Inspiring Excellence

Department of Mathematics and Natural Sciences

MAT120 Lab - Integral Calculus and Differential Equations

Assignment

*The solution **must** be done in Google Colaboratory and the **ipynb** file must be submitted in the Google Classroom.*

*The outputs must be **organized** and **readable**. It must show necessary steps of the calculation and properly display the solutions/results.*

Copying of codes from any websites or other classmates will not be tolerated. You are ONLY ALLOWED to take help from the colab files that I have provided but be careful NOT to copy any codes.

Be creative, use your intuition. Answer the questions by yourself and Do not Cheat. Plagiarized copies will receive zero points regardless of the circumstances.

Total marks is 60
Deadline: 23:59 23rd March, 2024

Attempt All Questions

1. **Derivatives and Plots:** $(5 \times 2 = 10)$

   The department of transportation finds that the rate at which cars cross a bridge can be approximated by the function

$$f(t) = \frac{22.8}{3.5 + (t - 1.25)^4}$$

   where t = 0 at 4pm, and is measured in hours, and f(t) is measured in cars per minute.

   a Plot this functions to visualize the rate of cars crossing the bridge from 4pm to 6pm. Include enough points so that the curve you plot appears smooth. Use gray lines, one horizontal at $f(t) = 0$ and the other vertical at $t = 0$. Limit the values of $f(t)$, if required, so that the plot is understandable. Label the axes.

   b Estimate the total number of cars that cross the bridge between 4 and 6pm. You must carry out the integration numerically using the Monte Carlo Method.

2. **Simpson's Rule:** $(5 \times 4 = 20)$

Simpson's 1/3 Rule is a numerical method used to approximate definite integrals. It's based on approximating the area under a curve using quadratic approximations. The formula is given as

$$\int_a^b f(x)\,dx = \frac{h}{3}\big[f(a) + 4f(x_1) + 2f(x_2) + ... + 4f(x_{n-1}) + 2f(x_{n-2}) + f(b)\big] \quad (1)$$

The algorithm:

- **Inputs:** A function $f(x)$, specified interval $[a, b]$ and a value for the number of sub-intervals N.
- Compute the width of each subinterval $h : h = \frac{(b-a)}{N}$
- Set new variables 'x_i = a' and 'sum = f(a) + f(b)'.
- Begin a loop that runs from $i = 1$ till $i = N - 1$. In each iterations, update the value of x : x_i += h check if i is even or odd.
- If odd then sum += 4*f(x_i) otherwise if even then sum += 2*f(x_i)
- Finally update sum = h/3 * sum and return the output.

For the following integrals:

$$I_1 = \int_{-1}^{2} x\sqrt{2 + x^3}\,dx$$

$$I_2 = \int_{1}^{2} (\ln x)^{\frac{3}{2}}\,dx$$

a) Implement the Simpson's $\frac{1}{3}^{rd}$ rule to evaluate both of the integrals. Do this for multiple values of sub-intervals, N. (e.g $N = 10, 100, 1000, ...$).

b) Evaluate both integrals using the trapezoidal rule for the same sub-interval values as part **a**.

c) Plot the results from part **a** and **b** as a **results vs N** graph to compare the accuracy. Label each line properly.

d) Use sympy to solve both the integrals symbolically without the limits. Plot the resulting functions in a single graph and label each function correctly using legends to show it on the plot.

3. **Fresnel Integral:** (5)

If one looks at the diffraction of a plane wave, the diffracted wave (in the Fresnel approximation) is given by the expression

$$\Psi(z) = \int_0^z \exp\{ix^2\} = \mathcal{C}(z) + i\mathcal{S}(z) \quad (2)$$

where

$$\mathcal{C}(z) = \int_0^z \cos(x^2)\,dx \;\; \mathcal{S}(z) = \int_0^z \sin(x^2)\,dx$$

Carry out these integrals numercially (Using any preferred algorithm e.g. Trapezoidal Rule or Monte Carlo) for values of $z \in [0, 20]$ and generate a parametric plot $\mathcal{S}$ vs $\mathcal{C}$. The graph plotted is known as the Cornu Spiral.

4. **Highschool Calculus:** $(5 + 5 = 10)$

   a) Plot the function
   $$f(x) = x^4 - x^3 - 20x^2 + 10x + 20$$
   in the interval [-5,5] with axis labelling and grid. Find **all** the roots of this function in the same interval using Newton's Method or Bisection Method.

   b) Find the maximum value of
   $$f(x) = x^3 \exp\left(-x^2\right) + \sin(3x)$$
   for $-2 \leq x \leq 2$.
   (EVALUATE THE FIRST AND SECOND DERIVATIVES TO FIND THE DESIRED EXTREMA)

5. **Special Functions:** $(5 \times 3 = 15)$

   The Beta function $\beta(x, y)$ is defined for all $x, y > 0$ by
   $$\beta(x, y) = 2 \int_0^{\frac{\pi}{2}} (\sin t)^{2x-1} (\cos t)^{2y-1} dt$$
   Write a function named Beta(x, y) that takes x and y as arguments and returns the result of the integral.

   a) Evaluate approximate value for $\beta(5, 2)$ numerically using the definition for the Beta function. Do the integration for different values of N e.g $N = 10, 100, 1000$.

   The Beta function is can be expressed using the Gamma function by the following expression:
   $$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$
   The gamma function $\Gamma(x)$ is defined for all $x > 0$ by
   $$\Gamma(x) = \int_0^{\infty} x^{x-1} e^{-x} dx$$

   b) Compute the exact value of $\beta(5, 2)$ using the given formula relating to the Gamma function, use the same values of N respectively when computing the integral for the Gamma function.

   c) Finally, compare the results and check the accuracy.