# Section 7.5: Iterated Rounding

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# Limitation of Threshold Rounding

- Problems like MIN-WVC have an optimal solution where all the clauses have at least one variable $\geq 1/2$.
- What if there are not enough variables taking values at least $1/2$ in the optimal fractional solution?

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- Problems like MIN-WVC have an optimal solution where all the clauses have at least one variable $\geq 1/2$.
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**Solution: Partial Rounding with Residual Linear Program**

# Iterated Rounding

***Basic Idea:***
Suppose the fractional optimal solution of the residual linear program always contains a component of value greater than or equal to $1/2$.
In that case, we can continue this rounding process and eventually obtain a feasible integer solution that is still a 2-approximation.

# Generalized Spanning Network(GSN)

**Problem Statement:**
Given a graph $G = (V, E)$ with edge costs $c_e$ for each edge $e \in E$, and an integer $k > 0$, we aim to find a $k$-edge-connected subgraph $F$ that minimizes the total edge cost.

**Subgraph:**
A subgraph $F$ is $k$-edge-connected if, for every partition $(S, V - S)$ of the vertex set $V$ of $G$, there are at least $k$ edges in $F$ between $S$ and $V - S$.

# Generalized Spanning Network(GSN)

Based on all the aforementioned concepts, the problem can be formulated as the following Integer Linear Program (ILP):

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta_G(S)} x_e \geq k, \quad \emptyset \neq S \subset V, \\ & x_e \in \{0, 1\}, \quad e \in E, \end{aligned}$$

where $\delta_G(S)$ denotes the set of edges with exactly one endpoint in $S$.

# Generalized Spanning Network(GSN)

The LP relaxation of this ILP is as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{e \in E} c_e x_e \\
\text{subject to} \quad & \sum_{e \in \delta_G(S)} x_e \geq k, \quad \emptyset \neq S \subset V, \\
& 0 \leq x_e \leq 1, \quad e \in E.
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**Perk of this algorithm:**

Can be solved in polynomial time in $|V|$ although having more than $2^{|V|}$ constraints.

# Necessary Concepts

**Ellipsoid Method:**

- The feasible region is enclosed within an ellipsoid in high-dimensional space.
- At each step, the algorithm checks if the center of the ellipsoid satisfies all constraints.
- If not, it uses a violated constraint to refine the ellipsoid, reducing its size while ensuring it still contains the feasible region.
- The ellipsoid shrinks iteratively until the center satisfies all constraints.

# Necessary Concepts-Continued

**Separation Oracle:**

- Given a candidate solution $x$, the oracle determines whether $x$ satisfies all constraints.
- If $x$ violates any constraint, the oracle returns that linear inequality to refine the ellipsoid.

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### Lemma

$$\Omega = \{x \mid Ax = b,\, x \geq 0\}.$$

If $\min_{x\in\Omega}(cx)$ has an optimal solution, then it can be found at one of its vertices.

# Key Concept

Mapping our problem to Max-Flow Min-Cut Problem.

**Why?**
To construct separation oracles in polynomial time.

## Problem Formulation

- Assign each edge $e$, a capacity $x_e$.

  Then $x$ is feasible if and only if, for every two nodes $s, t$ of graph $G$, the maximum flow from $s$ to $t$ is at least $k$.

# Feasibility in Flow Networks

- Considering the constraint: $\sum_{e \in \delta_G(S)} x_e < k$
- The minimum cut $(S, V - S)$ can be found in polynomial time with respect to $|V|$
- If we make a partial assignment to the variables of the LP , the residual LP remains polynomial-time solvable with respect to $|V|$.

## Residual LP Feasibility

Suppose for $e \in F \subset E$, $x_e$ is already assigned value $u_e$.

If the assignment $(x_e)_{e \in E-F}$ is not feasible for the residual LP, then $(x_e)_{e \in E-F}$, together with $(u_e)_{e \in F}$, forms an infeasible assignment for the original LP.

# Extending Supmodular Funtions

**Weakly Supmodular Function:**

A function $f : 2^V \to \mathbb{Z}$ is weakly supmodular if:

(a) $f(V) = 0$,

(b) For any two subsets $A, B \subseteq V$, either
$f(A) + f(B) \leq f(A \setminus B) + f(B \setminus A)$ or
$f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$.

# Understanding Lemma 7.21

**Statement of Lemma 7.21:**

- If $f : 2^V \to \mathbb{Z}$ is a weakly supmodular function, then for the linear program (LP) defined as:

$$\text{minimize} \sum_{e \in E} c_e x_e$$

subject to:

$$\sum_{e \in \delta_G(S)} x_e \geq f(S), \quad S \subseteq V,$$
$$0 \leq x_e \leq 1, \quad e \in E,$$

every basic feasible solution $x$ contains at least one component $x_e \geq \frac{1}{3}$.

# Additional Steps

Variables $x_e$ that are at least $\frac{1}{3}$ can be rounded to 1.

**After Rounding:**

$$\text{minimize} \quad \sum_{e \in E-F} c_e x_e$$

subject to:

$$\sum_{e \in \delta_{G-F}(S)} x_e \geq f(S) - |\delta_F(S)|, \quad S \subseteq V,$$
$$0 \leq x_e \leq 1, \quad e \in E.$$

- The function $f(S) - |\delta_F(S)|$ remains weakly supmodular.

# GSN

**Algorithm 1** Iterated Rounding Algorithm for GSN

**Require:** A graph $G = (V, E)$ with an edge-cost function $c : E \to \mathbb{Q}^+$, and an integer $k > 0$.

Construct an LP with $f(S)$ defined previously and $F = \emptyset$.<br>
**while** $F$ is not $k$-edge-connected **do**<br>
    Find an optimal basic feasible solution $x^*$ of the LP.<br>
    **for** each edge $e \in E$ **do**<br>
        **if** $x_e^* \geq \frac{1}{3}$ **then**<br>
            Set $x_e = 1$ and add edge $e$ to the set $F$.<br>
        **end if**<br>
    **end for**<br>
    Update the residual LP based on the edges in $F$.<br>
**end while**

# Theorem 7.22: 3-Approximation for GSN

**Formulation:**

> Suppose $F$ is the output obtained from Algorithm through $t$ iterations.
>
> let $F_i$ be the set of edges added to $F$ in the first $i$ iterations, thus, $F = F_t$.
>
> $F^i = E - F_i$. Also, let's $x_i$ denote the optimal fractional solution for $F = F_i$.

Thus, under the condition that $x_e = 1$ for $e \in F_i$, $x_i$ is a better solution to the residual LP than any other solution, including $x_{i-1}$.

# 3-Approximation for GSN-Continued

So this follows:

$$\begin{aligned}
\sum_{e \in F} c_e &\leq \sum_{e \in F_{t-1}} c_e + 3 \sum_{e \in F^i_{t-1}} c_e x_{t-1,e} \\
&\leq \sum_{e \in F_{t-1}} c_e + 3 \sum_{e \in F^i_{t-1}} c_e x_{t-2,e} \\
&\leq \sum_{e \in F_{t-2}} c_e + 3 \sum_{e \in F^i_{t-2}} c_e x_{t-2,e} \\
&\leq \cdots \leq 3 \sum_{e \in E} c_e x_{0,e} \leq 3 \cdot \mathrm{opt},
\end{aligned}$$

where opt is the value of the optimal integer solution for $F = \emptyset$.

# Lemma 7.23

- **Given:** A basic feasible solution $x$ of a linear program such that $0 < x_e < 1$ for all $e \in E$.
- **Goal:** Prove the existence of a laminar family $F$ of active sets satisfying:
  1. $|F| = |E|$,
  2. The vectors $a_S$ for all $S \in F$ are linearly independent,
  3. $f(S) \geq 1$ for all $S \in F$.

# Key Concepts

- **Laminar Family:** A collection $L$ of sets such that any two sets $A, B \in L$ are either disjoint or one is contained in the other.
- **Active Set:** A set $S \subseteq V$ such that $f(S) = \sum_{e \in E(S, V \setminus S)} x_e$ satisfies a constraint. That means, for inequality ,$x_e$ gives a tight bound .

# Proof Outline

1. Start with a maximal laminar family $L$ of active sets.
2. Show that $\text{rank}(\{a_S \mid S \in L\}) = |E|$. [**Focus Point**]
3. Assume, for contradiction, that $\text{rank}(\{a_S \mid S \in L\}) < |E|$.
4. Use weak submodularity of $f$ to derive the contradiction.

# Contradiction Setup

- Assume $\mathrm{rank}(\{a_S \mid S \in L\}) < |E|$.
- There exists an active set $A$ such that $a_A \notin \mathrm{Span}(\{a_S \mid S \in L\})$ .
- By maximality of $L$, $A$ must cross at least one set $B \in L$.
- We choose $A$ to be the active set that crosses the minimum number of sets in $L$.
- Decompose $A$ and $B$:

$$S_1 = A \setminus B, \quad S_2 = A \cap B, \quad S_3 = B \setminus A, \quad S_4 = V \setminus (A \cup B).$$

Where we consider the set $B \in L$, which is assumed to cross $A$ but not cross $S_1$.

# Using Weak Submodularity

Assume:

$$f(A) + f(B) \le f(A \setminus B) + f(B \setminus A).$$

Define:

$$m_{i,j} = \sum_{e \in E(S_i, S_j)} x_e.$$

Since $A$ and $B$ are both active, we have

$$f(A) = m_{1,3} + m_{1,4} + m_{2,3} + m_{2,4},$$

$$f(B) = m_{1,2} + m_{1,3} + m_{2,4} + m_{3,4}.$$

# Continued.

Moreover, for constraints $S_1$ and $S_3$, we have

$$f(S_1) \leq m_{1,2} + m_{1,3} + m_{1,4},$$

$$f(S_3) \leq m_{1,3} + m_{2,3} + m_{3,4}.$$

Thus,

$$f(S_1) + f(S_3) + 2m_{2,4} \leq f(A) + f(B).$$

By active constraints:

$$f(A) + f(B) = f(S_1) + f(S_3),$$

which implies $S_1$ and $S_3$ are active.

# Deriving the Contradiction

- It follows that:
$$a_A + a_B = a_{S_1} + a_{S_3}.$$
- Since $a_A \notin \mathrm{Span}(L)$ and $a_B \in \mathrm{Span}(L)$, either $a_{S_1}$ or $a_{S_3} \notin \mathrm{Span}(L)$.
- Sets crossing $S_1$ or $S_3$ must also cross $A$, contradicting the minimal crossing assumption for $A$.

# Case 1: $a_{S_1} \notin \mathrm{Span}(L)$

We begin by assuming that $a_{S_1}$ is not in the span of the set of vectors $\{a_S \mid S \in L\}$. The goal is to derive a contradiction from this assumption.

# Claim: Every Set $C \in L$ Crossing $S_1$ Must Also Cross $A$

- Let $C \in L$ be a set that crosses $S_1$, meaning:

$$S_1 \cap C \neq \emptyset \quad \text{and} \quad S_1 \setminus C \neq \emptyset.$$

- Since $A$ is a superset of $S_1$, we have:

$$A \cap C \neq \emptyset \quad \text{and} \quad A \setminus C \neq \emptyset.$$

- Hence, any set $C$ crossing $S_1$ must also cross $A$.

# Closing Points

- Since $B$ does not cross $S_1$, we have:

$$B \cap S_1 = \emptyset \quad \text{and} \quad B \setminus S_1 \neq \emptyset.$$

- This gives us a contradiction, as it implies $B$ crosses $A$ but does not cross $S_1$.

$$\text{Number of sets crossing } S_1 < \text{Number of sets crossing } A.$$

- This contradicts our assumption that $A$ was the set minimizing the number of crossings.

Similarly we can prove that our case 2 ,$a_{S_3} \notin \text{Span}(L)$ is also false.

# Lemma 7.24: Statement

**Lemma:** Suppose $x_e$ is fractional for every $e \in E$. Then the laminar family $F$ from Lemma 7.23 contains a set $S$ with $|\delta_G(S)| \leq 3$.

# Proof Overview

**Goal:** Prove that $F$ contains a set $S$ such that $|\delta_G(S)| \leq 3$.
**Assumption (for contradiction):** Suppose $|\delta_G(S)| \geq 4$ for every $S \in F$.
**Approach:**

1. Construct a **forest** $T$ over $F$.
2. Count the number of **endpoints** (vertex-edge pairs) contributed by sets in $F$.
3. Derive a contradiction based on the total number of endpoints.

# Step 1: Constructing the Forest $T$

**Forest Construction:** Each edge $(A, B)$ in $T$ satisfies:

- $A \supset B$.
- There is no intermediate set $C$ such that $A \supset C \supset B$.

**Properties of $T$:**

- $T$ respects the laminar structure of $F$.
- $T$ can be decomposed into subtrees.

**Endpoints Definition:**
For each edge $e$ in $G$, define its endpoints $(u, e)$:
- $u$ is an endpoint of $e$.
- Assign $(u, e)$ to a set $S \in F$ such that $u \in S$ but $u \notin S'$ for any proper subset $S' \subset S$

# Step 2: Counting Endpoints

**Leaf Sets:** Each leaf $S$ of $T$ satisfies $|\delta_G(S)| \geq 4$. Hence, $S$ contributes at least 4 endpoints.

**Subtree Endpoint Claim:** For any subtree $T'$ of $T$, the number of endpoints $|P(T')|$ satisfies:

$$|P(T')| \geq 2|V(T')| + 2,$$

where $V(T')$ is the set of nodes in $T'$.

**Proof by Induction:**

*Base Case:*

For a single leaf $S$, $|P(S)| \geq 4 = 2|V(T')| + 2$.

*Inductive Step:* For a subtree $T'$ with root $R$:

- If $R$ has $k$ children, apply the induction hypothesis to each child.
- Combine contributions from $R$ and its children to verify the claim.

# Step 3: Deriving a Contradiction

**Total Endpoints:**
- Total endpoints $|P| \geq 2|F| + 2 = 2|E| + 2$.
- However, each edge contributes at most 2 endpoints.Hence, the total number of endpoints in $E'$ is exactly $2|E|$.

**Contradiction:**
- The two calculations of endpoints contradict each other.
Therefore, the assumption $|\delta_G(S)| \geq 4$ for all $S \in F$ is false.
**Conclusion:** There exists a set $S \in F$ such that $|\delta_G(S)| \leq 3$.

Thank You.
Any Questions