Chapter- 2

Random Variables (RV)

R.V.

Capital X দ্বারা প্রকাশ করা হয়

$$P\{X = 1\} = q$$

Let, X = number of heads appearing in tossing 2 fair coins.

$$P\{X = 0\} = P\{(T, T)\} = \frac{1}{4}$$

$$P\{X = 1\} = P\{(T, H), (H, T)\} = \frac{2}{4}$$

$$P\{X = 2\} = P\{(H, H)\} = \frac{1}{4}$$

Here,

$$P\{X = 0\} + P\{X = 1\} + P\{X = 2\} = 1$$

Random Variables

- Discreet RV
  - → Probability Mass Function
- Continuous RV
  - → Probability Density Function (PDF)

Cumulative Distribution Function:

$$F(a) = P\{X \leq a\}$$

Discreet RV

(i) Bernoulli RV

Either "success" or "failure"

// failure $\{X = 0\}$

// success $\{X = 1\}$

// $p \to$ probability of "success"

$$p(0) = P\{X=0\} = 1 - p$$

$$p(1) = P\{X=1\} = p$$

$$\sum_{i=0}^{1} p(i) = 1 - p + p = 1$$

(ii) Binomial RV

success occur in n trials.

$$p(i) = P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}$$

$\downarrow$

success $= i$ in n trials

p = prob. of succes

$1-p$ = " " failure

$$\sum_{i=0}^{n} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \binom{n}{0}(1-p)^n + \binom{n}{1} p (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2} + \cdots + \binom{n}{n} p^n$$

$$= \{p + (1-p)\}^n$$ // Binomial expansion

$$= 1$$

③ Geometric RV

success না হওয়া পর্যন্ত trial দিতে থাকবো।

X = # trials before getting success (including success)

$$p(i) = P\{X=i\} = \underbrace{p}_{\text{success (last trial)}} \underbrace{(1-p)^{i-1}}_{\#i-1 \text{ failed trial}}$$

success না আসা পর্যন্ত

$$\sum_{i=1}^{\infty} p(i) = \sum_{i=1}^{\infty} p(1-p)^{i-1}$$

$$= p \sum_{k=0}^{\infty} (1-p)^{k} \quad // [k = i-1]$$

$$= p \frac{1}{1-(1-p)} \quad // \text{জ্যামিতিক ধারা}$$

$$= 1$$

(iv) <u>Poission R.V</u> (Parameter λ)

λ: arrival rate (per second)

time t passed,

$$P\{X=i\} = \frac{e^{-\lambda} \lambda^{i}}{i!}$$

i arrival at time t.

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} \frac{e^{-\lambda} \lambda^{i}}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \cdots \right]$$

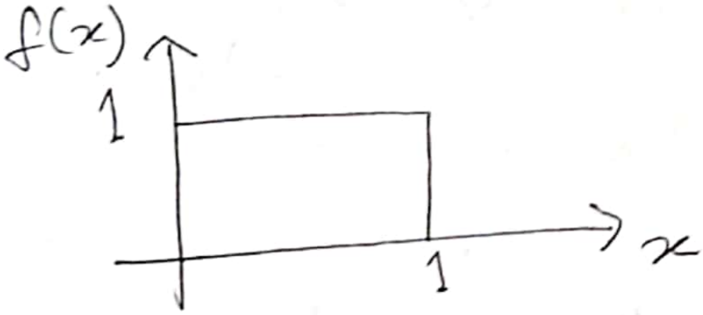
$$= e^{-\lambda} e^{\lambda}$$

$$= 1$$

Continuous R.V

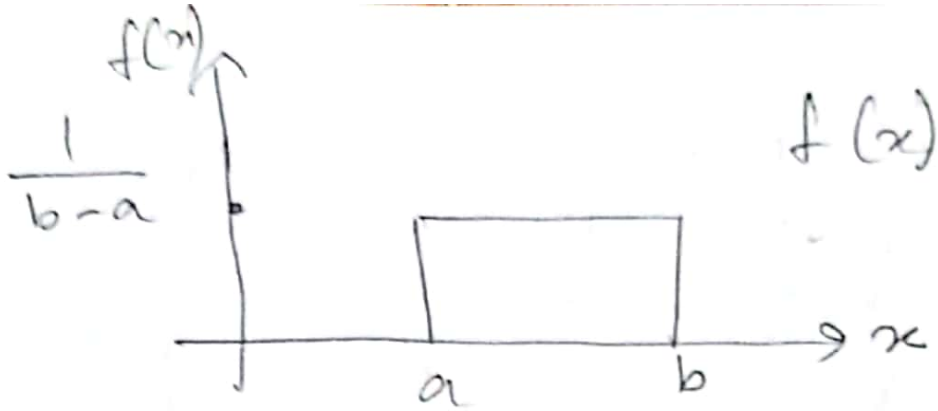
(i) Uniform RV

একটি Range এর মধ্যে সবার probability সমান।



$$f(x) = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

$f(x) \longrightarrow$ probability density function (PDF) in interval $[0, 1]$



$$f(x) = \begin{cases} \frac{1}{b-a} ; & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

এমনও হতে পারে।

$$\int_0^{\infty} f(x)\,dx = \int_0^1 1\,dx + \int_1^{\infty} 0\,dx$$

$$= 1$$

(ii) Exponential RV

$$f(x) = \begin{cases} \lambda e^{-\lambda x} ; & x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

cumulative distribution function,

$$F(a) = \int_0^a f(x)\,dx = 1 - e^{-\lambda a}$$

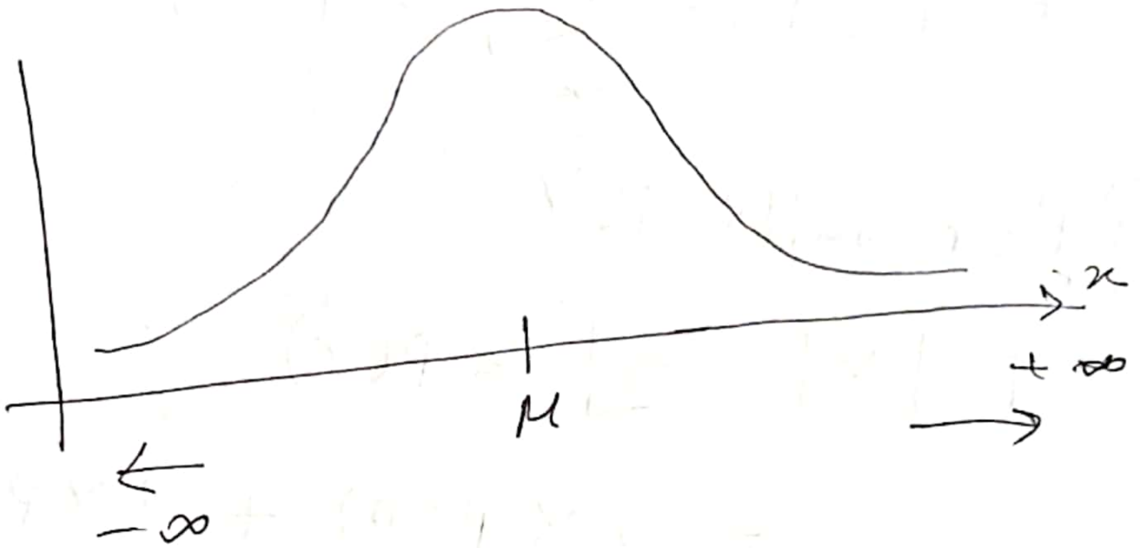
$$F(\infty) = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = 1$$

(iii) Gamma Random Variable

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} & ; \text{ if } x \geq 0 \\ 0 & ; \text{ if } x < 0 \end{cases}$$

(iv) Normal RV (Gaussian Distribution)

(μ, σ^2)



σ = standard deviation

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2 / 2\sigma^2}$$

$(-\infty < x < \infty)$

Expectation of a RV

Fair dice, $p(x) = \frac{1}{6}$ for all

$$E[X] = \left(\frac{1}{6} \times 1\right) + \left(\frac{1}{6} \times 2\right) + \left(\frac{1}{6} \times 3\right) + \left(\frac{1}{6} \times 4\right) + \left(\frac{1}{6} \times 5\right) + \left(\frac{1}{6} \times 6\right)$$

$$E[X] = \sum_x x\, P(x)$$

(i) Bernoulli RV

$$E[X] = \sum x\, p(x)$$

$$= 0 \times p(0) + 1 \times p(1)$$

$$= 0 + p$$

$$= p$$

$p(0) = 1 - p$

$p(1) = p$

(ii) Binomial RV

$$E[X] = \sum_{x=0}^{n} x \, p(x)$$

$$= \sum_{x=0}^{n} x \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{x \cdot n!}{x!\,(n-x)!} p^{x} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}$$

→ close form
নিলাম

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-k)!} p^{k} (1-p)^{n-k-1}$$

$[k = x-1]$

$$= np\,(p+1-p)^{n-1}$$

$$= np$$

(iii) Geometric RV

$$E[X] = \sum_{x=1}^{\infty} x\, p(x)$$

$$= \sum_{x=1}^{\infty} x\, p\, (1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} x\, q^{x-1} \qquad // \text{ Let, } [q = 1-p]$$

$$= p \sum_{x=1}^{\infty} \frac{\partial}{\partial q} q^{x}$$

$$= p \frac{\partial}{\partial q} \sum_{x=1}^{\infty} q^{x}$$

গুণোত্তর ধারা

$$= p \frac{\partial}{\partial q} \left(\frac{q}{1-q} \right)$$

$$= \frac{p}{(1-q)^2}$$

$$= \frac{p}{p^2}$$

$$= \frac{1}{p}$$

Poisson RV

$$E[X] = \sum_{x=0}^{\infty} x \, P(x)$$

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x}}{(x-1)!} \cancel{k}$$

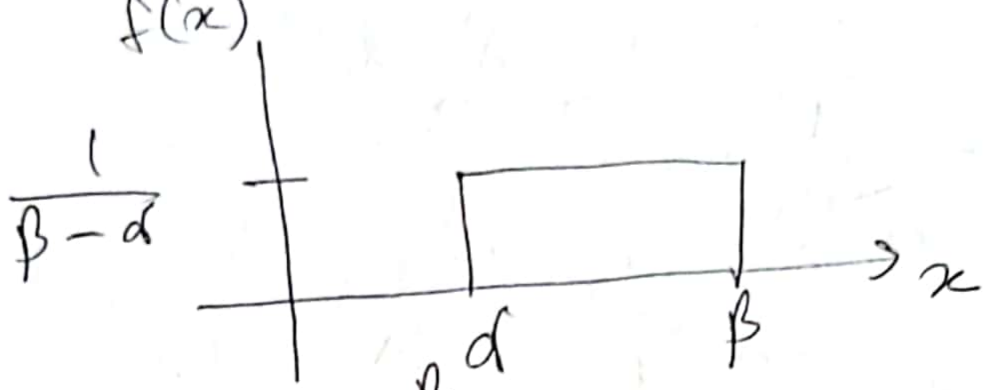
$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda}{k!} \qquad [k = x-1]$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

Continuous RV

Uniform RV



$$E[X] = \int_{\alpha}^{\beta} x f(x)\, dx$$

$$= \int_{\alpha}^{\beta} x \frac{1}{\beta - \alpha}\, dx$$

$$= \frac{1}{\beta - \alpha}\left[\frac{x^2}{2}\right]_{\alpha}^{\beta}$$

$$= \frac{\beta^2 - \alpha^2}{2(\beta - \alpha)}$$

$$= \frac{\alpha + \beta}{2}$$

Exponential R.V.

$$E[X] = \int_0^\infty x\, f(x)\, dx$$

$$= \int_0^\infty x \lambda e^{-\lambda x}\, dx$$

$$= \left[\lambda \left[x \frac{e^{-\lambda x}}{-\lambda} \right] - \lambda \left[\frac{e^{-\lambda x}}{(-\lambda)^2} \right] \right]_0^\infty$$

$\boxed{// \int uv\, dx = \cdots}$

$$= \left[-x e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda} \right]_0^\infty$$

$$= 0 - 0 + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}$$

বাকি 2টি বাতিল, লাগবে না।

Expected value of $g(x)$

$$E[g(x)] = \sum_x g(x)\, p(x)$$

$$E[ax+b] = \sum_x (ax+b)\, p(x)$$

$$= a\sum_x x\, p(x) + b\sum_x p(x)$$

$$= a\, E[X] + b \qquad // \sum_x p(x) = 1$$

Variance

$$Var(X) = E\left(X - E[X]\right)^2$$

$$= E\left[X^2 - 2XE[X] + \left(E[X]\right)^2\right]$$

$$= E[X^2] - 2\left(E[X]\right)^2 + \left(E[X]\right)^2$$

$$\boxed{Var(X) = E[X^2] - \left(E[X]\right)^2}$$

Jointly Distributed RV

Involving more than one RV

$$p(x,y) = P\{X = x, Y = y\}$$

$$F(x,y) = P\{X \le x, Y \le y\}$$

$$p_X(x, \cancel{y}) = \sum_y p(x,y) = P\{X = x\}$$

$$p_Y(\cancel{x}, y) = \sum_x p(x,y) = P\{Y = y\}$$

$$F_X(x) = P\{X \le x, y \le \infty\}$$

$$F_Y(y) = P\{X \le \infty, Y \le y\}$$

Now, $E[aX + bY] = a\,E[X] + b\,E[Y]$

$$= a \sum_{x,y} x\, p(x,y) + b \sum_{x,y} y\, p(x,y)$$

$$= a\,E[X] + b\,E[Y]$$

Co-Variance

$$Cov(X,Y) = E\left[(X - E[X])(Y - E[Y])\right]$$

$$= E\left[XY - XE[Y] - YE[X] + E[X]E[Y]\right]$$

$$\boxed{Cov(XY) = E[XY] - E[X]\,E[Y]}$$

$$E[XY] = E[X]\,E[Y]$$

→ // if X and Y are independent
i.e. Y, X doesn't depend on each other

$$\therefore \boxed{Cov(X,Y) = 0 \quad \text{// independent}}$$

→ No correlation betn X, Y
→ $Cov(X,Y) > 0 \rightarrow X\uparrow\ Y\uparrow$
→ $Cov(X,Y) < 0 \rightarrow X\uparrow\ Y\downarrow$

Variance

$$Var[X+Y] = E\left[(X+Y)^2\right] - \left(E[X+Y]\right)^2$$

$$= E\left[X^2+2XY+Y^2\right] - \left(E[X]+E[Y]\right)^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - \left(E[X]\right)^2 - \left(E[Y]\right)^2$$

$$- 2E[X]E[Y]$$

$$= E[X^2] - \left(E[X]\right)^2 + E[Y^2] - \left(E[Y]\right)^2$$

$$+ 2\left(E[XY] - E[X]E[Y]\right)$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Generalized

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, Y_j)$$

Binomial R.V. (n, p)

Variance (X)

$$X = X_1 + X_2 + X_3 + \cdots\cdots + X_n$$

$$Var(X) = \sum_{i=1}^{n} Var(X_i)$$

Now,

$$Var(X_i) = E[X_i^2] - (E[X_i])^2$$

$$= \sum_{x} x^2 p(x) - \left(\sum_{x} x\, p(x)\right)^2$$

$$= \left[0^2 (1-p) + 1^2 p\right] - \left[0(1-p) + 1 \cdot p\right]^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$\therefore Var(X) = \sum_{i=1}^{n} p(1-p) = np(1-p)$$

Sum of independent random variables

$X, Y \longrightarrow$ both are poisson distn

$\downarrow \quad \downarrow$

$\lambda_1 \;, \lambda_2 \longrightarrow$ parameters [arrival rate]

Q] $X + Y = \longrightarrow$ what kind of R.V.

Now,

$$P\{X+Y=n\} = \sum_{k=0}^{n} P\{X=k,\ Y=n-k\}$$

$$= \sum_{k=0}^{n} P\{X=k\}\ P\{Y=n-k\}$$

// independent now

$$= \sum_{k=0}^{n} \frac{e^{-\lambda_1}\lambda_1^{k}}{k!}\ \frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}$$

$$= \sum_{k=0}^{n} \frac{e^{-(\lambda_1+\lambda_2)}\ \lambda_1^{k}\ \lambda_2^{n-k}}{k!\ (n-k)!}$$

[close form -এ নেয়া লাগবে]

$$= \sum_{k=0}^{n} \frac{e^{-(\lambda_1+\lambda_2)}\ n!\ \lambda_1^{k}\ \lambda_2^{n-k}}{n!\ k!\ (n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n$$

So, $X+Y$ is a poisson distribution with parameter $\underline{(\lambda_1+\lambda_2)}$

এটা লিখতে লাগবে

$$\therefore E[X+Y] = \lambda_1 + \lambda_2$$

// poisson's Expected value $= \lambda$

Expectation or variance doesn't represent a distribution uniquely. So, moment generating function.

Moment Generating Function

$$E[X] = \sum_x x\, p(x)$$

$$Var[X] = E[X^2] - \left(E[X]\right)^2$$

$$= \sum_x x^2 p(x) - \left(\sum x\, p(x)\right)^2$$

$$\varphi(t) = E\left[e^{tX}\right]$$

$$= \begin{cases} \sum_x e^{tX} p(X) & ; \text{ if discrete} \\ \int_{-\infty}^{\infty} e^{tX} f(X)\, dx & ; \text{ if continuous} \end{cases}$$

If X is a RV, Moments of X are $E[X], E[X^2], \ldots\ldots, E[X^n]$.

$\varphi(t)$ দিয়ে এই moment গুলো বের করা যায়।

1st Moment, $E[X]$

$$\varphi'(t) = \frac{d}{dt}\varphi(t)$$

$$= \frac{d}{dt}\left(E\left[e^{tX}\right]\right)$$

$$= E\left[\frac{d}{dt}e^{tX}\right]$$

$$= E\left[Xe^{tX}\right]$$

$$t = 0 \rightarrow \varphi'(0) = E\left[Xe^{0}\right]$$

$$= E[X]$$

2nd Moment, $E[X^2]$

$$\varphi(t) = E\left[e^{tX}\right]$$

$$\varphi'(t) = E\left[Xe^{tX}\right]$$

$$\vdots$$

$$\varphi^{n}(t) = E\left[X^{n}e^{tX}\right]$$

$$\varphi''(t) = \frac{d}{dt}\varphi'(t)$$

$$= \frac{d}{dt} E[Xe^{tX}]$$

$$= E\left[\frac{d}{dt}(Xe^{tX})\right]$$

$$= E[X^2 e^{tX}]$$

$$\rightarrow \boxed{\varphi''(0) = E[X^2]}$$

<u>Binomial RV</u>

$$\varphi(t) = E[e^{tX}]$$

$$= \sum_x e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum (pe^t)^x (1-p)^{n-x} \binom{n}{x}$$

$$= (pe^t + 1 - p)^n$$

$\therefore \varphi(t) = (pe^t + 1 - p)^n$

$\varphi'(t) = n(pe^t + 1 - p)^{n-1} \cdot pe^t$

$\boxed{\varphi'(0) = E[X]}$

So, $\varphi'(0) = n(p + 1 - p)^{n-1} \cdot pe^0$

$\therefore \boxed{E[X] = np}$

Now,

$\varphi''(t) = \frac{d}{dt}\varphi'(t)$

$= \frac{d}{dt}\left[n(pe^t + 1 - p)^{n-1} pe^t\right]$

$= n(n-1)(pe^t + 1 - p)^{n-2}(pe^t)^2 +$

$n(pe^t + 1 - p)^{n-1} pe^t$

$\therefore \varphi''(0) = n(n-1)(p+1-p)^{n-2} \cdot (p)^2 +$

$n(p+1-p)^{n-1} p$

$\therefore\ E[X^2] = n(n-1)p^2 + np \quad // \varphi''(0) = E[X$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= \{n(n-1)p^2 + np\} - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

// $E[X^3]$, $E[X^4]$, ... eta syllabus

// এ নাই..

Poisson RV with mean λ

$$\varphi(t) = E\left[e^{tX}\right]$$

$$= \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda} e^{\lambda e^{t}} \qquad // \sum_{x=0}^{\infty} \frac{a^{x}}{x!} = e^{a}$$

$$\boxed{\varphi(t) = e^{\lambda(e^{t}-1)}}$$

$$\varphi'(t) = \frac{d}{dt}\left\{e^{\lambda(e^{t}-1)}\right\}$$

$$= e^{\lambda(e^{t}-1)} \lambda e^{t}$$

$$\varphi''(t) = \frac{\partial}{\partial t}\varphi'(t)$$

$$= e^{\lambda(e^t-1)}(\lambda e^t)^2 + \lambda e^t e^{\lambda(e^t-1)}$$

So,

$$\varphi'(0) = E[X] = \lambda$$

$$\varphi''(0) = E[X^2] = \lambda^2 + \lambda$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$= (\lambda^2 + \lambda) - (\lambda)^2$$

$$= \lambda$$

Sum of Independent RV

$\varphi_{X+Y}(t) = (X+Y)$ এর moment generating function

X and Y are independent

$$E[X].E[Y] = E[XY]$$

$$\varphi_{X+Y}(t) = E\left[e^{t(X+Y)}\right]$$

$$= E\left[e^{tX} \cdot e^{tY}\right]$$

$$= E\left[e^{tX}\right] \cdot E\left[e^{tY}\right]$$

$$\boxed{\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t)}$$

Sum of Indep. R.V.s

$X \rightarrow$ Binomially distributed (n, p)

$Y \rightarrow$ " " (m, p)

$X+Y \rightarrow$? (what kind of distbⁿ)

$$\varphi_X(t) = (pe^t + 1 - p)^n$$

$$\varphi_Y(t) = (pe^t + 1 - p)^m$$

// p same
// for X and Y

$\varphi_{X+Y}(t) = \varphi_X(t)\,\varphi_Y(t)$ // independent

$= (pe^t + 1 - p)^{n+m}$

// matches with binomial

$\therefore X+Y$ is Binomial RV with

parameter $\underbrace{(n+m, p)}$ → এটা না লিখলে 2 marks কাটে

$X \to$ Poisson RV (λ_1)

$Y \to$ " " (λ_2)

$X+Y \to$?

$\varphi_X(t) = e^{\lambda_1(e^t - 1)}$ // মুখস্ত নেয়া
// যাবেনা, বের করা
// লাগবে

$\varphi_Y(t) = e^{\lambda_2(e^t - 1)}$

$\varphi_{X+Y}(t) = e^{(\lambda_1 + \lambda_2)(e^t - 1)}$

$\therefore X+Y$ is Poisson RV with param. $(\lambda_1 + \lambda_2)$