Binomial Distribution or (Bernoulli's Distⁿ)

The distⁿ is concerned with trials of a repetitive nature in which only the occurrence or non-occurrence, success or failure, acceptance or rejection, Yes or no of a particular event is of Interest.

Let there be n independent trials in an Experiment. Let a random variable X denote the no. of successes in these n trials. Let p be the prob. of a success & q that of a failure in a single trial so that $p+q=1$

We assume that there are n trials & the happening of an event A is r times & it not happening is $n-r$ times.

This may be shown as follows:

$$\underbrace{AA\text{---}A}_{r \text{ times}} \quad \underbrace{\bar{A}\bar{A}\text{---}\bar{A}}_{n-r \text{ times}} \qquad \text{— (1)}$$

A indicates its happening, $\bar{A}$ its not happening & $P(A)=p$ and $P(\bar{A})=q$

$$\Rightarrow \underbrace{pp\text{--}p}_{r\text{-times}} \quad \underbrace{q\cdot q\text{--}\cdot q}_{n-r \text{ times}} = p^r q^{n-r} \qquad \text{— (2)}$$

Clearly (1) is merely one order of arranging r A's

The prob. of (1) $= p^r q^{n-r} \times$ No. of diffⁿ arrangements of r A's and $(n-r)$ $\bar{A}$'s

$\therefore$ Req. Prob. ie. $\boxed{P(r) = {}^nC_r\, p^r q^{n-r}}$

where $p+q=1$, $r=0,1,2,\text{---},n$

* **Remark**: (i) n, the no. of trials is finite.
(ii) each trial has only two possible outcomes usually succ & failure.
(iii) all the trials are independent.
(iv) p (or q) is constant for all trials.

Mean of B.D:-

$$\text{Mean } \mu = \sum_{x=0}^{n} x\, P(x) = \sum_{x=0}^{n} x \cdot {}^nC_x\, p^x q^{n-x}$$

$$= 0 + 1\cdot {}^nC_1\, q^{n-1} p + 2\cdot {}^nC_2\, p^2 q^{n-2} + \cdots + n\; {}^nC_n\, p^n q^0$$

$$= np\, q^{n-1} + 2\cdot\frac{n(n-1)}{2\cdot 1} p^2 q^{n-2} + \cdots + n\cdot p^n$$

$$= np\,[q^{n-1} + (n-1)q^{n-2} + \cdots + p^{n-1}]$$

$$= np\,[{}^{n-1}C_0\, q^{n-1} + {}^{n-1}C_1\, q^{n-2} p + \cdots + {}^{n-1}C_{n-1}\, p^{n-1}]$$

$$= np\,(p+q)^{n-1} = np.$$

Variance σ^2: $= \sum_{x=0}^{n} x^2 P(x) - \mu^2 = \sum_{x=0}^{n} [x + x(x-1)]\,P(x) - \mu^2$

$$= \sum_{x=0}^{n} x\,P(x) + \sum_{x=0}^{n} x(x-1)\,P(x) - \mu^2 = \mu + \sum_{x=2}^{n} x(x-1)\,{}^nC_x\, q^{n-x} p^x - \mu^2$$

$$= \mu + n(n-1)p^2 - \mu^2 \quad [\because p+q=1]$$

$$= np + n(n-1)p^2 - n^2p^2$$

$$= np\,[1 + (n-1)p - np] = np\,[1-p] = npq.$$

$$\boxed{\sigma^2 = npq}$$

find ... of a ... s.d of

Find the prob. of getting 4 heads in 6 tosses of a fair coin.

Sol^n

$n=6,\ r=4,\ p=\frac{1}{2},\ q=\frac{1}{2}$

$P(r) = {}^nC_r\, q^{n-r}\, p^r$

$P(4) = {}^6C_4\, q^{6-4}\, p^4$

$= \frac{6\times5}{1\times2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \frac{15}{64}$

Q2):- If 10% of bolts produced by a machine are defective, Determine the prob. that out of 10 bolts chosen at random. (i) 1 (ii) None (iii) At most 2 bolts will be defective.

Sol^n:

$p(\text{defective}) = \frac{10}{100} = \frac{1}{10}$

$\therefore q(\text{non-defective}) = 1-\frac{1}{10} = \frac{9}{10}$

$n=10$

(i) Here $r=1$

$P(r=1) = {}^{10}C_1 (p)^1 q^{10-1} = {}^{10}C_1\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^9$

$= (0.9)^9 = 0.3874$

(ii) $r=0$

$P(r=0) = {}^{10}C_0\, p^0 q^{10-0} = \left(\frac{9}{10}\right)^{10} = 0.3486$

(iii) At most 2 bolts will be defective

$P(r\le 2) = P(0)+P(1)+P(2)$

$= 0.3486 + 0.3874 + {}^{10}C_2\, p^2 q^{10-2}$

$= 0.3486 + 0.3874 + 0.1937 = 0.9297$

Ques: Fit a Binomial Distⁿ to the following frequency data

x	0	1	2	3	4
f	30	62	46	10	2

Solⁿ

x	0	1	2	3	4
f	30	62	46	10	2
fx	0	62	92	30	8

$\Sigma f = 150$, $\Sigma fx = 192$

Mean $= \dfrac{\Sigma fx}{\Sigma f} = \dfrac{192}{150} = 1.28$

$np = 1.28 \Rightarrow 4p = 1.28 \Rightarrow p = 0.32$

$q = 1 - p = 1 - 0.32 = 0.64$

$N = 150$

Hence binomial distⁿ is $N(p+q)^n$

$= 150\,(0.32 + 0.68)^4$

Ques: A binomial variable X satisfies the relation $9\,P(X=4) = P(X=2)$, when $n=6$, find the value of p & $P(X=1)$.

Soln: We know that $P(X=r) = {}^nC_r\, p^r q^{n-r}$

$$P(X=4) = {}^6C_4\, p^4 q^{6-4}$$

$$P(X=2) = {}^6C_2\, p^2 q^{6-2}$$

A.T.Q

$$9\left[{}^6C_4\, p^4 q^2\right] = {}^6C_2\, p^2 q^4$$

$$9p^2 = q^2 \Rightarrow 9p^2 = (1-p)^2$$

$$\Rightarrow 9p^2 = 1 + p^2 - 2p \Rightarrow 8p^2 + 2p - 1 = 0$$

$$\Rightarrow p = -\tfrac{1}{2}, \tfrac{1}{4} \Rightarrow \boxed{p = 1/4}$$

$$P(X=1) = {}^6C_1\, p^1 q^{6-1} = {}^6C_1 \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^5 = 0.3559.$$

Poisson Distribution

If the parameters n & p of a binomial distⁿ are known we can find the distⁿ. But in situation where n is very large & p is very small, application of binomial distⁿ is labourious. However if we assume that $n \to \infty$ & $p \to 0$ s.t. np is always remains finite say λ.

$$P(X=x) \text{ i.e } P(x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

Mean of Poisson's distⁿ = λ

Variance ,, ,, = λ

Applications of P. D.:

(i) Arrival pattern of defective vehicals in a workshop.

(ii) No. of deaths in a district by rare disease.

(iii) No. of cars passing through a street in some time.

Ques: 6 coins are tossed 6400 times Using the Poisson distribution, find the app. prob. of getting 6 heads x times

Solⁿ: Prob. of getting one head with one coin = $\frac{1}{2}$

∴ The Prob. of getting 6 heads with 6 coins $= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

∴ Average no. of 6 heads with 6 coins in 6400 throws

$$= np = 6400 \times \frac{1}{64} = 100$$

∴ The mean of the Poisson distⁿ. = 100

App. prob. of getting 6 heads x times when the distⁿ. is Poisson $= \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(100)^x e^{-100}}{x!}$

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Quesr Suppose that a book of 600 pages contains 40 printing mistakes. Assume that these Errors are randomly distributed throughout the book and r the number of Errors per page has a possion dist^n. What is the Probability that 10 Pages selected at random will be free from Errors

Sol $p = \frac{40}{600} = \frac{1}{15}$

$n = 10$

$\lambda = np = (10)\left(\frac{1}{15}\right) = \frac{2}{3}$

$P(r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-2/3}\left(-\frac{2}{3}\right)^r}{r!}$

$P(0) = \frac{e^{-2/3}\left(-\frac{2}{3}\right)^0}{0!} = e^{-2/3} = 0.51.$

Quesr If the Probability of a bad reaction from a certain injection is 0.0002 determine the chance that out of 1000 individuals more than two will get a bad reaction.

Sol $p = 0.0002, \quad n = 1000$

$\lambda = np = 0.0002 \times 1000 = 0.2$

$P(r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-0.2}(0.2)^r}{r!}$

$P(r>2) = 1 - P(r \leq 2)$

$= 1 - [P(r=0) + P(r=1) + P(r=2)]$

$= 1 - \left[e^{-0.2} + e^{-0.2}(0.2) + \frac{e^{-0.2}(0.2)^2}{2!}\right]$

$1-[0.8187+0.1637+0.0164] = 0.0012$ (6)

Quesr Six coins are tossed 6400 times. Using the Poisson distribution, determine the approximate probability of getting six heads x times.

(ii) A poisson distribution has a double mode at $x=3$ and $x=4$. What is the probability that x will have one or the other of these two values.

Sol Probability of getting one head with one coin $= \frac{1}{2}$

The Probability of getting six heads with six coins

$$= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$$\lambda = 6400 \times \frac{1}{64} = 100$$

Approximate probability getting six head x times when dist^n is poisson

$$= \frac{\lambda^x e^{-\lambda}}{x!} = \frac{(100)^x e^{-100}}{x!}$$

Sol (ii) Since two modes are given. When λ is an integer modes are $\lambda-1$ and λ

$$\lambda - 1 = 3 \Rightarrow \lambda = 4$$

Probability (When $x=3$) $= \frac{e^{-4}(4)^3}{3!}$

Probability (When $x=4$) $= \frac{e^{-4}(4)^4}{4!}$

Reqd Prob $P(x=3 \text{ or } x=4)$

$P(x=3) + P(x=4)$

$$= \frac{e^{-4}(4)^3}{3!} + \frac{e^{-4}(4)^4}{4!}$$

$$= \frac{64}{3} e^{-4} = 0.39073$$

Quest Show that in a poission distn with unit mean, mean deviation about mean is $\left(\frac{2}{e}\right)$ S.D

Sol $\lambda = 1$ (given)

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{e^{-1}}{x!} \quad ; \quad x = 0,1,2,\ldots$$

Mean deviation about Mean 1 is

$$\sum_{x=0}^{\infty} |x-1|\, p(x) = e^{-1} \sum_{x=0}^{\infty} \frac{|x-1|}{x!}$$

$$= e^{-1}\left[1 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} \cdots\right]$$

$$\left\{ \frac{1}{2!} + \frac{2}{3!} \cdots \quad \frac{n}{(n+1)!} = \frac{n+1-1}{(n+1)!} = \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) \right.$$

$$= e^{-1}\left[1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots\right]$$

$$= e^{-1}(1+1)$$

$$= \frac{2}{e} \times 1$$

[in Possion distn Mean = Variance = λ, S.D = 1.

Ques

A Car hire firm has two Cars, which it hires out day by day. The number of demands for a Car on each day a distributed as a poisson distn with mean 1.5. Calculate the proportion of days on which neither Car is used and the proportion of days on which some demand is refused ($e^{-1.5} = 0.2231$)

Sol $\lambda = 1.5$ (given)

$\therefore$ Proportion of days on which ~~not~~ neither car is used

$$P(x=0) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-\lambda}(\lambda)^0}{0!} = e^{-1.5} = 0.2231$$

Proportion of days on which some demand is refused.

$$P(x>2) = 1-P(x\le 2)$$

$$= 1-\left[e^{-\lambda}+\frac{\lambda e^{-\lambda}}{1!}+\frac{\lambda^2 e^{-\lambda}}{2!}\right]$$

$$= 1-e^{-1.5}\left[1+1.5+\frac{(1.5)^2}{2}\right]$$

$$= 0.1912625$$

Quest Suppose the number of telephone calls on an operator received from 9:00 to 9:05 follow a poisson distn with a mean 3. find the Probability that

(i) The operator will receive no calls in that time interval tomorrow.

(ii) In the next three days, the operator will receive a total of 1 call in that time interval ($e^{-3} = 0.04978$)

Sol here $\lambda = 3$.

(i) $P(x=0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-3} = 0.04978$

(ii) Reqd Prob :- $P(0)P(0)P(1) + P(0)P(1)P(0) + P(1)P(0)P(0)$

$$= 3\left[\frac{e^{-\lambda}\lambda^0}{0!}\right]^2 \times \frac{e^{-\lambda}\lambda^1}{1!}$$

$$= 9(e^{-3})^3 = 0.00111$$

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Ques. if the variance of the Poisson distribution is 2, find the Probabilities for $r = 1,2,3,4$. from the recurrence relation of the Poisson distn. Also find $P(X \geq 4)$ (13)

Sol λ is mean and variance of Possion distn.

$$\lambda = 2$$

$$P(r+1) = \frac{\lambda}{r+1} P(r)$$

$$= \frac{2}{r+1} \frac{e^{-\lambda} \lambda^{r}}{r!}$$

$$P(r) = \frac{e^{-\lambda} \lambda^{r}}{r!}$$

$$P(0) = \frac{e^{-\lambda} (\lambda)^0}{0!} = e^{-2} = 0.1353$$

Put $r = 0, 1, 2, 3, 4$.

$$P(1) = 2 P(0) = 2 \times 0.1353 = 0.2706$$

$$P(2) = \frac{2}{2} P(1) = 0.2706$$

$$P(3) = \frac{2}{3} P(2) = \frac{2}{3} \times 0.2706 = 0.1804.$$

$$P(4) = \frac{2}{4} P(3) = 0.0902.$$

$$P(X \geq 4) = 1 - P(X < 4)$$

$$= 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$= 1 - [0.1353 + 0.2706 + 0.2706 + 0.1804]$$

$$= 0.1431.$$