Chap - 4

Markov chain

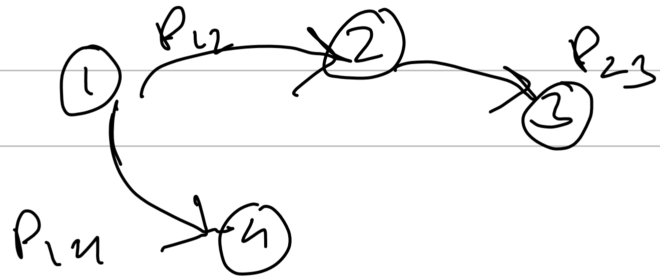
Chain → Previous state এর উপর Depend করে আগামী state predict করা

Stochastic Process

- a collection of RVs.
  - → time এর উপর define করা হয়
  - → $\{X(t), t \in T\}$
  - → Transition Probability নেই

- Markov Chain:

- Markov Chain is an stochastic process $\{X_n, n = 1, 2, 3, \ldots\}$ such that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j



$$P_{ij} = P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i-1, \ldots, \ldots X_1 = i, X_0 = i_0\}$$

The P_{ij} is independent of past states

and depends only on the present state.

(i) $P_{ij} \geq 0$

(ii) $\sum_j P_{ij} = 1$, at each step, there must be a transition.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & & n \end{array} \\ \begin{bmatrix} P_{00} & P_{01} & \cdots & P_{0n} \\ P_{10} & P_{11} & \cdots & P_{1n} \\ \vdots & & & \\ P_{n0} & P_{n1} & \cdots & P_{nn} \end{bmatrix} \end{array}$$

#row → # col ⌋ transition.

Ex:

Weather : Rainy (R) , Sunny (S)

(states)

$$P = \begin{array}{c|cc} & R & S \\ \hline R & \alpha & 1-\alpha \\ S & \beta & 1-\beta \end{array}$$

row → today
col → tomorrow

α → today rainy → tomorrow rainy

β → " sunny → " rainy

Communication System:

$$1/0 \to \boxed{\qquad\qquad\qquad} \to 1/0$$

$$P = \begin{array}{c|cc} & 0 & 1 \ \text{output} \\ \hline 0 & p & 1-p \\ 1 & 1-p & p \\ \text{input} & & \end{array}$$

Given

$$\boxed{\left.\begin{matrix} 1 \to 1 \\ 0 \to 0 \end{matrix}\right\} p}$$

Chapman-Kolmogorov Equation

state (i) ———————— state (j)

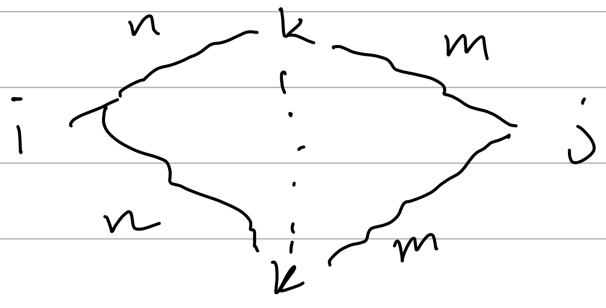
P_{ij} = one step transition probability

$P_{ij}^{(n)}$ = n " " " that a process is in a state, i will reach state, j after n transitions.

$$P_{ij}^{(n+m)} = \sum_{k} P_{ik}^{n} P_{kj}^{m}$$

$\downarrow$

i থেকে j তে (n+m) step এ যাওয়ার probability



Proof:

$$P_{ij}^{(n+m)} = P\{X_{n+m} = j \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P\{X_{n+m} = j \mid X_n = k, X_0 = i\} \, P\{X_n = k \mid X_0 = i\}$$

$$= \cancel{\sum} \sum_{k=}^{\infty} P\{X_{n+m} = j \mid X_n = k\} P\{X_n = k \mid X_0 = i\}$$

$$= \sum_{k=0}^{\infty} P^{m}_{\cancel{k}j} \, P^{n}_{jk} \quad \cancel{=}$$

$$P^{n+m} = P^{n} P^{m}$$

$n = 1, \; m = 1$

P^2 제곱

Example

$$\alpha = 0.7, \quad \beta = 0.4$$

$$P = \begin{array}{c c} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$$P^2 = P \cdot P$$

$$= \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$P_{00}^{(2)} = 0.61 \rightarrow$ Today Rainy, 2 din পরও Rainy

$P_{11}^{(2)} = 0.48 \rightarrow$ " Sunny, 2 din পর sunny

$$P^4 = P^2 \times P^2$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\{0, 1\} \longrightarrow$ Recurrent state

0, 1 এ একবার গেলে আর বের হতে পারবে না

0, 1 এই ঘুরবে।

$\{2\} \longrightarrow$ Transient state

(long term এ বের হয়ে যাবে)

$\{3\} \longrightarrow$ Absorbing state.

long term এ সেখানে

গিয়ে terminate করবে

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Markov chain is irreducable if there is only one class

i.e. all state communicate with each othe.

Markov chain

X_1 X_2 X_3

$$P = \begin{array}{c|cccc} {}_{MT} \backslash {}^{TW} & RR & RS & SR & SS \\ \hline RR & 0.7 & 0.3 & 0 & 0 \\ RS & 0 & 0 & 0.4 & 0.6 \\ SR & 0.5 & 0.5 & 0 & 0 \\ SS & 0 & 0 & 0.2 & 0.8 \end{array}$$

MT → Mon and Tue

TW → Tue and Wed

MT → RR ?

Thursday → Rainy → Prob ?

$$P^{(2)} = \begin{array}{c|cccc} {}_{MT} \backslash {}^{WTH} & RR & RS & SR & SS \\ \hline RR & 0.49 & 0.21 & 0.12 & 0.18 \\ RS & 0.20 & 0.20 & 0.12 & 0.48 \\ SR & 0.35 & 0.15 & 0.20 & 0.30 \\ SS & 0.1 & 0.1 & 0.16 & 0.64 \end{array}$$

$$P_{00} + P_{02} = 0.49 + 0.12$$
$$= 0.61$$

Limiting Probability

Theorem:

if $\lim_{n\to\infty} P_{ij}^n$ exists and is independent of i and $\pi_j = \lim_{n\to\infty} P_{ij}^n$, $j \geq 0$.

then π_j is the unique non negative sol^n of

$$\pi_j = \sum_{i=0}^{m} \pi_i P_{ij}, \quad j \geq 0$$

$$\sum_{j=0}^{m} \pi_j = 1$$

Ex

$$\begin{array}{c|cc} & R & S \\ \hline R & \alpha & 1-\alpha \\ S & \beta & 1-\beta \end{array}$$

$$\begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

$\pi_0 \qquad \pi_1$

$$\pi_0 + \pi_1 = 1$$

$$\pi_0 = \alpha \pi_0 + \beta \pi_1$$

$$\pi_1 = (1-\alpha)\pi_0 + (1-\beta)\pi_1$$

$$\pi_0 = \frac{\beta}{1-\alpha+\beta} \quad \text{(rainy) long run}$$

$$\pi_1 = \frac{1-\alpha}{1-\alpha+\beta} \quad \text{(sunny)}$$

$$\begin{bmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{bmatrix}$$

2 properties of Markov Chain

(i) Period → state i is said to have period of d if $P_{ii} = 0$ whenever n is not divisible by d and d is the largest integer with this property.

A state with period 1 is said to be aperiodic.

Markov Chain in Genetics

Hardy Weinburg Law

Male - Female ratio same
অনুপাত 1

$$P\{AA\} = p_0, \quad P\{aa\} = q_0, \quad P\{Aa\} = r_0$$

$$p_0 + q_0 + r_0 = 1$$

Let,
next generation probability

$$P\{AA\} = p, \quad P\{aa\} = q, \quad P\{Aa\} = r$$

$$P\{A\} = P\{A \mid AA\} p_0 + P\{A \mid aa\} q_0 + P\{A \mid Aa\} r_0$$

$$= p_0 + 0 + \frac{r_0}{2}$$

$$= p_0 + \frac{r_0}{2}$$

$$p = P\{AA\} = \left(p_0 + \frac{r_0}{2}\right)^2$$

$$q = P\{aa\} = \left(q_0 + \frac{r_0}{2}\right)^2$$

$$r = P\{Aa\} = \binom{2}{1}\left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$$

$$p + q + r = \left(p_0 + \frac{r_0}{2}\right)^2 + \left(q_0 + \frac{r_0}{2}\right)^2$$

$$+ 2\left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$$

$$= \left(p_0 + q_0 + r_0\right)^2 = 1^2 = 1$$

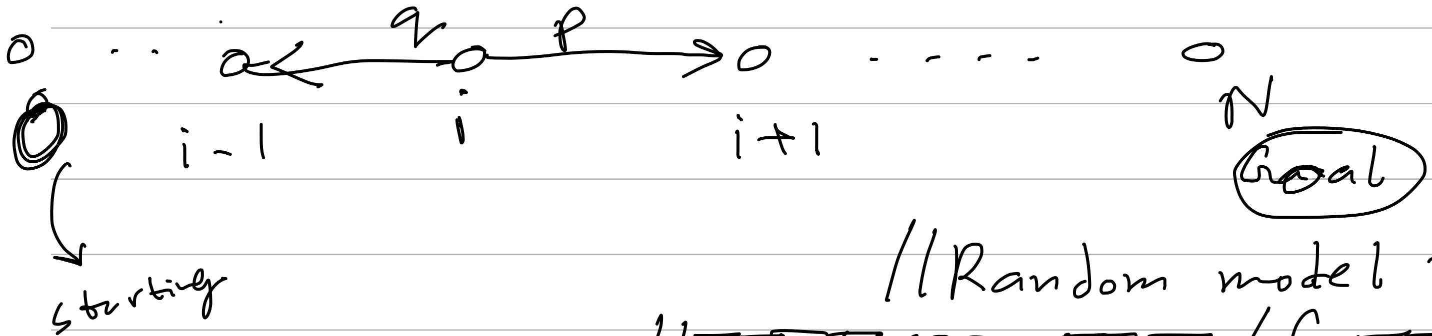
$$p + \frac{r}{2} = \left(p_0 + \frac{r_0}{2}\right)^2 + \left(p_0 + \frac{r_0}{2}\right)\left(q_0 + \frac{r_0}{2}\right)$$

$$= \left(p_0 + \frac{r_0}{2}\right)\left(p_0 + q_0 + r_0\right)$$

$$= p_0 + \frac{r_0}{2}$$

Markov Chain

The Gambler's Ruin Problem



// Random model

// ডরলেও যায় সামনে / পিছনে যায়

Let, $P_i \to$ Prob that starting from i the Gambler will eventually reach N.

$$P_i = p P_{i+1} + q P_{i-1}$$

$$\Rightarrow (p+q) P_i = p P_{i+1} + q P_{i-1}$$

// $p+q = 1$

$$\Rightarrow P_{i+1} - P_i = \frac{q}{p} (P_i - P_{i-1})$$

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p} P_1 \quad \left.\right\} \begin{matrix} P_0 = 0 \\ P_N = 1 \end{matrix}$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$

$$P_i - P_{i-1} = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$P_i - P_1 = P_1 \left[\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1} \right]$$

$$P_i = \begin{cases} P_1 \dfrac{\left(1 - \frac{q}{p}\right)^i}{1 - \frac{q}{p}} & ; \text{ if } q \neq p \\ i P_1 & \text{if } q = p \end{cases}$$

$$P_N = \begin{cases} \dfrac{P_1 \quad 1 - \left(\frac{q}{p}\right)^{N}}{\left(1 - \frac{q}{p}\right)} & , \ p \neq \frac{1}{2} \\ N P_1 & , \ p = \frac{1}{2} \end{cases}$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}, & \text{if } p \neq \frac{1}{2} \\ \dfrac{i}{N}, & p = \frac{1}{2} \end{cases}$$

$$P_i = \begin{cases} \left(1-\frac{q}{p}\right)^i & p > \frac{1}{2} \\ 0 & p < \frac{1}{2} \end{cases}$$