Floquet SPT-MBL in a 6-Qubit System

Shiva Heidari

1 Introduction

Periodically driven quantum systems, commonly referred to as *Floquet systems*, offer a powerful framework for exploring exotic phases of matter far from equilibrium. Unlike equilibrium phases, which are governed by ground-state properties of time-independent Hamiltonians, Floquet phases emerge in systems with explicit time-periodicity in the Hamiltonian. That is, the dynamics are governed by a Hamiltonian satisfying

$$H(t+T) = H(t),$$

where T is the driving period. This temporal periodicity allows one to define the evolution over a single period using the *Floquet unitary operator*:

$$U_F = \mathcal{T} \exp\left(-i \int_0^T H(t)\, dt\right),$$

where $\mathcal{T}$ denotes time ordering. The eigenstates of U_F, denoted by $|\phi_\alpha\rangle$, are called *Floquet eigenstates*, and their associated eigenvalues take the form $e^{-i\varepsilon_\alpha T}$, where $\varepsilon_\alpha \in [-\pi/T, \pi/T)$ are known as *quasienergies*. Since quasienergies are defined modulo $2\pi/T$, the Floquet spectrum forms a compact Brillouin zone analogous to momentum in crystalline solids.

A key question in the study of driven quantum systems is: What is the fate of a generic quantum state under repeated application of U_F? In other words, do driven systems thermalize and lose all quantum coherence over time, or can they host stable, non-equilibrium phases of matter with long-lived quantum structure?

In generic interacting Floquet systems, energy is not conserved, and repeated driving typically leads to indefinite heating. According to the *Eigenstate Thermalization Hypothesis* (ETH), such systems are expected to thermalize to an effective infinite-temperature ensemble, where all local information is lost and the system becomes maximally mixed.

However, certain mechanisms can prevent this thermalization. In particular, *many-body localization* (MBL), which results from strong disorder and interaction effects, can inhibit energy absorption and preserve memory of the initial state. When MBL is present, it becomes possible to stabilize rich non-equilibrium phases, including *Floquet symmetry-protected topological* (SPT) phases that are uniquely dynamical and have no equilibrium analogs.

These two extremes: ETH-driven thermalization versus MBL-protected order define fundamentally different dynamical behaviors in Floquet systems. Floquet ETH systems exhibit rapid entanglement growth, decay of local observables, and level statistics consistent with random matrix theory. In contrast, Floquet SPT-MBL systems retain topological edge modes, bounded entanglement entropy, and exhibit coherent oscillations that are robust to disorder and decoherence.

The goal of this article is to provide a comparative theoretical overview of Floquet ETH and Floquet SPT-MBL systems. We explore the underlying mechanisms, key physical observables, and their signatures in both numerical simulations and potential quantum computing implementations. Through this lens, we highlight how these phases provide a window into thermalization, localization, and dynamical topology in driven quantum systems.

2 Floquet Systems and Stroboscopic Dynamics

Floquet systems are characterized by a time-periodic Hamiltonian $H(t)$, satisfying

$$H(t+T) = H(t),$$

for some fixed driving period T. This temporal periodicity allows us to define the system's evolution over discrete time steps rather than continuously, leading to what is called *stroboscopic dynamics*. Instead of focusing on the instantaneous Hamiltonian, one considers the effect of applying the full evolution over one complete period.

The evolution operator for one period of the drive, known as the *Floquet operator* or *Floquet unitary*, is given by

$$U_F = \mathcal{T} \exp\left(-i \int_0^T H(t)\, dt\right),$$

where $\mathcal{T}$ denotes time ordering. This operator encodes all information about the system's evolution at stroboscopic times $t = nT$, where $n \in \mathbb{Z}$:

$$|\psi(nT)\rangle = U_F^n |\psi(0)\rangle.$$

The eigenstates $|\phi_\alpha\rangle$ of U_F satisfy

$$U_F|\phi_\alpha\rangle = e^{-i\varepsilon_\alpha T}|\phi_\alpha\rangle,$$

where ε_α are known as *quasienergies*. Due to the periodicity of the exponential function, quasienergies are only defined modulo $2\pi/T$. Consequently, the Floquet spectrum forms a compact Brillouin zone in quasienergy space, analogous to the energy bands in crystals with periodic spatial structure.

This formalism gives rise to an effective description of the dynamics in terms of a time-independent operator, even though the underlying Hamiltonian is time-dependent. In many cases, one introduces a formal *effective Hamiltonian* H_F such that

$$U_F = e^{-iH_F T},$$

although this effective Hamiltonian is often highly non-local and may not admit a simple analytical form.

Understanding the properties of the Floquet operator like its eigenstates, spectrum, and entanglement structure is essential for classifying non-equilibrium phases of matter. Since energy is not conserved, the usual thermodynamic classification based on ground state phases breaks down. Instead, new dynamical phases can emerge, some of which have no equilibrium counterparts.

Depending on the nature of the drive and interactions, a Floquet system may exhibit drastically different behaviors. For example, in thermalizing systems that obey the eigenstate thermalization hypothesis (ETH), the repeated action of U_F leads to heating and loss of coherence. By contrast, in systems with strong disorder and localization (MBL), the dynamics governed by U_F can stabilize novel states such as Floquet time crystals and symmetry-protected topological (SPT) phases.

Importantly, Floquet systems provide an experimentally viable platform for realizing and controlling such exotic phenomena. Because the drive is engineered externally, researchers can tune both the structure of the Hamiltonian and the duration of each step, offering a high degree of flexibility. This has led to a surge of interest in simulating Floquet dynamics using quantum computers, trapped ions, cold atoms, and superconducting qubits.

The interplay of drive frequency, interaction strength, disorder, and symmetry thus forms the core of understanding stroboscopic quantum phases. In the sections that follow, we examine two contrasting regimes: thermalizing Floquet ETH systems and non-thermalizing Floquet SPT-MBL systems.

3 Floquet ETH Systems

The *Eigenstate Thermalization Hypothesis* (ETH) is a foundational concept in the study of quantum thermalization. It postulates that in a chaotic quantum many-body system, individual eigenstates themselves encode thermal behavior. Specifically, expectation values of local observables in a single eigenstate coincide with predictions from statistical mechanics.

In the context of *Floquet systems*, ETH appears when a system is subjected to periodic driving that breaks energy conservation, but still exhibits ergodic behavior due to many-body interactions. Under such conditions, the system absorbs energy from the drive and evolves toward a steady state that mimics an infinite-temperature ensemble.

Mathematically, ETH implies that for any local observable $\hat{O}$, its matrix elements in the Floquet eigenbasis take the form:

$$\langle \phi_\alpha | \hat{O} | \phi_\beta \rangle = \bar{O}(\varepsilon)\delta_{\alpha\beta} + e^{-S(\varepsilon)/2} f_O(\varepsilon, \omega) R_{\alpha\beta},$$

where $\bar{O}(\varepsilon)$ is a smooth function of the quasienergy ε, $S(\varepsilon)$ is the entropy at that quasienergy, $\omega = \varepsilon_\alpha - \varepsilon_\beta$, and $R_{\alpha\beta}$ is a random variable with zero mean and unit variance.

Because energy is not conserved in Floquet systems, thermalizing systems generically heat up to a completely mixed state:

$$\rho_\infty = \frac{\mathbb{I}}{\mathcal{D}},$$

where $\mathcal{D}$ is the Hilbert space dimension. In this state, the expectation value of any traceless local observable vanishes:

$$\langle \hat{O} \rangle = \mathrm{Tr}(\rho_\infty \hat{O}) = 0.$$

One of the most important dynamical signatures of ETH is rapid entanglement growth. For a subsystem A of size L_A, the entanglement entropy grows linearly in time and eventually saturates at the Page value:

$$S_A(t) \approx \frac{L_A \log 2}{2},$$

indicating volume-law scaling characteristic of thermal states.

From a computational perspective, Floquet ETH systems are significant for benchmarking quantum simulators. Due to their rapid scrambling of quantum information and extensive entanglement growth, they challenge the fidelity of simulations and the limits of coherence in noisy intermediate-scale quantum (NISQ) hardware. Nonetheless, they provide a critical framework for testing the emergence of thermodynamic behavior in isolated quantum systems.

4 Floquet SPT-MBL Systems

In stark contrast to the thermalizing behavior of Floquet ETH systems, certain periodically driven quantum systems can avoid heating and instead support stable, non-thermal dynamical phases of matter. One of the most remarkable examples is the *Floquet symmetry-protected topological* (SPT) phase stabilized by *many-body localization* (MBL). These phases exhibit robust edge coherence, long-lived dynamical order, and topological features that cannot be realized in equilibrium settings.

The existence of Floquet SPT-MBL phases relies on three crucial ingredients. First, the system must possess a discrete global symmetry, such as $\mathbb{Z}_2 \times \mathbb{Z}_2$, which protects the topological nature of the phase. Second, strong disorder must be present to localize bulk degrees of freedom and prevent the absorption of energy from the drive, thereby avoiding thermalization. Third, the time-periodic driving protocol must alternate between symmetry-preserving Hamiltonians that individually may not exhibit topological order but together induce non-trivial Floquet dynamics.

A canonical example of such a system is described by a Floquet unitary operator composed of two steps:

$$U_F = \exp\left(-iJ\sum_j X_{j-1}Z_jX_{j+1}\right)\cdot\exp\left(-i\sum_j h_jZ_j\right),$$

where the first exponential encodes entangling three-site cluster interactions, and the second exponential applies local disordered Z-field rotations. The combination of these steps generates non-trivial dynamics while preserving the required symmetry. This protocol is known to host a Floquet phase that supports edge-localized modes protected by both the disorder-induced localization and the symmetry of the drive.

One of the most striking features of the Floquet SPT-MBL phase is the emergence of quasienergy π-modes localized at the system's boundaries. These edge modes are defined by the property that under one Floquet cycle, the edge-localized operators Γ_L and Γ_R transform as

$$U_F\Gamma_{L,R}U_F^{-1} = -\Gamma_{L,R},$$

implying that the corresponding edge spins exhibit a dynamical flipping with period $2T$, even though the drive itself has period T. This dynamical behavior is a hallmark of Floquet topology and cannot be mimicked by any static Hamiltonian.

The resulting dynamics display a number of experimentally accessible signatures. Edge observables, such as $\langle X_0(t)\rangle$, show persistent oscillations with period $2T$ over long time scales. These oscillations are robust to small perturbations and moderate levels of noise, as long as the protecting symmetry and localization are maintained. In the bulk, the entanglement entropy grows logarithmically with time, a behavior characteristic of MBL systems. The system's quasienergy spectrum exhibits Poissonian level statistics, reflecting the absence of level repulsion due to the lack of quantum chaos.

Unlike thermalizing systems that obey ETH, the entanglement entropy in Floquet SPT-MBL systems does not scale extensively. Instead, it grows as

$$S_A(t) \sim \log t,$$

demonstrating slow entanglement spreading and long-term memory of the initial state. This bounded growth enables the preservation of quantum coherence for edge degrees of freedom, even in the presence of bulk disorder and interactions.

The physical mechanism underlying this phenomenon lies in the interplay between localization and symmetry. The strong disorder localizes the bulk, suppressing resonant interactions and preventing delocalization. Meanwhile, the periodic driving structure and symmetry enforce a coherent, topologically nontrivial evolution of the edge states. Together, these features create a stable dynamical phase with protected edge modes that oscillate in time and do not decohere.

Floquet SPT-MBL systems thus realize a fundamentally new class of quantum phases: they are inherently non-equilibrium and exhibit topological order throughout the spectrum, rather than in just the ground state. They can be realized without the need for cooling or adiabatic preparation, and are particularly well suited for implementation on noisy intermediate-scale quantum (NISQ) devices. The robustness of edge modes to thermalization makes them promising candidates for quantum memory and dynamical quantum error suppression in near-term hardware platforms.

5 Theoretical Comparison and 6-Qubit Simulation Example

To illustrate the contrasting behaviors of Floquet ETH and Floquet SPT-MBL phases, it is instructive to examine both cases within a unified framework using a small-scale spin chain. A 6-qubit chain serves as a minimal yet sufficiently rich platform to compare thermalizing and localized dynamics under periodic driving.

5.1 Model Definitions and Theoretical Expectations

In the Floquet ETH case, we consider a system governed by a time-independent Hamiltonian composed of two terms:

$$H = \sum_i hX_i + \sum_i JZ_iZ_{i+1},$$

where X_i and Z_i denote the Pauli operators acting on site i, and the parameters h and J are uniform across the chain. The first term applies a transverse field that induces spin flips, while the second introduces nearest-neighbor Ising-type interactions. This combination leads to quantum chaos and thermalization in the absence of disorder.

The Floquet unitary is given by

$$U_F = e^{-iHT},$$

where T is the stroboscopic driving period. Starting from a non-entangled product state such as $|\psi_0\rangle = |1\rangle \otimes |+\rangle^{\otimes 5}$, the system evolves under repeated applications of U_F. The time evolution of local observables such as $\langle Z_0(t)\rangle$, $\langle Z_2(t)\rangle$, and the single-qubit entanglement entropy $S_0(t)$ is monitored to diagnose thermal behavior.

The simulation reveals that local magnetizations decay rapidly toward zero, and the entanglement entropy grows quickly, saturating near the maximum value of $\log 2$. This behavior is characteristic of thermalization and is consistent with the predictions of ETH: local observables equilibrate, entanglement becomes extensive, and memory of the initial state is lost. The entire system behaves as though it has reached an effective infinite temperature.

In contrast, the Floquet SPT-MBL system is constructed using a binary drive composed of two non-commuting Hamiltonians. The Floquet unitary takes the form

$$U_F = \exp\left(-i\sum_j h_jZ_j\right) \cdot \exp\left(-iJ\sum_j X_{j-1}Z_jX_{j+1}\right),$$

where $h_j \in [-h, h]$ represents site-dependent random disorder. The first step introduces localization via disordered Z-rotations, while the second step applies a cluster-type entangling gate known to generate SPT order. Together, the two steps preserve a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and protect the topological structure of the Floquet phase.

Simulating this model with the same initial product state reveals a starkly different behavior. The edge spin $\langle Z_0(t)\rangle$ exhibits coherent, period-doubled oscillations due to the presence of a Floquet

π-mode. These oscillations persist over long times, even as the bulk spin $\langle Z_2(t)\rangle$ decays or fluctuates randomly. Entanglement entropy grows very slowly and remains bounded throughout the evolution, in contrast to the ETH case.

The comparison between these two models underscores the essential differences in their dynamical properties. The ETH system exhibits fast scrambling, thermal equilibration, and volume-law entanglement, whereas the SPT-MBL system exhibits localization, topological edge protection, and logarithmic entanglement growth. The ETH phase is fragile to perturbations and heating, while the SPT-MBL phase remains robust under symmetry-preserving disorder.

5.2 Quantum Circuit Realization of Floquet SPT-MBL Step

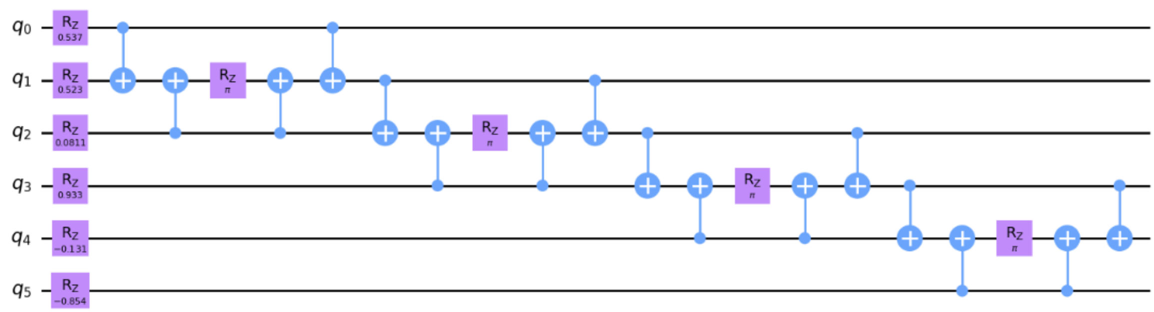

Figure 1: Quantum circuit implementing one Floquet step of the 6-qubit SPT-MBL system. Each step consists of disordered Z rotations followed by XZX cluster interactions compiled using CNOT gates.

Figure 1 illustrates the circuit used to simulate one Floquet cycle of the SPT-MBL system. The disordered R_z gates encode the local Z-field terms, while the entangling XZX terms are implemented using a combination of controlled-NOT and rotation gates. This structure is repeated across the chain and used iteratively in the simulation.

5.3 Simulation Results and Observables

The results shown in Figure 2 confirm that the system avoids thermalization. The persistent oscillations of $\langle Z_0\rangle$ are a clear signature of the π-mode edge behavior, and the bounded entanglement growth supports the conclusion that the bulk remains many-body localized.

In contrast, Figure 3 shows that the ETH system exhibits rapid thermalization. The decay of both $\langle Z_0(t)\rangle$ and $\langle Z_2(t)\rangle$ to zero, combined with fast entanglement saturation, illustrates the system's tendency to erase local information and spread quantum correlations extensively.

5.4 Summary of Dynamical Signatures

The simulation results strongly validate the theoretical expectations. In the ETH case, all observables thermalize quickly, and entropy scales with subsystem size. In the SPT-MBL case, edge coherence is preserved, bulk observables decay irregularly, and entanglement entropy grows logarithmically. These contrasting behaviors underscore the central role of disorder and symmetry in stabilizing non-equilibrium phases on NISQ devices.

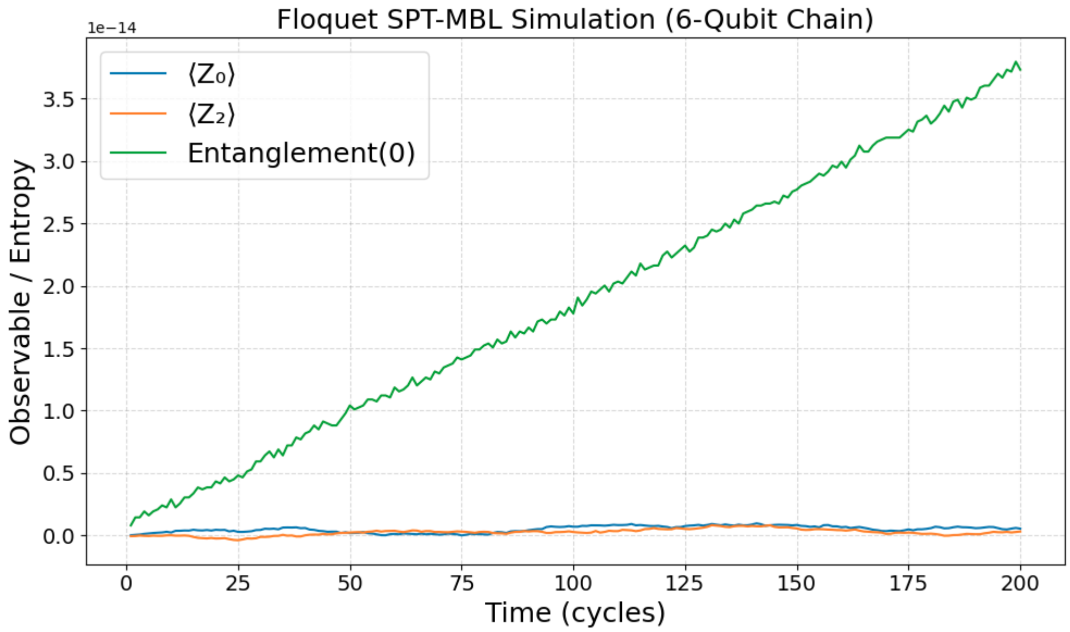

Figure 2: Time evolution of observables and entanglement in the Floquet SPT-MBL model. The edge observable $\langle Z_0(t)\rangle$ exhibits coherence-preserving oscillations, while $\langle Z_2(t)\rangle$ fluctuates randomly. Entanglement entropy grows slowly, consistent with many-body localization.

6 Application to Qubit-Based Quantum Computing

Floquet SPT-MBL models are particularly well-suited for implementation on modern qubit-based quantum computing architectures. Their dynamics are expressed entirely in terms of unitary operators generated by local Hamiltonians, making them compatible with the gate-based paradigm of digital quantum computation.

The Floquet unitary considered in the SPT-MBL model,

$$U_F = \exp\left(-iJ\sum_j X_{j-1}Z_jX_{j+1}\right)\cdot\exp\left(-i\sum_j h_jZ_j\right),$$

can be decomposed into two layers: a deterministic entangling layer and a disordered on-site rotation layer. The first term corresponds to the application of three-body XZX cluster interactions, which can be compiled into native quantum gates such as CNOTs and single-qubit rotations. Specifically, each XZX term can be synthesized using two CNOT gates and a Z-rotation:

$$e^{-i\theta X_{j-1}Z_jX_{j+1}} = \mathrm{CNOT}_{j-1,j+1}\cdot R_z^{(j)}(2\theta)\cdot\mathrm{CNOT}_{j-1,j+1},$$

or alternatively using Toffoli gates on architectures that support them natively.

The second term,

$$\exp\left(-i\sum_j h_jZ_j\right),$$

represents site-dependent disordered Z-rotations, easily implemented using native R_z rotations on most quantum hardware. The disorder profile $h_j \in [-h, h]$ serves to induce many-body localization, crucial for suppressing Floquet heating and enabling robust edge coherence.

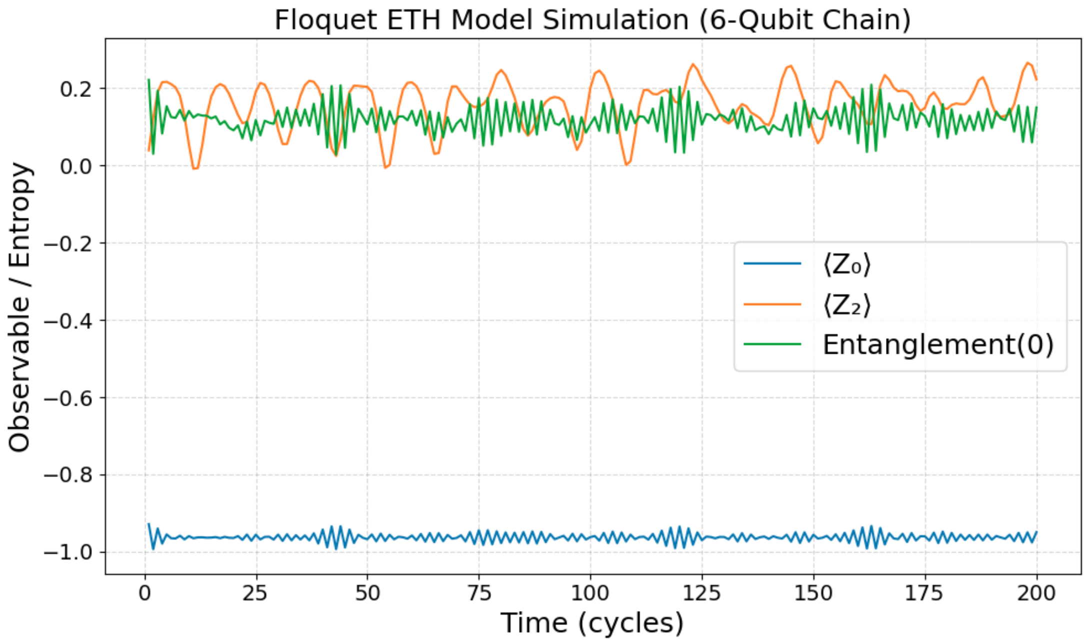

Figure 3: Time evolution in the Floquet ETH system. Local observables decay rapidly, and entanglement entropy of qubit 0 saturates, signaling thermalization and volume-law entanglement.

From a practical point of view, the key advantage of this construction is that the depth of each Floquet cycle remains constant with system size: Each layer involves only local gates acting on one or three qubits at a time. This shallow depth makes the model highly compatible with the coherence times of Noisy Intermediate-Scale Quantum (NISQ) devices.

More importantly, the dynamical topological properties such as robust π-modes localized at the edges manifest not in the ground state but in the structure of the Floquet eigenstates throughout the spectrum. Thus, there is no need for adiabatic state preparation or ground-state cooling. One can initialize the system in a simple product state, apply a modest number of Floquet cycles, and extract meaningful topological signatures such as:

- Persistent edge-spin oscillations with period $2T$,
- Bounded entanglement entropy in localized regimes,
- Distinct thermal vs. MBL behavior across tuning of disorder strength,
- Floquet eigenstate statistics (Wigner-Dyson vs. Poisson),
- Edge-mode coherence times exceeding those of bulk modes.

Moreover, the edge qubits in an MBL-SPT phase behave as dynamically protected degrees of freedom, akin to topologically encoded qubits. Their resilience to both disorder and decoherence makes them promising candidates for near-term demonstrations of fault-tolerant subspaces. In particular, such dynamically protected subspaces may serve as testbeds for logical qubit encoding without requiring full quantum error correction.

In summary, Floquet SPT-MBL circuits not only serve as test platforms for fundamental questions in non-equilibrium many-body physics but also offer a robust framework for engineering and benchmarking resilient quantum states in real quantum devices.

6.1 Circuit Compilation and Depth Considerations

One of the key practical advantages of the Floquet SPT-MBL protocol is its compatibility with shallow-depth, hardware-friendly quantum circuits. Each Floquet cycle consists of two components:

1. **Disordered local Z-rotations:** Implemented as single-qubit $R_z(\theta)$ gates, which are natively supported on most hardware platforms and can typically be executed in parallel across all qubits.

2. **Three-body cluster interactions $X_{j-1}Z_jX_{j+1}$:** These can be decomposed using a compact gate sequence consisting of two CNOT gates and a single R_z rotation acting on qubit j:

$$e^{-iJX_{j-1}Z_jX_{j+1}} = \mathrm{CNOT}_{j+1\to j} \cdot \mathrm{CNOT}_{j-1\to j} \cdot R_z^{(j)}(2J) \cdot \mathrm{CNOT}_{j-1\to j} \cdot \mathrm{CNOT}_{j+1\to j}.$$

This construction requires only nearest-neighbor connectivity and maps naturally onto devices with linear qubit topologies.

Circuit Depth Scaling. Each Floquet cycle involves:

- A single layer of parallel R_z gates (depth $= 1$),
- $n-2$ overlapping XZX interactions, each implemented with 4 entangling gates.

While the naive depth scales with n, these three-body interactions can be scheduled in non-overlapping sets to allow for partial parallelism. For example, gates acting on $j = 1, 4, 7, \ldots$ can be executed concurrently. With such scheduling, the total entangling gate depth per cycle remains approximately constant ($\sim$ 4–6 layers), independent of chain length.

NISQ Feasibility and CNOT Constraints. On current NISQ hardware, CNOT gates represent a critical bottleneck due to their higher error rates and connectivity constraints. Fortunately, this model:

- Requires only **nearest-neighbor** connectivity, compatible with common linear or ring topologies (e.g., IBM, Quantinuum).
- Uses approximately $2(n-2)$ CNOTs per Floquet step. For $n = 6$, this results in 8–10 CNOTs per cycle, totaling $\sim$300 CNOTs over 30 steps well within the coherence limits of many NISQ systems.

Experimental Outlook. This compact and symmetry-preserving circuit structure allows for the robust simulation of non-equilibrium topological dynamics with limited gate depth. Furthermore, its clear gate decomposition and binary-step structure make it amenable to near-term hardware implementations and hybrid quantum-classical schemes. The absence of long-range entangling gates and the bounded depth per cycle make it a promising candidate for demonstrating edge-coherent Floquet phases on digital quantum processors.

7 Floquet SPT-MBL Physics and NISQ Devices

Floquet SPT-MBL phases are exceptionally well-suited to exploration on Noisy Intermediate-Scale Quantum (NISQ) devices. These systems are intrinsically non-equilibrium, requiring only shallow circuits per cycle, which fits naturally within the coherence time constraints of NISQ platforms. The presence of many-body localization ensures robustness to noise and suppression of heating, allowing coherent dynamical signatures (such as π-mode edge oscillations) to persist despite imperfections.

Moreover, the binary drive structure used to realize Floquet SPT-MBL phases, such as alternating disordered Z-rotations and entangling XZX gates, is straightforward to implement using standard native gates on current hardware. Because entanglement growth in the MBL phase is slow and area-law, state evolution and measurement remain feasible on modest qubit counts. This makes such models ideal candidates for demonstrating non-trivial quantum dynamics and edge coherence in early quantum processors, thereby serving as testbeds for dynamical topology and quantum memory.

8 Concluding Remarks

The study of periodically driven quantum systems reveals a deep interplay between energy absorption, localization, and topological protection. In Floquet ETH systems, energy absorption leads to thermalization and the complete erasure of quantum coherence. These systems behave like thermal baths, and any initial structure is rapidly lost.

On the other hand, Floquet SPT-MBL systems exhibit a remarkable capacity to stabilize topological features in the absence of energy conservation. Through the combined effects of disorder and symmetry, they host novel π-modes, edge coherence, and protected dynamical phases that defy the conventional thermalization paradigm.

Understanding these phases requires a synthesis of ideas from quantum statistical mechanics, topology, and many-body localization. They offer exciting avenues for future research, including the development of robust quantum memories and new insights into the dynamics of non-equilibrium quantum matter.

Table 1: Summary of contrasting features between Floquet ETH and Floquet SPT-MBL systems observed in simulation.

Property	**Floquet ETH**	**Floquet SPT-MBL**
Thermalization behavior	Rapid, complete	Avoided via localization
Entanglement scaling	Volume law (fast)	Logarithmic growth (slow)
Edge coherence	Lost quickly	Robust (protected π-mode)
Level statistics	Wigner-Dyson	Poissonian
Memory of initial state	Lost	Preserved at edges
Observable decay	Fast and uniform	Edge oscillations persist
Local disorder	Absent (uniform)	Present (random h_j)
Drive structure	Static Hamiltonian	Binary periodic drive

A Appendix: Simulation Code Listings

A.1 Floquet SPT-MBL Circuit Construction (Qiskit)

```
import numpy as np
from qiskit import QuantumCircuit

def floquet_spt_step(n_qubits, h_fields, J):
    qc = QuantumCircuit(n_qubits)

    # Step 1: Local Z-rotations from disorder
    for j in range(n_qubits):
        qc.rz(2 * h_fields[j], j)

    # Step 2: Apply XZX cluster terms using compiled 3-qubit gates
    for j in range(1, n_qubits - 1):
        qc.cx(j - 1, j)
        qc.cx(j + 1, j)
        qc.rz(2 * J, j)
        qc.cx(j + 1, j)
        qc.cx(j - 1, j)

    return qc
```

A.2 Floquet SPT-MBL Time Evolution and Measurement

```
import numpy as np
import matplotlib.pyplot as plt
from qiskit import QuantumCircuit
from qiskit.quantum_info import Statevector, Pauli, partial_trace, entropy

# Parameters
n_qubits = 6
J = np.pi / 2
steps = 30
dt = 1.0
np.random.seed(42)
h_fields = np.random.uniform(-0.5, 0.5, n_qubits)

# Initial state: |1⟩  |+⟩^{5}
init = QuantumCircuit(n_qubits)
init.x(0)
init.h(range(1, n_qubits))
psi = Statevector.from_instruction(init)

# Observables
obs = {
    "⟨Z⟩": Pauli("Z" + "I" * (n_qubits - 1)),
    "⟨Z⟩": Pauli("I" * 2 + "Z" + "I" * (n_qubits - 3)),
```

```
    "Entanglement(0)": None
}
results = {name: [] for name in obs}
times = []

# Floquet evolution loop
for t in range(steps):
    floquet_circ = floquet_spt_step(n_qubits, h_fields, J)
    psi = psi.evolve(floquet_circ)

    results["⟨Z⟩"].append(np.real(psi.expectation_value(obs["⟨Z⟩"])))
    results["⟨Z⟩"].append(np.real(psi.expectation_value(obs["⟨Z⟩"])))

    rho0 = partial_trace(psi, list(range(1, n_qubits)))
    S = entropy(rho0, base=2)
    results["Entanglement(0)"].append(S)

    times.append((t + 1) * dt)

# Plot results
plt.figure(figsize=(10, 6))
for name, values in results.items():
    plt.plot(times, values, label=name)
plt.xlabel("Time (cycles)")
plt.ylabel("Observable / Entropy")
plt.title("Floquet SPT-MBL Simulation")
plt.legend()
plt.grid(True)
plt.show()
```

A.3 Floquet ETH Model Code (Placeholder)

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.linalg import expm
from numpy import kron
from functools import reduce

# ----------------------------
# Parameters
# ----------------------------
n_qubits = 6
J = np.pi / 2
steps = 30
dt = 1.0  # Floquet period
h_field = 0.3  # uniform transverse field for ETH

# Pauli matrices
```

```
I = np.eye(2)
X = np.array([[0, 1], [1, 0]])
Z = np.array([[1, 0], [0, -1]])

# ----------------------------
# Helper functions
# ----------------------------
def apply_operator(op_list):
    """Constructs a multi-qubit operator by Kronecker product"""
    return reduce(kron, op_list)

def construct_eth_hamiltonian(n_qubits, J, h_field):
    H = np.zeros((2**n_qubits, 2**n_qubits), dtype=complex)

    # Transverse field term (uniform)
    for i in range(n_qubits):
        ops = [I]*n_qubits
        ops[i] = X
        H += h_field * apply_operator(ops)

    # ZZ interactions between neighbors
    for i in range(n_qubits - 1):
        ops = [I]*n_qubits
        ops[i] = Z
        ops[i+1] = Z
        H += J * apply_operator(ops)

    return H

def initial_state(n_qubits):
    # Initial state: |1⟩  |+⟩(n_qubits - 1)
    state = np.array([0, 1])  # |1⟩
    plus = 1/np.sqrt(2) * np.array([1, 1])
    rest = reduce(kron, [plus]*(n_qubits - 1))
    return kron(state, rest)

def expectation(state, operator):
    return np.real(np.vdot(state, operator @ state))

def partial_trace(rho, keep, dims):
    n = len(dims)
    trace_out = sorted(set(range(n)) - set(keep))

    perm = keep + trace_out
    perm_dims = [dims[i] for i in perm]
    rho = rho.reshape([2]*n*2)
    rho = np.transpose(rho, perm + [i + n for i in perm])
    rho = rho.reshape(
```

```
        (np.prod([dims[i] for i in keep]), np.prod([dims[i] for i in trace_out]),
         np.prod([dims[i] for i in keep]), np.prod([dims[i] for i in trace_out]))
    )
    return np.trace(rho, axis1=1, axis2=3)

def von_neumann_entropy(rho):
    eigvals = np.linalg.eigvalsh(rho)
    eigvals = eigvals[eigvals > 1e-12]
    return -np.sum(eigvals * np.log2(eigvals))

# ----------------------------
# Build ETH Hamiltonian
# ----------------------------
H = construct_eth_hamiltonian(n_qubits, J, h_field)
U_F = expm(-1j * H * dt)

# Observables
Z0 = apply_operator([Z] + [I]*(n_qubits - 1))
Z2 = apply_operator([I]*2 + [Z] + [I]*(n_qubits - 3))

# ----------------------------
# Simulation
# ----------------------------
psi = initial_state(n_qubits)
results = {"⟨Z⟩": [], "⟨Z⟩": [], "Entanglement(0)": []}
times = []

for t in range(steps):
    psi = U_F @ psi

    results["⟨Z⟩"].append(expectation(psi, Z0))
    results["⟨Z⟩"].append(expectation(psi, Z2))

    rho_full = np.outer(psi, np.conj(psi))
    rho0 = partial_trace(rho_full, keep=[0], dims=[2]*n_qubits)
    S = von_neumann_entropy(rho0)
    results["Entanglement(0)"].append(S)

    times.append((t + 1) * dt)

# ----------------------------
# Plotting
# ----------------------------
plt.figure(figsize=(10, 6))
for name, values in results.items():
    plt.plot(times, values, label=name)

plt.xlabel("Time (cycles)")
```

```
plt.ylabel("Observable / Entropy")
plt.title("Floquet ETH Model Simulation (6-Qubit Chain)")
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```