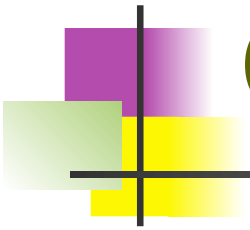
Constraint Satisfaction Problems

Chapter 5

Section 1 – 3



Outline

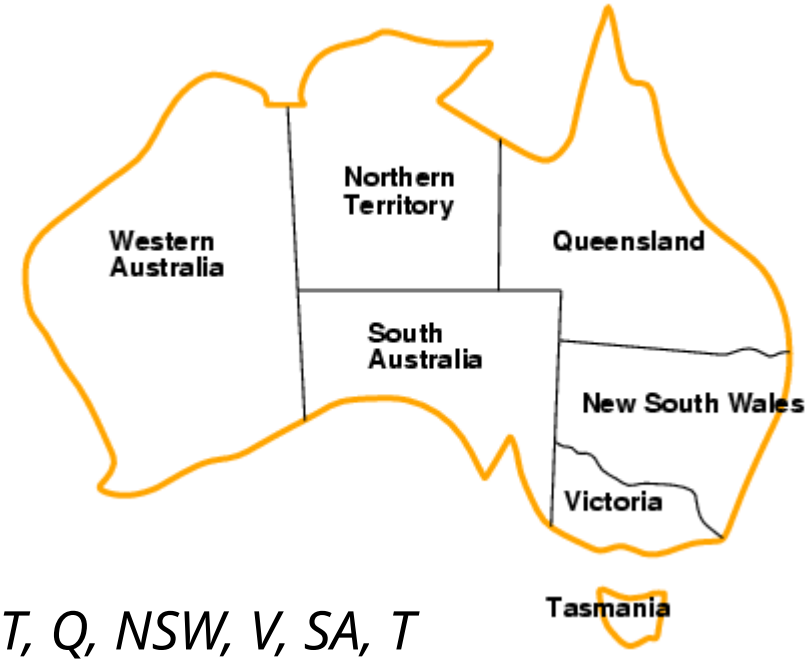
- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

Constraint satisfaction problems (CSPs)

Standard search problem:

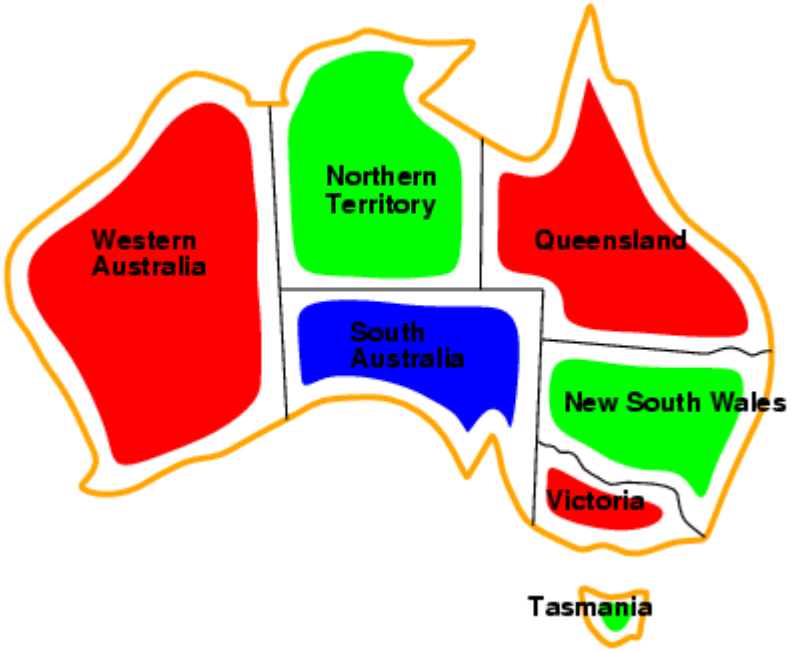
- state is a "black box“ – any data structure that supports successor function, heuristic function, and goal test
  - CSP:
- state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring



- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
-
- e.g., WA ≠ NT, or (WA,NT) in {(red,green),(red,blue),(green,red), (green,blue),(blue,red),(blue,green)}
-

Example: Map-Coloring

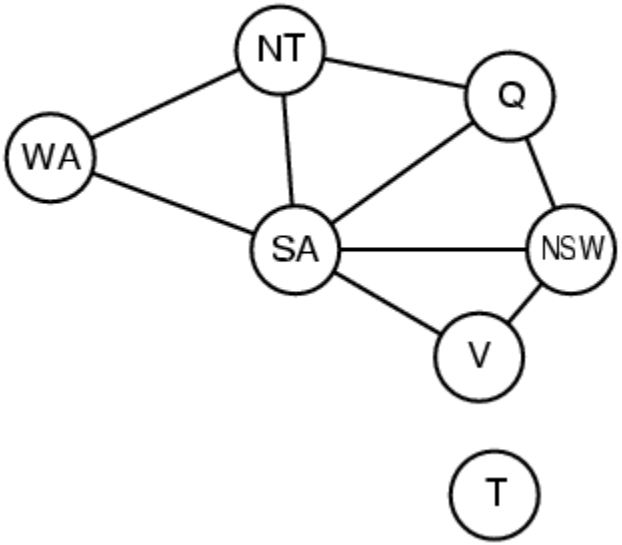


Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

Constraint graph

Binary CSP: each constraint relates two variables

- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

Discrete variables

- finite domains:
  - n variables, domain size d ☾ $O(d^n)$ complete assignments
  - e.g., Boolean CSPs, incl.~Boolean satisfiability (NP-complete)
- infinite domains:
  - integers, strings, etc.
  - e.g., job scheduling, variables are start/end days for each job
  - need a constraint language, e.g., $StartJob_1 + 5 \leqslant StartJob_3$

Continuous variables

- e.g., start/end times for Hubble Space Telescope observations
- linear constraints solvable in polynomial time by linear programming

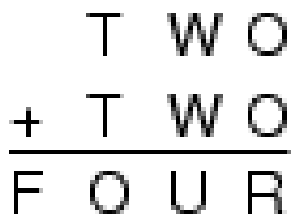
Varieties of constraints

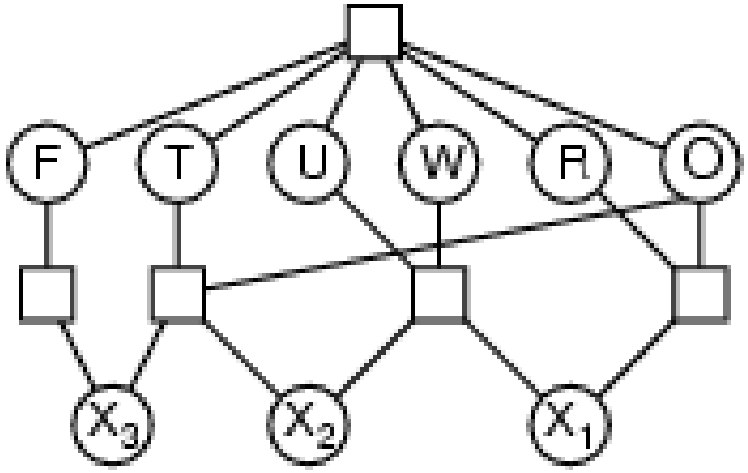
- Unary constraints involve a single variable,
- e.g., SA ≠ green

- Binary constraints involve pairs of variables,
- e.g., SA ≠ WA

- Higher-order constraints involve 3 or more variables,
- e.g., cryptarithmetic column constraints

Example: Cryptarithmetic





- Variables: *F T U W R O* X_1 X_2 X_3
- Domains: {*0,1,2,3,4,5,6,7,8,9*}
- Constraints: *Alldiff (F,T,U,W,R,O)*
-
- $O + O = R + 10 \cdot X_1$
  -
  - $X_1 + W + W = U + 10 \cdot X_2$
  -
  - $X_2 + T + T = O + 10 \cdot X_3$

Real-world CSPs

- Assignment problems
- e.g., who teaches what class
  - Timetabling problems
- e.g., which class is offered when and where?
  - Transportation scheduling
- Factory scheduling
- Notice that many real-world problems involve real-valued variables

Standard search formulation (incremental)

Let's start with the straightforward approach, then fix it

States are defined by the values assigned so far

- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
- ☾ fail if no legal assignments

- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables
   ☾ use depth-first search
3. Path is irrelevant, so can also use complete-state formulation
4. b = (n - l)d at depth l, hence n! · d^n leaves
5.

Backtracking search

- Variable assignments are commutative}, i.e.,
- [WA = red then NT = green] same as [NT = green then WA = red]

- Only need to consider assignments to a single variable at each node
- b = d and there are d^n leaves

Depth-first search for CSPs with single-variable assignments is called backtracking search

Backtracking search is the basic uninformed algorithm for CSPs

Can solve *n*-queens for $n \approx 25$

Backtracking search

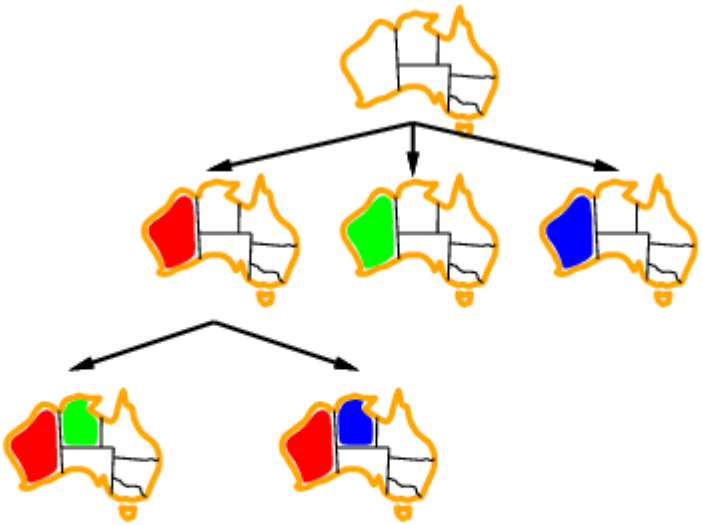
```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
    return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment according to Constraints[csp] then
            add { var = value } to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failue then return result
            remove { var = value } from assignment
    return failure
```

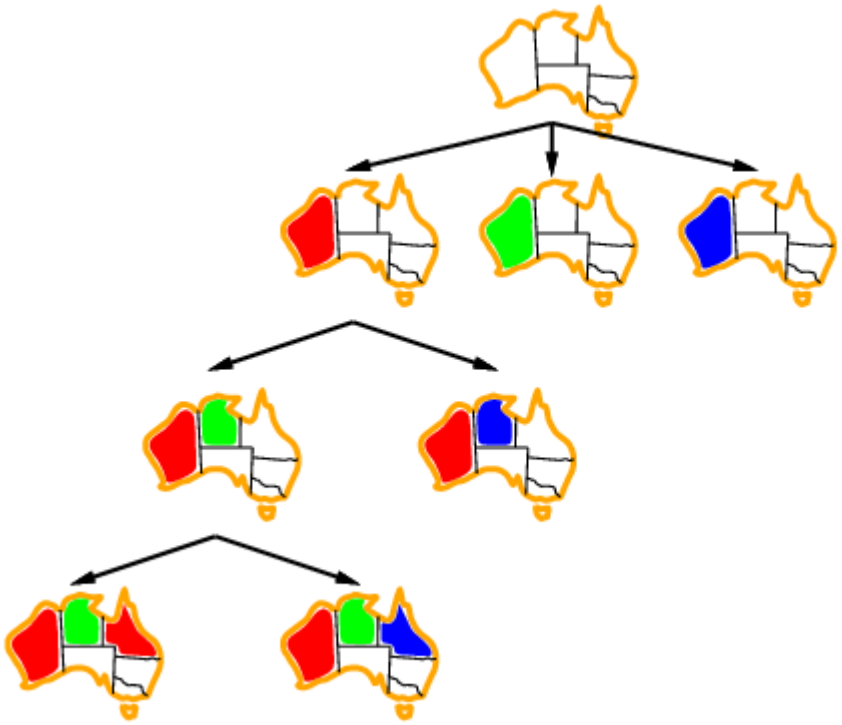
Backtracking example

Backtracking example

Backtracking example



Backtracking example



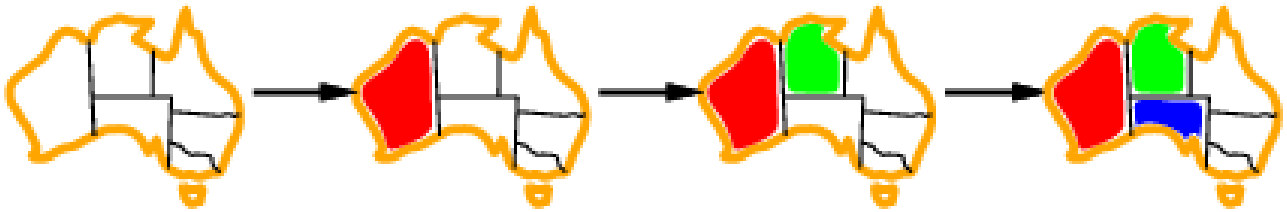
Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

- Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

Most constrained variable

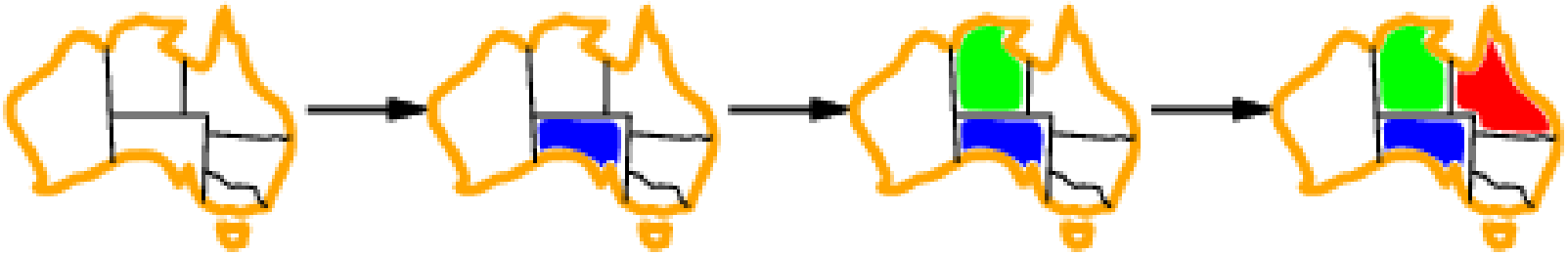
- Most constrained variable:
- choose the variable with the fewest legal values



a.k.a. minimum remaining values (MRV) heuristic

Most constraining variable

- Tie-breaker among most constrained variables
- Most constraining variable:
-
- choose the variable with the most constraints on



Least constraining value

Given a variable, choose the least constraining value:

-
- the one that rules out the fewest values in the remaining variables
  - 

Combining these heuristics makes 1000 queens feasible

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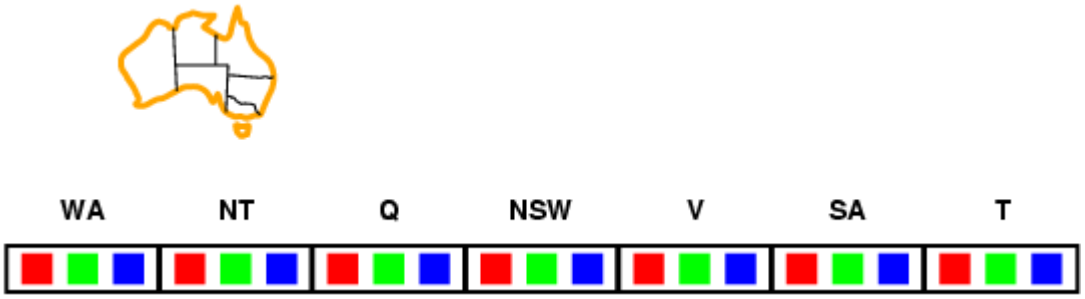
Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values
  -



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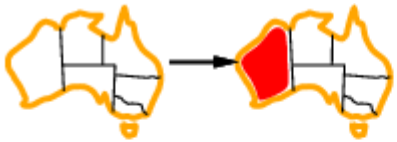

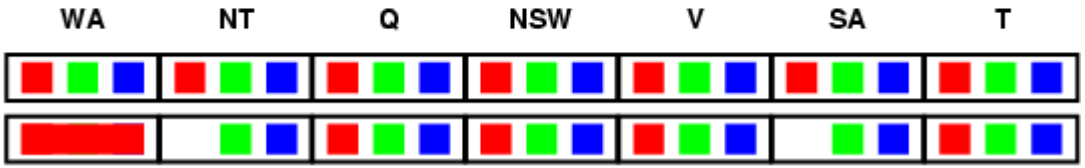

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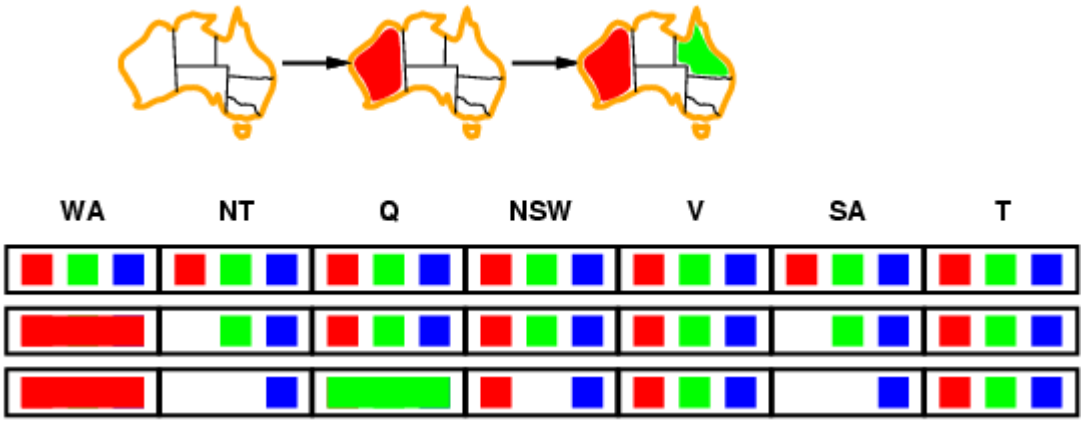

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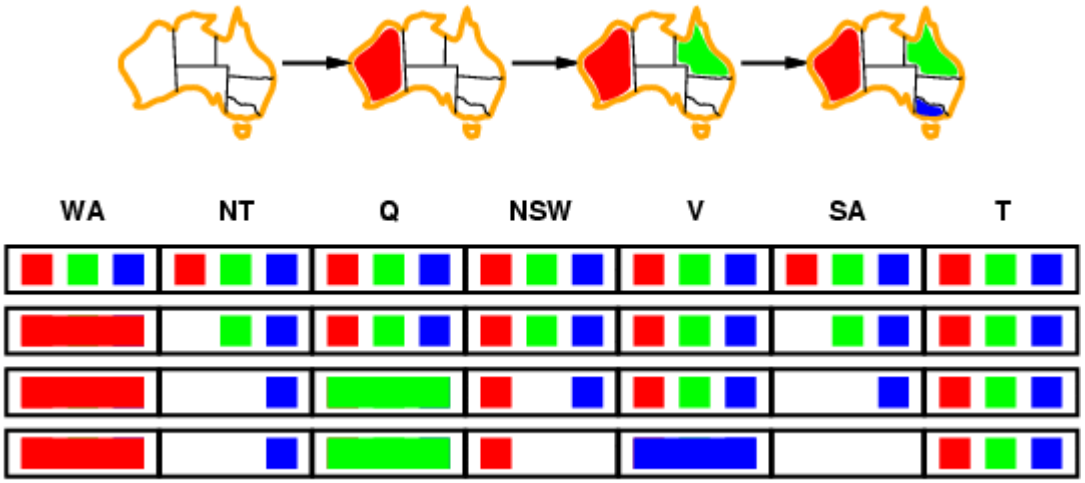
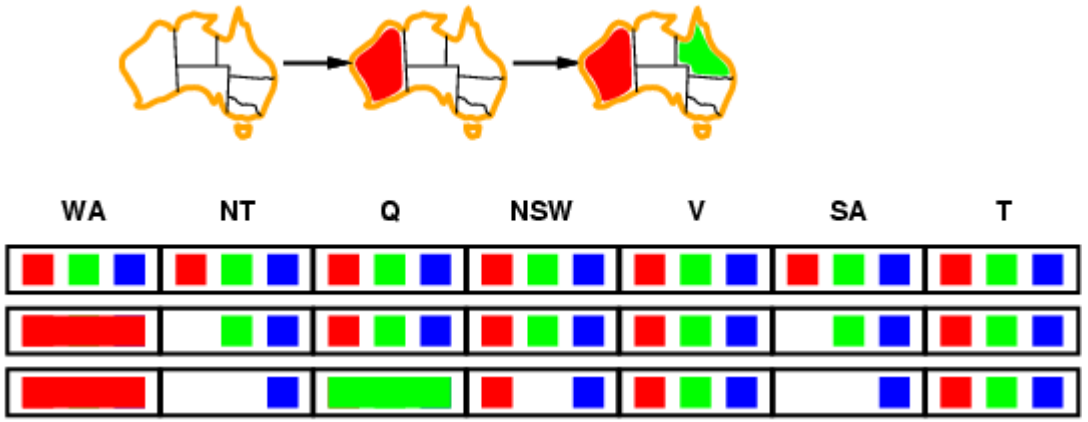


Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

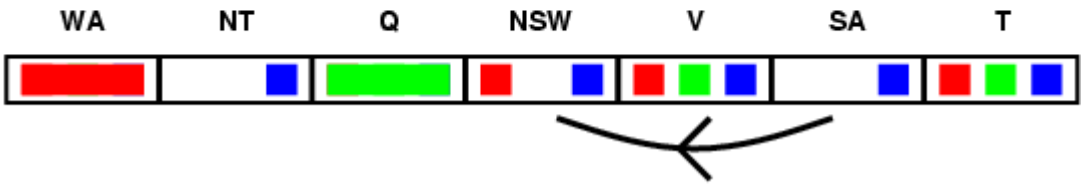


NT and SA cannot both be blue!

- Constraint propagation repeatedly enforces constraints locally

Arc consistency

- Simplest form of propagation makes each arc consistent
- X ☾ Y is consistent iff
-
- for every v



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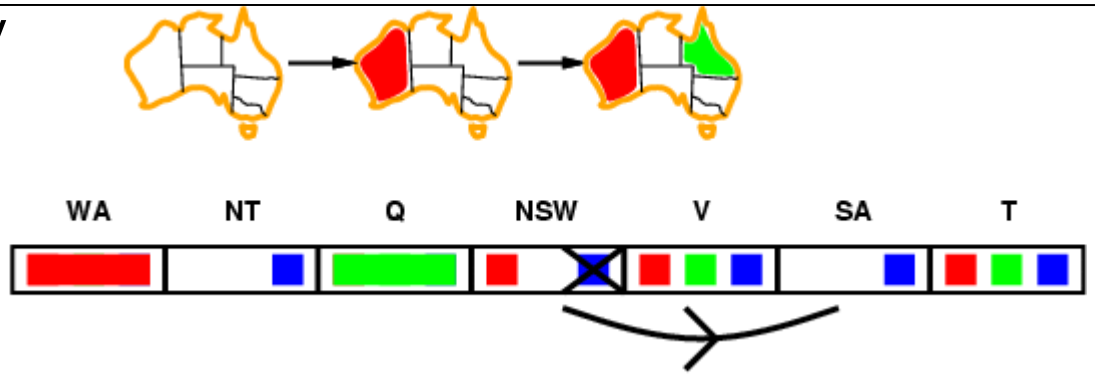

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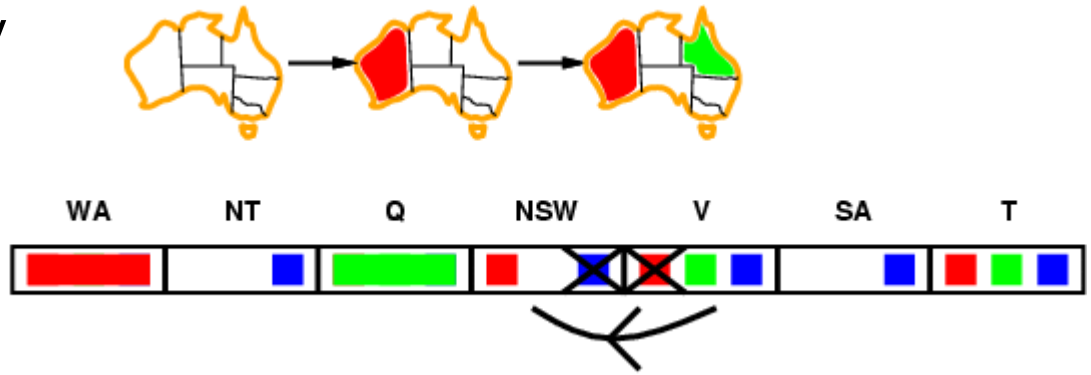

If X loses a value, neighbors of X need to be rechecked

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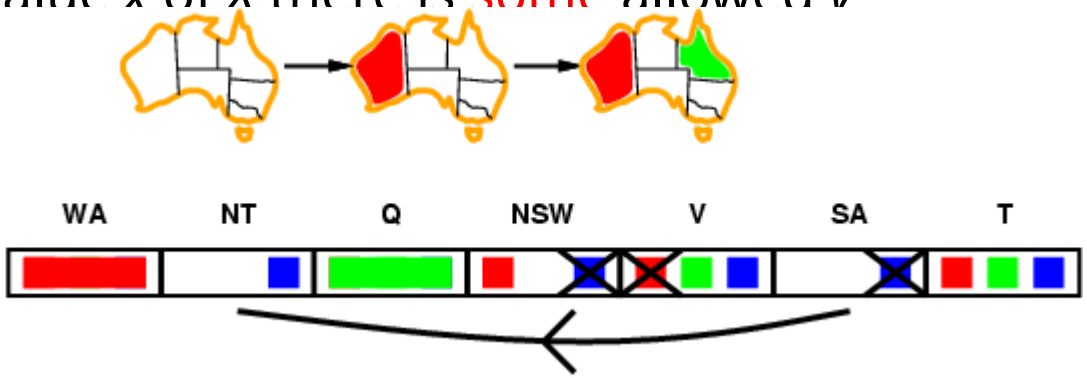
Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \subset Y$ is consistent iff
-
- for every value x of X there is some allowed y



- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
-
- Can be run as a preprocessor or after each assignment

Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
  inputs: csp, a binary CSP with variables {X_1, X_2, ..., X_n}
  local variables: queue, a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (X_i, X_j) ← REMOVE-FIRST(queue)
    if RM-INCONSISTENT-VALUES(X_i, X_j) then
      for each X_k in NEIGHBORS[X_i] do
        add (X_k, X_i) to queue

---

function RM-INCONSISTENT-VALUES( X_i, X_j) returns true iff remove a value
  removed ← false
  for each x in DOMAIN[X_i] do
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy constraint(X_i, X_j)
      then delete x from DOMAIN[X_i]; removed ← true
  return removed
```

Time complexity: $O(n^2d^3)$

Local search for CSPs

Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned

To apply to CSPs:

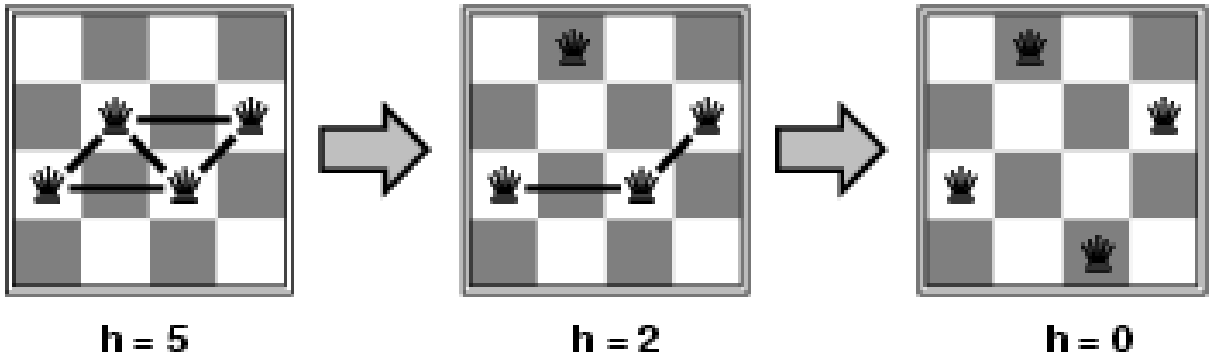
- allow states with unsatisfied constraints
  - operators reassign variable values

Variable selection: randomly select any conflicted variable

Value selection by min-conflicts heuristic:

Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)
Actions: move queen in column
Goal test: no attacks
Evaluation: *h(n)* = number of attacks



- Given random initial state, can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)

Summary

CSPs are a special kind of problem:

- states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice