CSE 301 Concrete Mathematics Lecture Note Scribes

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CSE, BUET

Lect-1 (Section A)
CSE 301 M. A. for C. S.
Part A: Tahmid Hasan

Textbook: Concrete Mathematics (Graham, Knuth, Patashnik)

Syllabus:
1) Recurrence (Ch 1)
2) Sums (Ch 2)
3) Number Theory (Ch 4)
4) Special Numbers (Ch 6)
5) Generating Functions (Ch 7)

Recurrence

$f_n = f_{n-1} + f_{n-2}$

Tower of Hanoi

$T_0 = 0$

$T_1 = 1$

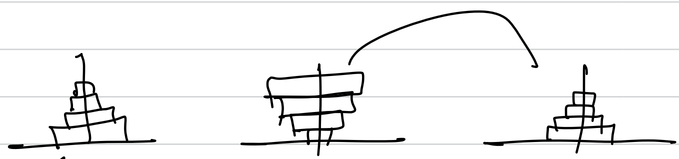
$T_2 = 3$

$T_3 = 7$

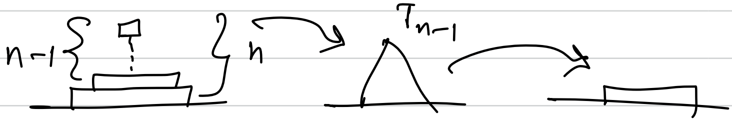
$T_4 = 15$

$T_5 = 31$

$T_6 = 63$



$T_n \le T_{n-1} + 1 + T_{n-1} = 2T_{n-1} + 1$



$T_n \geq 1 + T_{n-1} + T_{n-1}$

$$\boxed{\begin{aligned} T_0 &= 0 \\ T_n &= 2T_{n-1} + 1 \end{aligned}}$$

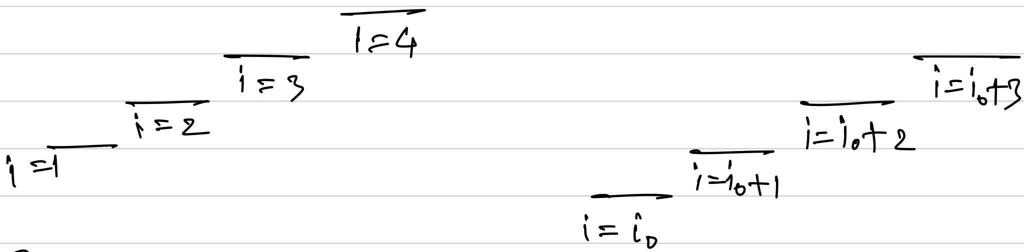
$T_n = 2^n - 1$

Mathematical Induction: $P(K)$ (Sipser p. 23)

induction hypothesis

1) Basis: ~~w~~ $p(1)$

2) Induction step: for any $i \geq 1$, if $p(i)$ is true, then $p(i+1)$ true



1) Basis: $p(i_0)$

2) Induction step: for any $i \geq i_0$ if $p(i_0), p(i_0+1), \ldots, p(i)$ is true, then $p(i+1)$ is true.

$$\left.\begin{array}{l} T_0 = 0 \\ T_n = 2T_{n-1} + 1 \end{array}\right\} \quad T_n = 2^n - 1$$

1) Basis: $T_0 = 2^0 - 1 = 0$

Hypo: given $p(i_0), p(i_0+1), \ldots, p(i)$ true

$$T_{i+1} = 2T_i + 1 = 2(2^i - 1) + 1$$
$$= 2^{i+1} - 2 + 1 = 2^{i+1} - 1$$

CS E 301: M. A. for C. S. Lect - 1 : Section B

Part A: Tahmid Hasan

Textbook: Concrete Mathematics (Graham, Knuth, Patashnik)

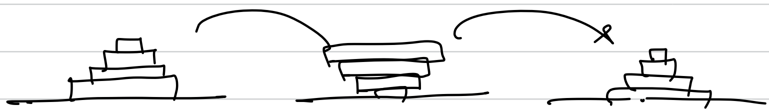
Syllabus: 1) Recurrence (Ch 1)
2) Sums (Ch 2)
3) Number Theory (Ch 4)
4) Special Numbers (Ch 6)
5) Generating functions (Ch 7)

Recurrence

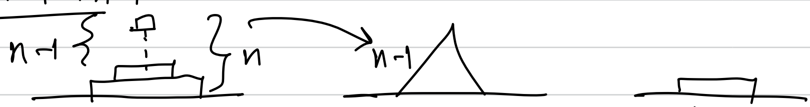
$S_n = n + S_{n-1}$

$f_n = f_{n-1} + f_{n-2}$

Tower of Hanoi



$T_n \leq T_{n-1} + 1 + T_{n-1}$



$T_1 = 1$ $T_0 = 0$

$T_2 = 3$

$T_3 = 7$

$T_4 = 15$

$T_5 = 31$

$T_6 = 63$

$T_7 = 127$

$T_n \geq 1 + T_{n-1} + T_{n-1}$

$\boxed{T_n = 2T_{n-1} + 1}$

$T_n = 2^n - 1$

Mathematical Induction: $P(K) \to \in \mathbb{N}$ (Sipser P.23)

Basis: $P(1)$ is true.

Induction: for any $i \geq 1$, $\underbrace{\text{if } P(i) \text{ is true}}_{\text{Induction hypothesis}}$, then $P(i+1)$ is true

$K=1$ $K=2$ $K=3$ $K=4$

$K=i_0$ $K=i_0+1$ $K=i_0+2$ $K=i_0+3$

Basis: $P(i_0)$ is true.

Induction: for any $i \geq i_0$, if $P(i_0), P(i_0+1), \ldots, P(i)$ is true, then $P(i+1)$ is true.

$T_n = 2^n - 1$

$T_0 = 0$

$T_n = 2T_{n-1} + 1$

Basis: $i=0$, $T_0 = 2^0 - 1 = 0$

Induction: for any $i \geq 0$, if $T(i)$ is true, then we need to prove $T(i+1)$ is true

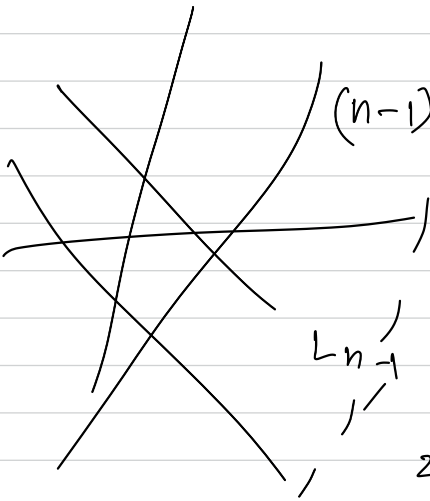
$$T_{i+1} = 2T_i + 1 = 2 \times (2^i - 1) + 1$$

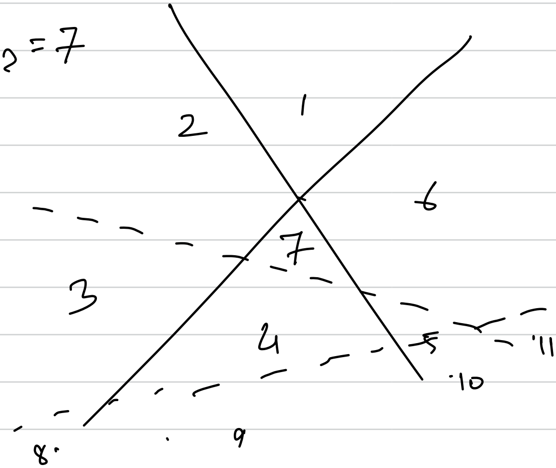
$$= 2^{i+1} - 2 + 1 = 2^{i+1} - 1$$

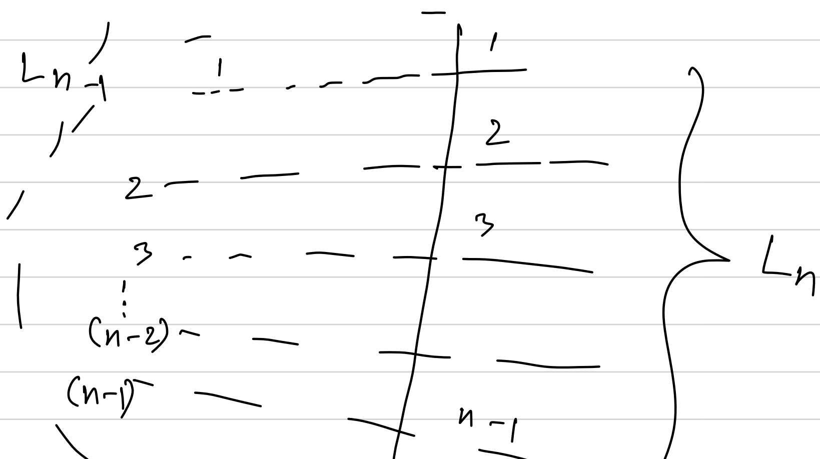
Lect-2: Lines on a plane Section: B

$L_n \quad L_0 = 1, \; L_1 = 2, \; L_2 = 4, \; L_3 = 7$

$\cancel{L_n = 2^n} \quad L_4 = 11$

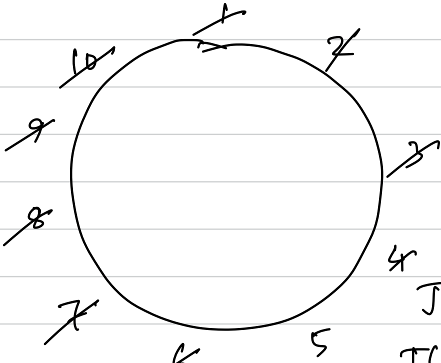


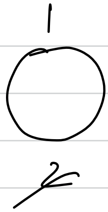




$$
\begin{cases}
L_0 = 1 \\
L_n = L_{n-1} + n \\
\quad = L_{n-2} + (n-1) + n \\
\quad = L_{n-3} + (n-2) + (n-1) + n \\
\quad \vdots \\
\quad = L_0 + \underbrace{1 + 2 + 3 \cdots\cdots + n} = 1 + \frac{n(n+1)}{2}
\end{cases}
$$

Josephus Problem:

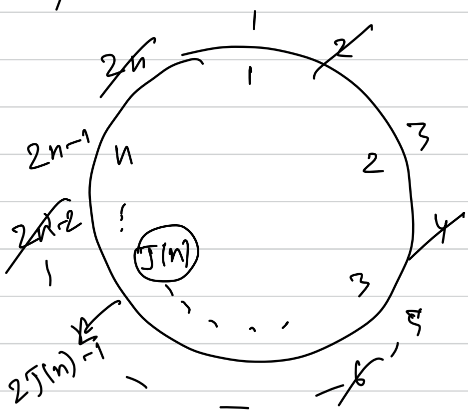


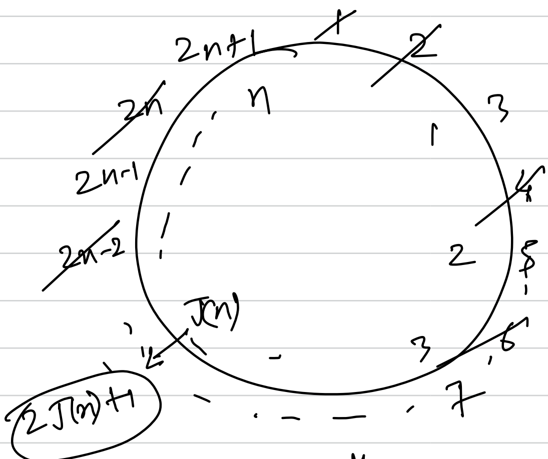


n	1	2	3	4	5	6
$J(n)$	1	1	3	1	3	5

$$J(2n) = 2J(n) - 1 \qquad J(1) = 1$$
$$J(2n+1) = 2J(n) + 1$$
$\Big\}\; O(\lg n)$

$J(2n)$





n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2^0	2^1		2^2				2^3								2^4
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1
								l_0	l_1	l_2	l_3	l_4	l_5	l_6	l_7	

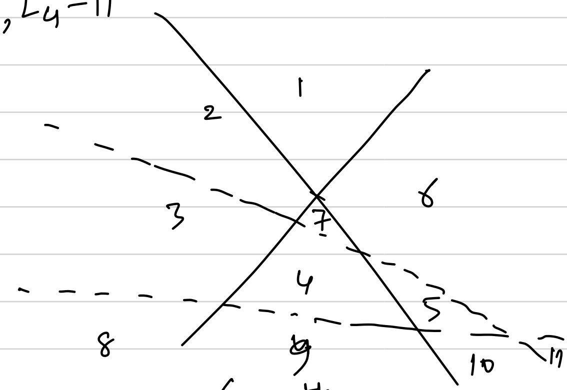
$$n = 2^m + l \qquad J(n) = 2l+1 \Rightarrow J(2^m + l) = 2l+1 \quad m \geq 0$$

$$n < 2^{m+1} \Rightarrow 2^m + l < 2^{m+1} \Rightarrow l < 2^{m+1} - 2^m = 2^m \Rightarrow \underline{0 \leq l < 2^m}$$

Induction on m:

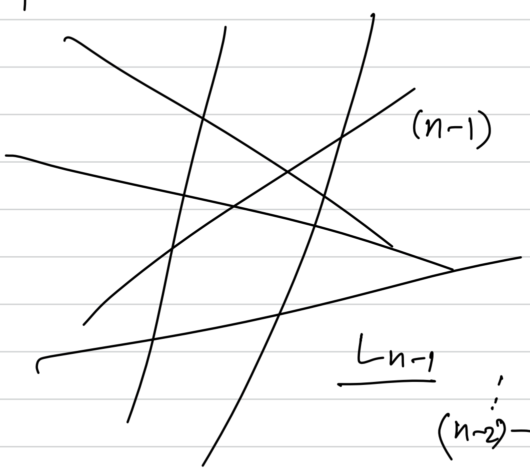
Lect 2: Lines on a plane — Sec. A

$L_n \quad L_0=1,\ L_1=2,\ L_2=4,\ L_3=7,\ L_4=11$

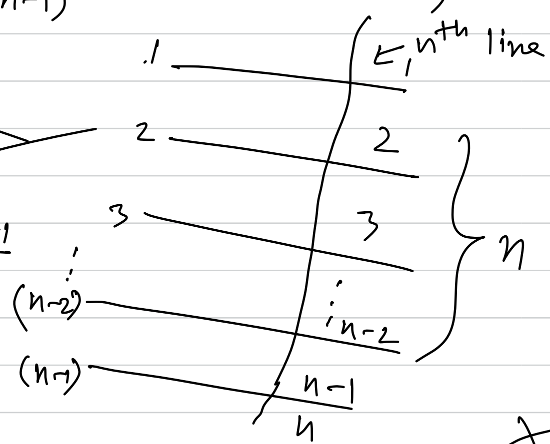


$L_n = 2^n$

L_{n-1}



$\underline{L_{n-1}}$



$L_0 = 1$

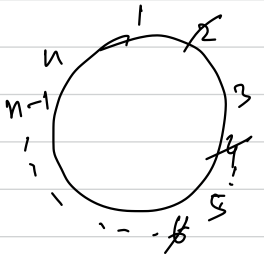
$$
\begin{aligned}
L_n &= L_{n-1} + n \\
&= L_{n-2} + (n-1) + n \\
&= L_{n-3} + (n-2) + (n-1) + n \\
&\ \vdots \\
&= L_0 + 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
&= 1 + \frac{n(n+1)}{2}
\end{aligned}
$$

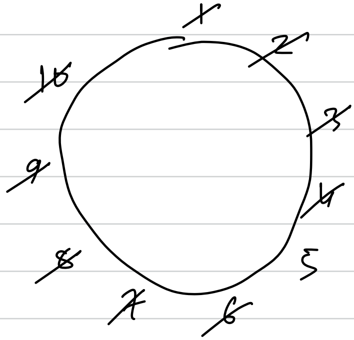
Josephus Problem:

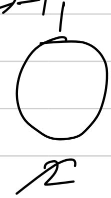
$J(n)$

$J(10)=5$

$J(2)=1$







$J(n) = n/2$

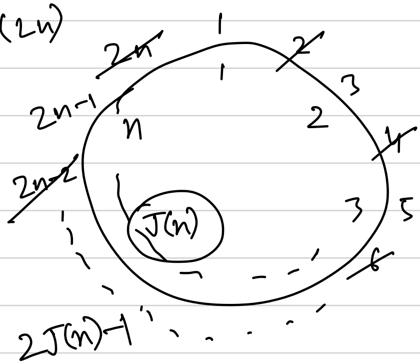
n	1	2	3	4	5	6
$J(n)$	1	1	3	1	3	5

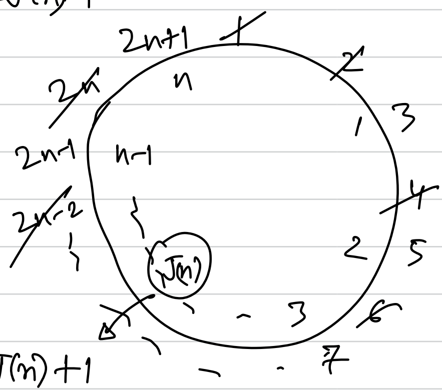
$J(1) = 1$

$O(\lg n)$

$J(2n) = 2J(n)-1$

$J(2n)$





$J(2n+1) = \underline{2J(n)+1}$

n	$\overset{2^0}{1}$	$\overset{2^1}{2}$	3	$\overset{2^2}{4}$	5	6	7	$\overset{2^3}{8}$	9	10	11	12	13	14	15	$\overset{2^4}{16}$
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1
								0	1	2	3	4	5	6	7	

$15 = 2^3+7$

$25 = 2^4+9$

$n = 2^m + l$ $\quad J(n) = 2l+1$

$$n < 2^{m+1} \Rightarrow 2^m + l < 2^{m+1} \Rightarrow l < 2^{m+1} - 2^m = 2^m$$

$0 \le l < 2^m$

$J(1)=1$

$J(2n) = 2J(n)-1$

$J(2n+1) = 2J(n)+1$

$J(2^m + l) = 2l+1; \quad m \geq 0, \; 0 \le l < 2^m$

Induction on m

Lect-3: Josephus contd. Sec. A

$J(1)=1$

$J(2n)=2J(n)-1$

$J(2n+1)=2J(n)+1$

$n=2^m+l;\ m\geq 0,\ 0\leq l<2^m$

$J(2^m+l)=2l+1$

$0\leq l<2^0 \Rightarrow 0\leq l<1$

Basis: $m=0$ $J(2^m+l)=J(2^0+0)=2\cdot 0+1=2l+1$

$\Rightarrow J(1)=1$

Induction:

Assume true for $0,\ldots,m-1$

$$J(\underbrace{\overbrace{2^m}^{e}+\overbrace{l}^{e}}_{2n})=2J(2^{m-1}+l/2)-1=2(\cancel{2}\cdot l/\cancel{2}+1)-1=\underline{2l+1}$$

$$J(\overbrace{\underbrace{2^m}_{e}+\underbrace{l}_{o}}^{2n\ o})=J(\underbrace{2^m+l-1}_{2n}+1)=2J\left(2^{m-1}+\frac{l-1}{2}\right)+1$$

$$=2\left(\cancel{2}\cdot\frac{l-1}{\cancel{2}}+1\right)+1=\underline{2l+1}$$

$n=(\overset{1}{b_m}b_{m-1}b_{m-2}\ldots b_1b_0)_2$ $n=b_m2^m+b_{m-1}2^{m-1}+\ldots+b_1\times 2+b_0$

$n=(1\,b_{m-1}b_{m-2}\ldots b_1b_0)_2$

$2^m=(1\ 0\ 0\ \ldots\ 0\ 0)_2$

$l=(0\,b_{m-1}b_{m-2}\ldots b_1b_0)_2$

$2l=(b_{m-1}b_{m-2}\ldots b_1b_0 0)_2$

$2l+1=(b_{m-1}b_{m-2}\ldots b_1b_0\underset{\hookrightarrow b_m}{1})_2$

$n=2^m+l\Rightarrow l=n-2^m$

$J((b_mb_{m-1}\ldots b_1b_0)_2)=(b_{m-1}b_{m-2}\ldots b_1b_0b_m)_2$

$J((11101)_2) = \underline{(1011)_2} \to (111)_2 \to (111)_2$

fixed point $f(n) = n$

$J(n) = n/2$
$\Rightarrow 2l+1 = (2^m+l)/2$
$\Rightarrow 4l+2 = 2^m + l \Rightarrow l = \frac{1}{3}(2^m - 2)$

$m=0 \to l$ ✗
$m=1 \to l=0$
$m=2 \to l$ ✗
$m=3 \to l=2$

m	l	$n=2^m+l$	$J(n)=2l+1=n/2$	n (binary)
1	0	2	1	10
3	2	10	5	1010
5	10	42	21	101010
7	42	170	85	10101010

$J(1) = 1$
$J(2n) = 2J(n) - 1$
$J(2n+1) = 2J(n) + 1$

$f(1) = \alpha$
$f(2n) = 2f(n) + \beta$
$f(2n+1) = 2f(n) + \gamma$

n	$f(n)$
1.	α
2.	$2\alpha + \beta$
3.	$2\alpha + \gamma$
4.	$4\alpha + 3\beta$
5.	$4\alpha + 2\beta + \gamma$
6.	$4\alpha + \beta + 2\gamma$
7.	$4\alpha + 3\gamma$
8.	$8\alpha + 7\beta$
9.	$8\alpha + 6\beta + \gamma$

$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$

$n = 2^m + l$

$$\boxed{\begin{aligned} A(n) &= 2^m \\ B(n) &= 2^m - l - 1 \\ C(n) &= l \end{aligned}}$$

$B(n) + \underbrace{C(n)}_{l} = 2^m - 1$

$\frac{3x+2}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

Lect-3: Josephus cont. Sec. B

$J(1)=1$ $\quad n=2^m+l;\ m\geq 0\ \&\ 0\leq l<2^m$

$J(2n)=2J(n)-1$ $\quad J(n)=2l+1 \;\; \cdots\cdots (1)$

$J(2n+1)=2J(n)+1$

$0\leq l<2^0 \Rightarrow 0\leq l<1$

$\hookrightarrow l=0$

Basis: $m=0$; $J(2^m+l)=2l+1$

$\Leftrightarrow J(2^0+0)=2\cdot 0+1$

$\Leftrightarrow J(1)=1$

Induction: Assume (1) is true for $0, 1, \ldots, m-1$

$$J(\overbrace{\underbrace{2^m}_{e}+\underbrace{l}_{e}}^{2n}) = \underbrace{2J(2^{m-1}+l/2)}_{0}-1 = 2(2\cdot l/2+1)-1 = 2l+1$$

$$J(\overbrace{\underbrace{2^m}_{e}+\underbrace{l}_{0}}^{0}) = J(\overbrace{\underbrace{2^m+l-1}_{e}+1}^{0}) = 2J\left(2^{m-1}+\frac{l-1}{2}\right)+1$$

$$= 2\left(2\cdot\frac{l-1}{2}+1\right)+1 = 2l+1$$

$n=(b_m b_{m-1}\cdots b_1 b_0)_2$, $\quad b_m = 1$ $\quad n=b_m 2^m+b_{m-1}2^{m-1}+\cdots+2b_1+b_0$

$n=(1\,b_{m-1}\cdots b_1 b_0)_2$ $\quad n=2^m+l \Rightarrow l=n-2^m$

$2^m=(1\,0\cdots\cdots 0\,0)_2$

$l=(0\,b_{m-1}\cdots\cdots b_1 b_0)_2$

$2l=(b_{m-1}b_{m-2}\cdots b_0 0)_2$

$2l+1=(b_{m-1}b_{m-2}\cdots b_0 1)_2$

$J((b_m b_{m-1}\cdots b_1 b_0)_2) = (b_{m-1}b_{m-2}\cdots b_1 b_0 b_m)_2$

$(1011)_2 \to (0111)_2 \to (111)_2 \to (111)_2$

$\underbrace{1011001010111101}$

111, 111, 111

$J(n)=n \to$ fixed point

$f(n)=n$

$J(n) = n/2 \qquad n = 2^m + l, \Rightarrow J(n) = 2l+1$

$\Rightarrow 2l+1 = (2^m+l)/2 \Rightarrow 4l+2 = 2^m + l \Rightarrow \underline{\underline{l}} = \frac{1}{3}(2^m - 2)$

m	l	$n = 2^m + l$	$J(n) = n/2 = 2l+1$	n (binary)
1	0	2	1	10
3	2	10	5	1010
5	10	42	21	101010
7	42	170	85	10101010

$$\left.\begin{aligned} J(1) &= 1 \\ J(2n) &= 2J(n) - 1 \\ J(2n+1) &= 2J(n) + 1 \end{aligned}\right\} \quad \begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n) + \beta \\ f(2n+1) &= 2f(n) + \gamma \end{aligned}$$

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

n	$f(n)$
1.	α
2.	$2\alpha + \beta$
3.	$2\alpha + \gamma$
4.	$4\alpha + 3\beta$
5.	$4\alpha + 2\beta + \gamma$
6.	$4\alpha + \beta + 2\gamma$
7.	$4\alpha + 3\gamma$
8.	$8\alpha + 7\beta$
9.	$8\alpha + 6\beta + \gamma$

$n = 2^m + l$

$A(n) = \underline{\underline{2^m}}$

$C(n) = \underline{l}$

$B(n) + C(n) = 2^m - 1$

$\Rightarrow B(n) + l = 2^m - 1$

$\Rightarrow B(n) = \underline{\underline{2^m - l - 1}}$

Lect. 4: Generalized Josephus

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \quad \ldots (1)$$

$f(1) = \alpha$ $\qquad A(n) = 2^m$

$f(2n) = 2f(n) + \beta$ $\qquad B(n) = 2^m - l - 1$ $\qquad n = 2^m + l$

$f(2n+1) = 2f(n) + \gamma$ $\qquad C(n) = l$

$$\frac{3x+2}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$$

$(\alpha, \beta, \gamma) \equiv (1, 0, 0)$ $\qquad A(1) = 1$

$A(2n) = 2A(n)$

$A(2n+1) = 2A(n)$

n	1	2	3	4	5	6	7	8	9
$A(n)$	1	2	2	4	4	4	4	8	8

$f(n) = A(n)$

$\boxed{A(n) = 2^m}$ ✓

$f(n) = 1 \Rightarrow$
$$\left.\begin{array}{l} 1 = \alpha \\ 1 = 2\cdot 1 + \beta \\ 1 = 2\cdot 1 + \gamma \end{array}\right\} (\alpha, \beta, \gamma) = (1, -1, -1)$$

$\boxed{A(n) - B(n) - C(n) = 1} \Rightarrow B(n) = A(n) - C(n) - 1 = \underline{2^m - l - 1}$ ✓

$f(n) = n \Rightarrow$
$$\left.\begin{array}{l} 1 = \alpha \\ 2n = 2\cdot n + \beta \\ 2n+1 = 2\cdot n + \gamma \end{array}\right\} (\alpha, \beta, \gamma) = (\underline{1}, \underline{0}, \underline{1})$$

$\boxed{A(n) + C(n) = n}$ $\qquad C(n) = n - 2^m = \underline{l}$ ✓

Repertoire method: ex. 16 & 20

$J((b_m b_{m-1} \cdots b_1 b_0)_2) = (b_{m-1} \cdots b_1 b_0 b_m)_2$; $\quad b_m = 1$

$\underline{f(1) = \alpha}$

$f(2n+j) = 2f(n) + \beta_j$; $\quad j \in \{0, 1\}$, $\quad \beta_0 = \beta$, $\beta_1 = \gamma$

$$d \,\big|\, \frac{a}{\frac{dq}{r}} \,\big|\, q$$

$a = dq + r$

$$2 \,\big|\, \frac{2n+j}{\frac{2n}{j}} \,\big|\, n$$

$$
\begin{aligned}
f((b_m b_{m-1} \cdots b_1 b_0)_2) &= 2f((b_m b_{m-1} \cdots b_1)_2) + \beta_{b_0} \\
&= 2(2f((b_m b_{m-1} \cdots b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 4f((b_m b_{m-1} \cdots b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&= 8f((b_m b_{m-1} \cdots b_3)_2) + 4\beta_{b_2} + 2\beta_1 + \beta_{b_0} \\
&\ \vdots \\
&= 2^m f((b_m)_2) + 2^{m-1}\beta_{b_{m-1}} + 2^{m-2}\beta_{b_{m-2}} + \cdots + 2\beta_{b_1} + \beta_{b_0} \\
&= 2^m \alpha + 2^{m-1}\beta_{b_{m-1}} + 2^{m-2}\beta_{b_{m-2}} + \cdots + 2\beta_{b_1} + \beta_{b_0} \\
&= (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_1} \beta_{b_0})_2 \text{ — notation abuse!}
\end{aligned}
$$

$J(100) = J(2^6 + 36) = 2 \cdot 36 + 1 = 73$

$\alpha = 1$

$\beta = -1 = \beta_0$

$\gamma = 1 = \beta_1$

$100 \rightarrow \underline{(1100100)_2}$

$2^6 \times 1 + 2^5 \times 1 + 2^4 \times (-1) + 2^3 \times (-1) + 2^2 \times 1 + 2^1 \times (-1) + 2^0 \times (-1)$

$= \underline{\underline{73}}$

$$
\begin{aligned}
f(j) &= \alpha_j && \text{for } 1 \le j < d \\
f(dn + j) &= cf(n) + \beta_j && \text{for } 0 \le j < d;\ n \ge 1
\end{aligned}
$$

$$f((b_m b_{m-1} \cdots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_1} \beta_{b_0})_c$$

Lect. 4: Generalized Josephus sec. B

$f(1) = \alpha$
$f(2n) = 2f(n) + \beta$
$f(2n+1) = 2f(n) + \gamma$

$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \quad \cdots (1)$
$A(n) = 2^m, \; B(n) = 2^m - l - 1, \; C(n) = l$

$\frac{3x+2}{(x-1)(x-2)} \equiv \frac{A}{x-1} + \frac{B}{x-2}$

$(\alpha, \beta, \gamma) \equiv (1, 0, 0)$
$f(n) = A(n)$

$A(1) = 1$
$A(2n) = 2A(n)$
$A(2n+1) = 2A(n)$

n	1	2	3	4	5	6	7	8	9
$A(n)$	1	2	2	4	4	4	4	8	8

$n = 2^m + l \Rightarrow \boxed{A(n) = 2^m}$ ✓

$f(n) = 1 \Rightarrow$
$1 = \alpha$
$1 = 2 \cdot 1 + \beta$
$1 = 2 \cdot 1 + \gamma$
$\Big\} \; (\alpha, \beta, \gamma) \equiv (1, -1, -1)$

$\boxed{A(n) - B(n) - C(n) = 1}$

$f(n) = n \Rightarrow$
$1 = \alpha$
$2n = 2 \cdot n + \beta$
$2n+1 = 2 \cdot n + \gamma$
$\Big\} \; (\alpha, \beta, \gamma) \equiv (1, 0, 1)$

$\boxed{A(n) + C(n) = n} \Rightarrow C(n) = n - 2^m = \underline{l}$ ✓

$A(n) - B(n) - C(n) = 1 \Rightarrow B(n) = A(n) - C(n) - 1 = 2^m - l - 1$ ✓

Repertoire method: Ex. 16 & 20.

$J((b_m b_{m-1} \cdots b_1 b_0)_2) = (b_{m-1} \cdots b_1 b_0 b_m)_2 \quad \underline{\underline{b_m = 1}}$

$$\begin{array}{r|c|l} d & a & q \\ & dq & \\ \hline & r & \end{array}$$

$a = dq + r; \; 0 \leq r < d$

$f(1) = \alpha$
$f(2n+j) = 2f(n) + \beta_j; \; j \in \{0, 1\}, \beta_0 = \beta, \beta_1 = \gamma$

$$\begin{array}{r|c|l} 2 & 2n+j & n \\ & 2n & \\ \hline & j & \end{array}$$

$$f((b_m b_{m-1} \cdots b_1 b_0)_2) = 2f((b_m b_{m-1} \cdots b_1)_2) + \beta_{b_0}$$

$$= 2(2f(b_m b_{m-1} \cdots b_2)_2 + \beta_{b_1}) + \beta_{b_0}$$

$$= 4f((b_m b_{m-1} \cdots b_2)_2) + 2\beta_{b_1} + \beta_{b_0}$$

$$= 8f((b_m b_{m-1} \cdots b_3)_2) + 4\beta_{b_2} + 2\beta_{b_1} + \beta_{b_0}$$

$$\vdots$$

$$= 2^m f((b_m)_2) + 2^{m-1}\beta_{m-1} + \cdots + 4\beta_{b_2} + 2\beta_{b_1} + \beta_{b_0}$$

$$= 2^m \alpha + 2^{m-1}\beta_{m-1} + \cdots + 4\beta_{b_2} + 2\beta_{b_1} + \beta_{b_0}$$

$$= (\alpha \beta_{b_{m-1}} \cdots \beta_{b_2} \beta_{b_1} \beta_{b_0})_2$$ → notation abuse!

$J(100) = J(2^6 + 36) = 36 \cdot 2 + 1 = 73$

$$\begin{array}{ccccccc} b_m & b_{m-1} & b_{m-2} & \cdots & & b_1 & b_0 \\ (1 & 1 & 0 & 0 & 1 & 0 & 0)_2 \end{array}$$

$J(1) = 1$ $\alpha = 1$

$J(2n) = 2J(n) - 1$ $\beta = -1 = \beta_0$

$J(2n+1) = 2J(n) + 1$ $\gamma = 1 = \beta_1$

$$2^6 \times 1 + 2^5 \times 1 + 2^4 \times (-1) + 2^3 \times (-1) + 2^2 \times 1 + 2^1 \times (-1) + 2^0 \times (-1)$$

$$= 64 + 32 - 16 - 8 + 4 - 2 - 1 = 73$$

$f(j) = \alpha_j$ $1 \le j < d$

$f(dn+j) = cf(n) + \beta_j$ $0 \le j < d$; $n \ge 1$

$$f((b_m b_{m-1} \cdots b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_1} \beta_{b_0})_c$$

$f(1) = 1$

$f(2) = 2$

$f(3n) = 4f(n) + 1$

$f(3n+1) = 4f(n) + 2$

$f(3n+2) = 4f(n) + 3$

Lect5: Sums Sec. B

Sum of first n integers: $1+2+3+\cdots+(n-1)+n$

$1+2+\cdots+n$ ↳ term

$1+\cdots+n$

$a_1+a_2+\cdots+a_n \to$ general term: a_k

$1+2+\cdots+2^{n-1} \to a_k = 2^{k-1}$ | $2^0+2^1+\cdots+2^{n-1}$

Sigma notation: $\sum_{k=1}^{n} a_k \to$ delimited form

Generalized sigma notation: $\sum_{1\le k\le n} a_k \longrightarrow \sum_{P(k)} a_k$: sum of all terms satisfying $P(k)$

$\downarrow$

$\sum_k a_k [P(k)]$

$P(k) \equiv 1\le k\le n$

$$\sum_{\substack{1\le k\le 100\\ k\ \text{odd}}} k^2 \equiv \sum_{k=0}^{49}(2k+1)^2 \quad \Big| \quad \sum_{\substack{1\le p\le N\\ p\ \text{prime}}} \frac{1}{p} = \sum_{k=1}^{\pi(N)} \frac{1}{p_k}$$

$$\to \sum_p \frac{1}{p}[p\ \text{prime}][p\le N]$$

$$\sum_{1\le k\le n} a_k = \sum_{1\le k+1\le n} a_{k+1};$$

$$\sum_{k=2}^{n-1} k(k-1)(n-k) = \sum_{k=0}^{n} k(k-1)(n-k)$$

$$[\text{statement}] = \begin{cases} 1; & \text{if statement is true} \\ 0; & \text{if statement is false} \end{cases}$$

↳ Iverson notation

Sum $\to$ Recurrence: $S_n = \sum_{k=0}^{n} a_k \to S_0 = a_0;\ S_n = \sum_{k=0}^{n-1} a_k + a_n$

$\Rightarrow S_n = S_{n-1} + a_n$

$a_k = \beta + \gamma k$ $\quad \boxed{R_0 = \alpha,\ R_n = R_{n-1} + \beta + \gamma n}$

$R_1 = \alpha+\beta+\gamma,\ R_2 = \alpha + 2\beta + 3\gamma \to R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

$R_n = 1;\ 1 = \alpha;\ 1 = 1 + \beta + \gamma n \Rightarrow \beta + \gamma n = 0 + 0\cdot n \quad (\alpha,\beta,\gamma) = (1,0,0)$

$\boxed{A(n) = 1}$

$R_n = n;\ 0 = \alpha,\ n = n-1+\beta+\gamma n \Rightarrow \beta + \gamma n = 1 + 0\cdot n \quad (\alpha,\beta,\gamma) = (0,1,0)$

$\boxed{B(n) = n}$

$R_n = n^2$; $0 = \alpha$, $n^2 = (n-1)^2 + \beta + \gamma n \Rightarrow \beta + \gamma n = 2n - 1$; $(\alpha, \beta, \gamma) = (0, -1, 2)$

$2C(n) - B(n) = n^2 \Rightarrow \boxed{C(n) = (n^2+n)/2}$

$R_n = \alpha + \beta n + \gamma(n^2+n)/2$; $a_k = \beta + \gamma k$; $a_0 = \alpha$

$$\sum_{k=0}^{n} (a + bk)$$

Recurrence $\rightarrow$ Sum

$T_0 = 0$, $T_n = 2T_{n-1} + 1 \rightarrow T_n/2^n = T_{n-1}/2^{n-1} + 1/2^n$; $S_n = T_n/2^n$

$\hookrightarrow T_0/2^0 = 0/2^0$;

$\hookrightarrow S_n = S_{n-1} + 1/2^n$

$= S_{n-2} + 1/2^{n-1} + 1/2^n$

$= \vdots$

$= S_0 + \frac{1}{2} + \frac{1}{2^2} + \cdots + 1/2^n$

$= 0 + \dfrac{\frac{1}{2}(1 - 1/2^n)}{1 - 1/2} = 1 - \frac{1}{2^n}$

$S_n = 1 - 1/2^n$

$\Rightarrow T_n/2^n = 1 - 1/2^n$

$\Rightarrow T_n = 2^n - 1$

Lect. 5: Sums Sec. A

Sum of first n integers: $1+2+3+\cdots+(n-1)+n$

$1+2+\cdots+n$ ↳ term

$1+\cdots+n$

$a_1+a_2+\cdots+a_n$. General term: a_k

$1+2+\cdots+2^{n-1}$; $a_k=2^{k-1}$

$\sum_{k=1}^{n} a_k$ (sigma notation) $\sum_{1\le k\le n} a_k$ (generalized sigma notation)

$\sum_{\substack{1\le k\le 100\\ k \text{ odd}}} k^2$ $\sum_{k=0}^{49}(2k+1)^2$ | $\sum_{\substack{p\le N\\ p \text{ prime}}} \frac{1}{p}$ $\sum_{k=1}^{\pi(N)} \frac{1}{p_k}$

$\sum_{1\le k\le n} a_k = \sum_{1\le k+1\le n} a_{k+1}$; $\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1}$

$\sum_{P(k)} a_k$ sum over all k satisfying property $P(k)$

$\sum_{k=2}^{n-1} k(k-1)(n-k) = \sum_{k=0}^{n} k(k-1)(n-k)$

$[\text{statement}] = \begin{cases} 1, & \text{if statement is true} \\ 0, & \text{if statement is false} \end{cases}$ Iverson notation

$\sum_{k} a_k[P(k)]$ $\sum_{p} \frac{1}{p}[p\le N][p \text{ prime}]$

Sums → Recurrence $S_n=\sum_{k=0}^{n} a_k$, $S_0=a_0$; $S_n=\sum_{k=0}^{n-1} a_k + a_n$

$S_n = S_{n-1}+a_n$

$a_n=\beta+\gamma n$; $R_0=\alpha$, $R_n=R_{n-1}+\beta+\gamma n$

$R_1=\alpha+\beta+\gamma$, $R_2=\alpha+2\beta+3\gamma$ $R_n=A(n)\alpha+B(n)\beta+C(n)\gamma$

$R_n = 1$; $1 = \alpha$, $1 = 1 + \beta + \gamma n \Rightarrow \beta + \gamma n = 0 + 0 \cdot n$ $(\alpha, \beta, \gamma) = (1, 0, 0)$

$\boxed{A(n) = 1}$

$R_n = n$; $0 = \alpha$, $n = n - 1 + \beta + \gamma n \Rightarrow \beta + \gamma n = 1 + 0 \cdot n$ $(\alpha, \beta, \gamma) = (0, 1, 0)$

$\boxed{B(n) = n}$

$R_n = n^2$; $0 = \alpha$, $n^2 = (n-1)^2 + \beta + \gamma n \Rightarrow \beta + \gamma n = -1 + 2n$ $(\alpha, \beta, \gamma) = (0, -1, 2)$

$-B(n) + 2C(n) = n^2 \Rightarrow C(n) = (n^2 + n)/2$

$$\sum_{k=0}^{n} \underline{(a + bk)}$$

$a = \alpha$
$a = \beta$
$b = \gamma$

$R_0 = a \; (\alpha)$
$R_n = \beta + \gamma k$
$\beta \to a$, $\gamma \to b$

Recurrence $\rightarrow$ Sum

$T_0 = 0$, $T_n = 2T_{n-1} + 1 \Rightarrow T_n/2^n = T_{n-1}/2^{n-1} + 1/2^n$; $S_n = T_n/2^n$

$\hookrightarrow T_0/2^0 = 0/2^0$

$S_{n-1} = T_{n-1}/2^{n-1}$

$$\Rightarrow S_n = S_{n-1} + 1/2^n$$
$$= S_{n-2} + 1/2^{n-1} + 1/2^n$$
$$\vdots$$
$$= S_0 + \underbrace{1/2 + 1/2^2 \cdots + 1/2^n}$$

$$T_n/2^n = \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - 1/2} = 1 - 1/2^n$$

$$\Rightarrow T_n = 2^n - 1$$

$T_0 = 0,\ T_n = 2T_{n-1} + 1$

$a_n T_n = b_n T_{n-1} + C_n \Rightarrow s_n a_n T_n = s_n b_n T_{n-1} + s_n C_n$

$[s_n b_n = s_{n-1} a_{n-1}] \quad \Rightarrow s_n a_n T_n = s_{n-1} a_{n-1} T_{n-1} + s_n C_n$

$[s_n a_n T_n = S_n] \quad \Rightarrow S_n = S_{n-1} + s_n C_n$

$\qquad\qquad = S_{n-2} + s_{n-1} C_{n-1} + s_n C_n$

$\vdots$

$$\Rightarrow s_n a_n T_n = S_0 + \sum_{k=1}^{n} s_k C_k = s_0 a_0 T_0 + \sum_{k=1}^{n} s_k C_k = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k C_k$$

$$\Rightarrow T_n = \frac{1}{s_n a_n}\left(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k C_k\right)$$

$$s_n b_n = s_{n-1} a_{n-1} \Rightarrow s_n = \frac{a_{n-1}}{b_n} s_{n-1} = \frac{a_{n-1}}{b_n} \times \frac{a_{n-2}}{b_{n-1}} \times s_{n-2} = \cdots = \frac{a_{n-1} \cdots a_1}{b_n \cdots b_2} s_1$$

ToH: $a_n = 1,\ b_n = 2,\ C_n = 1$; $s_n = \underbrace{\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2}}_{(n-1)} \times \frac{1}{2} = 1/2^n$

$$T_n = 2^n \times 1 \times \sum_{k=1}^{n} \frac{1}{2^k} \times 1 = 2^n - 1$$

Quicksort: 

$a_i < a_k > a_j$

$C_0 = 0$

$C_n = (n+1) + C_0 + C_{n-1}$

$C_n = (n+1) + C_1 + C_{n-2}$

$C_n = (n+1) + C_2 + C_{n-3}$

$\vdots$

$\underline{C_n = (n+1) + C_{n-1} + C_0}$

$$nC_n = n(n+1) + 2\sum_{k=0}^{n-1} C_k \Rightarrow (n-1)C_{n-1} = n(n-1) + 2\sum_{k=0}^{n-2} C_k$$

$$\Rightarrow nC_n - (n-1)C_{n-1} = 2n + 2C_{n-1}$$

$$\Rightarrow C_n = n+1 + \frac{2}{n}\sum_{k=0}^{n-1} C_k \qquad \Rightarrow nC_n = (n+1)C_{n-1} + 2n$$

$$\qquad\qquad a_n T_n = b_n T_{n-1} + C_n$$

$$s_n = \frac{a_{n-1}a_{n-2}\cdots a_1}{b_n b_{n-1}\cdots b_2} = \frac{\cancel{(n-1)}(n-2)\cdots \cancel{3}\cdot 2\cdot 1}{(n+1)\,n\,\cancel{(n-1)}\cdots \cancel{3}} = \frac{2}{n(n+1)}$$

$$C_n = \cancel{n}(n+1)/2 \times \frac{1}{\cancel{n}} \sum_{k=1}^{n} \frac{2}{\cancel{k}(k+1)} \times 2\cancel{k} = 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1}$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n}\frac{1}{k}$$

$$\sum_{k=1}^{n}\frac{1}{k+1} = \sum_{1\le k\le n}\frac{1}{k+1} = \sum_{1\le k-1\le n}\frac{1}{k} = \sum_{2\le k\le n+1}\frac{1}{k} = \sum_{2\le k\le n}\frac{1}{k} + \frac{1}{n+1}$$

$$= \sum_{1\le k\le n}\frac{1}{k} + \frac{1}{n+1} - 1$$

$$C_n = 2(n+1)\left(H_n - \frac{n}{n+1}\right) \qquad = H_n - \frac{n}{n+1}$$

$$= 2(n+1)H_n - 2n \qquad \lim_{n\to\infty} H_n = \ln n$$

$$= O(n \lg n)$$

Manipulation of sums: ($\mathbb{K}$ is a finite set of integers)

1) $\sum_{k\in\mathbb{K}} c a_k = c\sum_{k\in\mathbb{K}} a_k$ (distributive law)

2) $\sum_{k\in\mathbb{K}} (a_k + b_k) = \sum_{k\in\mathbb{K}} a_k + \sum_{k\in\mathbb{K}} b_k$ (associative law)

3) $\sum_{k\in\mathbb{K}} a_k = \sum_{p(k)\in\mathbb{K}} a_{p(k)}$ (commutative law) $P(k)$ is a permutation over $\mathbb{K}$.

$\mathbb{K} = \{-1, 0, 1\}$, $P(k) = -k$ $\quad -1 \to 1,\ 0 \to 0,\ 1 \to -1$

$a_{-1} + a_0 + a_1 = a_1 + a_0 + a_{-1}$

$P(k) = n-k \quad 0 \to n,\ 1 \to n-1,\ \vdots,\ n \to 0$

$0 \le n-k \le n \to k \le n,\ 0 \le k \to 0 \le k \le n$

$$S = \sum_{0\le k\le n}(a + bk)$$

$$= \sum_{0\le n-k\le n}(a + b(n-k)) = \sum_{0\le k\le n}(a + b(n-k))$$

$$2S = \sum_{0\le k\le n}(a + bk + a + b(n-k)) = \sum_{0\le k\le n}(2a + bn)$$

$$= (2a + bn)\sum_{0\le k\le n} 1 = (2a + bn)(n+1) \Rightarrow S = \left(a + \frac{1}{2}bn\right)(n+1)$$

Lect. 6: summation factors and manipulation of sums — Sec. B

$T_0 = 0,\quad T_n = 2T_{n-1}+1$

$a_nT_n = b_nT_{n-1}+c_n \Rightarrow s_na_nT_n = s_nb_nT_{n-1}+s_nc_n$

$[s_nb_n = s_{n-1}a_{n-1}] \Rightarrow s_na_nT_n = s_{n-1}a_{n-1}T_{n-1}+s_nc_n$

$[s_na_nT_n = S_n] \Rightarrow S_n = S_{n-1}+s_nc_n$

$= S_{n-2}+s_{n-1}c_{n-1}+s_nc_n$

$\vdots$

$= S_0 + s_1c_1 + s_1c_2 + \cdots + s_nc_n = S_0 + \sum_{k=1}^{n} s_kc_k$

$s_na_nT_n = s_0a_0T_0 + \sum_{k=1}^{n} s_kc_k = s_1b_1T_0 + \sum_{k=1}^{n} s_kc_k$

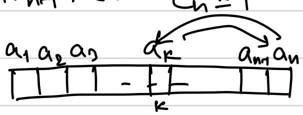
$\Rightarrow T_n = \frac{1}{s_na_n}\left(s_1b_1T_0 + \sum_{k=1}^{n} s_kc_k\right)$

$s_nb_n = s_{n-1}a_{n-1} \Rightarrow s_n = \frac{a_{n-1}}{b_n}\times s_{n-1} = \frac{a_{n-1}}{b_n}\times\frac{a_{n-2}}{b_{n-1}}\times s_{n-2} = \cdots = \frac{a_{n-1}\cdots a_1}{b_n\cdots b_2}s_1$

ToH: $T_n = 2T_{n-1}+1$ $\quad a_n = 1,\ b_n = 2,\ c_n = 1$ $\quad s_n = \frac{1\times 1\cdots\times 1}{2\times 2\cdots\times 2}s_1 = \frac{1}{2^{n-1}}\times\frac{1}{2} = \frac{1}{2^n}$

$a_nT_n = b_nT_{n-1}+c_n$

Quicksort:



$C_0 = 0$ $\quad a_i < a_k > a_j$

$C_n = (n+1)+C_0+C_{n-1}$

$C_n = (n+1)+C_1+C_{n-2}$

$\vdots$

$C_n = (n+1)+C_{n-1}+C_0$

$nC_n = n(n+1)+2\sum_{k=0}^{n-1}C_k \longrightarrow (n-1)C_{n-1} = n(n-1)+2\sum_{k=0}^{n-2}C_k$

$\Rightarrow C_n = (n+1)+\frac{2}{n}\sum_{k=0}^{n-1}C_k \Rightarrow nC_n-(n-1)C_{n-1} = 2n+2C_{n-1}$

$\Rightarrow nC_n = (n+1)C_{n-1}+2n$

$a_nT_n = b_nT_{n-1}+c_n$

$$S_n = \frac{a_{n-1}a_{n-2}\cdots a_1}{b_n b_{n-1}\cdots b_2} = \frac{\cancel{(n-1)}(n-2)\cdots\cancel{3}\cdot 2\cdot 1}{(n+1)n\cancel{(n-1)}\cdots\cancel{3}} = \frac{2}{n(n+1)}$$

$$C_n = \cancel{n}(n+1)/2 \times \frac{1}{\cancel{n}} \sum_{k=1}^{n} \frac{2}{\cancel{k}(k+1)} \times 2\cancel{k} = 2(n+1)\sum_{k=1}^{n}\frac{1}{k+1}$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n}\frac{1}{k}$$

$$\sum_{k=1}^{n}\frac{1}{k+1} = \sum_{1\le k\le n}\frac{1}{k+1} = \sum_{1\le k-1\le n}\frac{1}{k-1+1} = \sum_{2\le k\le n+1}\frac{1}{k} = \sum_{2\le k\le n}\frac{1}{k} + \frac{1}{n+1}$$

$$= \sum_{1\le k\le n}\frac{1}{k} + \frac{1}{n+1} - 1 = H_n - \frac{n}{n+1} \qquad \lim_{n\to\infty} H_n = \ln n$$

$$C_n = 2(n+1)\left(H_n - \frac{n}{n+1}\right) = 2(n+1)H_n - 2n = O(n\lg n)$$

Manipulation of sums: $\mathbb{K}$ is a finite set of integers

1) $\sum_{k\in\mathbb{K}} c a_k = c\sum_{k\in\mathbb{K}} a_k$ (distributive law)

2) $\sum_{k\in\mathbb{K}} (a_k + b_k) = \sum_{k\in\mathbb{K}} a_k + \sum_{k\in\mathbb{K}} b_k$ (associative law)

3) $\sum_{k\in\mathbb{K}} a_k = \sum_{p(k)\in\mathbb{K}} a_{p(k)}$ (commutative law) $p(k)$ is a permutation over $\mathbb{K}$.

$\mathbb{K} = \{-1, 0, 1\}$ $p(k) = -k$ $\begin{matrix} -1 \to 1 \\ 0 \to 0 \\ 1 \to -1 \end{matrix}$ $a_{-1} + a_0 + a_1 = a_1 + a_0 + a_{-1}$

$S = \sum_{0\le k\le n}(a+bk)$ $p(k) = n-k$ $\begin{matrix} 0 \to n \\ 1 \to n-1 \\ \vdots \\ n \to 0 \end{matrix}$ $S = \sum_{0\le n-k\le n}(a + b(n-k)) = \sum_{0\le k\le n}(a+b(n-k))$

$2S = \sum_{0\le k\le n}(a+bk) + \sum_{0\le k\le n}(a+b(n-k))$ $\quad \frac{0\le n-k}{\Rightarrow k\le n} \,\&\, \frac{n-k\le n}{0\le k} \Rightarrow 0\le k\le n$

$$= \sum_{0\le k\le n}(a + \cancel{bk} + a + bn - \cancel{bk}) = \sum_{0\le k\le n}(2a+bn) = (2a+bn)\sum_{0\le k\le n} 1$$

$$= (2a+bn)(n+1)$$

$$\Rightarrow S = \left(a + \frac{1}{2}bn\right)(n+1)$$