SUPPLEMENTAL MATERIAL

Aharonov–Bohm interference from coherent spin-polarized edge transport in Fe(Te,Se) superconducting rings

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S1. THEORETICAL FRAMEWORK

We begin with the Hamiltonian of the helical Luttinger liquid (HLL) model [1–3], which describes interacting one-dimensional fermions with spin-momentum locking:

$$H = \frac{\hbar\nu}{2\pi}\int_0^L dx\left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\varphi)^2\right], \tag{S1}$$

where the bosonic fields satisfy the canonical commutation relation $[\varphi(x), \partial_y\theta(y)] = i\pi\delta(x-y)$. Here, $\varphi(x)$ and $\theta(x)$ represent collective charge and current fluctuations, respectively, and serve as bosonic analogs of fermionic degrees of freedom. The parameter ν denotes the propagation velocity of bosonic excitations, and L is the circumference of the ring. The Luttinger parameter K characterizes the strength and nature of interactions: $K < 1$ corresponds to repulsive interactions, $K > 1$ to attractive ones, and $K = 1$ to the non-interacting (free fermion) limit.

For a helical edge, the fermionic fields for right- and left-moving modes are bosonized as

$$\psi_R(x) = \frac{1}{\sqrt{2\pi\zeta}}e^{i[\varphi(x)+\theta(x)]}, \qquad \psi_L(x) = \frac{1}{\sqrt{2\pi\zeta}}e^{i[\varphi(x)-\theta(x)]}, \tag{S2}$$

where ζ is a short-distance cutoff that regularizes the theory and ensures proper normalization. The prefactor $1/\sqrt{2\pi\zeta}$ accounts for the spatial smearing of fermionic operators intrinsic to bosonization. Due to spin-momentum locking, $\psi_R(x)$ describes right-moving (spin-up) electrons, while $\psi_L(x)$ corresponds to left-moving (spin-down) electrons.

A magnetic flux Φ threading the ring induces an Aharonov–Bohm (AB) phase for electrons traveling around the edge. This phase shift is given by

$$\delta\Phi_{\text{AB}} = \frac{2\pi\Phi}{\Phi_0}, \tag{S3}$$

where $\Phi_0 = h/e$ is the flux quantum. This phase modifies the boundary condition of the edge state wavefunction. For AB interference, we focus on the coherence of a single species, for example right-moving (spin-up) electrons described by $\psi_R(x)$, as they traverse the ring. The AB oscillation amplitude is directly linked to the quantum coherence of an electron propagating around the ring, which is encoded in the correlation Green's function:

$$\mathcal{G}(x,t) = \langle\psi^\dagger(x,t)\psi(0,0)\rangle. \tag{S4}$$

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This function quantifies the amplitude for a fermion, introduced at position 0 and time 0, to be detected at position x and time t, thus providing insight into how interactions and temperature affect coherence. Using the bosonized form of the fermionic operator for right-moving modes: $\psi_R(x,t) = \frac{1}{\sqrt{2\pi\zeta}} e^{i[\varphi(x,t)+\theta(x,t)]}$, the Green's function becomes

$$\mathcal{G}(x,t) = \frac{1}{2\pi\zeta} \left\langle e^{-i[\varphi(x,t)+\theta(x,t)]} e^{i[\varphi(0,0)+\theta(0,0)]} \right\rangle . \tag{S5}$$

Since $\varphi(x,t)$ and $\theta(x,t)$ are Gaussian bosonic fields, we can apply the identity

$$\langle e^{iA} e^{-iB} \rangle = e^{-\frac{1}{2}\langle (A-B)^2 \rangle}, \tag{S6}$$

which yields

$$\mathcal{G}(x,t) = \frac{1}{2\pi\zeta} \exp\left[-\frac{1}{2} \left\langle ([\varphi(x,t)+\theta(x,t)] - [\varphi(0,0)+\theta(0,0)])^2 \right\rangle \right]. \tag{S7}$$

The next step involves evaluating the correlation functions $\langle[\varphi(x,t)-\varphi(0,0)]^2\rangle$ and $\langle[\theta(x,t)-\theta(0,0)]^2\rangle$ at finite temperature. We then examine how magnetic flux modifies the behavior of the edge modes by altering the boundary conditions and zero-mode structure. The bosonic field $\varphi(x,t)$ on a ring of length L at temperature T can be mode-expanded as [4]:

$$\varphi(x,t) = \varphi_0 + \frac{2\pi x}{L} N + \sum_{n\neq 0} \frac{1}{\sqrt{|n|}} \left(a_n e^{2\pi i n x/L} e^{-i\omega_n t} + \text{h.c.} \right). \tag{S8}$$

The first two terms represent the zero mode ($n = 0$), with $N \in \mathbb{Z}$ representing the winding number. The frequencies $\omega_n = \frac{2\pi n}{\beta}$ are the bosonic Matsubara frequencies, where $\beta = \frac{1}{k_B T}$, and a_n, $a_n^\dagger$ are the bosonic annihilation and creation operators, respectively.

In the presence of a magnetic flux Φ threading the ring, the boundary condition for $\varphi(x)$ becomes

$$\varphi(x+L) = \varphi(x) + 2\pi \frac{\Phi}{\Phi_0}, \tag{S9}$$

which modifies the zero mode as

$$\varphi(x,t) = \varphi_0 + \frac{2\pi}{L} \left(N + \frac{\Phi}{\Phi_0} \right) x + \tilde{\varphi}(x,t). \tag{S10}$$

The oscillatory part is given by

$$\tilde{\varphi}(x,t) = \sum_{n\neq 0} \frac{1}{\sqrt{|n|}} \left(a_n e^{2\pi i n x/L} e^{-i\omega_n t} + \text{h.c.} \right), \tag{S11}$$

and satisfies periodic boundary conditions: $\tilde{\varphi}(x+L,t) = \tilde{\varphi}(x,t)$. These Fourier modes ($n \neq 0$) describe local particle–hole fluctuations and are insensitive to the topological phase accumulated due to magnetic flux. However, the zero-mode contribution

$$\varphi_0(x) = \frac{2\pi}{L}\left(N + \frac{\Phi}{\Phi_0}\right)x, \tag{S12}$$

solely determines the flux-dependent boundary condition. Consequently, magnetic flux enters the theory only through the zero-mode sector. As an electron encircles the ring, this term accumulates a total phase $2\pi\Phi/\Phi_0$, yielding the relation

$$e^{i[\varphi(L)-\varphi(0)]} = (-1)^N e^{-i2\pi\Phi/\Phi_0}. \tag{S13}$$

In a 2D topological superconductor with a ring-shaped edge, the winding number $N = 1$ reflects the system's nontrivial topology and encodes a Berry phase $\gamma = \pi$ associated with a spin-1/2 fermion encircling the flux [1, 5]. This topological phase appears explicitly as a prefactor in the fermionic Green's function discussed in the next section. With the flux dependence clarified, we now examine how finite temperature impacts the coherence encoded in the Green's function.

To understand how temperature influences coherence, we first derive the finite-temperature correlation functions of the bosonic fields $\varphi(x,t)$ and $\theta(x,t)$ that enter the Green's function in Eq. S7. Since these fields are Gaussian, the correlators can be computed by thermal averaging over their normal mode expansions. The oscillatory component of the field is given by:

$$\tilde{\varphi}(x,t) = \sum_{q>0}\sqrt{\frac{2\pi}{qL}}\left[a_q e^{i(qx-\omega_q t)} + a_q^\dagger e^{-i(qx-\omega_q t)}\right], \tag{S14}$$

where $\omega_q = vq$ with $q = 2\pi n/L$. The thermal average $\langle a_q^\dagger a_q\rangle = n_B(\omega_q)$ follows the Bose–Einstein distribution:

$$n_B(\omega_q) = \frac{1}{e^{\beta\hbar\omega_q} - 1}. \tag{S15}$$

The field correlation function then becomes:

$$\langle\varphi(x,t)\varphi(0,0)\rangle_T = \int_0^\infty \frac{dq}{q}\cos[q(x-\nu t)]\coth\left(\frac{\beta\hbar\nu q}{2}\right). \tag{S16}$$

This integral can be evaluated using contour integration or conformal mapping techniques, yielding:

$$\langle\varphi(x,t)\varphi(0,0)\rangle_T = -\frac{1}{4K}\ln\left[\frac{\sinh^2\left(\frac{\pi}{\beta\nu}(x+\nu t)\right)}{\left(\frac{\pi\zeta}{\beta\nu}\right)^2}\right], \tag{S17}$$

and similarly for $\theta(x,t)$:

$$\langle \theta(x,t)\theta(0,0)\rangle_T = -\frac{K}{4}\ln\left[\frac{\sinh^2\left(\frac{\pi}{\beta\nu}(x+\nu t)\right)}{\left(\frac{\pi\zeta}{\beta\nu}\right)^2}\right]. \tag{S18}$$

The coefficients K and $1/K$ reflect the strength of the interactions and their influence on the respective field fluctuations. A large K implies stronger phase fluctuations (in θ) and weaker density fluctuations (in φ). We neglect the mixed correlator $\langle \varphi(x,t)\,\theta(0,0)\rangle$ and $\langle \theta(x,t)\,\varphi(0,0)\rangle$ since it contributes only an overall phase to the Green's function and does not affect the temperature-dependent amplitude of Aharonov–Bohm (AB) oscillations.

Using these temperature-dependent correlation functions, we evaluate the quantity entering Eq. S7:

$$\langle [\varphi(x,t)+\theta(x,t)]\,[\varphi(0,0)+\theta(0,0)]\rangle = -\left(\frac{1}{4K}+\frac{K}{4}\right)\ln\left[\frac{\sinh^2\left(\frac{\pi k_B T}{\hbar\nu}(x+\nu t)\right)}{\left(\frac{\pi\zeta k_B T}{\hbar\nu}\right)^2}\right]. \tag{S19}$$

Defining the interaction parameter:

$$\gamma = \frac{1}{2}\left(K+\frac{1}{K}\right)-1. \tag{S20}$$

The -1 ensures $\gamma=0$ in the non-interacting limit ($K=1$). This normalization isolates the contribution of electron–electron interactions to the suppression of coherence. So we can simplify the above as:

$$\langle [\varphi(x,t)+\theta(x,t)]\,[\varphi(0,0)+\theta(0,0)]\rangle = -\frac{\gamma}{2}\ln\left[\frac{\sinh\left(\frac{\pi k_B T}{\hbar\nu}(x+\nu t)\right)}{\frac{\pi\zeta k_B T}{\hbar\nu}}\right]. \tag{S21}$$

Thus, the finite-temperature Green's function becomes:

$$\mathcal{G}(x,t) = \frac{1}{2\pi\zeta}\left[\frac{\frac{\pi\zeta k_B T}{\hbar\nu}}{\sinh\left(\frac{\pi k_B T}{\hbar\nu}(x+\nu t)\right)}\right]^\gamma. \tag{S22}$$

To study the AB interference, we evaluate the correlation function around the full loop of the ring:

$$\mathcal{G}(L,T) = \frac{(-1)^N}{2\pi\zeta}\left(\frac{\frac{\pi\zeta k_B T}{\hbar\nu}}{\sinh\left(\frac{\pi L k_B T}{\hbar\nu}\right)}\right)^\gamma e^{i2\pi\Phi/\Phi_0} = -\mathcal{A}_{AB}(T)\,e^{i2\pi\Phi/\Phi_0}, \tag{S23}$$

where the minus sign originates from the nontrivial topology of the system with winding number $N=1$, and $\mathcal{A}_{AB}$ denotes the temperature-dependent AB amplitude:

$$\mathcal{A}_{AB}(T) = \frac{1}{2\pi\zeta}\left(\frac{\frac{\pi\zeta k_B T}{\hbar\nu}}{\sinh\left(\frac{\pi L k_B T}{\hbar\nu}\right)}\right)^\gamma. \tag{S24}$$

In the non-interacting case ($\gamma = 0$), the AB amplitude remains constant with temperature, consistent with theoretical predictions for non-interacting one-dimensional channels [6, 7]. The characteristic temperature scale,

$$T_L = \frac{\hbar\nu}{\pi k_B L}, \tag{S25}$$

marks the crossover point beyond which dephasing becomes significant.

In the low-temperature limit $T < T_L$, we use the approximation $\sinh(x) \approx x$ to obtain:

$$\mathcal{A}_{AB} \approx \frac{1}{2\pi\zeta}\left(\frac{\zeta}{L}\right)^{\gamma}, \tag{S26}$$

which indicates saturation of the AB amplitude and maximal coherence as $T \to 0$. Here, the dimensionless ratio ζ/L encodes the scaling of microscopic interactions with system size, while the exponent γ controls the strength of interaction-induced dephasing.

In the higher temperature regime $T > T_L$ and for weak interactions $\gamma \ll 1$, we expand $\sinh^{-\gamma}(x)$ as:

$$\sinh^{-\gamma}(x) \approx 1 - \gamma(x - \log 2) + \mathcal{O}(\gamma^2), \tag{S27}$$

which captures leading-order interaction corrections. Substituting this into the amplitude expression yields:

$$\mathcal{A}_{AB} \approx \frac{1}{2\pi\zeta}\left(\frac{T}{T_\zeta}\right)^{\gamma}\left[1 - \gamma\frac{T}{T_L}\right], \tag{S28}$$

where $T_\zeta = \hbar\nu/\pi k_B \zeta$. This result demonstrates the linear decay of the AB amplitude with increasing temperature, originating from interaction-induced dephasing.

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