HW #5

1 a) $\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} y_i - \hat{y}_i = \sum_{i=1}^{n} y_i - \bar{y} + r \frac{\sigma_y}{\sigma_x} (x_i - \bar{x})$

$= -n\bar{y} - r \frac{\sigma_y}{\sigma_x} n\bar{x} + \sum_{i=1}^{n} y_i + r \frac{\sigma_y}{\sigma_x} \sum_{i=1}^{n} x_i$

$= \left(- \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} y_i \right) + \left(- r \frac{\sigma_y}{\sigma_x} \sum_{i=1}^{n} x_i + r \frac{\sigma_y}{\sigma_x} \sum_{i=1}^{n} x_i \right)$

$= 0 + 0 = 0$

b) Taking the average of the residuals

$\frac{1}{n} \sum_{i=1}^{n} e_i = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - \hat{y}_i \right) = \bar{y} - \bar{\hat{y}}$

We know the sum of residuals is 0, so the average is also 0. (part a)

So $\bar{y} - \bar{\hat{y}} = 0$

thus $\bar{y} = \bar{\hat{y}}$

c) Evaluating the regression line function at $x = \bar{x}$ yields $\hat{y} = \bar{y}$

$\hat{y} = \bar{y} + r \frac{\sigma_y}{\sigma_x} (\bar{x} - \bar{x}) = \bar{y}$

so the regression line contains point $(\bar{x}, \bar{y})$

2 a) Recall that $\hat{Y}$ is the closest point

the true response vector Y that exists

in the column space of X (which is span $\{\vec{1}, \vec{x}\}$)

This means that $e = Y - \hat{Y}$ is the component

of Y that is orthogonal to the column space

of X. That means that the dot product

between the residual e and any vector in the

column space is 0.

Note that the first column of X is $\vec{1}$

so $e^T \vec{1} = \sum_{i=1}^{n} e_i = 0$

b) As stated in part a), $\hat{Y} = X\hat{\theta}$

is in the column space of X, or in other words,

it is in the span of X. This means $e = Y - \hat{Y}$

contains the component of Y that is orthogonal

to the span of X. So e is orthogonal to every

column vector in X, which includes $\vec{x}$.

c) As stated in parts a and b, $\hat{Y}$ is in the column

space of X, which is span $\{\vec{1}, \vec{x}\}$, e contains

the component of Y orthogonal to the column space of

X (which is where $\hat{Y}$) exists since $e = Y - \hat{Y}$

So e is orthogonal to $\hat{Y}$

3. $\frac{\partial R(\gamma)}{\partial \gamma} = \frac{-2}{n}\left(\sum_{i=1}^{n} (y_i - \hat{\gamma} x_i) x_i\right) = 0$

$$= \sum_{i=1}^{n} x_i y_i - \hat{\gamma} \sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow \hat{\gamma} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

4 a) This doesn't hold anymore. The proof in part 2 a required that $\vec{1}$ be in the column space of X. Since it is not, we can't prove $\vec{1}^T e = \sum_{i=1}^{n} e_i = 0$

b) This still holds. $\hat{Y}$ still exists in our coulumn space, which is given by span $\{\vec{x}\}$ meaning $e = Y - \hat{Y}$ is still orthogonal to $\hat{Y}$ as it is the component of Y orthogonal to the column space of X, which is span $\{\vec{x}\}$. So $\hat{Y}$ is still orthogonal to $\vec{x}$.

c) This still holds.

By the previous part, since $\hat{Y} \in$ span $\{\vec{x}\}$ and the residual e is orthogonal to the span $\{\vec{x}\}$ e must also be orthogonal to $\hat{Y}$ as well.

d) This doesn't hold anymore.

Since we no longer have an intercept term our predicted response equation no longer has a $\hat{\theta}_0$ term meaning our new model evaluated at $x = \bar{x}$, $\hat{y} = \hat{\gamma} \bar{x}$ may not always equal $\bar{y}$.