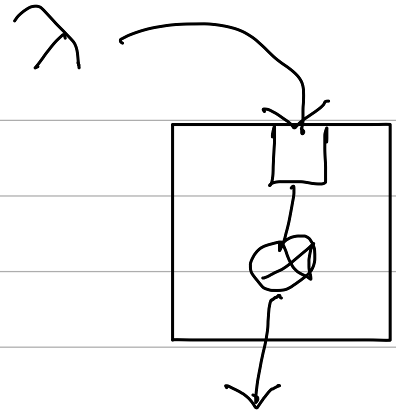
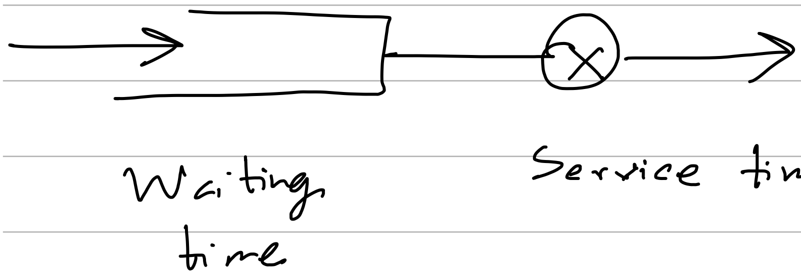
Queueing Theory:





L = Average # of customers in the system.

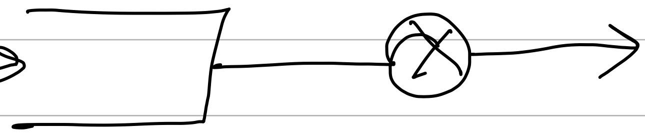
W = Avg amount of time a customer spends in the system.

L_Q = Avg # of customers in the queue.

W_Q = Avg amount of time a customer spends in the queue.

Single Server Exponential Queueing System

Poisson
Process
rate λ



time
exponential

Interarrival time is exp. distr. with

mean $\frac{1}{\lambda}$.

Service time is exp. dist. with

mean $\frac{1}{\mu}$

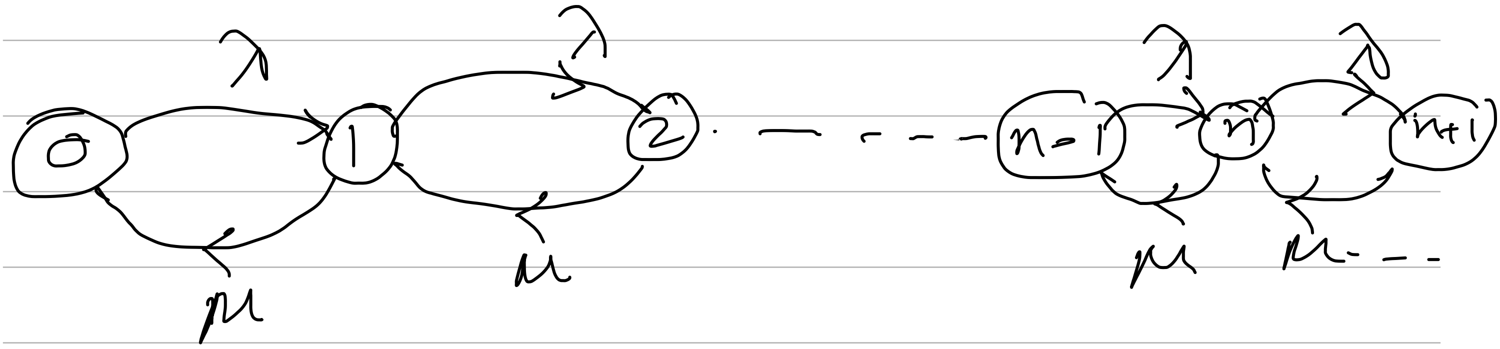
This systeme. $\longrightarrow$ M / M / 1

Interarrival time is Memoryless

Service time is also Memoryless

Single Server

M/M/1 Queue:



balance Equation

P_n = Prob that there are n customers in the system.

State: Rate at which process leaves = Rate at which process enters.

state 0 $\rightarrow$ $\lambda P_0 = \mu P_1$

state n $\rightarrow$ $\lambda P_n + \mu P_n = \lambda P_{n-1} + \mu P_{n+1}$

$n \geq 1$

$$\Rightarrow P_{n+1} = \frac{\lambda}{\mu} P_n + P_n - \frac{\lambda}{\mu} P_{n-1}$$

$$P_2 = \frac{\lambda}{\mu} P_1 + P_1 - \frac{\lambda}{\mu} P_0$$

$$= \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 + \left(P_2 - \frac{\lambda}{\mu} P_1\right) = \frac{\lambda}{\mu} P_2$$

$$= \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\therefore \boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n P_0}$$

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow \frac{1}{1 - \frac{\lambda}{\mu}} P_0 = 1 \qquad // \frac{\lambda}{\mu} < 1;$$

$$// \ \lambda < \mu$$

$$\Rightarrow \boxed{P_0 = 1 - \frac{\lambda}{\mu}}$$

$$\therefore \; P_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

Expected value

Average # of customer in the system

$$\sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n$$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1}$$

$// x = \frac{\lambda}{\mu}$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$// \; x < 1$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{d}{dx} \frac{1}{1-x}$$

$\boxed{x<1}$

$$= \frac{\lambda}{\mu}\left(1 - \frac{\lambda}{\mu}\right) \frac{1}{(1-x)^2},$$

$$= \frac{\lambda/\mu}{1 - \lambda/\mu} = \frac{\lambda}{\mu - \lambda}$$

$$L = \lambda W$$

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$$

$$W_Q = W - E[S]$$

$$= \frac{1}{\mu - \lambda} - \frac{1}{\mu}$$

$$= \frac{\lambda}{\mu(\mu - \lambda)}$$

$$L_Q = \lambda W_Q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

a

M/M/1 Queue with Finite Capacity N

$$\sum_{n=0}^{N} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n P_0 = 1$$

$$\Rightarrow P_0 \frac{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}{1-\frac{\lambda}{\mu}} = 1$$

$$P_0 = \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$$

there is no cond^n on $\frac{\lambda}{\mu} < 1$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$$

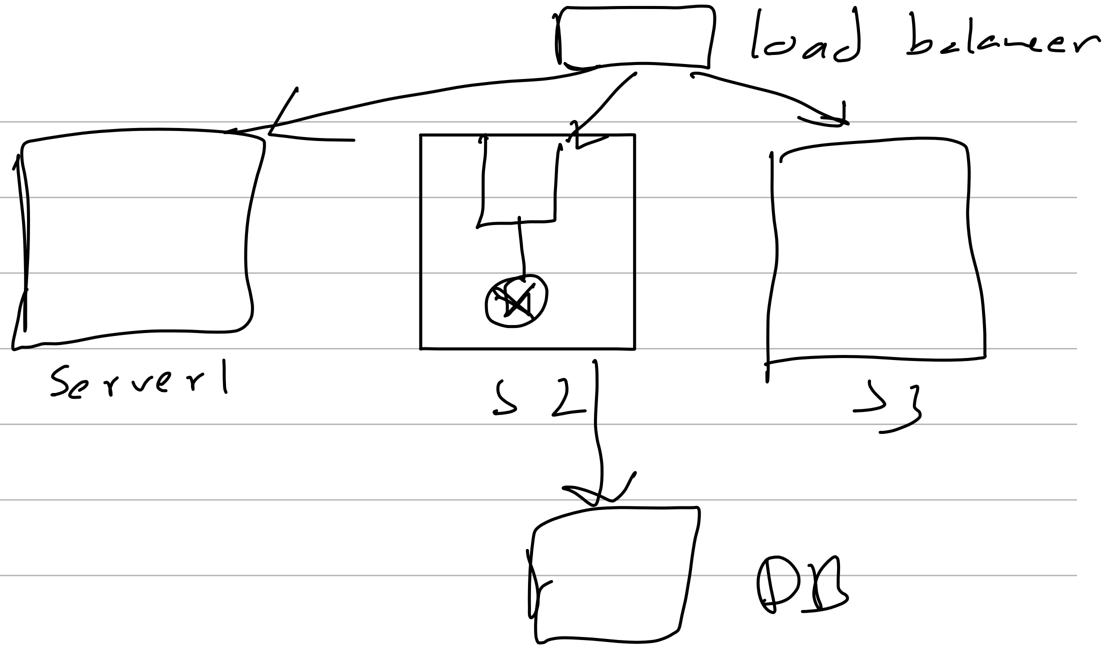
$$L = \sum_{n=0}^{N} n P_n$$

$$= \sum_{n=0}^{N} n \left(\frac{\lambda}{\mu}\right)^n \frac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$$

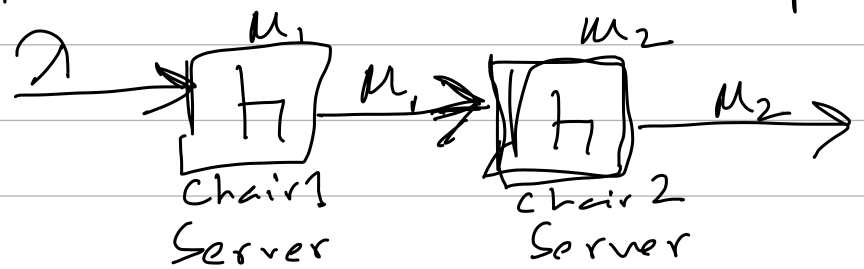
$$= \cdots$$

$$W = \frac{L}{\lambda_a} = \frac{L}{\lambda(1-P_N)}$$

$$\lambda_{actual} = \lambda_a = \lambda - \lambda P_N = \lambda(1-P_N)$$



A Shoe-Shine Shop



ধুলা ঝাড়ে   polish করে

$c_1 \to \mu_1$ rate → service dey

$c_2 \to \mu_2$ " " " "

<u>State</u>

$(0,0) \to$ both server/chair empty

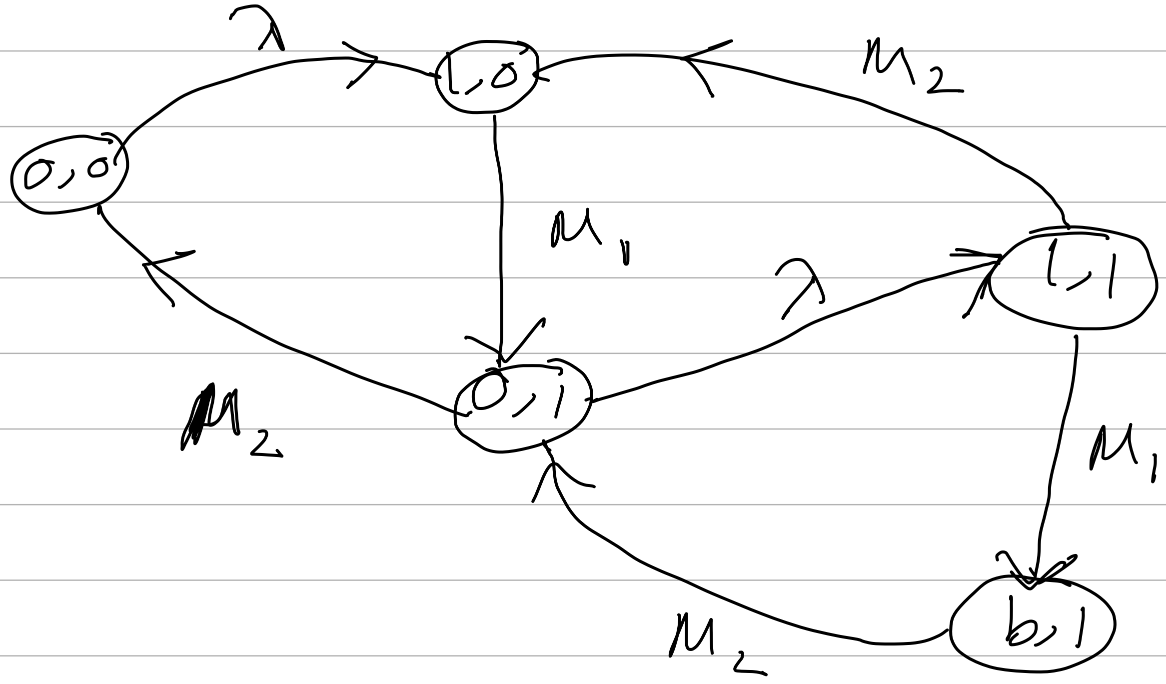
$(0,1) \to$ one customer in chair 2

$(1,0) \to$ " " " chair 1

$(1,1) \to$ both beig served

$(b,1) \to$ 1st jon boshe ache, service neya shesh, 2nd jon service nichche

Transition Diagram:



Balance Equation:

Rate that process leaves =

" " " enters

State

(0,0) $\lambda P_{0,0} = \mu_2 P_{0,1}$

(0,1) $(\lambda + \mu_2) P_{0,1} = \mu_1 P_{1,0} + \mu_2 P_{b,1}$

(1,0) $\mu_1 P_{1,0} = \lambda P_{0,0} + \mu_2 P_{1,1}$

(1,1) $(\mu_1 + \mu_2) P_{1,1} = \lambda P_{0,1}$

(b,1) $\mu_2 P_{b,1} = \mu_1 P_{1,1}$

$$P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1} + P_{b,1} = 1$$

6 equation $\longrightarrow$ 5 unknown.

L = Avg # of customer in system

$$= 1\,(P_{0,1} + P_{1,0}) + 2\,(P_{1,1} + P_{b,1})$$

W = Avg tim a customer spends in a system.

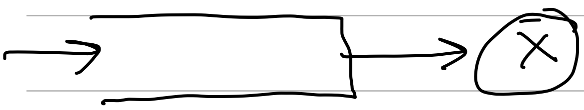
$$= \frac{L}{\lambda_a}$$

Proportion of arriving customer entering system $(P_{0,0} + P_{0,1})$ $\rightarrow$ 1st chair empty

$$\lambda_a = \lambda\,(P_{0,0} + P_{0,1})$$

Queueing system with Bulk Service.

Serves 2 customer at a time.



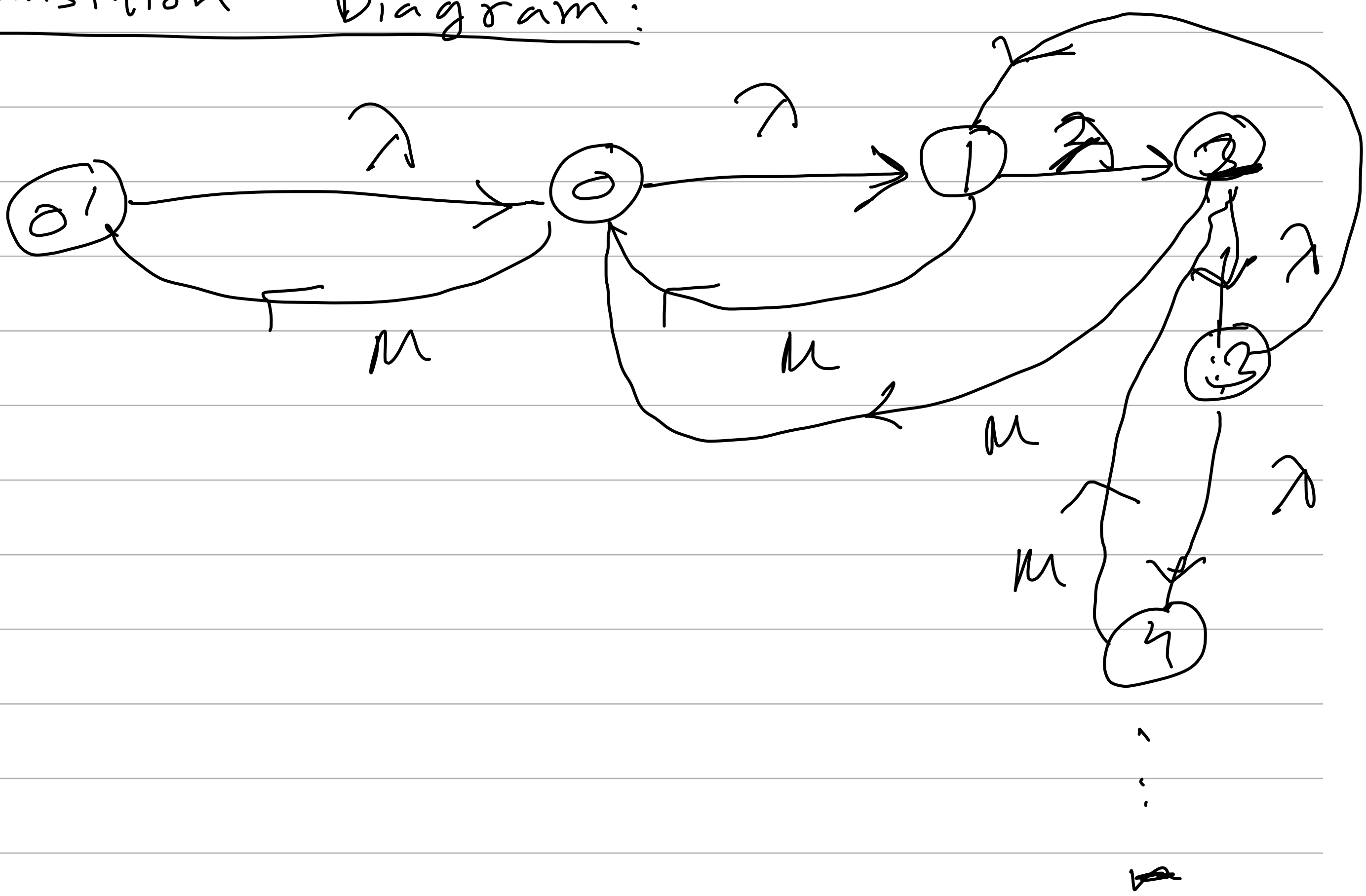
State:

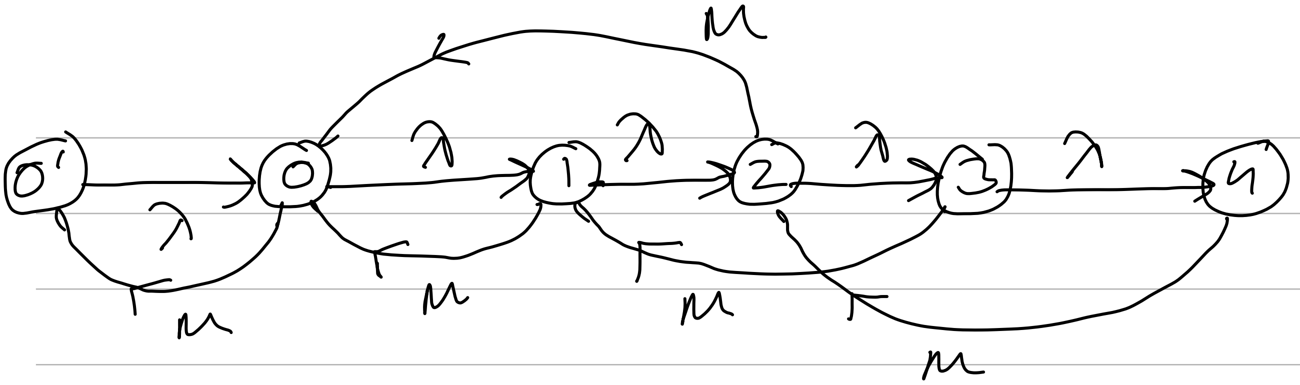
$0' \rightarrow$ no one in service

$0 \rightarrow$ server busy, no one waiting

$n, n>0, \rightarrow n$ customer waiting.

Transition Diagram:





Balance Equation:

<u>State:</u>

$0'$ $\qquad \lambda P_{0'} = \mu P_0$

0 $\qquad (\lambda+\mu) P_0 = \lambda P_{0'} + \mu P_1 + \mu P_2$

n $\qquad (\lambda+\mu) P_n = \lambda P_{n-1} + \mu P_{n+2}$

It has a solution of the form,

$P_n = \alpha^n P_0$

$$\Rightarrow (\lambda+\mu)\alpha^n P_0 = \lambda \alpha^{n-1} P_0 + \mu \alpha^{n+2} P_0$$

$$\Rightarrow (\lambda+\mu)\alpha = \lambda + \mu \alpha^3$$

$$\Rightarrow \mu \alpha^3 - (\lambda+\mu)\alpha + \lambda = 0$$

$$\Rightarrow \mu\alpha^2(\alpha-1) + \mu\alpha(\alpha-1) - \lambda(\alpha-1) = 0$$

$$\Rightarrow (\alpha-1)\quad(\mu\alpha^2 + \mu\alpha - \lambda) = 0$$

$$\therefore \alpha = 1 \quad \text{or} \quad \alpha = \frac{-\mu \pm \sqrt{\mu^2 + 4\mu\lambda}}{2\mu}$$

$$\sum_i P_i = 1$$

$\alpha \neq 1$, $\quad P_0 + P_1 + \cdots + P_n = 1$

$$n P_0 = 1$$

$$P_0 = \frac{1}{n}$$

$$n \to \infty, \quad P_0 = 0$$

$$\alpha = \frac{-\mu + \sqrt{\mu^2 + 4\mu\lambda}}{2\mu} < 1$$

$$\Rightarrow -1 + \sqrt{1 + \frac{4\lambda}{\mu}} < 2$$

$$\Rightarrow \sqrt{1 + \frac{4\lambda}{\mu}} < 3$$

$$\Rightarrow 1 + \frac{4\lambda}{\mu} < 9$$

$$\Rightarrow \frac{4\lambda}{\mu} < 8$$

$$\Rightarrow \frac{\lambda}{\mu} < 2$$

$$\therefore \lambda < 2\mu$$

$$P_n = \sigma^n P_0$$

$$P_{0'} + P_0 + \sum_{n=1}^{\infty} P_n = 1$$

$$P_0 = \frac{\lambda(1-\sigma)}{\lambda + \mu(1-\sigma)}$$

$$L_Q = \sum_{n=1}^{\infty} n P_n = \frac{\lambda \sigma}{(1-\sigma)\left[\lambda + \mu(1-\sigma)\right]}$$

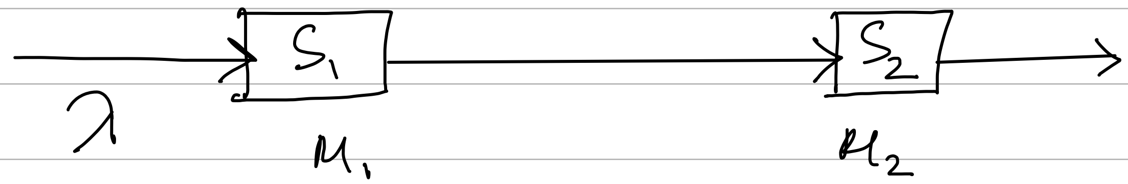
$$W_Q = \frac{L_Q}{\lambda}$$

$$W = W_Q + \frac{1}{\mu}$$

$$L = W\lambda$$

11/9/23

Network of queues: infinite queue in both servers



$\lambda, \mu_1, \mu_2 \rightarrow$ poisson process

$\lambda < \mu_1, \mu_2$, so [S_2 te arrival λ as $\mu_1 > \lambda$]

State

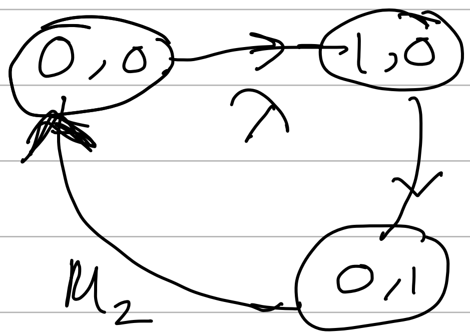
(0 , 0)

(n, 0)

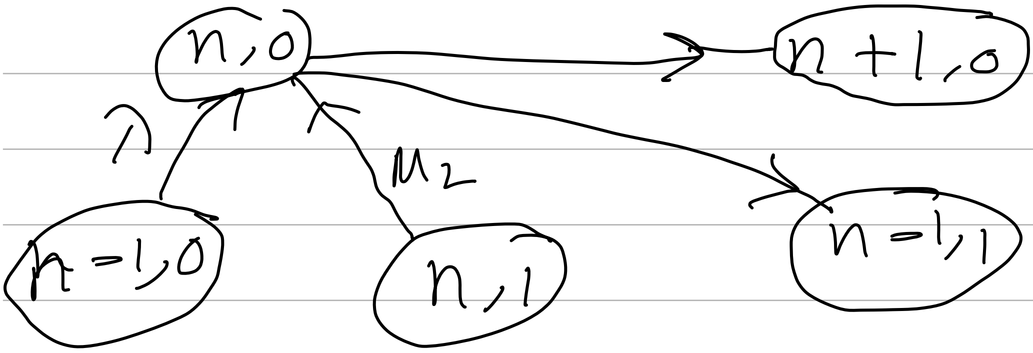
(0, m)

(n, m)

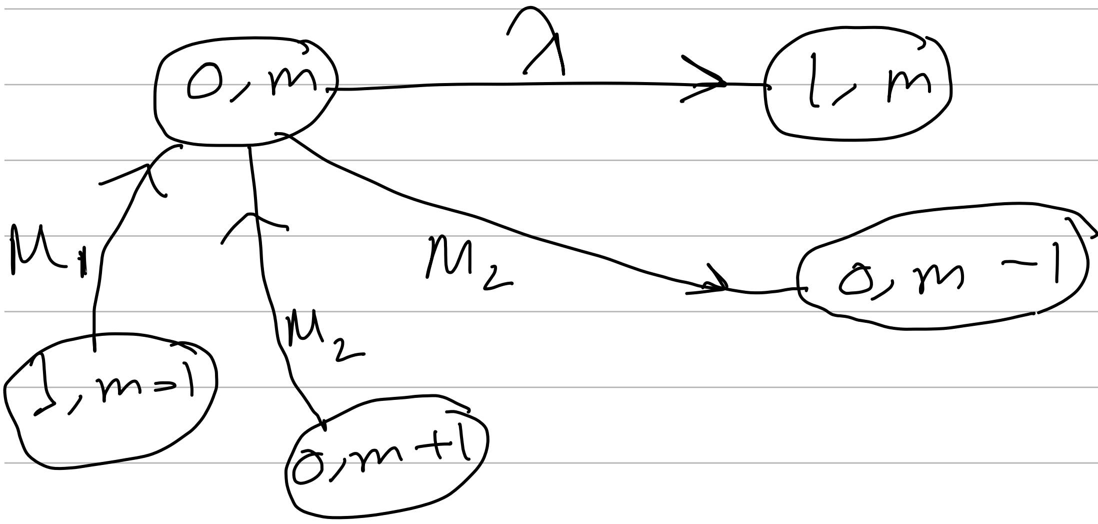
Transition Diagram for (0,0)



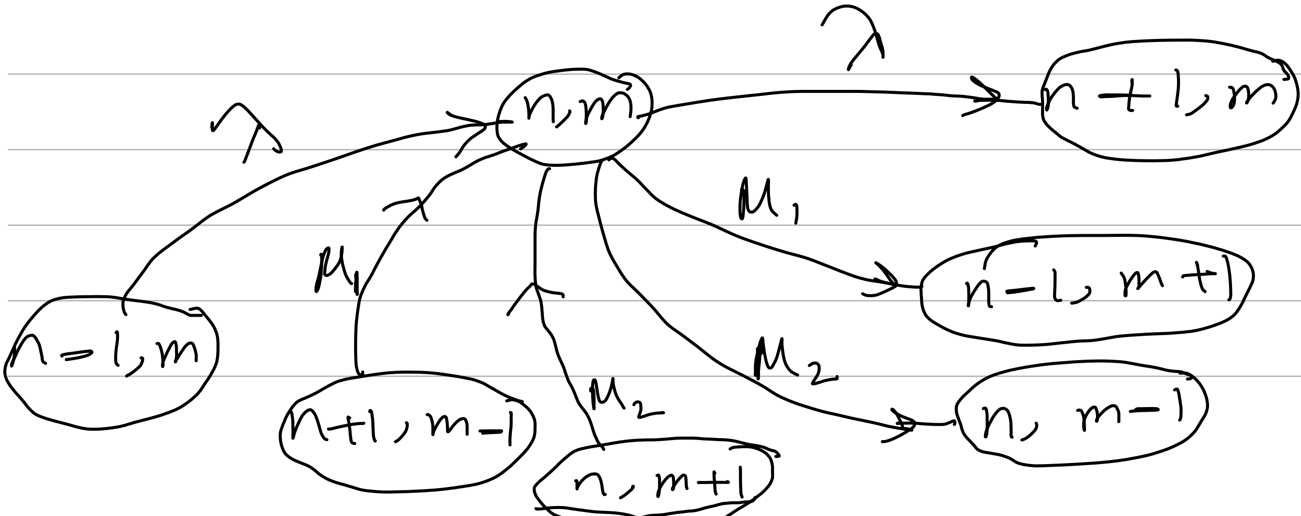
for $(n,0)$


n,0
n+1,0
λ
μ2
n-1,0
n,1
n-1,1

$(0,m)$


0,m
λ
1,m
μ1
μ2
μ2
0,m-1
1,m-1
0,m+1

(n,m)


λ
n,m
n+1,m
λ
μ1
μ1
n-1,m+1
n-1,m
μ2
μ2
n+1,m-1
n,m-1
n,m+1

State = Rate that process leaves
= " " " enters

0,0 $\quad \lambda P_{0,0} = \mu_2 P_{0,1}$

n,0 $\quad (\lambda+\mu_1) P_{n,0} = \lambda P_{n-1,0} + \mu_2 P_{n,1}$

0,m $\quad (\lambda+\mu_2) P_{0,m} = \mu_1 P_{1,m-1} + \mu_2 P_{0,m+1}$

n,m $\quad (\lambda+\mu_1+\mu_2) P_{n,m} = \lambda P_{n-1,m} +$

$\mu_1 P_{n+1,m-1} + \mu_2 P_{n,m+1}$

For M/M/1 System,

$$P\{n \text{ customer in Server } 1\} = \left(\frac{\lambda}{\mu_1}\right)^n \left(1 - \frac{\lambda}{\mu_1}\right)$$
$P_{n,0}$

$$P\{m \text{ customer in Server } 2\} = \left(\frac{\lambda}{\mu_2}\right)^m \left(1 - \frac{\lambda}{\mu_2}\right)$$
$P_{0,m}$

$$P_{n,m} = P_{n,0} \times P_{0,m}$$
$$= \left(\frac{\lambda}{\mu_1}\right)^n \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^m \left(1 - \frac{\lambda}{\mu_2}\right)$$

[if no of customers in S_1 & S_2 are independent.]

L = Avg # of customer in the system

$$= \sum_{n,m} (n+m)\, P_{n,m}$$

$$= \sum_{n,m} (n+m) \left(\frac{\lambda}{\mu_1}\right)^n \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^m \left(1 - \frac{\lambda}{\mu_2}\right)$$

$$= \sum_n n \left(\frac{\lambda}{\mu_1}\right)^n \left(1-\frac{\lambda}{\mu_1}\right) \underbrace{\sum_m \left(\frac{\lambda}{\mu_2}\right)^m \left(1-\frac{\lambda}{\mu_2}\right)}_{1}$$

$$+ \sum_m m \left(\frac{\lambda}{\mu_2}\right)^m \left(1-\frac{\lambda}{\mu_2}\right) \underbrace{\sum_n \left(\frac{\lambda}{\mu_1}\right)^n \left(1-\frac{\lambda}{\mu_1}\right)}_{1}$$

, all possible

$$= \frac{\lambda/\mu_1}{\left(1-\frac{\lambda}{\mu_1}\right)^2}\left(1-\frac{\lambda}{\mu_1}\right) + \frac{\lambda/\mu_2}{\left(1-\frac{\lambda}{\mu_2}\right)^2}\left(1-\frac{\lambda}{\mu_2}\right)$$

$$= \frac{\lambda/\mu_1}{1-\frac{\lambda}{\mu_1}} + \frac{\lambda/\mu_2}{1-\frac{\lambda}{\mu_2}}$$

$$= \frac{\lambda}{\mu_1-\lambda} + \frac{\lambda}{\mu_2-\lambda}$$

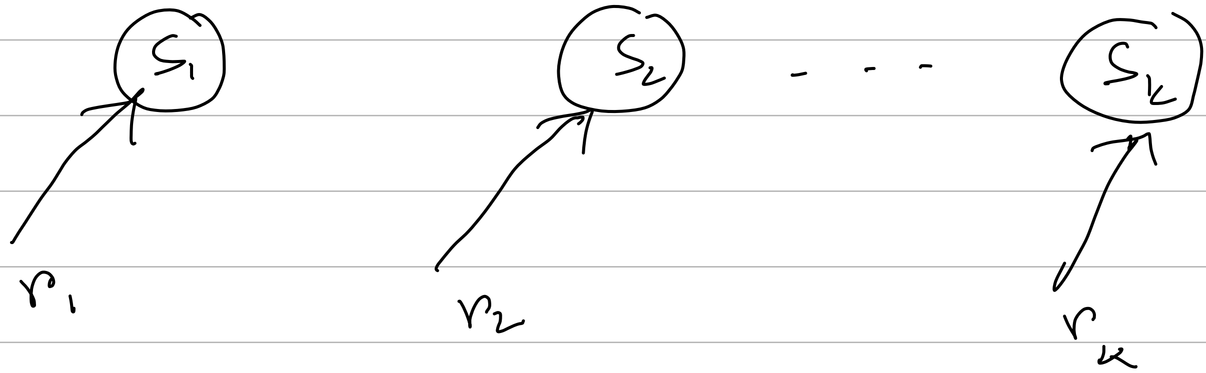
$$= L_1 + L_2$$

Avg time a cust. spends in a system

$$W = \frac{L}{\lambda} = \frac{L_1}{\lambda} + \frac{L_2}{\lambda} = W_1 + W_2$$

$$= \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}$$

k servers



After being served at S_i, customer has probability P_{ij} of going to server S_j $(j = 1, 2, \ldots)$

$$\sum_{j=1}^{k} P_{ij} \leq 1$$ [1 না. cause, baireo jaite pare]

Probability that cust. departs system after being served by S_i,

$$1 - \sum_{j=1}^{k} P_{ij}$$

Let,

λ_j = Total arrival rate at S_j

$$\lambda_j = r_j + \sum_i \left(\underbrace{\lambda_i}_{S_i \text{ থেকে ber hoise}} \; \underbrace{P_{ij}}_{\text{কত অংশ গেছে}} \right)$$

যেটা,

$$P\{n \text{ customer at } S_j\} = \left(\frac{\lambda_j}{\mu_j}\right)^n \left(1 - \frac{\lambda_j}{\mu_j}\right),$$

μ_j exponentially distributed

service rate at S_j

$\frac{\lambda_j}{\mu_j} < 1$ for stability

$$P(n_1, n_2, \ldots, n_k) = \prod_{j=1}^{k} \left(\frac{\lambda_j}{\mu_j}\right)^{n_j} \left(1 - \frac{\lambda_j}{\mu_j}\right)$$

Avg no of customer in the system
(যোগ করে দিব, যেহেতু, independent)

$$L = \sum_{j=1}^{k} \left(\text{Avg \# of cust. in } S_j \right)$$

$$= \sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}$$

[Σ প্রতিটি queue তে avg #cust]

Avg time a customer spends in the system, $W = \dfrac{L}{\lambda_a}$

$\lambda_a \rightarrow$ বাইরে থেকে আসা অ্যারাইভাল.

$$= \frac{\sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}}{\sum_{j=1}^{k} r_j}$$

1, 2

3

4, 5

8