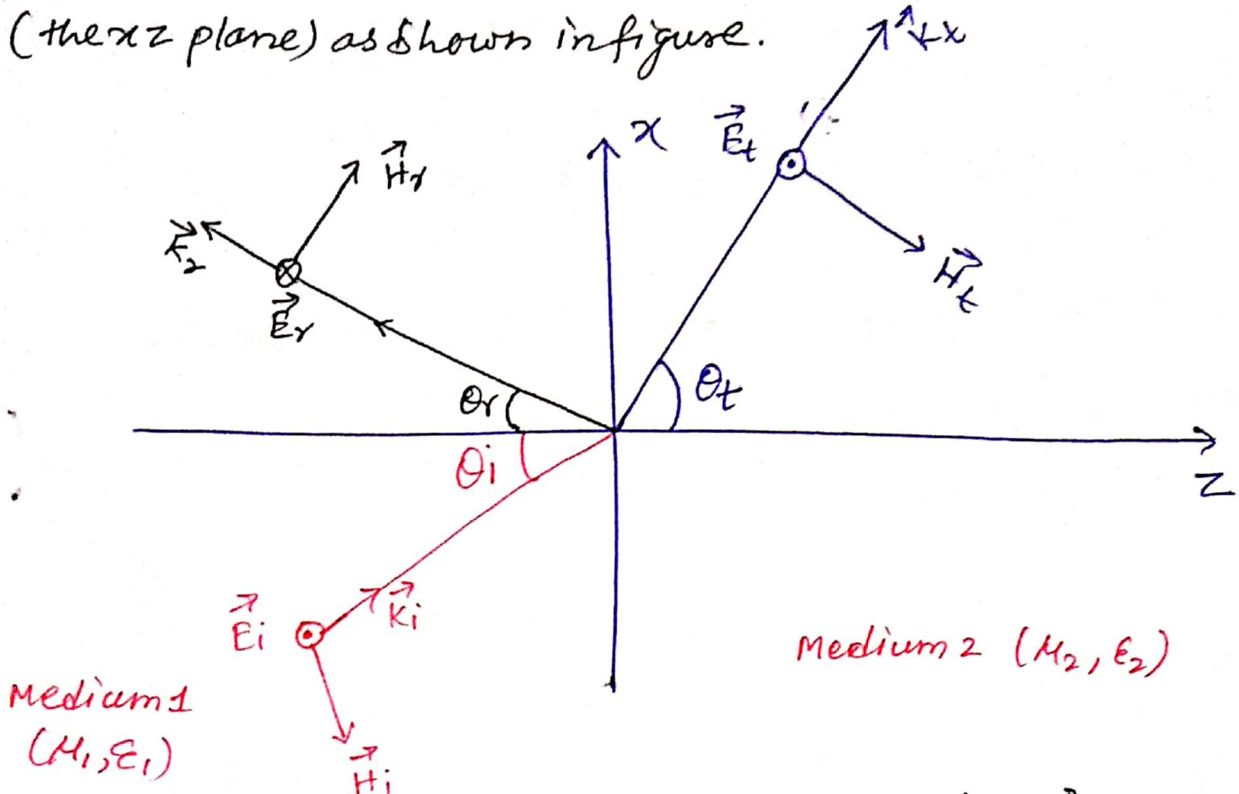


fig:- oblique incidence with $\vec{E}$ perpendicular to the plane of incidence.

This may also be said that $\vec{H}$ field is parallel to the plane of incidence. The incident and reflected fields in medium 1 are given by

$$\vec{E}_{is} = E_{io}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\, \hat{a}_y$$

$$\vec{H}_{is} = \frac{E_{io}}{\eta_1}\left(-\cos\theta_i\, \hat{a}_x + \sin\theta_i\, \hat{a}_z\right) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_{rs} = E_{ro}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\, \hat{a}_y$$

$$\vec{H}_{rs} = \frac{E_{ro}}{\eta_1}\left(\cos\theta_r\, \hat{a}_x + \sin\theta_r\, \hat{a}_z\right) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

while the transmitted fields in medium 2 are given by

$$\vec{E}_{ts} = E_{to}\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\, \hat{a}_y$$

$$\vec{H}_{ts} = \frac{E_{to}}{\eta_2}\left(-\cos\theta_t\, \hat{a}_x + \sin\theta_t\, \hat{a}_z\right) e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

(279) Again, requiring that the tangential components of $\vec{E}$ and $\vec{H}$ be continuous at $z=0$ and setting $\theta_r = \theta_i$ we get

$$E_{io} + E_{ro} = E_{to}$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro})\cos\theta_i = \frac{1}{\eta_2} E_{to}\cos\theta_t$$

(279) Expressing E_{ro} & E_{to} in terms of E_{io} leads to

$$\boxed{\Gamma_1 = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}}$$

or, $E_{ro} = \Gamma_1 E_{io}$

and $$\boxed{\tau_1 = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}}$$

Fresnel's equations.

or, $E_{to} = \tau_1 E_{io}$

We can also show that $\boxed{1 + \Gamma_1 = \tau_1}$

For no reflection $\Gamma_1 = 0$ (or $E_r = 0$), we can find $\theta_i = \theta_{B,\parallel}$ the Brewster angle ($\theta_{B\parallel}$). This condition is also said that of total transmission ($\tau_1 = 1$).

If medium 2 is conductor then $\Gamma_1 = -1$ & $\tau_1 = 0$

then, $\vec{E}_{is} = 2E_{io}\, e^{-j\beta_1 x\sin\theta_i} \cdot e^{-j\pi/2} \sin(\beta_1 z\cos\theta_i)$

& $\vec{H}_{is} = (\cos\theta_i \cdot \hat{a}_x + \sin\theta_i\, \hat{a}_z) \cdot \frac{2E_{io}}{\eta_1}$

A plane interface bet^n two perfect dielectrics is located at $z=0$. A 4 GHz uniform plane wave travelling in the $\hat{a}_z$ direction is incident from region 1, $z<0$ onto region 2, $z>0$. The wavelength in the dielectrics are $\lambda_1 = 6$ cm & $\lambda_2 = 4$ cm. Both the materials are non-magnetic. What percentage of the energy incident on the boundary is: ⓐ reflected ⓑ transmitted ⓒ what is the standing wave ratio in the region 1?

Sol.^n Here; region 1, $z<0, \sigma_1 = 0; \lambda_1 = 6$ cm, μ_0 & region 2, $z>0, \sigma_2 = 0$, $\lambda_2 = 4$ cm, μ_0. The freq^n is $f = 4\times10^9$ Hz. Let the permitivities of mediums be ϵ_1 & ϵ_2

ⓐ The phase constant (β_1) of region 1

$$\beta_1 = \omega\sqrt{\mu_0\epsilon_1} = 8\pi\times10^9\sqrt{\mu_0\epsilon_1}$$

But $\beta_1 = \frac{2\pi}{\lambda_1}$

thus, $8\pi\times10^9\sqrt{\mu_0\epsilon_1} = \frac{2\pi}{0.06}$

or, $\sqrt{\mu_0\epsilon_1} = \frac{10^{-9}}{0.24}$

For region 2

$$8\pi\times10^9\sqrt{\mu_0\epsilon_2} = \frac{2\pi}{0.04}$$

or, $\sqrt{\mu_0\epsilon_2} = \frac{10^{-9}}{0.16}$

The intrinsic impedances for two regions are

$$\eta_1 = \sqrt{\frac{\mu_0}{\epsilon_1}} = 0.24\times10^9\mu_0 \;\&\; \eta_2 = \sqrt{\frac{\mu_0}{\epsilon_2}} = 0.16\times10^9\mu_0$$

From these values we get ratio of intrinsic impedance.

$$\frac{\eta_2}{\eta_1} = \frac{2}{3}$$

The reflection co-efficent, $\Gamma = \frac{(\eta_2/\eta_1)-1}{(\frac{\eta_2}{\eta_1})+1} = -0.2$

% of power reflected $= |\Gamma|^2 = |-0.2|^2 = 0.04 = 4\%$

ⓑ Since 4% power is reflected as such the transmitted power is $100 - 4 = 96\%$

the standing wave ratio in region 1 is $S = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.2}{1-0.2}$

$= 1.52$ //

Qn: A brass ($\sigma = 10^7$ ℧/m) pipe with inner and outer radii of 1.7 and 2 cm carries a total current of 100 A d.c. Find $\vec{E}$, $\vec{H}$ & $\vec{P}$ within the brass.

Sol^n Here, $a = 0.017$ m & $b = 0.02$ m, $\sigma = 10^7$ ℧/m, $I = 100$ A dc.

The current density is

$$\vec{J} = \frac{100}{\pi(0.02^2 - 0.017^2)} \hat{a}_z = 2.87 \times 10^5 \hat{a}_z \text{ A/m}^2$$

The electric field intensity,

$$\vec{E} = \frac{\vec{J}}{\sigma} = \frac{2.87 \times 10^5}{10^7} \hat{a}_z = 0.0287 \hat{a}_z \text{ V/m}$$

Applying Ampere's circuital law, within the brass pipe,

$$2\pi\rho H_\phi = 100 \frac{\rho^2 - 0.017^2}{0.02^2 - 0.017^2} \Rightarrow H_\phi = 143455.6\left(\rho - \frac{0.017^2}{\rho}\right)$$

$$\therefore \vec{H} = \left(143455.6\rho - \frac{41.46}{\rho}\right) \hat{a}_\phi \text{ A/m}$$

The poynting vector

$$\vec{P} = \vec{E} \times \vec{H}$$

$$= 0.0287 \times \left(143455.6\rho - \frac{41.46}{\rho}\right)(-\hat{a}_\rho)$$

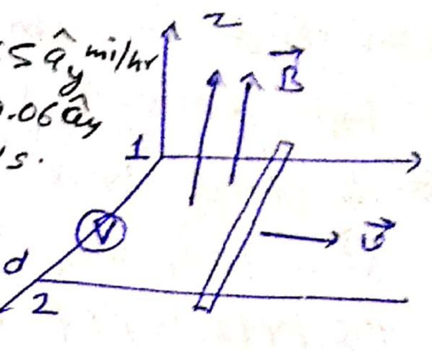
$$= \left(4117.18\rho - \frac{1.1899}{\rho}\right) -\hat{a}_\rho \text{ W/m}^2$$

The flow of power is towards centre of the pipe

Qn: In figure shown replace the uniform $\vec{B}$ field By $\vec{B} = 2e^{-50y}\hat{a}_z$ T. Find $V_{12}(t)$ if $d = 4$cm & $\vec{v} = 65\hat{a}_y$ mi/h with $y=0$ at $t=0$.

Sol.^{n}

Given, $\vec{B} = 2e^{-50y}\hat{a}_z$ T, $d = 4$cm and $\vec{v} = 65\hat{a}_y$ mi/hr $= 29.06\hat{a}_y$ m/s.



the boundary condition is $y(0) = 0$

Then, the flux cut at any position,

$$\phi = \int_0^d \int_0^y B.dy\,dx = d.\int_0^y 2e^{-50y}.dy = -\frac{0.08}{50}(e^{-50y} - 1)$$

the rate of change of flux,

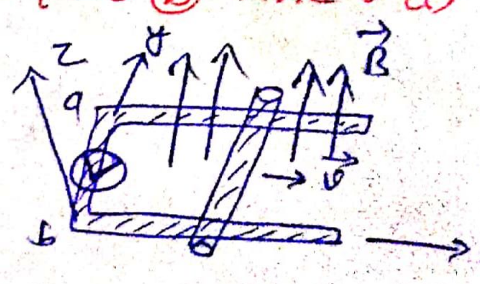
$$\frac{d\phi}{dt} = \frac{0.08}{50}e^{-50y}v = \frac{-0.08}{50} \times e^{-50y}(\cancel{-50}) \times 29.06 = 2.32e^{-50y}$$

At any time t, the position $y = 29.06t$

The emf induced is, $V_{12} = -\frac{d\phi}{dt} = -2.32e^{-50\times 26.09t}$

$$= -2.32e^{-1453t} \text{ V}$$

Qn: In figure given Let the separation of the rails be 8cm and let $\vec{B} = 0.5e^{0.1x}\hat{a}_z$ T (a) Find $V_{ab}(t)$ if the bar moves with a velocity $\vec{v} = 5\hat{a}_x$ m/s and $x=0$ at $t=0$ (b) Find $\vec{v}(t)$ if $V_{ab}(t) = -0.1t$ V and $x=0$ at $t=0$.



Sol.^{n} Given, $d = 0.08$m,

$\vec{B} = 0.5e^{0.1x}\hat{a}_z$ T

& $\vec{v} = 5\hat{a}_x$ m/s.

Initial condition is $x(0) = 0$, then the flux cut at any position

$$\phi = \int_0^d \int_0^x B\,dy\,dx = d\int_0^x 0.5e^{0.1x}dx = \frac{0.08 \times 0.5}{0.1}[e^{0.1x}]_0^x$$

$$= 0.04 \times 10(e^{0.1x} - 1) = 0.4(e^{0.1x} - 1)$$

emf induced betⁿ: ⓐ & ⓑ is

$$V_{ab} = \frac{-\partial \phi}{\partial t} = -0.04\,(e^{0.1x})\frac{dx}{dt} = -0.04\, e^{0.1x} \times v$$

$$= -0.04 \times e^{0.1x}\, v.$$

In t seconds the incremental distance covered, $dx = 5dt$

or, $x = 5t$

thus the emf induced

$$V_{ab} = -0.04 \times e^{0.1x} \times v = -0.04 \times e^{0.1 \times 5t} \times 5$$

$$= -0.2\, e^{0.5t}\ V$$

ⓑ Let $V_{ab}(t) = -0.1t$ V and $x = 0$ at $t = 0$

therefore $$-\frac{d\phi}{dt} = -0.1t$$

or, $$\frac{d\phi}{dt} = 0.1t$$

Integrating both sides

$$\phi = \int 0.1t = 0.1\frac{t^2}{2} + k = 0.05t^2 + k = 0.4.(e^{0.1x} - 1)$$

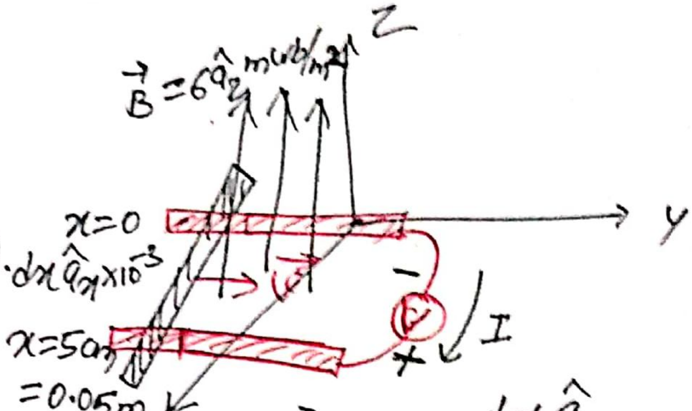
At $x = 0$ at $t = 0$ then, $k = 0$ & $x = 10\ln(1 + 0.125t^2)$

& the velocity $$v(t) = \frac{dx}{dt} = \frac{10 \times 0.125 \times 2t}{1 + 0.125t^2} = \frac{20t}{t^2+8}\ \text{m/s}.$$

Qn: Consider two parallel conductors placed at $x=0$ and $x=5cm$ in a magnetic field $\vec{B}=6\hat{a}_z$ mwb/m². A high resistance voltmeter is connected at one end & a conducting bar is sliding at other end with velocity $\vec{v}=18\hat{a}_y$ m/s. Calculate the induced voltage & show the polarity of induced voltage across the voltmeter.
[2073 chaitra]

<u>sol^n</u>

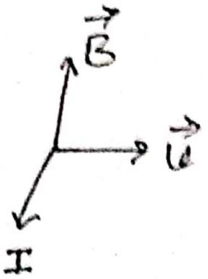
$$V_{emf} = \oint_L (\vec{v}\times\vec{B})\cdot d\vec{l}$$



$$= \int_{x=0}^{0.05} (18\hat{a}_y \times 6\hat{a}_z)\cdot dx\,\hat{a}_x \times 10^{-3}$$

$$= \int_{x=0}^{0.05} 18\times 6\, dx \times 10^{-3}$$

$$d\vec{S} = dy \cdot dx\, \hat{a}_z$$
$$d\vec{l} = dx\, \hat{a}_x$$



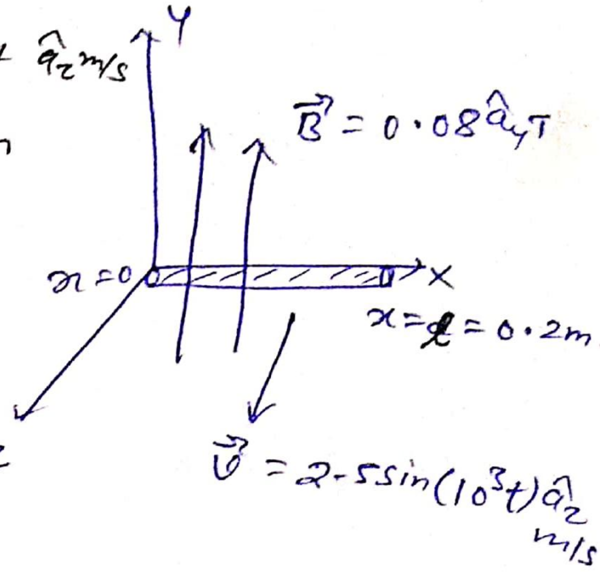
$$= 6\times 18\times 0.05 \times 10^{-3}$$
$$= 5.4\times 10^{-3}\,V$$

The direction of emf is as shown in figure. from the bar placed at $x=0$ to $x=5cm$. bar through voltmeter.

A straight conductor of 0.2m lies along x-axis with one end at origin. If this conductor is subjected to the magnetic flux density $\vec{B} = 0.08\,\hat{a}_y$ T and $\vec{v} = 2.5 \sin 10^3 t\,\hat{a}_z$ m/s. Calculate the emf induced in the conductor. [2071 Chaitra]

Sol.^{n} Given, $\vec{B} = 0.08\,\hat{a}_y$ T, $\vec{v} = 2.5 \sin 10^3 t\,\hat{a}_z$ m/s

& length of conductor $\ell = 0.2$m. Then



$$V_{emf} = \int_{x=0}^{0.2} (\vec{v} \times \vec{B}) \cdot \vec{d\ell}$$

$$= \int_{x=0}^{0.2} 2.5 \sin 10^3 t \times 0.08\,(-\hat{a}_x)\, dx\, \hat{a}_x$$

$$= -\int_{x=0}^{0.2} 0.2 \sin 10^3 t\, dx$$

$$= -0.2 \times 0.2 \sin 10^3 t$$

$$= -0.04 \sin 10^3 t \text{ V}$$

Qn: Express $E_y = 100 \cos(10^8 t - 0.5z + 30^\circ)$ V/m as a phasor.

Sol.^{n}

Given, $E_y = 100 \cos(10^8 t - 0.5z + 30^\circ)$ V/m.

Writing in exponential form.

$$E_y = \mathrm{Re}\{100\, e^{j(10^8 t - 0.5z + 30^\circ)}\}$$

Dropping Re and omitting ωt term $(10^8 t)$, we obtain the phasor as:

$$E_{ys} = 100\, e^{j(-0.5z + 30^\circ)} \text{ V/m.}$$

∵ Field intensity vector in phasor form is $\vec{E}_s = 100\angle 30^\circ \hat{a}_x + 20\angle -50^\circ \hat{a}_y + 40\angle 210^\circ \hat{a}_z$ V/m, Find the vector in real function of time, where the frequency is given as 1 MHz.

Sol.ⁿ Given, Field intensity vector in phasor form is

$$\vec{E}_s = 100\angle 30^\circ \hat{a}_x + 20\angle -50^\circ \hat{a}_y + 40\angle 210^\circ \hat{a}_z \text{ V/m.}$$

$$= 100\, e^{j30^\circ} \hat{a}_x + 20\, e^{-j50^\circ} \hat{a}_y + 40\, e^{j210^\circ} \hat{a}_z \text{ V/m.}$$

Reinserting the $e^{j\omega t}$ term

$$\vec{E}_s(t) = \left(100\, e^{j30^\circ} \hat{a}_x + 20\, e^{-j50^\circ} \hat{a}_y + 40\, e^{j210^\circ} \hat{a}_z\right) \times e^{j\omega t}$$

where, $\omega = 2\pi f = 2\pi \times 10^6$ rad/s.

$$\therefore \vec{E}_s(t) = 100^{j(30^\circ + 2\pi\times 10^6 t)} \hat{a}_x + 20\, e^{j(-50^\circ + 2\pi\times 10^6 t)} \hat{a}_y + 40\, e^{j(210^\circ + 2\pi\times 10^6 t)} \hat{a}_z$$

Now, taking real part only.

$$\vec{E}(t) = Re\left\{100^{j(2\pi\times 10^6 t + 30^\circ)} \hat{a}_x + 20\, e^{j(2\pi\times 10^6 t - 50^\circ)} \hat{a}_y + 40\, e^{j(2\pi\times 10^6 t + 210^\circ)} \hat{a}_z\right\}$$

$$= 100 \cos(2\pi\times 10^6 t + 30^\circ)\hat{a}_x + 20 \cos(2\pi\times 10^6 t - 50^\circ)\hat{a}_y + 40^\circ \cos(2\pi\times 10^6 t + 210^\circ)\hat{a}_z \text{ V/m.}$$

Within a certain region, $\varepsilon = 10^{-11}$ F/m and $\mu = 10^{-5}$ H/m. If $B_x = 2\times10^{-4}\cos 10^5 t \sin 10^{-3} y$ T; (a) Use $\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$ to find $\vec{E}$. (b) find the total magnetic flux passing through the surface $x = 0, 0 < y < 40$m, $0 < z < 2$m at $t = 1\mu s$ (c) Find the value of closed line integral of $\vec{E}$ around the perimeter of the given surface. [2072 Kartik]

Sol.^{n}

Given, $\varepsilon = 10^{-11}$ F/m & $\mu = 10^{-5}$ H/m.

$B_x = 2\times10^{-4}\cos 10^5 t \sin 10^{-3} y$ T $\Rightarrow \vec{B} = 2\times10^{-4}\cos 10^5 t \cdot \sin 10^{-3} y\, \hat{a}_x$ T

(a) $\nabla \times \vec{H} = \nabla \times \left(\frac{\vec{B}}{\mu}\right)$

$$= \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \times \left(\frac{2\times10^{-4}\cos 10^5 t \sin 10^{-3} y\, \hat{a}_x}{10^{-5}}\right)$$

$$= -\frac{\partial}{\partial y}\left[\frac{2\times10^{-4}\cos 10^5 t \sin 10^{-3} y}{10^{-5}}\right]\hat{a}_z$$

$$= -\frac{2\times10^{-4}}{10^{-5}}\cos 10^5 t \times 10^{-3} \cos 10^{-3} y\, \hat{a}_z$$

$$= -2\times10^{-7}\times10^{5}\cos 10^5 t \cos 10^{-3} y\, \hat{a}_z$$

$$= -0.02 \cos 10^5 t \cos 10^{-3} y\, \hat{a}_z$$

again, $-0.02\cos 10^5 t \cos 10^{-3} y\, \hat{a}_z = \varepsilon \frac{\partial \vec{E}}{\partial t}$

or, $\partial \vec{E} = -2\times10^{9}\cos 10^5 t \cdot \cos 10^{-3} y\, \hat{a}_z\, dt$

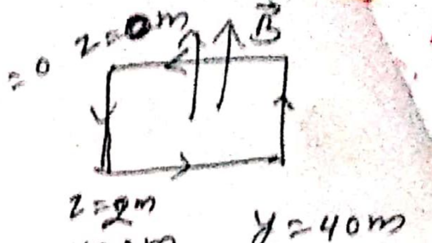
or, $\vec{E} = \int -2\times10^{9}\cos 10^5 t \cdot \cos 10^{-3} y\, \hat{a}_z\, dt$

or, $\vec{E} = \frac{-2\times10^{9}\cos 10^{-3} y \cdot \sin 10^5 t}{10^5}\hat{a}_z$

$\therefore \vec{E} = -20000 \sin 10^5 t \cos 10^{-3} y\, \hat{a}_z$ V/m.

Again, (b)

$$\phi = \int_S \vec{B}\cdot d\vec{s}$$



$$= \int_{y=0}^{40m}\int_{z=0}^{2m} 2\times10^{-4}\cos 10^5 t \sin 10^{-3} y \, dy\, dz$$

$(x=0)$

$$= 4\times10^{-4}\cos 10^5 t \int_{y=0}^{40m} \sin 10^{-3} y \, dy$$

$$= -4\times10^{-4}\cos 10^5 t \cdot \left. \frac{\cos 10^{-3} y}{10^{-3}} \right|_{y=0}^{40}$$

$$= -4\times10^{-1}\cos 10^5 t \left[\cos 10^{-3}\times 40 - \cos 10^{-3}\times 0\right]$$

$$= -0.40 \cos 10^5 t \left[\cos 0.04 - \cos 0\right]$$

$$= 0.0003199 \cos 10^5 t \text{ wb.}$$

at $t = 1\mu s$, $\phi(t=1\mu s) = 0.0003199 \times \cos 10^5 \times 10^{-6}$ wb.

$= 0.3183$ mwb.

(c) $$\oint_L \vec{E}\cdot d\vec{l} \; (t=1\mu s) = \int_{z=0\,(y=0,\,x=0)}^{2} -20000 \sin 10^5 t \cos 10^{-3} y \, dz + \int_{y=0\,(x=0,\,z=2)}^{40} -20000 \sin 10^5 t \cos 10^{-3} y \, dy$$

$$+ \int_{z=2\,(x=0,\,y=40)}^{0} -20000 \sin 10^5 t \cos 10^{-3} y \, dz + \int_{y=40\,(z=0,\,x=0)}^{0} -20{,}000 \sin 10^5 t \cos 10^{-3} y \, dy$$

$$= -3993.3366 + 3990.142 = -3.1942 \text{ V}$$

Q: The electric field amplitude of a uniform plane wave propagating in the $\hat{a}_z$ direction is 250 V/m. If $\vec{E} = E_x \hat{a}_x$ and $\omega = 1.00$ Mrad/s, find ⓐ the freq.ⁿ ⓑ wavelength ⓒ the period ⓓ the amplitude of $\vec{H}$.

[2071 Shrawan]

Sol.ⁿ Given, $\omega = 1$ M rad/s.

ⓐ We have, $\omega = 2\pi f$

or, $f = \dfrac{10^6}{2\pi} = 159.15$ KHz

ⓑ $\lambda = \dfrac{c}{f} = \dfrac{3\times10^8}{159.15\times10^3} = 1.885$ Km.

ⓒ We have, Period $(T) = \dfrac{1}{f} = \dfrac{1}{159.15\times10^3} = 6.283\times10^{-6}$ s $= 6.283\ \mu s$.

ⓓ Here, $\vec{E} = E_x \hat{a}_x$ where, $E_x = 250$ V/m.

& $H_y = \dfrac{E_x}{\eta}$

where, $\eta = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 120\pi = 377\Omega$, Assuming free space

So, $H_y = \dfrac{250}{377} = 0.663$ A/m.

Q: Let $\vec{H_s} = (2\angle{-40^\circ}\,\hat{a}_x - 3\angle{20^\circ}\,\hat{a}_y)e^{-j0.07z}$ A/m for a uniform plane wave travelling in free space. Find ⓐ ω ⓑ H_x at P(1, 2, 3) at t = 31 ns ⓒ $|\vec{H}|$ at t = 0 at the origin.

Sol.ⁿ Given, $\vec{H_s} = (2\angle{-40^\circ}\,\hat{a}_x - 3\angle{20^\circ}\,\hat{a}_y)e^{-j0.07z}$ A/m

where, $\gamma = \beta = 0.07$ [∵ free space]

where, $|\beta| = \omega\sqrt{\mu_0\varepsilon_0}$

or, $0.07 = \omega \times \dfrac{1}{c}$ or, $\omega = 3\times10^8 \times 0.07 = 21$ Mrad/s.

Again,

$$\vec{H_s} = \left(2e^{-j40^\circ}\hat{a}_x - 3e^{j20^\circ}\hat{a}_y\right)e^{-j0.07z}$$

$$= 2e^{j(-40^\circ-0.07z)}\hat{a}_x - 3e^{j(20^\circ-0.07z)}\hat{a}_y$$

or, $\vec{H_s}(t) = 2e^{j(21\times10^6 t-40^\circ-0.07z)}\hat{a}_x - 3e^{j(21\times10^6 t+20^\circ-0.07z)}\hat{a}_y$

so, $\vec{H}(t) = Re\left\{2e^{j(21\times10^6 t-40-0.07z)}\hat{a}_x - 3e^{j(21\times10^6 t+20^\circ-0.07z)}\hat{a}_y\right\}$

$$= 2\cos(21\times10^6 t-40^\circ-0.07z)\hat{a}_x - 3\cos(21\times10^6 t+20^\circ-0.07z)$$

So, H_x at $P(1,2,3)$ at $t=31ns$ is

$$H_x = \left|2\cos(21\times10^6\times31\times10^{-9}-40^\circ-0.07\times3)\right|$$

$$= \left|2\cos(0.651-0.6981-0.21)\right|$$

$$= \left|2\cos(-0.2571)\right|$$

$$= 1.9342\ A/m.$$

ⓒ Again,

$$|\vec{H}| = \sqrt{4\cos^2(21\times10^6 t-40^\circ-0.07z)+9\cos^2(21\times10^6 t+20^\circ-0.07z)}$$

at $t=0$ & $O(0,0,0)$

$$|\vec{H}| = \sqrt{4\cos^2(0-40^\circ-0)+9\cos^2(0+20^\circ-0)}$$

$$= \sqrt{4\cos^2 40^\circ + 9\cos^2 20^\circ}$$

$$= \sqrt{2.34729+7.9472}$$

$$= \sqrt{10.2945} = 3.21\ A/m.$$

: Given a nonmagnetic material having $\varepsilon_r = 3.2$ and $\sigma = 1.5\times10^{-4}$ S/m find numerical values at 3 MHz for the: (a) loss tangent (b) attenuation constant (c) Phase constant (d) intrinsic impedance.

Sol.^{n} Given, $\varepsilon_r = 3.2$, $\sigma = 1.5\times10^{-4}$ S/m. and $f = 3$ MHz:

(a) Loss tangent, $\tan\theta = \dfrac{\sigma}{\omega\varepsilon} = \dfrac{1.5\times10^{-4}}{2\pi\times3\times10^{6}\times\varepsilon_0\varepsilon_r} = \dfrac{1.5\times10^{-4}}{2\pi\times3\times10^{6}\times8.85\times10^{-12}\times3.2}$

Now,

$= 0.281$

(b) attenuation constant $(\alpha) = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\sqrt{\sqrt{1+\dfrac{\sigma^2}{\omega^2\varepsilon^2}}-1}$

here, $\mu_r = 1$

$= \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\sqrt{\sqrt{1+\tan^2\theta}-1}$

$= 2\pi\times3\times10^{6}\sqrt{\dfrac{3.2}{2}\times\dfrac{1}{c^2}}\cdot\sqrt{\sqrt{1+(0.281)^2}-1}$

$= \dfrac{6\pi\times10^{6}}{3\times10^{8}}\cdot\sqrt{\dfrac{3.2}{2}}\cdot\sqrt{0.03873}$

$= 2\pi\times10^{-2}\times0.1968\times1.2649$

$= 1.564\times10^{-2}$

$= 0.01564$ Np/m (Nepers/m)

Again,

(c) $\beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\sqrt{\sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}+1}$

$= \dfrac{6\pi\times10^{6}}{3\times10^{8}}\sqrt{\dfrac{3.2}{2}}\cdot\sqrt{\sqrt{(1+0.281^2)}+1} = 2\pi\times10^{-2}\times1.42784\times1.2649$

$= 0.1135$ rad/m

(d) Intrinsic impedance $(\eta) = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

रविन ... इन्जिनियर

where,

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}} \quad \& \quad \theta_\eta = \frac{\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right)}{2}$$

Then,

$$|\eta| = \frac{\sqrt{\frac{1\times\mu_0}{\varepsilon_0\times 3.2}}}{\sqrt{\sqrt{1+0.281^2}}} = 206.68\,\Omega$$

$$\& \quad \theta_\eta = \frac{\tan^{-1}(0.281)}{2} = 7.84^\circ$$

$$\therefore \eta = 206.68 \angle 7.84^\circ\,\Omega$$

Qn: Consider a material for which $\mu_r = 1$, $\varepsilon_r = 2.5$ and the loss tangent is 0.12. If these three values are constant with freq^n in the range $0.5\,MHz \le f \le 100\,MHz$. Calculate (a) σ at 1 & 75 MHz (b) λ at 1 and 75 MHz (c) v_p at 1 & 75 MHz.

Sol^n Given, $\mu_r = 1$, $\varepsilon_r = 2.5$ & $\tan\theta = 0.12$ which are constants for freq^n range $0.5\,MHz \le f \le 100\,MHz$.

(a) $\tan\theta = \dfrac{\sigma}{\omega\varepsilon}$

for 1 MHz, $\omega = 2\pi \times 10^6$ rad/s.

& $\varepsilon = 8.85\times10^{-12} \times 2.5$ F/m.

~~then~~, $0.12 = \dfrac{\sigma}{2\pi\times10^6 \times 8.85\times10^{-12}\times 2.5}$ $\Rightarrow \sigma = 16.68\times10^{-6}$

$= 1.668\times10^{-5}$ S/m

for freq^n 75 MHz

$$0.12 = \frac{\sigma}{2\pi \times 75\times10^{6} \times 8.85\times10^{-12} \times 2.5}$$

or, $\sigma = 1.251\times10^{-3}$ S/m.

Again,

(b) phase constant, $\beta = \frac{2\pi}{\lambda} = \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\sqrt{1+\tan^2\theta}+1}$

for 1 MHz

$$\text{or, } \frac{2\pi}{\lambda} = 2\pi\times10^{6}\sqrt{\frac{\mu_0\epsilon_0\times2.5}{2}}\sqrt{\sqrt{1+0.12^2}+1}$$

$$\text{or, } \frac{1}{\lambda} = \frac{10^6}{3\times10^8}\sqrt{\frac{2.5}{2}} \times 1.41675$$

$$\text{or, } \frac{1}{\lambda} = 0.52799\times10^{-2}$$

$$\text{or, } \lambda = 189.397 \text{ m.}$$

and for 75 MHz

$$\lambda = \frac{189.397}{75} = 2.525 \text{ m.}$$

(c) v_p at 1 MHz $= \lambda \times f = 189.397\times10^{6}$
$= 1.89\times10^{8}$ m/s.

& v_p at 75 MHz $= \lambda\times f = 2.525\times75\times10^{6}$
$= 1.89\times10^{8}$ m/s.

Find the amplitude of displacement current density in metallic conductor at 60 Hz if $\varepsilon = \varepsilon_0, \mu = \mu_0, \sigma = 5.8 \times 10^7$ S/m and $\vec{J} = \sin(377t - 117.1z)\hat{a}_n$ MA/m² [2073 Shrawan]

Sol.ⁿ Given,

$$\vec{J} = \sin(377t - 117.1z)\hat{a}_n \text{ MA/m}^2$$

freq? $f = 60$ Hz, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ & $\sigma = 5.8 \times 10^7$ S/m.

Now, from definition of Loss tangent

$$\tan\theta = \frac{|\vec{J}_s|}{|\vec{J}_{ds}|} = \frac{\sigma}{\omega\varepsilon} = \frac{5.8 \times 10^7}{2\pi \times 60 \times 8.85 \times 10^{-12}} = 1.738 \times 10^{16}$$

$$\text{or, } |\vec{J}_{ds}| = \frac{|\vec{J}_s|}{1.738 \times 10^{16}}$$

where, $|\vec{J}_s|$ can be calculated as:

$$\vec{J} = \sin(377t - 117.1z)\hat{a}_n \text{ MA/m}^2$$

$$\text{or, } \vec{J} = \cos(377t - 117.1z - 90^\circ)\hat{a}_n \text{ MA/m}^2 \quad [A\sin(\omega t + \theta) = A\cos(\omega t + \theta - 90^\circ)]$$

$$\text{or, } \vec{J} = \text{Re}\{e^{j377t} \cdot e^{-j117.1z} \cdot e^{-j90^\circ}\}\hat{a}_n \text{ MA/m}^2$$

$$\text{or, } \vec{J}_s = e^{j(-90^\circ - 117.1z)}\hat{a}_n \text{ MA/m}^2$$

Also, $e^{j\theta} = \cos\theta + j\sin\theta$, so [Euler's formula]

$$\vec{J}_s = \cos(-90^\circ - 117.1z) + j\sin(-90^\circ - 117.1z)$$

$$= \cos(90^\circ + 117.1z) - j\sin(90^\circ + 117.1z) \quad [\cos(-\theta) = \cos\theta \text{ \& } \sin(-\theta) = -\sin\theta]$$

$$\text{So, } |\vec{J}_s| = \sqrt{\cos^2(90^\circ + 117.1z) + \{-\sin(90^\circ + 117.1z)\}^2}$$

$$= 1 \times 10^6 \text{ A/m}^2$$

$$\text{\& Hence, } |\vec{J}_d| = \frac{10^6}{1.738 \times 10^{16}} = 0.5754 \times 10^{-10} = 57.5 \text{ pA/m}^2$$

Qn: The phasor component of electric field intensity in free space is given by $\vec{E_s} = (100\angle 45^\circ) e^{-j50z} \hat{a}_x$ V/m. Determine frequency of the wave, wave impedance, $\vec{H_s}$ and magnitude of $\vec{E}$ at $z = 10mm$, $t = 20ps$. [2068 Chaitra]

Soln: Given,

$$\vec{E_s} = (100\angle 45^\circ) e^{-j50z} \hat{a}_x \text{ V/m}$$

$$\text{or, } \vec{E_s} = 100 e^{j45^\circ} \cdot e^{-j50z} \hat{a}_x$$

$$\text{or, } \vec{E} = Re\left\{ 100 e^{j(\omega t + 45^\circ - 50z)} \right\} \hat{a}_x$$

$$\text{or, } \vec{E} = 100 \cos(\omega t + 45^\circ - 50z) \hat{a}_x$$

Comparing with $\vec{E} = E_0 \cos(\omega t + \theta + \beta z) \hat{a}_x$

$$\beta = |-50| = \frac{2\pi}{\lambda}$$

$$\text{or, } \lambda = \frac{2\pi}{50} \text{ m}$$

& frequency, $f = \frac{c}{\lambda} = \frac{3 \times 10^8}{2\pi} \times 50 = 2.38 \times 10^9$

$= 2.38$ GHz

Again, wave impedance, $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi = 377\,\Omega$

and $\vec{H_s}$ can be found as:

$$\vec{H_s} = \frac{E_{sx}}{\eta_0} \hat{a}_y = \frac{100\angle 45^\circ e^{-j50z}}{377} \hat{a}_y = 0.265\angle 45^\circ e^{-j50z} \hat{a}_y \text{ A/m.}$$

Now,

$|\vec{E}|$ at $z = 10mm$ & $t = 20ps$ can be calculated as:

$$\omega = 2\pi f = 2\pi \times \frac{3 \times 10^8}{2\pi} \times 50 = 150 \times 10^8 \text{ rad/s} = 1.5 \times 10^{10} \text{ rad/s.}$$

So, $|\vec{E}| = 100 \cos(1.5 \times 10^{10} \times 20 \times 10^{-12} + 45^\circ - 50 \times 0.01) = 100\cos(0.0856) = 99.63$ V/m

Qn: Assume that dry soil has conductivity equal to 10^{-4} S/m, $\varepsilon = 3\varepsilon_0$ and $\mu = \mu_0$. Determine the freq^n at which the ratio of magnitude of the conduction current density and displacement current density is unity. [2069 chaitra]

Sol^n:

Given, $\sigma = 10^{-4}$ S/m, $\varepsilon = 3\varepsilon_0$ & $\mu = \mu_0$

Also, $\dfrac{|\vec{J}_s|}{|\vec{J}_{ds}|} = 1 = \dfrac{\sigma}{\omega\varepsilon}$

or, $\omega = \dfrac{\sigma}{\varepsilon} = \dfrac{10^{-4}}{3 \times 8.85 \times 10^{-12}}$

or, $f = \dfrac{10^{8}}{3 \times 8.85 \times 2\pi} = 599453.6 = 0.5994$ MHz

Qn: Select the value of K so that each of the following pairs of fields satisfies Maxwell's equations in a region where $\sigma = 0$ and $\rho_v = 0$: $\vec{D} = 5x\,\hat{a}_x - 2y\,\hat{a}_y + kz\,\hat{a}_z$ $\mu C/m^2$;

$\vec{B} = 2\hat{a}_y$ mT

if $\mu = 0.25$ H/m and $\varepsilon = 0.01$ F/m. [2070 Ashad]

Sol^n: Given, $\sigma = 0$, $\rho_v = 0$, $\vec{D} = 5x\hat{a}_x - 2y\hat{a}_y + kz\hat{a}_z\,\mu C/m^2$

$\vec{B} = 2\hat{a}_y$ mT, $\mu = 0.25$ H/m & $\varepsilon = 0.01$ F/m.

From Maxwell's equation

$\nabla \cdot \vec{D} = \rho_v = 0 \quad [\because \rho_v = 0]$

or, $\left(\dfrac{\partial}{\partial x}\hat{a}_x + \dfrac{\partial}{\partial y}\hat{a}_y + \dfrac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(5x\,\hat{a}_x - 2y\,\hat{a}_y + kz\,\hat{a}_z\right) = 0$

or, $5 - 2 + K = 0$

or, $K = -3$ $\mu C/m^2$

n: Find the amplitude of displacement current density inside a typical metallic conductor where $f = 1\,KHz$, $\sigma = 5\times10^7$ mho/m, $\varepsilon_r = 1$ and the conduction current density is $\vec{J} = 10^7 \sin(6283t - 444z)\,\hat{a}_y\ A/m^2$ [2070 chaitra] [2071 shrawan]

Sol.ⁿ We have,

$$f = 1\,KHz,\ \sigma = 5\times10^7\ \text{mho/m},\ \varepsilon_r = 1$$

$$\vec{J} = 10^7 \sin(6283t - 444z)\,\hat{a}_y\ A/m^2$$

We have, the ratio of magnitude of conduction current to the magnitude of displacement current is

$$\frac{|\vec{J}_c|}{|\vec{J}_{ds}|} = \frac{\sigma}{\omega\varepsilon}$$

$$\text{or, } \frac{|\vec{J}_c|}{|\vec{J}_{ds}|} = \frac{5\times10^7}{2\pi\times10^3\times8.85\times10^{-12}} = \frac{5\times10^{16}}{2\pi\times8.85}$$

$$\text{or, } \frac{|\vec{J}_c|}{|\vec{J}_{ds}|} = 8.992\times10^{14}$$

Again,

$$\vec{J} = 10^7 \sin(6283t - 444z)\,\hat{a}_y\ A/m^2$$

$$= 10^7 \cos(6283t - 444z - 90^\circ)\,\hat{a}_y\ A/m^2$$

$$\text{or, } \vec{J} = Re\left\{10^7\, e^{j6283t} \cdot e^{-j444z} \cdot e^{-j\frac{\pi}{2}}\right\}\hat{a}_y$$

$$\text{or, } \vec{J}_s = 10^7\, e^{-j444z} \cdot e^{-j\frac{\pi}{2}}\,\hat{a}_y$$

$$\text{or, } \vec{J}_s = 10^7\, e^{-j(444z + \frac{\pi}{2})}\,\hat{a}_y\ A/m^2$$

Again, Using $e^{i\theta} = \cos\theta + i\sin\theta$

$$\vec{J}_s = [\cos(-444z - \tfrac{\pi}{2}) + j\sin(-444z - \tfrac{\pi}{2})] \times 10^7 \, \hat{a}_y \; A/m^2$$

$$= [\cos(44z + \tfrac{\pi}{2}) - j\sin(444z + \tfrac{\pi}{2})] \times 10^7 \, \hat{a}_y \; A/m^2$$

$$\therefore |\vec{J}_s| = \sqrt{\cos^2(44z + \tfrac{\pi}{2}) + \sin^2(44z + \tfrac{\pi}{2})} \times 10^7 \; A/m^2$$

$$= 10^7 \; A/m^2$$

$$\therefore \frac{10^7}{|\vec{J}_{ds}|} = 8.992 \times 10^{14}$$

$$\text{or, } |\vec{J}_{ds}| = \frac{1112099.64}{10^{14}} = 11.121 \times 10^{-9} = 11.121 \; nA/m^2$$

Qn: Find the skin depth for copper at 10000 MHz and 60 Hz. The conductivity 5.8×10^7 S/m. (μ_0)

Soln: Given, $f_1 = 10000$ MHz & $f_2 = 60$ Hz

& $\sigma = 5.8 \times 10^7$ S/m.

Then, Skin depth $(\delta_{cu}) = \dfrac{1}{\sqrt{\pi f \mu \sigma}} = \dfrac{1}{\sqrt{\pi \times 4 \times \pi \times 10^{-7} \times 5.8 \times 10^7 f}}$

$$= \frac{0.066119}{\sqrt{f}} \text{ mm.}$$

At $f_1 = 10000$ MHz

$$\therefore \delta_{cu} = \frac{0.066119}{\sqrt{10^{10}}} = 6.61 \times 10^{-4} \text{ mm}$$

At $f_2 = 60$ Hz

$$\therefore \delta_{cu} = \frac{0.066119}{\sqrt{60}} = 8.53 \text{ mm.}$$

Qn: Find the attenuation and phase constants, velocity and wavelength within good conductor copper with conductivity 5.8×10^{7} S/m at 60 Hz.

रविन दासुला
ईन्जिनियर

Sol.ⁿ Given, $\sigma_{cu} = 5.8\times10^{7}$ S/m & $f = 60$ Hz.

then, For conductor, $\alpha = \beta = \frac{1}{\delta_{cu}} = \sqrt{\pi f \mu \sigma}$

$= \sqrt{3.14\times60\times4\times3.14\times10^{-7}\times5.8\times10^{7}}$

$= 117.15\ m^{-1}$

Again, $\beta = \frac{2\pi}{\lambda}$

or, $\lambda = \frac{2\pi}{\beta} = 0.0536m = 5.36cm$.

And velocity of wave propagation is

$v = \frac{\omega}{\beta} = \frac{2\pi\times60}{117.15} = 3.216$ m/s.

Qn: Find the resistance of copper wire of 1mm radius and 1km length at dc current and 1MHz frequency. $\sigma = 5.8\times10^{7}$ S/m.

Sol.ⁿ Given, radius of copper wire $(r) = 1mm = 0.001m$

length of copper wire $(l) = 1km = 1000m$.

frequency $(f) = 1MHz = 10^{6}$ Hz

then, $R_{dc} = \frac{l}{\sigma S} = \frac{1000}{5.8\times10^{7}\times3.14\times(0.001)^{2}} = 5.49\ \Omega$

At 1MHz the skin depth of copper is,

$\delta_{cu} = \frac{1}{\sqrt{\pi f \mu \sigma}} = 0.0661mm << a = 1mm$, therefore the resistance of wire changes due to skin effect at high freqⁿ

∴ At 1MHz, $R_{ac} = \frac{l}{2\pi a \sigma \delta} = \frac{1000}{2\pi\times0.001\times5.8\times10^{7}\times0.0661\times10^{-3}} = 41.5\ \Omega$

Qn: A uniform plane wave in non-magnetic medium has $\vec{E} = 50\cos(10^8 t + 2z)\hat{a}_y$ V/m. Find :

(i) The direction of propagation.

(ii) Phase constant, β, wavelength, λ velocity, v_p, relative permitivity, ε_r, intrinsic impedance, η.

(iii) $\vec{H}$.

[2073 Shrawan]

Sol.ⁿ Given,

$$\vec{E} = 50\cos(10^8 t + 2z)\hat{a}_y \text{ V/m}$$ is in non-magnetic medium.

(i) Since the phase varies with z values, the wave propagates in z-dirⁿ. [-ve z-dirⁿ as it is $\cos(\omega t + \beta z)$]

(ii) we have, $\beta = 2$ comparing with $\vec{E} = E_0 \cos(\omega t + \beta z)\hat{a}_y$

& $\omega = 10^8$ rad/s.

so, $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2} = 3.142\text{m}$.

Also, $v_p = \frac{\omega}{\beta} = \frac{10^8}{2} = 5\times10^7$ m/s.

And, $\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_r}$ [$\because \mu_r = 1$, non-magnetic]

or, $2 = 10^8\sqrt{\mu_0\varepsilon_0}\cdot\sqrt{\varepsilon_r}$

or, $2 = \frac{10^8}{c}\sqrt{\varepsilon_r}$

or, $\sqrt{\varepsilon_r} = \frac{3\times2\times10^8}{10^8} = 6$

or, $\varepsilon_r = 36$

Again,

Intrinsic impedance, $\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}} = \frac{1}{6} \times 377 = 62.83\,\Omega$

(iii) Now,

As wave is propagating in $-z$ dir^n & $\vec{E}$ is in $+\hat{a}_y$ dir^n the dir^n of $\vec{H}$ will be $\hat{a}_p = \hat{a}_E \times \hat{a}_H \Rightarrow -\hat{a}_z = \hat{a}_y \times \hat{a}_H$

So, $\hat{a}_H$ must be $\hat{a}_x$ to satisfy $-\hat{a}_z = \hat{a}_y \times \hat{a}_x$.

Now, $\vec{H} = \frac{E_0}{|\eta|} \cos(10^8 t + 2z + \theta_\eta)\, \hat{a}_x$ A/m.

Since, $|\eta| = 62.83\,\Omega$ & $\theta_\eta = 0$; $E_0 = 50$

$\vec{H} = \frac{\hat{a}_p \times \vec{E}}{\eta}$

$\vec{H} = \frac{50}{62.83} \cos(10^8 t + 2z)\, \hat{a}_x$ A/m.

$= 0.7958 \cos(10^8 t + 2z)\, \hat{a}_x$ A/m.

Qn: A lossless dielectric material has $\sigma = 0$, $\mu_r = 1$, $\varepsilon_r = 4$. An electromagnetic wave has magnetic field expressed as: $\vec{H} = -0.1 \cos(\omega t - z)\,\hat{a}_x + 0.5 \sin(\omega t - z)\,\hat{a}_y$ A/m. Find:

ⓐ Angular frequency (ω)

ⓑ ~~wavel~~ impedance (η)

ⓒ $\vec{E}$.

[2072 Chaitra]

Sol.^{n} Given, A lossless dielectric material has $\sigma = 0$, $\mu_r = 1$, $\varepsilon_r = 4$ & $\vec{H} = -0.1 \cos(\omega t - z)\,\hat{a}_x + 0.5 \sin(\omega t - z)\,\hat{a}_y$ A/m.

ⓐ Here, Comparing with $\vec{H} = H_{01} \cos(\omega t - \beta z)\,\hat{a}_x + H_{02} \sin(\omega t - \beta z)\,\hat{a}_y$ A/m ; $\beta = 1$

So, $\beta = \omega\sqrt{\mu\varepsilon}$ or, $1 = \omega\sqrt{\mu\varepsilon}$

or, $\omega = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\cdot\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{1\times4}} = \frac{3\times10^8}{2} = 1.5\times10^8$ rad/s.

Let,
$\vec{H} = -0.1\cos(\omega t - z)\hat{a}_x + 0.5\sin(\omega t - z)\hat{a}_y$ A/m.

$= \vec{H}_1 + \vec{H}_2$

$$\boxed{\vec{E} = (\vec{H}\times\hat{a}_p)\cdot\eta}$$

where, $\vec{H}_1 = -0.1\cos(\omega t - z)\hat{a}_x$

$\vec{H}_2 = 0.5\sin(\omega t - z)\hat{a}_y$

And corresponding electric fields will be

$$\boxed{\begin{aligned}\hat{a}_H &= \hat{a}_p\times\hat{a}_E \\ \vec{H} &= \frac{\hat{a}_p\times\vec{E}}{\eta}\end{aligned}}$$

$\vec{E} = \vec{E}_1 + \vec{E}_2$

where, $\vec{E}_1 = E_{10}\cos(\omega t - z)\hat{a}_{E_1}$

$\vec{E}_2 = E_{20}\sin(\omega t - z)\hat{a}_{E_2}$

since, $\hat{a}_{p_1} = \hat{a}_{E_1}\times\hat{a}_{H_1}$

or, $\hat{a}_z = \hat{a}_{E_1}\times(\hat{a}_x)$

Then, $\hat{a}_{E_1}$ must be $\hat{a}_y$

$\hat{a}_{p_2} = \hat{a}_{E_2}\times\hat{a}_{H_2}$

or, $\hat{a}_z = \hat{a}_{E_2}\times\hat{a}_y$

Then, $\hat{a}_{E_2}$ must be $\hat{a}_x$

$\therefore\ \vec{E}_1 = E_{10}\cos(\omega t - z)\hat{a}_y$ & $\vec{E}_2 = E_{20}\sin(\omega t - z)\hat{a}_x$

(b) the wave impedance $(\eta) = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\cdot\sqrt{\frac{\mu_r}{\varepsilon_r}}$

$= 120\pi\times\sqrt{\frac{1}{4}}$

$= 60\pi\ \Omega$

$\therefore\ E_{10} = \eta * H_{10} = 60\pi\times(0.1) = 6\pi$ V/m

& $E_{20} = \eta\times H_{20} = 60\pi\times0.5 = 30\pi$ V/m

Now

(c) $\therefore\ \vec{E} = \vec{E}_1 + \vec{E}_2 = 6\pi\cos(\omega t - z)\hat{a}_y + 30\pi\sin(\omega t - z)\hat{a}_x$ V/m

$= 94.25\,\cancel{\cos}\sin(1.5\times10^8t - z)\hat{a}_x + 18.85\cos(1.5\times10^8t - z)\hat{a}_y$ V/m

Qn: The magnetic field intensity ($\vec{H}$) in free space is given as: $\vec{H}(x,t) = 10\cos(10^8 t + \beta x)\,\hat{a}_y$ A/m. Find:

(a) Phase constant (β)

(b) wavelength (λ)

(c) $|\vec{E}(x,t)|$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ ns.

[2072 Kartik]

रविन ... इन्जिनियर

Sol.ⁿ

Given, $\vec{H}(x,t) = 10\cos(10^8 t + \beta x)\,\hat{a}_y$ A/m. in free space

Since this is a uniform plane wave, $\omega = 10^8$ rad/s.

and $c = 3\times 10^8$ m/s.

(a) $\therefore \beta = \omega/c = \dfrac{10^8}{3\times 10^8} = 0.33$ rad/m.

(b) $\lambda = \dfrac{2\pi}{\beta} = 18.9$ m.

(c) Now, $\vec{E}(x,t) = \eta_0 \vec{H}(x,t)\,\hat{a}_E$

Since, $\hat{a}_p$ (propagation) $= \hat{a}_E \times \hat{a}_H$

or, $-\hat{a}_x = \hat{a}_E \times \hat{a}_y$

then, $\hat{a}_E = \hat{a}_z$

$\therefore \vec{E}(x,t) = \eta_0 \vec{H}(x,t)\,\hat{a}_z$

$= 377 \times 10 \cos(10^8 t + 0.33x)\,\hat{a}_z$

$= 3.77\times 10^3 \cos(10^8 t + 0.33x)\,\hat{a}_z$

& $|\vec{E}(x,t)|$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ns is

$|\vec{E}(x,t)| = 3.77\times 10^3 \cos(10^8 \times 10^{-9} + 0.33\times 0.1)$

$= 3.736 \times 10^3$ V/m.

Qn: A 9.4GHz uniform plane wave is propagating in a medium with $\varepsilon_r = 2.25$ and $\mu_r = 1$. If the magnetic field intensity is 7mA/m and the material is lossless find: (i) velocity of propagation (ii) The wavelength (iii) Phase Constant (iv) Intrinsic impedance (v) Magnitude of electric field intensity. [2074 Ashwin]

Soln

(iii) $\beta = ?$, $f = 9.4 \times 10^9$ Hz, $\varepsilon_r = 2.25$, $\mu_r = 1$

$|\vec{H}| = 7$ mA/m.

then,

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$\text{or, } \beta = \omega\sqrt{\mu_0\varepsilon_0} \times \sqrt{\mu_r\varepsilon_r} = \frac{2\pi \times 9.4 \times 10^9}{3 \times 10^8} \times \sqrt{2.25}$$

$$= 295.31 \text{ rad/m.}$$

Again,

(i)

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\cdot\sqrt{\mu_r\varepsilon_r}} = \frac{c}{\sqrt{2.25}} = 2 \times 10^8 \text{ m/s.}$$

(ii)

$$\lambda = \frac{v}{f} = \frac{2 \times 10^8}{9.4 \times 10^9} = 0.0213 \text{ m.}$$

(iv)

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \sqrt{\frac{\mu_r}{\varepsilon_r}} = 377 \times \sqrt{2.25} = 565.5\,\Omega$$

(v) $|\vec{H}| = 7$ mA/m

$$|\vec{E}| = |\vec{H}| \times \eta = 7 \times 10^{-3} \times 565.5 = 3.9585 \text{ V/m.}$$

Qn: A uniform plane wave in free space at a frequency 12 MHz is given by $\vec{E} = 200\cos(\omega t + 120x + 30^\circ)\,\hat{a}_y$ V/m. find (a) $|E_{max}|$ (b) $\vec{H}$ at $x = 40$ mm and $t = 340$ ps.

[2071 Chaitra]

Sol.^{n} Given, $\vec{E} = 200\cos(\omega t + 120x + 30^\circ)\,\hat{a}_y$ V/m.

$f = 12$ MHz $= 12\times10^6$ Hz

(a) For $\vec{E}$ to be maximum $\cos(\omega t + 120x + 30^\circ) = 1$ must be occured. So, $|E_{max}| = 200$ V/m.

(b) Again, $\vec{H} = \dfrac{\vec{E}}{\eta_0}\,\hat{a}_H$

where, $\eta_0 = 377\,\Omega$ [for free space]

$\hat{a}_p = \hat{a}_E \times \hat{a}_H$

or, $-\hat{a}_x = \hat{a}_y \times \hat{a}_H$

$\therefore \hat{a}_H = -\hat{a}_z$

$\therefore \vec{H} = -\dfrac{\vec{E}}{\eta_0}\,\hat{a}_z = -\dfrac{200}{377}\cos(\omega t + 120x + 30^\circ)\,\hat{a}_z$ A/m.

& $\omega = 2\pi\times f = 2\pi\times12\times10^6$

$= 75.398$ rad/s. $\times10^6$

so, $\vec{H}$ at $x = 40$ mm $= 0.04$ m & $t = 340$ ps $= 3.4\times10^{-10}$ s

if $\vec{H} = -\dfrac{200}{377}\cos\left(75.398\times10^6\times3.4\times10^{-10} + 120\times0.04 + 30\times\dfrac{\pi}{180}\right)\hat{a}_z$

$= -0.5305\times\cos(0.02563 + 4.8 + 0.5236)\,\hat{a}_z$

$= -0.3155\,\hat{a}_z$ A/m.

Qn: A time harmonic uniform plane wave $\vec{E}(x,y,z,t)$ with polarization in $\hat{a}_z$ direction and frequency 150 MHz is moving in free space in negative y-direction and has maximum amplitude 2V/m. Determine (a) The angular frequency, ω (b) Phase constant, β (c) Expression for $\vec{E}(x,y,z,t)$ and (d) Expression for $\vec{H}(x,y,z,t)$. [2068 Shrawan]

Soln: Given,

frequency of wave, $f = 150\ MHz = 15\times10^{7}\ Hz$

Maxm amplitude of $\vec{E}(x,y,z,t) = |\vec{E}(x,y,z,t)|_{max} = 2V/m$.

Since, the wave is polarized in $\hat{a}_z$ dirn the $\vec{E}(x,y,z,t)$ varies along +ve z-direction and propagation is along −ve y-direction.

(a) angular freqn, $\omega = 2\pi \times f = 2\pi\times15\times10^{7} = 94.25\times10^{7}$ rad/s

(b) Phase constant, $\beta = \frac{\omega}{c} = \frac{94.25\times10^{7}}{3\times10^{8}} = 3.142$ rad/m

(c) The Expression $\vec{E}(x,y,z,t)$ is

$$\vec{E}(x,y,z,t) = |\vec{E}(x,y,z,t)|_{max}\cos(\omega t+\beta y)\,\hat{a}_z$$
$$= 2\cos(94.25\times10^{7}t + 3.142y)\,\hat{a}_z$$

(d) The expression $\vec{H}(x,y,z,t)$ is

$$\vec{H}(x,y,z,t) = \frac{\vec{E}(x,y,z,t)}{\eta_0}\,\hat{a}_H$$

where, $\eta_0 = 120\pi$ or 377Ω

and $\hat{a}_p = \hat{a}_E \times \hat{a}_H$

or, $-\hat{a}_y = \hat{a}_z \times \hat{a}_H$

or, $-\hat{a}_y = \hat{a}_z \times (-\hat{a}_x)$

$\Rightarrow \hat{a}_H = -\hat{a}_x$

$\therefore \vec{H}(x,y,z,t) = -\frac{2}{377}\cos(94.25\times10^{7}t + 3.142y)\,\hat{a}_x$ A/m.

$= -5.305\times10^{-3}\cos(94.25\times10^{7}t + 3.142y)\,\hat{a}_x$ A/m.

Q: An Electric field $\vec{E}$ in free space is given as:
$\vec{E} = 200\cos(10^8 t - \beta y)\,\hat{a}_z$ V/m. Find:
(a) Phase constant (β); [2070 Ashad]
(b) Wavelength (λ);
(c) Magnetic field intensity $\vec{H}$ at $P(-0.1, 1.5, -0.4)$ at $t = 4$ ns.

Sol^n: Given, $\vec{E} = 200\cos(10^8 t - \beta y)\,\hat{a}_z$ V/m in free space

Comparing with $\vec{E} = E_0 \cos(\omega t - \beta y)\,\hat{a}_z$ V/m

$\omega = 10^8$ rad/s.

and velocity, $c = 3\times10^8$ m/s

(a) So, phase constant, $\beta = \frac{\omega}{c}$ or $\beta = \omega\sqrt{\mu_0\varepsilon_0}$
$= \frac{1\times10^8}{3\times10^8}$
$= 0.33$ rad/m.

Again,
(b) wavelength, $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.33} = 18.83$ m

(c) $\vec{H} = \frac{\vec{E}}{\eta_0}\,\hat{a}_H$

where, $\hat{a}_p = \hat{a}_E \times \hat{a}_H$ & $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\,\Omega$

or, $\hat{a}_y = \hat{a}_z \times \hat{a}_x$

$\Rightarrow \hat{a}_H = \hat{a}_x$

$\therefore \vec{H} = \frac{200\cos(10^8 t - 0.33y)}{377}\,\hat{a}_x$

$= 0.5305\cos(10^8 t - 0.33y)\,\hat{a}_x$

& $\vec{H}$ at $P(-0.1, 1.5, -0.4)$ at $t = 4$ns $= 0.5305\times\cos(10^8\times4\times10^{-9} - 0.33\times1.5)\,\hat{a}_x$

$= 0.5281\,\hat{a}_x$ A/m

Qn: Find the reflection co-efficient for the interface betn air and fresh water ($\epsilon = 81\epsilon_0, \sigma \cong 0$) in case of normal incidence.

[2072 Kartik]

Soln:

Given,

For water, $\epsilon = 81\epsilon_0$ & $\sigma \cong 0$

Also, for water, $\mu = \mu_0$

Since, the incidence is normal

reflection co-efficient, $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$

where, $\eta_2 = \sqrt{\dfrac{\mu}{\epsilon}} = \sqrt{\dfrac{\mu_0}{\epsilon_0}} \cdot \sqrt{\dfrac{1}{81}} = \dfrac{377}{9}\ \Omega$

& $\eta_1 = \eta_0 = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = 377\ \Omega$.

So, $\Gamma = \dfrac{\frac{377}{9} - 377}{\frac{377}{9} + 377} = \dfrac{-8}{10} = -0.8$ //

Qn: The electric field amplitude of a uniform plane wave propagating in the free space in $\hat{a}_z$ dirn is 250 V/m. If $\vec{E} = E_x \hat{a}_y$ and $\omega = 1$ M rad/s, find: ⓐ the frequency ⓑ the wavelength ⓒ the period ⓓ the amplitude of $\vec{H}$. [2071 Shrawan]

[see in page 285]

Qn: A time harmonic uniform plane wave $\vec{E}(x,y,z,t)$ with polarization in $\hat{a}_x$ direction, and freqn 150 MHz is moving in free space in −ve y-dirn and has maxm amplitude 2 V/m. Determine ⓐ angular freqn (ω) ⓑ phase constant β ⓒ Expression for $\vec{E}(x,y,z,t)$ and ⓓ Expression for $\vec{H}(x,y,z,t)$. [2068 Shrawan]

Soln: ⓐ $\omega = 2\pi f$ ⓑ $\beta = \frac{\omega}{c}$ ⓒ $\vec{E}(x,y,z,t) = |E_{max}| \cos(\omega t - \beta y)\hat{a}_x$

ⓓ $\vec{H}(x,y,z,t) = \frac{|E_{max}|}{\eta_0} \cos(\omega t - \beta y)\hat{a}_z$

Qn: A plane wave in nonmagnetic medium has $\vec{E} = 50\sin(10^8 t + 2z)\hat{a}_y$ V/m. Find ⓐ The dirn of wave propagation. ⓑ λ, f & ϵ_r ⓒ $\vec{H}$.

Qn: A 9.375 GHz uniform plane wave is propagating in polyethylene ($\epsilon_r = 2.26$, $\sigma/\omega\epsilon = 0.0002$) [$\mu_r = 1$]. If the amplitude of the electric field intensity is 500 V/m and the material is assumed to be lossless, find; ⓐ the phase constant ⓑ the wavelength in the polyethylene ⓒ the velocity of propagation ⓓ the intrinsic impedance ⓔ the amplitude of the magnetic field intensity.

Ans: 295 rad/m, 2.13 cm, 1.99×10^8 m/s, 251 Ω, 1.99 A/m.

Soln: ⓐ $\beta = \omega\sqrt{\mu\epsilon}$ ⓑ $\lambda = \frac{2\pi}{\beta}$ ⓓ $\eta = \sqrt{\frac{\mu}{\epsilon}}$ ⓒ $v = \lambda \times f$

ⓔ $|\vec{H}| = \frac{|\vec{E}|}{\eta} = \frac{500}{251} = 1.99$ A/m.

Qn: An EM wave travels in free space with the electric field component $\vec{E} = (15\hat{a}_y - 5\hat{a}_z)\cos(\omega t - 3y + 5z)$ V/m. Find (a) ω and λ (b) the magnetic field component.
[2073 Chaitra]

Sol.^{n}

$$\vec{K}_i = K_x\hat{a}_x + K_y\hat{a}_y + K_z\hat{a}_z$$
$$= 0\hat{a}_x + 3\hat{a}_y - 5\hat{a}_z$$
$$= 3\hat{a}_y - 5\hat{a}_z$$
$$\beta = |\vec{K}_i| = \sqrt{3^2 + (-5)^2} = \sqrt{34}$$

(a) For free space

$$\beta = \omega\sqrt{\mu_0\varepsilon_0}$$

$$\& \ \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{34}} = 1.077 \text{ m}.$$

or, $\sqrt{34} = \dfrac{\omega}{c}$

or, $\omega = 3\times10^8 \times \sqrt{34} = 1.75\times10^9$ rad/s.

(b) For magnetic field component, we have

$$\vec{H} = \frac{\hat{a}_{Ki} \times \vec{E}}{\eta_0} \quad \text{or} \quad \frac{\vec{K}_i \times \vec{E}}{\omega\mu_0}.$$

$$= \frac{(3\hat{a}_y - 5\hat{a}_z)}{\sqrt{34}\times377} \times (15\hat{a}_y - 5\hat{a}_z)\cos(\omega t - 3y + 5z)$$

$$= \frac{(-15\hat{a}_x + 75\hat{a}_x)}{\sqrt{34}\times377}\cos(\omega t - 3y + 5z)$$

$$= 0.0227\hat{a}_x\cos(\omega t - 3y + 5z) \text{ A/m}.$$

$$= 0.0227\hat{a}_x\cos(1.75\times10^9 t - 3y + 5z) \text{ A/m}.$$

Qn: An EM wave travels in free space with the electric field component $\vec{E} = (10\hat{a}_y + 5\hat{a}_z)\cos(\omega t + 2y - 4z)$ V/m. Find (a) ω & λ (b) the magnetic field component (c) the time average power in the wave. [2069 chaitra].

रविन कासुला
ईन्जिनियर

Sol.ⁿ

Given, $\vec{E} = (10\hat{a}_y + 5\hat{a}_z)\cos(\omega t + 2y - 4z)$ V/m.

From the incident $\vec{E}$, it is seen that the propagation vector is

$$\vec{k}_i = -2\hat{a}_y + 4\hat{a}_z$$

So, $k_i = |\vec{k}_i| = \sqrt{2^2 + 4^2} = \sqrt{20} = \beta$

(a) Angular velocity, ω can be calculated as:

$$\beta = k_i = \omega\sqrt{\mu_0 \epsilon_0}$$

or, $\omega = k_i c = \sqrt{20} \times 3 \times 10^8 = 1.342 \times 10^9$ rad/s.

and $\lambda = \dfrac{2\pi}{k_i} = \dfrac{2\pi}{\sqrt{20}} = 1.404$ m

(b) the magnetic field component can be calculated as:

$\vec{H} = \dfrac{\hat{a}_{k_i} \times \vec{E}}{\eta_0}$ or $\dfrac{\vec{k}_i \times \vec{E}}{\omega \mu_0} \rightarrow$ time domain,

$= \dfrac{(-2\hat{a}_y + 4\hat{a}_z) \times (10\hat{a}_y + 5\hat{a}_z)}{\sqrt{20} \times 377} \cos(\omega t + 2y - 4z)$

$= -29.66\cos(1.342 \times 10^9 t + 2y - 4z)\,\hat{a}_x \times 10^{-3}$ A/m.

$\vec{H} = \dfrac{\vec{k}_i \times \vec{E}}{\omega \mu_0} = \dfrac{(-2\hat{a}_y + 4\hat{a}_z) \times (10\hat{a}_y + 5\hat{a}_z)}{1.342 \times 10^9 \times 4\pi \times 10^{-7}} \cos(\omega t + 2y - 4z)$

$= \dfrac{-10\hat{a}_x - 40\hat{a}_x}{1686.4069} \cos(\omega t + 2y - 4z)$

$= -0.0296\cos(\omega t + 2y - 4z)$

(c) the time average power in the wave is

$$P_{avg} = \frac{|E_0|^2}{2\eta_0}\hat{a}_{k_i} = \frac{125}{2(120\pi)} \frac{(-2\hat{a}_y + 4\hat{a}_z)}{\sqrt{20}} = -74.15\hat{a}_y + 148.9\hat{a}_z \text{ mW/m}^2$$

(304)

Qn: An EM wave travels in free space with electric field component $\vec{E_s} = 100\, e^{j(0.866y+0.5z)}\,\hat{a}_x$ V/m.

Determine:

(a) ω & λ (b) The magnetic field component (c) The time average power in the wave.

Sol.ⁿ Comparing with $\vec{E_s} = E_0 e^{j\vec{k}\cdot\vec{r}} = E_0 e^{j(k_x x + k_y y + k_z z)}\,\hat{a}_x$

$k_x = 0,\ k_y = 0.866$ & $k_z = 0.5$

So, $k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{0.866^2 + 0.5^2} = 1$

But in free space

(a) $k = \beta = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

So, $\omega = kc = 3\times10^8$ rad/s.

& $\lambda = \frac{2\pi}{k} = 2\pi = 6.283$ m

(b) The magnetic field corresponding is given by

$\vec{H_s} = \frac{1}{\mu\omega}\vec{k}\times\vec{E_s}$ ⟶ phasor form.

$= \frac{(0.866\hat{a}_y + 0.5\hat{a}_z)}{4\pi\times10^{-7}\times3\times10^8} \times 100\, e^{j\vec{k}\cdot\vec{r}}\,\hat{a}_x$

$= (1.33\hat{a}_y - 2.3\hat{a}_z)\, e^{j(0.866y+0.5z)} \times 10^{-2}$ A/m

(c) The time average power is

$P_{avg} = \frac{1}{2}\mathrm{Re}\{\vec{E_s}\times\vec{H_s}^*\} = \frac{E_0^2}{2\eta_0}\hat{a}_k$

$= \frac{100^2}{2(120\pi)}(0.866\hat{a}_y + 0.5\hat{a}_z)$

$= 11.49\,\hat{a}_y + 6.631\,\hat{a}_z$ mW/m²