MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(An Autonomous Institution – UGC, Govt.of India)

Recognizes under 2(f) and 12(B) of UGC ACT 1956

(Affiliated to JNTUH, Hyderabad, Approved by AICTE –Accredited by NBA & NAAC-"A" Grade-ISO 9001:2015 Certified)

LINEAR ALGEBRA AND ORDINARY DIFFERENTIAL EQUATIONS

B.Tech – I Year – I Semester

DEPARTMENT OF HUMANITIES AND SCIENCES

LINEAR ALGEBRA AND ORDINARY DIFFERENTIAL EQUATIONS

(common to all branches)

Course Objectives: To learn

1. The concept of a Rank of the matrix and applying the concept to know the consistency and solving the system of linear equations.
2. The concept of Eigen values, Eigen vectors and Diagonolisation.
3. The maxima and minima of functions of several variables.
4. The Applications of first order ordinary differential equations.
5. The methods to solve higher order differential equations.

UNIT I: Matrices

Introduction, Types of matrices, Rank of a matrix - Echelon form and Normal form, Consistency of system of linear equations (Homogeneous and Non-Homogeneous)-Gauss elimination method and LU Decomposition method.

UNIT II: Eigen values and Eigen vectors

Linear dependence and independence of vectors, Eigen values and Eigen vectors and their properties (without proof), Diagonalisation of a matrix. Cayley-Hamilton theorem(without proof), finding inverse and power of a matrix by Cayley-Hamilton Theorem; Quadratic forms and Nature of the Quadratic Forms; Reduction of Quadratic form to canonical forms by Orthogonal Transformation.

UNIT III: Multi Variable Calculus (Differentiation)

Functions of two variables- Limit, Continuity, Partial derivatives, Total differential and differentiability, Derivatives of composite and implicit functions, Jacobian-functional dependence and independence, Maxima and minima and saddle points, Method of Lagrange multipliers, Taylors theorem for two variables.

UNIT IV: First Order Ordinary Differential Equations

Exact, Equations reducible to exact form, Applications of first order differential equations - Orthogonal Trajectories(Cartesian form), Newton's law of cooling, Law of natural growth and decay,.

UNIT V: Differential Equations of Higher Order

Linear differential equations of second and higher order with constant coefficients: Non-homogeneous term of the type $f(x) = e^{ax}$, sinax, cosax, x^n, e^{ax} V and x^n V - Method of variation of parameters.

Text Books :

i) Higher Engineering Mathematics by B V Ramana., Tata McGraw Hill.
ii) Higher Engineering Mathematics by B.S. Grewal, Khanna Publishers.
iii)Advanced Engineering Mathematics by Kreyszig ,John Wiley & Sons .

Reference Books :

i)Advanced Engineering Mathematics by R.K Jain & S R K Iyenger, Narosa Publishers.
ii) Ordinary and Partial Differential Equations by M.D. Raisinghania, S.Chand Publishers
iii)Engineering Mathematics by N.P Bali and Manish Goyal.

Course Outcomes: After learning the concepts of this paper the student will be able to

1. Analyze the solution of the system of linear equations and to find the Eigen values and Eigen vectors of a matrix.
2. Reduce the quadratic form to canonical form using orthogonal transformations.
3. Find the extreme values of functions of two variables with / without constraints.
4. Solve first order, first degree differential equations and their applications.
5. Solve higher order differential equations.

UNIT-I

MATRICES

Introduction: The influence of Matrices in mathematical world is spread wide because it provides an important base to many of the principles and practices. The origin of mathematical matrices lies with the study of systems of simultaneous linear equations. Some of the things Matrices is used for are to solve systems of linear format, to find least-square best fit lines to predict future outcomes or find trends, to encode and decode messages.

There are many uses of matrices in Engineering such as Graph theory, Linear combinations of quantum states in Physics, Computer animation, for writing secret messages and Cryptography.

Basic Definitions:

Matrix: A matrix is a two dimensional array of numbers or expressions arranged in a set of rows and columns. An $m \times n$ matrix A has m rows and n columns and is written

$$\text{Eg: } A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{12} \ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots & \\ a_{m1} & a_{m2} \ldots\ldots a_{mn} \end{bmatrix}_{\text{m x n}} = [a_{\text{ij}}]mxn \qquad \text{where } 1 \le i \le m, 1 \le j \le n.$$

Where $a'_{ij}s$ are scalars.

Order of the Matrix: The number of rows and columns represents the order of the matrix. It is denoted by mxn, where m is number of rows and n is number of columns.

Types of Matrices:

Row Matrix: A Matrix having only one row is called a “Row Matrix”.

Eg: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1x3}$

Column Matrix: A Matrix having only one column is called a “Column Matrix” .

Eg: $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3x1}$

Null Matrix: A= $\left[a_{ij}\right]_{mxn}$ such that $a_{ij} = 0 \ \forall \ i$ and j. Then A is called a "Zero Matrix". It is denoted by $Omxn$.

Eg: $O_{2x3}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rectangular Matrix: If A=$\left[a_{ij}\right]_{mxn}$, and $m \neq n$ then the matrix A is called a "Rectangular Matrix"

Eg : $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ is a 2x3 matrix

Square Matrix: If $A = \left[a_{ij}\right]_{mxn}$ and $m = n$ then A is called a "Square Matrix".

Eg : $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is a 2x2 matrix

Lower Triangular Matrix: A square Matrix $A_{nXn} = \left[a_{ij}\right]_{nxn}$ is said to be lower Triangular if $a_{ij} = 0$ if $i < j$ i.e. if all the elements above the principal diagonal are zeros.

Eg: $\begin{bmatrix} 4 & 0 & 0 \\ 5 & 2 & 0 \\ 7 & 3 & 6 \end{bmatrix}$ is a Lower triangular matrix.

Upper Triangular Matrix: A square Matrix $A = \left[a_{ij}\right]_{nxn}$ is said to be upper triangular if $a_{ij} = 0$ if $i > j$. i.e. all the elements below the principal diagonal are zeros.

Eg: $\begin{bmatrix} 1 & 3 & 8 \\ 0 & 4 & -5 \\ 0 & 0 & 2 \end{bmatrix}$ is an Upper triangular matrix

Triangular Matrix: A square matrix which is either lower triangular or upper triangular is called a triangle matrix.

Principal Diagonal of a Matrix: In a square matrix, the set of all a_{ij}, for which $i = j$ are called principal diagonal elements. The line joining the principal diagonal elements is called principal diagonal.

Note: Principal diagonal exists only in a square matrix.

Diagonal elements in a matrix: $A = [a_{ij}]_{nxn}$, the elements a_{ij} of A for which $i = j$.

i.e. $a_{11}, a_{22} \dots . a_{nn}$ are called the diagonal elements of A

Eg: A= $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ diagonal elements are 1, 5, 9

Diagonal Matrix: A Square Matrix is said to be diagonal matrix, if $a_{ij} = 0$ for $i \neq j$ i.e. all the elements except the principal diagonal elements are zeros.

Note: 1. Diagonal matrix is both lower and upper triangular.

2. If $d_1, d_2 \ldots \ldots d_n$ are the diagonal elements in a diagonal matrix it can be represented as **diag** $[d_1, d_2, \ldots\ldots, d_n]$

Eg : A = diag (3,1,-2)= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Scalar Matrix: A diagonal matrix whose leading diagonal elements are equal is called a "Scalar Matrix". Eg : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Unit/Identity Matrix: If A = $[a_{ij}]_{nxn}$ such that $a_{ij} = 1$ for $i = j$, and $a_{ij} = 0$ for $i \neq j$ then A is called a "Identity Matrix" or Unit matrix. It is denoted by I_n

Eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Trace of Matrix: The sum of all the diagonal elements of a square matrix A is called Trace of a matrix A, and is denoted by Trace A or $tr\ A$.

Eg : A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then $tr\ A = a + b + c$

Singular & Non Singular Matrices: A square matrix A is said to be "Singular" if the determinant of A ($|A|$) $= 0$, Otherwise A is said to be "Non-singular".

Note: 1. Only non-singular matrices possess inverse.

2. The product of non-singular matrices is also non-singular.

Inverse of a Matrix: Let A be a non-singular matrix of order n if there exist a matrix B such that $AB = BA = I$ then B is called the inverse of A and is denoted by A^{-1}

If inverse of a matrix exist, it is said to be invertible.

Note: 1. The necessary and sufficient condition for a square matrix to posses inverse is that $|A| \neq 0$.

2 .Every Invertible matrix has unique inverse**.**

3. If A, B are two invertible square matrices then AB is also invertible and

$(AB)^{-1} = B^{-1}A^{-1}$

4. $A^{-1} = \dfrac{AdjA}{\det A}$ $\quad where\ detA \neq 0$,

Theorem: The inverse of a Matrix if exists is Unique.

Note: 1. $(A^{-1})^{-1} = A$ 2. $I^{-1} = I$

Theorem: If A, B are invertible matrices of the same order, then

(i). $(AB)^{-1} = B^{-1}A^{-1}$

(ii). $(A^T)^{-1} = (A^{-1})^T$

Sub Matrix: - A matrix obtained by deleting some of the rows or columns or both from the given matrix is called a sub matrix of the given matrix.

Eg: Let $A = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 8 & 9 & 10 & 5 \\ 3 & 4 & 5 & -1 \end{bmatrix}$. Then $\begin{bmatrix} 8 & 9 & 10 \\ 3 & 4 & 5 \end{bmatrix}_{2x3}$ is a sub matrix of A obtained by deleting first row and 4th column of A.

Minor of a Matrix: Let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'.

Eg: $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ be a 4x3 matrix

Here $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ is a sub-matrix of order '2'

$|B|$ = 2-3 = -1 is a minor of order '2

And $C = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 2 \\ 5 & 6 & 7 \end{bmatrix}$ is a sub-matrix of order '3'

$\det C = 2(7-12) - 1(21-10) + (18-5) = -9$

Properties of trace of a matrix: Let A and B be two square matrices and λ be any scalar

1) $tr\ (\lambda A) = \lambda (tr\ A)$; 2) $tr(A+B) = trA + trB$; 3) $tr\ (AB) = tr(BA)$

Idempotent Matrix: A square matrix A Such that $A^2 = A$ then A is called "Idempotent Matrix".

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Involuntary Matrix: A square matrix A such that $A^2 = I$ then A is called an Involutary Matrix.

Eg: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Nilpotent Matrix: A square matrix A is said to be Nilpotent if there exists a positive integer n such that $A^n = 0$ here the least n is called the Index of the Nilpotent Matrix.

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Transpose of a Matrix: The matrix obtained by interchanging rows and columns of the given matrix A is called as transpose of the given matrix A. It is denoted by A^T or A'

Eg: A = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Properties of transpose of a matrix: If A and B are two matrices and A^T , B^T are their transposes then

1) $\left(A^T\right)^T = A$; 2) $(A+B)^T = A^T + B^T$; 3) $(KA)^T = KA^T$; 4) $(AB)^T = B^T A^T$

Symmetric Matrix: A square matrix A is said to be symmetric if $A^T = A$

If $A = \left[a_{ij}\right]_{nxn}$ *then* $A^T = \left[a_{ji}\right]nxn$ where $a_{ij} = a_{ji}$

Eg: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

Skew-Symmetric Matrix: A square matrix A is said to be Skew symmetric If $A^T = -A$.

If $A = \left[a_{ij}\right]_{nxn}$ *then* $A^T = \left[a_{ji}\right]_{nxn}$ *where* $a_{ij} = -a_{ji}$.

Eg : $\begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is a skew – symmetric matrix

Note: All the principle diagonal elements of a skew symmetric matrix are always zero.

Since $a_{ij} = -a_{ij} \Rightarrow a_{ij} = 0$

Theorem: Every square matrix can be expressed uniquely as the sum of symmetric and skew symmetric matrices.

Proof: Let A be a square matrix, A = $\frac{1}{2}(A+A)$ = $\frac{1}{2}(A+A^T+A-A^T)$ = $\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$ = $P + Q$, where P = $\frac{1}{2}(A+A^T)$; Q = $\frac{1}{2}(A-A^T)$

Thus every square matrix can be expressed as a sum of two matrices.

Consider $P^T=\left[\frac{1}{2}(A+A^T)\right]^T=\frac{1}{2}(A+A^T)^T=\frac{1}{2}\left(A^T+(A^T)^T\right)=\frac{1}{2}(A+A^T)$=P, since $P^T=P$,

P is symmetric

Consider $Q^T=\left[\frac{1}{2}(A-A^T)\right]^T=\frac{1}{2}(A-A^T)^T=\frac{1}{2}\left(A^T-(A^T)^T\right)$= - $\frac{1}{2}(A-A^T)$= - Q

Since $Q^T=-Q$, Q is Skew-symmetric.

To prove the representation is unique: Let $A = R + S \rightarrow (1)$ be the representation, where R is symmetric and S is skew symmetric. i.e. $R^T=R, S^T=-S$

Consider $$A^T=(R+S)^T=R^T+S^T=R-S \qquad \rightarrow (2)$$

$$(1)-(2) \Rightarrow A-A^T=2S \Rightarrow S=\frac{1}{2}(A-A^T)=Q$$

Therefore every square matrix can be expressed as a sum of a symmetric and a skew symmetric matrix

Ex. Express the given matrix A as a sum of a symmetric and skew symmetric matrices

where $A=\begin{bmatrix} 2 & -4 & 9 \\ 14 & 7 & 13 \\ 9 & 5 & 11 \end{bmatrix}$

Solution: $A^T=\begin{bmatrix} 2 & 14 & 3 \\ -4 & 7 & 5 \\ 9 & 3 & 11 \end{bmatrix}$

$$A+A^T=\begin{bmatrix} 4 & 10 & 12 \\ 10 & 14 & 18 \\ 12 & 18 & 22 \end{bmatrix} \Rightarrow P=\frac{1}{2}(A+A^T)=\begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix}; P \text{ is symmetric}$$

$$A-A^T=\begin{bmatrix} 0 & -18 & 6 \\ 18 & 0 & 8 \\ -6 & -8 & 0 \end{bmatrix} \Rightarrow Q=\frac{1}{2}(A-A^T)=\begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}; Q \text{ is skew-symmetric}$$

Now $A = P + Q = \begin{bmatrix} 2 & 5 & 6 \\ 5 & 7 & 9 \\ 6 & 9 & 11 \end{bmatrix} + \begin{bmatrix} 0 & -9 & 3 \\ 9 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$

Orthogonal Matrix: A square matrix A is said to be an Orthogonal Matrix if $AA^{T} = A^{T}A = I$.

Note: 1. If A, B are orthogonal matrices, then AB and BA are orthogonal matrices.

2. Inverse and transpose of an orthogonal matrix is also an orthogonal matrix.

Result: If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

Result: The inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal

Solved Problems:

1. Show that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Sol: Given A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then $A^{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Consider $A.A^{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is orthogonal matrix.

2. Prove that the matrix $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Sol: Given A = $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ Then $A^{T} = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

Consider $A.A^{T} = \frac{1}{9}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$\Rightarrow A.A^{T} = I$

Similarly $A^{T}.A = I$

Hence A is orthogonal matrix

3. Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ **is orthogonal.**

Sol: - For orthogonal matrix $AA^T = I$

So, $AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} = I$

$$\begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving $2b^2-c^2=0$, $a^2-b^2-c^2=0$

We get $c = \pm\sqrt{2}b$ $\quad a^2 = b^2+2b^2 = 3b^2$

⇨ $a = \pm\sqrt{3}b$

From the diagonal elements of I

$4b^2+c^2 = 1$ ⇨ $4b^2+2b^2 = 1$ (since $c^2=2b^2$) ⇨ $b = \pm\frac{1}{\sqrt{6}}$

$a = \pm\sqrt{3}b = \pm\frac{1}{\sqrt{2}}$; $\quad b = \pm\frac{1}{\sqrt{6}}$; $c = \pm\sqrt{2}b = \pm\frac{1}{\sqrt{3}}$

4. Is matrix $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ **Orthogonal?**

Sol:- Given $A = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$$\Rightarrow AA^T = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I_3$$

$AA^T \neq A^TA \neq I_3$

$\therefore$ Matrix is not orthogonal.

Complex matrix: A matrix whose elements are complex numbers is called a complex matrix.

Conjugate of a complex matrix: A matrix obtained from A on replacing its elements by the corresponding conjugate complex numbers is called conjugate of a complex matrix. It is denoted by $\overline{A}$

If $A = [a_{ij}]_{mxn}$, $\overline{A} = [\overline{a_{ij}}]_{mxn}$, where $\overline{a_{ij}}$ is the conjugate of a_{ij}.

Eg: If A=$\begin{bmatrix} 2+3i & 5 \\ 6-7i & -5+i \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 2-3i & 5 \\ 6+7i & -5-i \end{bmatrix}$

Note: If $\overline{A}$ and $\overline{B}$ be the conjugate matrices of A and B respectively, then

(i) $\overline{\left(\overline{A}\right)}$=A (ii) $\overline{A+B}=\overline{A}+\overline{B}$ (iii) $\overline{(KA)}=\overline{K}\ \overline{A}$

Transpose conjugate of a complex matrix: Transpose of conjugate of complex matrix is called transposed conjugate of complex matrix. It is denoted by A^{θ} *or* A^{*}.

Note: If A^{θ} and B^{θ} be the transposed conjugates of A and B respectively, then

(i) $\left(A^{\theta}\right)^{\theta}$ = A (ii) $\left(A\pm B\right)^{\theta}=A^{\theta}\pm B^{\theta}$

(iii) $\left(KA\right)^{\theta}=\overline{K}A^{\theta}$ (iv) $\left(AB\right)^{\theta}=A^{\theta}B^{\theta}$

Hermitian Matrix: A square matrix A is said to be Hermitian Matrix iff $A^{\theta}=A$.

Eg: A=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and A^{θ}=$\begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Note: 1. In Hermitian matrix the principal diagonal elements are real.

2. The Hermitian matrix over the field of Real numbers is nothing but real symmetric matrix.

3. In Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij}=\overline{a_{ji}}\ \forall i,j$.

Skew-Hermitian Matrix: A square matrix A is said to be Skew-Hermitian Matrix if $A^{\theta}=-A$.

Eg: Let A=$\begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ then $\overline{A}=\begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$ and $\left(\overline{A}\right)^{T}=\begin{bmatrix} 3i & -2-i \\ 2-i & i \end{bmatrix}$

$\therefore$ $\left(\overline{A}\right)^{T}$=-A $\therefore$ A is skew-Hermitian matrix.

Note: 1. In Skew-Hermitian matrix the principal diagonal elements are either Zero or Purely Imaginary.

2. The Skew- Hermitian matrix over the field of Real numbers is nothing but real Skew - Symmetric matrix.

3. In Skew-Hermitian matrix A= $\left[a_{ij}\right]_{nxn}$, $a_{ij}=-\overline{a_{ji}}\ \forall i,j$.

Unitary Matrix: A Square matrix A is said to be unitary matrix if $AA^{\theta}=A^{\theta}A=I$ *or* $A^{\theta}=A^{-1}$

Eg: $B=\frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$

Theorem1: Every square matrix can be uniquely expressed as a sum of Hermitian and skew – Hermitian Matrices.

Proof: - Let A be a square matrix write

$$A=\frac{1}{2}(2A)=\frac{1}{2}(A+A)=\frac{1}{2}(A+A^{\theta}+A-A^{\theta})$$

$$A=\frac{1}{2}(A+A^{\theta})+\frac{1}{2}(A-A^{\theta}) i.e A=P+Q$$

Let $P=\frac{1}{2}\left(A+A^{\theta}\right);Q=\frac{1}{2}\left(A-A^{\theta}\right)$

Consider $P^{\theta}=\left[\frac{1}{2}(A+A^{\theta})^{\theta}\right]=\frac{1}{2}(A+A^{\theta})^{\theta}=(A+A^{\theta})=P$

I.e. $P^{\theta}=P, P$ is Hermitian matrix.

$$Q^{\theta}=\left[\frac{1}{2}(A-A^{\theta})\right]^{\theta}=\frac{1}{2}\left(A^{\theta}-A\right)=-\frac{1}{2}(A-A^{\theta})=-Q$$

Ie $Q^{\theta}=-Q, Q$ is skew – Hermitian matrix.

Thus every square matrix can be expressed as a sum of Hermitian & Skew Hermitian matrices.

To prove such representation is unique:

Let $A = R + S$------- (1) be another representation of A where R is Hermitian matrix & S is skew – Hermitian matrix.

$\therefore R=R^{\theta};S^{\theta}=-S$

Consider $A^{\theta}=(R+S)^{\theta}=R^{\theta}+S^{\theta}=R-S-$. Ie $A^{\theta}=R-A-----(2)$

$$\left(1\right)+\left(2\right)\Rightarrow A+A^{\theta}=2R \text{ ie } R=\frac{1}{2}\left(A+A^{\theta}\right)=P$$

$$\left(1\right)-\left(2\right)\Rightarrow A-A^{\theta}=2S \text{ ie } S=\frac{1}{2}\left(A-A^{\theta}\right)=Q$$

Thus every square matrix can be uniquely expressed as a sum of Hermitian & skew Hermitian matrices.

Solved Problems:

1. If $A=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then show that A is Hermitian and iA is skew-Hermitian.

Sol: Given A= $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$ And $\left(\overline{A}\right)^T = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$

$\therefore A = \left(\overline{A}\right)^T$ Hence A is Hermitian matrix.

Let B= iA

i.e B= $\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix}$ then

$$\overline{B} = \begin{bmatrix} -3i & 4-7i & -5+2i \\ -4-7i & 2i & -1-3i \\ 5+2i & 1-3i & -4i \end{bmatrix}$$

$$\left(\overline{B}\right)^T = \begin{bmatrix} -3i & -4-7i & 5+2i \\ 4-7i & 2i & 1-3i \\ -5+2i & -1-3i & -4i \end{bmatrix} = (-1)\begin{bmatrix} 3i & 4+7i & -5-2i \\ -4+7i & -2i & -1+3i \\ 5-2i & 1+3i & 4i \end{bmatrix} = -B$$

$\therefore \left(\overline{B}\right)^T$ =-B

$\therefore B = iA$ is a skew Hermitian matrix.

2. If A and B are Hermitian matrices, prove that $AB - BA$ is a skew-Hermitian matrix.

Sol: Given A and B are Hermitan matrices

$\therefore \left(\overline{A}\right)^T = A$ And $\left(\overline{B}\right)^T = B$ ------------- (1)

Now $\left(\overline{AB-BA}\right)^T = \left(\overline{AB}-\overline{BA}\right)^T$

$= \left(\overline{A}\,\overline{B}-\overline{B}\,\overline{A}\right)^T$

$= \left(\overline{A}\,\overline{B}\right)^T - \left(\overline{B}\,\overline{A}\right)^T = \left(\overline{B}\right)^T\left(\overline{A}\right)^T - \left(\overline{A}\right)^T\left(\overline{B}\right)^T$

$= BA - AB$ (By (1))

$= -(AB - BA)$

Hence $AB - BA$ is a skew-Hemitian matrix.

3. Show that $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ *is* unitary if and only if $a^2+b^2+c^2+d^2 = 1$

Sol: Given A=$\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} a-ic & -b-id \\ b-id & a+ic \end{bmatrix}$

Hence $A^{\theta} = \left(\overline{A}\right)^{T} = \begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$

$$\therefore AA^{\theta} = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}\begin{bmatrix} a-ic & b-id \\ -b-id & a+ic \end{bmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

$\therefore AA^{\theta} = I$ if and only if $a^2+b^2+c^2+d^2 = 1$

4. Given that A=$\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that $(I-A)(I+A)^{-1}$ is a unitary matrix.

Sol: we have $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \text{ And}$$

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\therefore (I+A)^{-1} = \frac{1}{1-\left(4i^2-1\right)}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Let $B = (I-A)(I+A)^{-1}$

$$B=\frac{1}{6}\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}\begin{bmatrix}1 & -1-2i\\ 1-2i & 1\end{bmatrix}=\frac{1}{6}\begin{bmatrix}1+(1-2i)(-1-2i) & -1-2i-1-2i\\ 1-2i+1-2i & (-1-2i)(1-2i)+1\end{bmatrix}$$

$$B=\frac{1}{6}\begin{bmatrix}-4 & -2-4i\\ 2-4i & -4\end{bmatrix}$$

Now $\overline{B}=\frac{1}{6}\begin{bmatrix}-4 & -2+4i\\ 2+4i & -4\end{bmatrix}$ and $\left(\overline{B}\right)^T=\frac{1}{6}\begin{bmatrix}-4 & 2+4i\\ -2+4i & -4\end{bmatrix}$

$$B\left(\overline{B}\right)^T=\frac{1}{36}\begin{bmatrix}-4 & -2-4i\\ 2-4i & -4\end{bmatrix}\begin{bmatrix}-4 & 2+4i\\ -2+4i & -4\end{bmatrix}$$

$$=\frac{1}{36}\begin{bmatrix}36 & 0\\ 0 & 36\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

$$\left(\overline{B}\right)^T=B^{-1}$$

i.e. B is unitary matrix.

$\therefore (I-A)(I+A)^{-1}$ is a unitary matrix.

5. Show that the inverse of a unitary matrix is unitary.

Sol: Let A be a unitary matrix. Then $AA^{\theta}=I$

i.e $\left(AA^{\theta}\right)^{-1}=I^{-1}$

$\Rightarrow \left(A^{\theta}\right)^{-1}A^{-1}=I$

$\Rightarrow \left(A^{-1}\right)^{\theta}A^{-1}=I$

Thus A^{-1} is unitary.

Rank of a Matrix:

Let A be mxn matrix. If A is a null matrix, we define its rank to be ‘0’. If A is a non-zero matrix, we say that ‘$\boldsymbol{r}$’ is the rank of A if

i. Every $(r+1)th$ order minor of A is '0' (zero) &

ii. At least one rth order minor of A which is not zero.

It is denoted by ρ (A) and read as rank of A.

Note: 1. Rank of a matrix is unique.

2. Every matrix will have a rank.

3. If A is a matrix of order mxn, then Rank of $A \leq min\,(m,n)$

4. If $\rho\,(A) = r$ then every minor of A of order $r+1$, or minor is zero.

5. Rank of the Identity matrix In is n.

6. If A is a matrix of order n and A is non-singular then $\rho\,(A) = n$

7. If A is a singular matrix of order n then $\rho(A) < n$

Important Note:

1. The rank of a matrix is $\leq r$ if all minors of $(r+1)th$ order are zero.

2. The rank of a matrix is $\geq r$, if there is at least one minor of order 'r' which is not equal to zero.

1. Find the rank of the given matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Sol: Given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$$det\,A = 1(48-40) - 2(36-28) + 3(30-28) = 8 - 16 + 6 = -2 \neq 0$$

We have minor of order 3 $\quad \therefore \rho\,(A) = 3$

2. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Given the matrix is of order 3x4

Its Rank $\leq$ min(3,4) = 3

Highest order of the minor will be 3.

Let us consider the minor $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$

Determinant of minor is 1(-49)-2(-56) + 3(35-48) = -49+112-39 = 24 $\neq$ 0.

Hence rank of the given matrix is '3'.

Elementary Transformations on a Matrix:

i). Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

(ii). If i^{th} row is multiplied with k then it is denoted by $R_i \to kR_i$

(iii). If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then it is denoted by $R_j \to R_j + kR_i$

Note: 1. The corresponding column transformations will be denoted by writing 'c'. i.e

$c_i \leftrightarrow c_j$, $c_i \to k\, c_j$, $c_j \to c_j + kc_i$

2. The elementary operations on a matrix do not change its rank.

Equivalance of Matrices: If B is obtained from A after a finite number of elementary transformations on A, then B is said to be equivalent to A.It is denoted as $B \sim A$.

Note : 1. If A and B are two equivalent matrices, then rank A = rank B.

2. If A and B have the same size and the same rank, then the two matrices are equivalent.

Elementary Matrix or E-Matrix: A matrix is obtained from a unit matrix by a single elementary transformation is called elementary matrix or E-matrix.

Notations: We use the following notations to denote the E-Matrices.

1) $E_{ij} \to$ Matrix obtained by interchange of i^{th} and j^{th} rows (columns).

2) $E_{i(k)} \to$ Matrix obtained by multiplying i^{th} row (column) by a non- zero number k.

3) $E_{ij(k)} \to$ Matrix obtained by adding k times of j^{th} row (column) toi^{th} row (column).

Echelon form of a matrix:

A matrix is said to be in Echelon form, if

(i) Zero rows, if any exists, they should be below the non-zero row.

(ii) The first non-zero entry in each non-zero row is equal to '1'.

(iii) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

Note : 1. The number of non-zero rows in echelon form of A is the rank of 'A'.

1. The rank of the transpose of a matrix is the same as that of original matrix.
2. The condition (ii) is optional.

Eg: 1. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

2. $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is a row echelon form.

Solved Problems :

1.Find the rank of the matrix $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ by reducing it to Echelon form.

Sol: Given $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & -2 & 4 \\ 1 & -3 & -1 \end{bmatrix}$ Applying row transformations on A.

$R_1 \leftrightarrow R_3$ $$A \sim \begin{bmatrix} 1 & -3 & -1 \\ 3 & -2 & 4 \\ 2 & 3 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1;\ R_3 \to R_3 - 2R_1$ $$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 7 & 7 \\ 0 & 9 & 9 \end{bmatrix}$$

$R_2 \to R_2/7, R_3 \to R_3/9$ $$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $$\sim \begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the Echelon form of matrix A.

The rank of a matrix A= Number of non – zero rows =2

2.For what values of k the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$ has rank '3'.

Sol: The given matrix is of the order 4x4

If its rank is 3 ⇨ det A =0

$$A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & -2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

Applying $R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - kR_1$, $R_4 \to 4R_4 - 9R_1$

We get $A \sim \begin{bmatrix} 4 & 4 & -3 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 8-4k & 8+3k & 8-k \\ 0 & 0 & 4k+27 & 3 \end{bmatrix}$

Since Rank $A = 3 \Rightarrow det\ A = 0$

$\Rightarrow 4 \begin{vmatrix} 0 & -1 & -1 \\ 8-4k & 8+3k & 8-k \\ 0 & 4k+27 & 3 \end{vmatrix} = 0$

$\Rightarrow 1[(8-4k)\,3] - 1(8-4k)\,(4k+27)] = 0$

$\Rightarrow (8-4k)\,(3-4k-27) = 0$

$\Rightarrow (8-4k)(-24-4k) = 0$

$\Rightarrow (2-k)(6+k) = 0$

$\Rightarrow k = 2 \; or \; k = -6$

3).Find the rank of the matrix using echelon form $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

By applying $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 4R_1$; $R_4 \to R_4 - 4R_1$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$

$R_1 \to \dfrac{R_1}{-1}, R_2 \to \dfrac{R_2}{-1}, R_3 \to \dfrac{R_3}{-3}$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 5 & 7 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$ $A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ A is in echelon form $\quad\therefore$ Rank of A = 2

4).Find the rank of the matrix $A = \begin{bmatrix} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 3 & -3 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$ **by reducing into echelon form.**

Sol: By applying $R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 3R_1;\ R_4 \to R_4 + R_1$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \end{bmatrix}$

$R_3 \to R_3 - R_2$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & 2 \end{bmatrix}$

$R_3 \leftrightarrow R_4$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_3 \to R_3 + R_2$ A ~ $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clear it is in echelon form, rank of A = 2

Normal form/Canonical form of a Matrix:

Every non-zero Matrix can be reduced to any one of the following forms. $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} I_r & 0 \end{bmatrix}; \begin{bmatrix} I_r \\ 0 \end{bmatrix}; \begin{bmatrix} I_r \end{bmatrix}$ Known as normal forms or canonical forms by using Elementary row or column or both transformations where I_r is the unit matrix of order 'r' and 'O' is the null matrix.

Note: 1.In this form "the rank of a matrix is equal to the order of an identity matrix.

2. Normal form another name is "canonical form"

Solved Problems :

1. By reducing the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ **into normal form, find its rank.**

Sol: Given A = $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - 3R_1$ $\qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & 5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$

$R_3 \to R_3/-2$ $\qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 3 & 2 & 11 \end{bmatrix}$

$R_3 \to R_3+R_2$ $\qquad A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_2 \to c_2 - 2c_1,\ c_3 \to c_3-3c_1,\ c_4 \to c_4-4c_1$ $\qquad A \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

$c_3 \to 3c_3 - 2c_2,\ c_4 \to 3c_4-5c_2$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 18 \end{bmatrix}$

$c_2 \to c_2/-3,\ c_4 \to c_4/18$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$c_4 \leftrightarrow c_3$ $\qquad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

This is in normal form $[I_3\ 0]$, $\therefore$ Hence Rank of A is '3'.

2).Find the rank of the matrix A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$ **by reducing into canonical form or normal form.**

Sol: Given A = $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

By applying $R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1, R_4 \to R_4 - 3R_1$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$

$R_3 \to R_3 - R_2, R_4 \to R_4 + 7R_2$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$

$R_4 \to R_4 - 6R_3$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$

$R_4 \to \frac{R_4}{-18}$ $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Apply $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_3 \to C_3 - 2C_2; C_4 \to C_4 + 5C_2$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$C_4 \to C_4 + 2C_3$ $A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Clearly it is in the normal form $[I_4]$ $\therefore$ Rank of $A = 4$

3).Define the rank of the matrix and find the rank of the following matrix $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 4R_1$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - 3R_2$

$$A \sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is in echelon form. So, rank of matrix = no. of non zero rows in echelon form. $\therefore Rank \quad \rho(A) = 2$

4).Reduce the matrix A to normal form and hence find its rank $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

$C_1 \to \frac{1}{2}C_1$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 1 & 3 & 7 & 5 \\ 1 & 5 & 11 & 6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$A \sim \begin{bmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 0 & 4 & 8 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$
$R_4 \to R_4 - 4R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 8 & 2 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow 4C_4 - C_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \rightarrow \frac{1}{3} C_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow A \sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$

This is in normal form. Thus Rank of matrix = Order of identify matrix. $\therefore$ Rank $\rho(A) = 3$

5). Reduce the matrix A = $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ **into canonical form and then find its rank.**

Sol: Apply $C_1 \leftrightarrow C_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_1$ $\quad A \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$C_3 \rightarrow C_3 - 2C_1; C_4 \rightarrow C_4 + 2C_1$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$R_2 \rightarrow \frac{R_2}{2}$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \leftrightarrow C_3$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \rightarrow C_3 - 2C_2; C_4 \rightarrow C_4 - 3C_2$ $\quad A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Which is in the normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, $\quad \therefore \rho$ (A) = 2

System of linear equations:

In this chapter we shall apply the theory of matrices to study the existence and nature of solutions for a system of m linear equations in 'n' unknowns.

The system of m linear equations in 'n' unknowns $x_{1,}x_{2,}x_3----x_n$ given by

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots..+a_{1n}x_n=0 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots..+a_{2n}x_n=0 \\ ------------------- \\ ------------------- \\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\ldots..+a_{mn}x_n=0 \end{array}\right\}------(1)$$

The above set of equations can be written in the Matrix form as $A\,X = B$

$$\begin{bmatrix} a_{11} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}\to(2)$$

A-Coefficient Matrix; X-Set of unknowns; B-Constant Matrix

Homogeneous Linear Equations: If $b_1=b_2=\ldots\ldots\ldots.=b_m=0$ then $B = 0$

Hence equation (2) Reduces to $AX = 0$ which are known as homogeneous linear equations

Non-Homogeneous Linear equations:

It at least one of b_1, b_2----b_m is non zero. Then $B\neq 0$, the system Reduces to $AX = B$ is known as Non-Homogeneous Linear equations.

Solution: A set of numbers $x_1, x_2 --x_n$ which satisfy all the equations in the system is known as solution of the system.

Consistent: If the system possesses a solution then the system of equations is said to be consistent.

Inconsistent: If the system has no solution then the system of equations is said to be Inconsistent.

Augmented Matrix: A matrix which is obtained by attaching the elements of B as the last column in the coefficient matrix A is called Augmented Matrix. It is denoted by $[A|B]$

$$[A/B]=C=\begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & - & - & a_{2n} & : & b_2 \\ a_{m1} & a_{m2} & a_{m3} & - & - & a_{mn} & : & b_3 \end{bmatrix}$$

1. If $\rho(A/B)=\rho(\text{A})$, then the system of equations $AX = B$ is consistent (solution exits).

a). If $\rho(A/B)=\rho(\text{A})=r = n$ (no. of unknowns) system is consistent and have a unique solution b). If $\rho(A/B)=\rho(A)=r < n$ (no. of unknowns) then the system of equations $AX = B$ will have an infinite no. of solutions. In this case $(n-r)$ variables can be assigned arbitrary values.

2. If $\rho(A/B)\neq\rho(A)$ then the system of equations $AX = B$ is inconsistent (no solution).

In case of homogeneous system $AX = 0$, the system is always consistent.

(or) $x_1 = 0$, $x_2 = 0$, --------, $x_n = 0$ is always the solution of the system known as the" zero solution ".

Non-trivial solution:

If $\rho(A/B)=\rho(A)=r < n$ (no. of unknowns) then the system of equations $AX = 0$ will have an infinite no. of non zero (non trivial) solutions. In this case $(n-r)$ variables can be assigned arbitrary values.

Also we use some direct methods for solving the system of equations.

Note: The direct methods are Cramer's rule, Matrix Inversion, Gaussian Elimination, Gauss Jordan, Factorization Tridiagonal system. These methods will give a unique solution.

Procedure to solve AX = B (Non Homogeneous equations)

Let us first consider n equations in n unknowns ie. $m = n$ then the system will be of the form

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots+a_{1n}x_n=0 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots+a_{2n}x_n=0 \\ -------------------- \\ -------------------- \\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\ldots\ldots+a_{mn}x_n=0 \end{array}\right\} ------(1)$$

The above system can be written as $AX = B$ --------- (1)

Where A is an $n\times n$ matrix.

Solving $AX = B$ using Echelon form:

Consider the system of m equations in n unknowns given by

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots\ldots+a_{1n}x_n=0 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots\ldots+a_{2n}x_n=0 \\ -------------------- \\ -------------------- \\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\ldots\ldots+a_{mn}x_n=0 \end{array}\right\} ------(1)$$

We know this system (1) can be we write as $AX = B$

The augmented matrix of the above system is [A / B] = $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

The system AX = B is consistent if $\rho\,(A) = \rho[A/B]$

i). $\rho\,(A) = \rho\,[A/B] = r < n$ (no. of unknowns).Then there is infinite no. of solutions.

ii). $\rho\,(A) = \rho\,[A/B]$ = number of unknowns then the system will have unique solution.

iii). $\rho\,(A) \neq \rho\,[A/B]$ the system has no solution.

Solved Problems :

1). Show that the equations $x + y + z = 4, 2x + 5y - 2z = 3, x + 7y - 7z = 5$ are not consistent.

Sol: Write given equations is of the form $AX = B$ i.e; $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$

Consider the Augment matrix is $[A\,/B]$ ⇨ $[A/B] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$

Applying $R_3 \to R_3 - 2R_2$, we get $[A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$

$\therefore \rho\,(A) = 2 \; and \; \rho\,(A/B) = 3$

The given system is inconsistent as $\rho\,(A) \neq \rho[A/B]$.

2). Show that the equations given below are consistent and hence solve them

$$x - 3y - 8z = -10, 3x + y - 4z = 0, 2x + 5y + 6z = 3$$

Sol: Matrix notation is $\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 3 \end{bmatrix}$

Augmented matrix [A/B] is [A/B] = $\begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 3 \end{bmatrix}$

$R_2 \to R_2-3R_1$ $R_3 \to R_3-2R_1$ $\sim\begin{bmatrix}1 & -3 & -8 & -10\\0 & 10 & 20 & 30\\0 & 11 & 22 & 23\end{bmatrix}$

$R_2 \to 1/10\, R_2$ $\sim\begin{bmatrix}1 & -3 & -8 & -10\\0 & 1 & 2 & 3\\0 & 11 & 22 & 23\end{bmatrix}$

$R_3 \to R_3-11R_2$ $\sim\begin{bmatrix}1 & -3 & -8 & -10\\0 & 1 & 2 & 3\\0 & 0 & 0 & -10\end{bmatrix}$

This is the Echelon form of [A/B] $\therefore \rho(A)=2$, $\rho(A/B)=3$

$\rho(A) \neq \rho[A/B]$.

The given system is inconsistent.

3).Find whether the following equations are consistent, if so solve

$\boldsymbol{x+y+2z=4\,,\, 2x-y+3z=9,\, 3x-y-z=2}$

Sol: We write the given equations in the form $AX=B$ i.e; $\begin{bmatrix}1 & 1 & 2\\2 & -1 & 3\\3 & -1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\9\\2\end{bmatrix}$

The Augmented matrix $[A/B]=\begin{bmatrix}1 & 1 & 2 & 4\\2 & -1 & 3 & 9\\3 & -1 & -1 & 2\end{bmatrix}$

Applying $R_2 \to R_2-2R_1$ and $R_3 \to R_3-3R_1$, we get $[A/B]\sim\begin{bmatrix}1 & 1 & 2 & 4\\0 & -3 & -1 & 1\\0 & -4 & -7 & -10\end{bmatrix}$

Applying $R_3 \to 3R_3-4R_2$, we get $[A/B]\sim\begin{bmatrix}1 & 1 & 2 & 4\\0 & -3 & -1 & 1\\0 & 0 & -17 & -34\end{bmatrix}$

this matrix is in Echelon form. $\rho(A)=3$ and $\rho(A/B)=3$

Since $\rho(A)=\rho[A/B]$. $\therefore$ The system of equations is consistent.

Here the number of unknowns is 3

Since $\rho(A)=\rho[A/B]$ = number of unknowns

$\therefore$ The system of equations has a unique solution

$We\ have \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -34 \end{bmatrix}$

$\Rightarrow -17z = -34 \Rightarrow z = 2$

$-3y - z = 1 \Rightarrow -3y = z + 1 \Rightarrow -3y = 3 \Rightarrow y = -1$

$and\ x + y + 2z = 4 \Rightarrow x = 4 - y - 2z = 4 + 1 - 4 = 1$

$\therefore x = 1, y = -1, z = 2$ is the solution.

4). Show that the equations $x + y + z = 6, x + 2y + 3z = 14, x + 4y + 7z = 30$ are consistent and solve them.

Sol: We write the given equations in the form AX=B i.e. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$

The Augmented matrix [A/B] = $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get [A/B] ~ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$

Applying $R_3 \to R_3 - 3R_2$, we get [A/B] ~ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This matrix is in Echelon form. ρ(A) = 2 and ρ(A/B) = 2

Since $\rho\ (A) = \rho\ [A/B]$.

The system of equations is consistent. Here the no. of unknowns are 3

Since rank of A is less than the no. of unknowns, the system of equations will have infinite number of solutions in terms of n-r=3-2=1 arbitrary constant.

The given system of equations reduced form is $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$

$\Rightarrow x + y + z = 6 \ldots\ldots\ldots (1), \quad y + 2z = 8 \ldots\ldots (2)$

Let $z = k$, put $z = k\ in\ (2)$ we get $y = 8 - 2k$

Put $z = k\ y = 8 - 2k$ in (1), we get

$x = 6 - y - z = 6 - 8 + 2k = -2 + k$

$\therefore x = -2 + k, y = 8 - 2k, z = k$ is the solution, where k is an arbitrary constant.

5). Show that $x+2y-z=3$ **;** $3x-y+2z=1$ **;** $2x-2y+3z=2; x-y+z=-1$ **are consistent and solve them**

Sol: The above system in matrix notation is $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}_{4\times3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}_{4\times1}$

$A \quad X \quad = B$

The Augmented matrix is $[AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

$$\begin{matrix} R_2 \to R_2 \to 3R_1 \\ R_3 \to R_3 \to 2R_1 \\ R_4 \to R_4 \to R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -0 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$$R_2 \to R_2 - R_3 \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$$

$$\begin{matrix} R_3 \to R_3 \to 3R_1 \\ R_4 \to R_4 \to 3R_2 \end{matrix} \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & +20 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$$R_3 \to \frac{R_3}{5} \quad \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix}$$

$$R_4 \to R_4 - 2R_3 \quad \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho\,(A) \;=\; 3 \;=\; \rho\,(A/B$

$\therefore \rho\,(A) \;= \rho\,(A/B)$ = No. of unknowns = 3

$\therefore$ The given system has unique solution.

The systems of equations equivalent to given system are

$x+2y-z=3 \qquad -y=-4; z=4$

$x+8-4=3 \qquad y=4; z=4$

$x=3-4=-1$

$\therefore x=-1, y=4, z=4.$

6). Solve $x+y+z=3; 3x-5y+2z=8; 5x-3y+4z=14$

Sol: - $\begin{bmatrix}1 & 1 & 1\\ 3 & -5 & 2\\ 5 & -3 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 8\\ 14\end{bmatrix}$

Augmented Matrix is $[AB]=\begin{bmatrix}1 & 1 & 1 & 3\\ 3 & -5 & 2 & 8\\ 5 & -3 & 4 & 14\end{bmatrix}$

$R_2 \to R_2-3R_1$
$R_3 \to R_3-5R_1$ $\sim\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & -8 & -1 & -1\\ 0 & -8 & -1 & -1\end{bmatrix}$

$R_3 \to R_3-R_2$ $\sim\left[\begin{array}{ccc|c}1 & 1 & 1 & 3\\ 0 & -8 & -1 & -1\\ 0 & 0 & 0 & 0\end{array}\right]$

$\rho[A]=\rho[AB]=2<$ Number of unknowns (3)

$\therefore$ The system has infinite number of solutions.

$x+y+z=-3, -8y-z=-1 \quad \Rightarrow 8y+z-1$

$Let\ z=k \quad \Rightarrow y=\frac{1-k}{8} \quad and \quad x=3-\frac{(1-k)}{8}-k=\frac{24-1+k-8k}{8}=\frac{23-7k}{8}$

$\Rightarrow X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\frac{23}{8}-\frac{7}{8}k\\ \frac{1}{8}-\frac{k}{8}\\ 0+k\end{bmatrix} \Rightarrow \quad X=\begin{bmatrix}\frac{23}{8}\\ \frac{1}{8}\\ 1\end{bmatrix}$ where k is any real number.

7). Find whether the following system of equations is consistent. If so solve them.

$x+2y+2z=2, \quad 3x-2y+z=5, \quad 2x-5y+3z=-4, \quad x+4y+6z=0.$

Sol: In Matrix form it is $\begin{bmatrix}1 & 2 & 2\\ 3 & -2 & 1\\ 2 & -5 & 3\\ 1 & 4 & 6\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}2\\ 5\\ -4\\ 0\end{bmatrix}$

$AX=B$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - R_1$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & -8 & -5 & -1 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

$R_2 \to R_2 - R_3$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & -9 & -1 & -8 \\ 0 & 2 & 4 & -2 \end{bmatrix}$

$R_3 \to R_3 + 9R_2$
$R_4 \to R_4 - 2R_2$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 12 & -16 \end{bmatrix}$

$R_4 \to \frac{1}{4} R_4$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 3 & -4 \end{bmatrix}$

$R_4 \to 37R_4 + 3R_3$ $\sim \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -37 & 55 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ is in echelon form

$\rho[A] = 3 \; and \; \rho[AB] = 4 \Rightarrow \rho[A] \neq \rho[AB]$ $\therefore$The given system is in consistent.

8). Discuss for what values of λ, μ the simultaneous equations $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu$ have

(i). No solution

(ii). A unique solution

(iii). An infinite number of solutions.

Sol: The matrix form of given system of Equations is A X = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$ = B

The augmented matrix is [A/B] = $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$ $\quad [A/B] \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\quad \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$

Case (i): let $\lambda \neq 3$ the rank of $A = 3$ and rank $[A/B] = 3$

Here the no. of unknowns is '3' $\quad \therefore \rho(A) = \rho(A/B) =$ No. of unknowns

The system has unique solution if $\lambda \neq 3$ and for any value of 'μ'.

Case (ii): Suppose $\lambda = 3$ and $\mu \neq 10$.

We have $\rho(A) = 2, \rho(A/B) = 3$

The system has no solution.

Case (iii): Let $\lambda = 3$ and $\mu = 10$.

We have $\rho(A) = 2, \rho(A/B) = 2$

Here $\rho(A) = \rho(A/B) \neq$ No. of unknowns $= 3$

The system has infinitely many solutions.

9). Find the values of a and b for which the equations $x+y+z=3$; $x+2y+2z=6$; $x+ay+3z=b$ have (i) No solution

(ii) A unique solution

(iii) Infinite no of solutions.

Sol: The above system in matrix notation is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ b \end{bmatrix}$

Augmented matrix $\quad [AB] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{bmatrix}$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$ $\quad \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-2 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\quad \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-3 & 0 & b-9 \end{bmatrix}$

• For $a=3\,\&\,b=9$ $\sim\left[\begin{array}{ccc|c}1&1&1&3\\0&1&1&3\\0&0&0&0\end{array}\right]$

$\therefore \rho[A]=\rho[AB]=2<3$ $\Rightarrow$ It has infinite no of solutions.

• *For* $a\neq 3\,\&\,b=any\ value$ $\sim\left[\begin{array}{ccc|c}1&1&1&3\\0&1&1&3\\0&a-3&0&b-9\end{array}\right]$

$\therefore \rho[A]=\rho[AB]=3$ $\Rightarrow$ It has a unique solution.

• *For* $a=3\,\&\,b\neq 9$ $\sim\left[\begin{array}{ccc|c}1&1&1&3\\0&1&1&3\\0&0&0&b-9\end{array}\right]$

$\therefore \rho[A]=2\neq\rho[AB]=3$ $\Rightarrow$ Inconsistent $\Rightarrow$ no solution

10). Solve the following system completely. $x+y+z=1; x+2y+4x=\alpha; x+4y+10z=\alpha^2$

Sol: The above system in matrix notation is $\begin{bmatrix}1&1&1\\1&2&4\\1&4&10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\\alpha\\\alpha^2\end{bmatrix}$

$A \quad X \quad = \quad B$

Augmented Matrix is $[AB]=\begin{bmatrix}1&1&1&1\\1&2&4&\alpha\\1&4&10&\alpha^2\end{bmatrix}$

$R_2\to R_2-R_1$
$R_3\to R_3-R_1$ $\sim\begin{bmatrix}1&1&1&1\\0&1&3&\alpha-1\\0&3&9&\alpha^2-1\end{bmatrix}$

$R_3\to R_3-3R_2$ $\sim\left[\begin{array}{ccc|c}1&1&1&1\\0&1&3&\alpha-1\\0&0&0&\alpha^2-3\alpha+2\end{array}\right]$

Here $\rho[A]=2\ and\ \rho[AB]=3$ $\Rightarrow$ The given system of equations is consistent if

$\alpha^2-3\alpha+2=0\Rightarrow\alpha^2-2\alpha-\alpha+2=0\Rightarrow(\alpha-2)(\alpha-1)=0\Rightarrow\alpha=2,\alpha=1$

Case (i): When $\alpha=1$

$\rho[A]=\rho[AB]=2<$ Number of unknowns.

$\therefore$ The system has infinite number of solutions.

The equivalent matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The equivalent systems of equations are $x+y+z=1; y+3z=0$

$\Rightarrow Let\ z=k \quad \Rightarrow y=-3k \quad and \quad x+(-3k)+K=1 \Rightarrow x-2k=1 \Rightarrow x=1+2k$

$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1+2k \\ 0-3k \\ 0+k \end{bmatrix} \Rightarrow X=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}+k\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ where k is any arbitrary constant.

Case (ii): When $\alpha=2$

$\rho[A]=\rho[AB]=2<$ no. of unknowns.

$\therefore$ The system has infinite number of solutions.

The equivalent matrix is $\sim\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The system of equations equivalent to the given system is $x + y + z = 1;\ y + 3z = 1$

$Let\ z=k \quad \Rightarrow y=1-3k\ and\ x+(1-3K)+k=1 \Rightarrow x=2k$

$\Rightarrow X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0+2k \\ 1-3k \\ 0+k \end{bmatrix} \Leftrightarrow X=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}+k\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ where k is any arbitrary constant.

11). Show that the equations $3x+4y+5z=a; 4x+5y+6z=b; 5x+6y+7z=c$ don't have a solution unless $a+c=2b$. solve equations when $a = b = c = -1$.

Sol: The Matrix notation is $\begin{bmatrix} 3 & 4 & 5 \\ 4 & 5 & 6 \\ 5 & 6 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$A \quad X \quad = \quad B$

Augment Matrix is $[AB]=\begin{bmatrix} 3 & 4 & 5 & a \\ 4 & 5 & 6 & b \\ 5 & 6 & 7 & c \end{bmatrix}$

$R_2 \to 3R_2-4R_1$
$R_3 \to 3R_3-5R_1$ $\sim\begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & -2 & -4 & 3c-5a \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\sim \begin{bmatrix} 3 & 4 & 5 & a \\ 0 & -1 & -2 & 3b-4a \\ 0 & 0 & 0 & 3a-6b+3c \end{bmatrix}$

Here $\rho[A]=2$ *and* $\rho[AB]=3$

$\therefore$ The given system of equations is consistent if $3a-6b+3c=0 \Rightarrow 3a+3c=6b \Rightarrow a+c=2b$

Thus the equations don't have a solution unless $a+c=2b, when\ a=b=c=-1$

The equivalent matrix is $\begin{bmatrix} 3 & 4 & 5 & -1 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho[A]=\rho[(AB)]=2<$ No. of unknowns.

$\therefore$ The system has infinite number of solutions. The system of equations equivalent to the given system $3x+4y+5z=-1; -y-2z=1 \Rightarrow y+2z=-1$

$Let\ z=k \Rightarrow y=-1-2k\ and\ 3x-4-8k+5k=-1 \Rightarrow x=1+k$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1+k \\ -1-2k \\ 0+k \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Linearly dependent set of vectors: A set $\{x_1, x_2, ---------------x_r\}$ of r vectors is said to be a linearly dependent set, if there exist r scalars k_1, k_2---k_r not all zero, such that $k_1x_1+k_2x_2+---------+k_rx_r = 0$

Linearly independent set of vectors: A set $\{x_1, x_2, ---------------x_r\}$ of r vectors is said to be a linearly independent set, if $k_1x_1+k_2x_2+---------+k_rx_r = 0$

then $k_1=0, k_2= 0-------k_r=0.$

Linear combination of vectors:

A vector x which can be expressed in the form $x = k_1x_1+k_2x_2+------+k_rx_r$ is said to be a linear combination of $\{x_1, x_2, -----------x_r\}$ here k_1, k_2---k_r are any scalars.

Linear dependence and independence of Vectors:

Solved Problems:

1). Show that the vectors $(1, 2, 3), (3, -2, 1), (1, -6, -5)$ from a linearly dependent set.

Sol: The Given Vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} X_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} X_3 = \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}$

The Vectors X_1, X_2, X_3 from a square matrix.

$$Let\ A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix} \quad Then\ |A| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{vmatrix}$$

$$= 1(10+6) - 2(15-1) + 3(-18+2)$$

$$= 16 + 32 - 48 = 0$$

The given vectors are linearly dependent $\because |A| = 0$

2). Show that the Vector $X_1 = (2, 2, 1), X_2 = (1, 4, -1)$ *and* $X_3 = (4, 6, -3)$ are linearly dependent.

Sol: Given Vectors $\boldsymbol{X_1} = (\mathbf{2}, \mathbf{2}, \mathbf{1}), \boldsymbol{X_2} = (\mathbf{1}, \mathbf{4}, -\mathbf{1})$ ***and*** $\boldsymbol{X_3} = (\mathbf{4}, \mathbf{6}, -\mathbf{3})$The Vectors X_1, X_2, X_3 form a square matrix.

$$Let\ A = \begin{bmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{bmatrix} \quad Then\ |A| = \begin{vmatrix} 2 & 1 & 4 \\ -2 & 4 & 6 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= 2(-12+6) + 2(-3+4) + 1(6-16) \quad = -20 \neq 0$$

$\therefore$ The given vectors are linearly dependent $\because |A| \neq 0$

Consistency of system of Homogeneous linear equations:

A system of m homogeneous linear equations in n unknowns, namely

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots.. + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots.. + a_{2n}x_n = 0 \\ ------------------- \\ ------------------- \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots.. + a_{mn}x_n = 0 \end{array}\right\} ------(1)$$

i.e. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots. a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots. a_{2n} \\ a_{m1} & a_{m2} & a_{m3} \ldots. a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ .. \\ .0 \end{bmatrix}$ Here A is called Co –efficient matrix.

Note: 1. Here $x_1 = x_2 = ---------- x_n = 0$ is called trivial solution or zero solution of $AX = 0$

2. A zero solution always linearly dependent.

Theorem: The number of linearly independent solutions of the linear system $AX = 0$ is $(n-r)$, r being the rank of the matrix A and n being the number of variables.

Note: 1.if A is a non-singular matrix then the linear system $AX = 0$ has only the zero solution.

2. The system $AX = 0$ possesses a non-zero solution if and only if A is a singular matrix.

Working rule for finding the solutions of the equation $AX = 0$

(i). Rank of A = No. of unknowns i.e. $r = n$

∴ The given system has zero solution.

(ii). Rank of A < No of unknowns $(r < n)$ and No. of equations < No. of unknowns $(m < n)$ then the system has infinite no. of solutions.

Note: If $AX = 0$ has more unknowns than equations the system always has infinite solutions.

Solved Problems :

1). Solve the system of equations

$\boldsymbol{x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0}$

Sol: We write the given system is AX = 0 i.e. $\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & -4 \\ 1 & -11 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1;\ R_3 \to R_3 - R_1$ $\quad A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$

$R_3 \to R_3 - 2R_2$ $\quad A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

The Rank of the $A = 2$ $i.e. \rho(A) = 2$ < No. of unknowns = 3

We have infinite No. of solution

Above matrix can we write as $x + 3y - 2z = 0$ $-7y + 8z = 0,$ $0 = 0$

$$Let\ z = k\ then\ y = 8/7k\ \&\ x = -10/7\ k$$

Giving different values to k, we get infinite no. of values of x, y, z.

2). Find all the non-trivial solution $2x - y + 3z = 0; 3x + 2y + z = 0; x - 4y + 5z = 0.$

Sol: In Matrix form it is $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$AX = 0$$

$$\text{The Augumented matrix}\left[A/O\right] = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \qquad \sim \begin{bmatrix} 1 & -4 & 5 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix} \quad \sim \begin{bmatrix} 1 & -4 & 5 & 0 \\ 0 & 14 & -14 & 0 \\ 0 & 7 & -7 & 0 \end{bmatrix}$$

$$R_3 \to 2R_3 - R_2 \qquad \sim \begin{bmatrix} 1 & -4 & 5 & 0 \\ 0 & 14 & -14 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{it is echelon form.}$$

The Rank of the A = 2 i.e. ρ(A) = 2 < No. of unknowns = 3

Hence the system has non trivial solutions. From echelon form, reduced equations are

$x - 4y + 5z = 0$ *and* $14y - 14z = 0$

Let $z = k$ *then* $y = k$ *and* $x - 4k + 5k = 0 \Rightarrow x = -k.$

Thus, the solution set is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \forall K.$

3). Show that the only real number λ for which the system $x + 2y + 3z = \lambda x, 3x + y + 2z = \lambda y, 2x + 3y + z = \lambda z$, has non-zero solution is 6 and solve them.

Sol: Above system can we expressed as AX = 0 i.e. $\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Given system of equations possess a non –zero solution i.e. ρ (A) < no. of unknowns.

⇨ For this we must have det A = 0

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$R_1 \to R_1 + R_2 + R_3 \quad \Rightarrow \begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

$$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1 \ \Rightarrow (6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

⇨ $(6-\lambda)[(-2-\lambda)(-1-\lambda)+1] = 0$

⇨ $(6-\lambda)\,(\lambda 2 + 3\lambda + 3) = 0$

⇨ $\lambda = 6$ only real values.

When λ = 6, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2+3R_1, R3 \to 5R_3+2R_1$ $\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3+R_2$ $\sim \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

⇨ $-5x + 2y + 3z = 0$ *and* $-19y + 19z = 0$ ⇨ $y = z$

Let $z = k$ ⇨ $y = k$ *and* $x = k$.

∴ *The solution is* $x = y = z = k$.

Gauss elimination method:

This method of solving a system of n liner equations in n unknowns consists of eliminating the coefficients in such a way that the system reduces to upper triangular system which may be solved by backward substitution. We discuss the method here for $n = 3$. The method is analogous for $n > 3$.

Consider the system

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$ ……………… (1)
$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

The augmented matrix of this system is

$[A, B] \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} b_1 \\ a_{21} & a_{22} & a_{23} b_2 \\ a_{31} & a_{32} & a_{33} b_3 \end{bmatrix}$ …………………..(2)

Performing $R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$ and $R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$, we get

$$[A,B]\sim\begin{bmatrix} a_{11} & a_{12} & a_{13}\, b_1 \\ 0 & \alpha_{22} & \alpha_{23}\beta_2 \\ 0 & \alpha_{32} & \alpha_{33}\beta_3 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Where $\alpha_{22} = a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right)$; $\alpha_{23} = a_{23} - a_{13}\left(\frac{a_{21}}{a_{11}}\right)$

$$\alpha_{32} = a_{32} - a_{12}\left(\frac{a_{31}}{a_{11}}\right) ;\ \alpha_{33} = a_{33} - a_{13}\left(\frac{a_{31}}{a_{11}}\right)$$

$\beta_2 = b_2 - \left(\frac{a_{21}}{a_{11}}\right) b_1;\ \beta_3 = b_3 - \left(\frac{a_{31}}{a_{11}}\right) b_1$

Here we assume $a_{11} \neq 0$

We call $\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}$ as multipliers for the first stage. a_{11} is called first pivot.

Now applying $R_3 \to R_3 - \frac{\alpha_{32}}{\alpha_{22}}(R_2)$, we get

$$[A,B]\sim\begin{bmatrix} a_{11} & a_{12} & a_{13}\, b_1 \\ 0 & \alpha_{22} & \alpha_{23}\beta_2 \\ 0 & 0 & \gamma_{33}\,\Delta_3 \end{bmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Where

$\gamma_{33} = \alpha_{33} - \left(\frac{\alpha_{32}}{\alpha_{22}}\right)\alpha_{23};\ \Delta_3 = \beta_3 - \left(\frac{\alpha_{32}}{\alpha_{22}}\right)\beta_2$

We have assumed $\alpha_{22} \neq 0$

Here the multiplier is $-\frac{\alpha_{32}}{\alpha_{22}}$ and new pivot is α_{22}.

The augmented matrix (4) corresponds to an upper triangular system which can be by backward substitution. The solution obtained is exact.

1.Solve the equations $3x + y + 2z = 3$; $2x - 3y - z = 3$; $x + 2y + z = 4$ using Gauss elimination method.

Sol. The given system of equations can be written as AX = B

Where A= $\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$

The Augmented matrix is

$$[A,B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix} \text{(Applying } R_1 \leftrightarrow R_3\text{)}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix} \text{(Applying } R_2 - 2R_1\text{ , } R_3 - 3R_1\text{)}$$

$$\sim\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{bmatrix} \text{(Applying } 7R_3 - 5R_2\text{)}$$

This corresponds to upper triangular system which gives

$$8z = -8 \Rightarrow z = -1$$

$$7y + 3z = 11 \Rightarrow 7y = 11 - 3z = 14 \Rightarrow y = 2$$

$$x + 2y + z = 4 \Rightarrow x = 4 - 2y - z = 4 - 4 + 1 \Rightarrow x = 1$$

The solution is $x = 1, y = 2, z = -1$.

2. Express the following system in matrix form and solve by Gauss elimination method $x+y+z=6$; $3x+3y+4z=20$; $2x+y+3z=13$.

Sol. The Augmented matrix of the given equations is

$$[A,B]=\begin{bmatrix}1 & 1 & 1 & 6\\ 3 & 3 & 4 & 20\\ 2 & 1 & 3 & 13\end{bmatrix}$$

Performing the row operations $R_2 \to R_2-3R_1$ and $R_3 \to R_3-2R_1$, we get

$$\sim\begin{bmatrix}1 & 1 & 16\\ 0 & 0 & 12\\ 0 & -1 & 11\end{bmatrix}$$

Using the operation $R_2 \leftrightarrow R_3$,we get $[A,B]\sim\begin{bmatrix}1 & 1 & 1 & 6\\ 0 & -1 & 1 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}$

This corresponds to upper triangular system which gives

$$z=\ 2$$
$$-y+z=1 \Rightarrow -y=1-z=-1\ \Rightarrow y=1$$
$$x+y+z=6 \Rightarrow x=6-y-z=6-1-2 \Rightarrow x=3$$

The solution is $x=3, y=1, z=2.$

LU decomposition

Suppose we have the system of equations AX = B. The motivation for an LU decomposition is based on the observation that systems of equations involving triangular coefficient matrices are easier to deal with. Indeed, the whole point of Gaussian elimination is to replace the coefficient matrix with one that is triangular. The LU decomposition is another approach designed to exploit triangular systems.

We suppose that we can write A = LU where L is a lower triangular matrix and U is an upper triangular matrix. An LU decomposition of a matrix A is the product of a lower triangular matrix and an upper triangular matrix that is equal to A.

1. Find the LU decomposition of the matrix Where A= $\begin{bmatrix}1 & 2 & 4\\ 3 & 8 & 14\\ 2 & 6 & 13\end{bmatrix}$

Sol: Let A= $\begin{bmatrix}1 & 2 & 4\\ 3 & 8 & 14\\ 2 & 6 & 13\end{bmatrix}$ = LU , where L = $\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$ and

U = $\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$.

Now we use this to find the entries in L and U.

$$\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\ l_{31}u_{11} & l_{31}u_{12}+l_{32}\,u_{22} & l_{31}u_{13}+l_{32}\,u_{23}+u_{33}\end{bmatrix}=\begin{bmatrix}1 & 2 & 4\\ 3 & 8 & 14\\ 2 & 6 & 13\end{bmatrix}$$

$$\Rightarrow \quad \begin{matrix} u_{11}=1 & u_{12}=2 & u_{13}=4 \\ l_{21}=3 & u_{22}=2 & u_{23}=2 \\ l_{31}=2 & l_{32}=1 & u_{33}=3 \end{matrix}$$

We have shown that

$$\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

and this is the LU decomposition of A.

2.Solve the system of equations $7x_1 - 2x_2 + x_3 = 12$, $14x_1 - 7x_2 - 3x_3 = 17$, $-7x_1 + 11x_2 + 18x_3 = 5$ using LU decomposition.

Sol: $$\begin{bmatrix} 7 & -2 & 1 \\ 14 & -7 & -3 \\ -7 & 11 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 5 \end{bmatrix}$$

LU decomposition of the coefficient matrix is

$$\begin{bmatrix} 7 & -2 & 1 \\ 14 & -7 & -3 \\ -7 & 11 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 5 \end{bmatrix}$$

Let $$\begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -7 \\ -4 \end{bmatrix}.$$

$$\Rightarrow \quad \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -7 \\ -4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}.$$

UNIT-II

EIGEN VALUES AND EIGEN VECTORS

Introduction

Let $A = \left[a_{ij}\right]_{nxn}$ be a square Matrix. Suppose the linear transformation $Y = AX$ transforms X into a scalar multiple of itself i.e. $AX = Y = \lambda X$, Then the unknown scalar λ is known as an "Eigen value" of the Matrix A and the corresponding non-zero vector X is known as "Eigen Vector" of A. Corresponding to Eigen value λ. Thus the Eigen values (or) characteristic values (or) proper values (or) latent roots are scalars λ which satisfy the equation.

$AX = \lambda$ X for $X \neq 0$, $AX - \lambda IX = 0 \Rightarrow (A - \lambda I)X = 0$

Which represents a system of 'n' homogeneous equations in 'n' variables x_1, x_2, ----, x_n this system of equations has non-trivial solutions If the coefficient matrix (A-λI) is singular i.e.

$$|A - \lambda I| = 0 \quad \Rightarrow \begin{vmatrix} a_{11-\lambda} & a_{12} & - & - & - & a_{1n} \\ a_{21} & a_{22-\lambda} & - & - & - & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{n1} & a_{n2} & - & - & - & a_{nn-\lambda} \end{vmatrix} = 0$$

Expansion of the determinant is $(-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + ----+K_n$ is the n^{th} degree of a polynomial $P_n(\lambda)$ which is known as **"Characteristic Polynomial"**. Of A $(-1)^n \lambda^n + K_1 \lambda^{n-1} + K_2 \lambda^{n-2} + ----+K_n$ =0 is known as **"Characteristic Equation"**. Thus the Eigen values of a square Matrix A are the roots of the characteristic equation.

Eg: Let $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ $\quad X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1. \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.X$

Here Characteristic vector of A is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and Characteristic root of A is "1".

Eigen Value: The roots of the characteristic equation are called Eigen values or characteristic roots or latent roots or proper values.

Eigen Vector: Let A=$\left[a_{ij}\right]_{nxn}$ be a Matrix of order n. A non-zero vector X is said to be a characteristic vector (or) Eigen vector of A if there exists a scalar λ such that $AX = \lambda X$.

Method of finding the Eigen vectors of a matrix.

Let $A = [a_{ij}]$ be a nxn matrix. Let X be an eigen vector of A corresponding to the eigen value λ.

Then by definition $AX = \lambda X$.

$\Rightarrow AX = \lambda IX$

$\Rightarrow AX - \lambda IX = 0$

$\Rightarrow (A - \lambda I)X = 0$ ------- (1)

This is a homogeneous system of n equations in n unknowns.

Will have a non-zero solution X if and only $|A - \lambda I| = 0$

- $A - \lambda I$ is called characteristic matrix of A
- $|A - \lambda I|$ is a polynomial in λ of degree n and is called the characteristic polynomial of A.
- $|A - \lambda I| = 0$ is called the characteristic equation
- Solving characteristic equation of A, we get the roots, $\lambda_1, \lambda_2, \lambda_3, \ldots \ldots \lambda_n,$ These are called the characteristic roots or eigen values of the matrix.
- Corresponding to each one of these n eigen values, we can find the characteristic vectors.

Procedure to find Eigen values and Eigen vectors

Let $A = \begin{bmatrix} a_{11} & a_{12} \ldots \ldots & a_{1n} \\ a_{21} & a_{22} \ldots \ldots & a_{2n} \\ \ldots \ldots & \ldots \ldots & \ldots \ldots \\ a_{n1} & a_{n2} \ldots \ldots & a_{nn} \end{bmatrix}$ be a given matrix

Characteristic matrix of A is $A - \lambda I$

$$i.e., A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

Then the characteristic polynomial is $|A - \lambda I|$

$$say\, \phi(\lambda) = |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$ we solve the $\emptyset(\lambda) = |A - \lambda I| = 0$, we get n roots, these are called eigen values or latent values or proper values.

Let each one of these Eigen values say λ their Eigen vector X corresponding the given value λ is obtained by solving Homogeneous system

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \\ x_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$ and determining the non-trivial solution.

Solved Problems

1. Find the Eigen values and the corresponding Eigen vectors of $\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Sol: $Let\ A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

Characteristic matrix = $[A-\lambda I] = \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$

$$\Longrightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(2-\lambda)+8=0$$

$$\Longrightarrow 16+\lambda^2-10\lambda+8=0$$

$$\Longrightarrow \lambda^2-10\,\lambda+24=0$$

$$\Longrightarrow (\,\lambda-6)(\lambda-4) = 0$$

$\Longrightarrow \lambda = 6, 4$ are eigen values of A

Consider the system $\begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

Eigen vector corresponding to $\boldsymbol{\lambda = 4}$

Put $\lambda = 4$ in the above system, we get $\begin{pmatrix} 4 & -4 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Longrightarrow 4x_1 - 4x_2 = 0 - - - (1)$ $\qquad$ $2x_1 - 2x_2 = 0 - - - (2)$

from (1)and (2) we have $x_1 = x_2$

Let $x_1 = \alpha$

$$Eigen\ vector\ is \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a Eigen vector of matrix A, corresponding eigen value $\lambda = 4$

Eigen Vector corresponding to $\boldsymbol{\lambda = 6}$

$put\ \lambda = 6$ in the above system, we get $\begin{pmatrix} 2 & -4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Longrightarrow 2x_1 - 4x_2 = 0 - - - (1)$ $\qquad$ $2x_1 - 4x_2 = 0 - - - (2)$

from (1) and (2) we have $x_1 = 2x_2$

Let $x_2 = \alpha \Rightarrow x_1 = 2\alpha$

$$Eigen\ vector = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is eigen vector of matrix A corresponding eigen value $\lambda = 6$

2. Find the eigen values and the corresponding eigen vectors of matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

i.e. |A-λI| = $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Longrightarrow (2-\lambda)(2-\lambda)^2 - 0 + [-(2-\lambda)] = 0$$

$$\Longrightarrow (2-\lambda)^3 — (\lambda - 2) = 0$$

$$\Longrightarrow \lambda - 2\ [-(\lambda-2)^2 - 1] = 0$$

$$\Longrightarrow \lambda - 2\ \ [-\lambda^2 + 4\lambda - 3] = 0$$

$$\Longrightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

⇨ λ=1, 2, 3

The eigen values of A is 1, 2, 3.

$For\ finding\ eigen\ vector\ the\ system\ is\ (A - \lambda \mathrm{I})\mathrm{X} = 0$

$$\Longrightarrow \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to λ=1

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$x_2 = 0$$

$$x_1 + x_3 = 0$$

$$x_1 = -x_3, x_2 = 0$$

$$Let\ x_3 = \alpha$$

$$\Rightarrow x_1 = -\alpha \qquad x_2 = 0, \qquad x_3 = \alpha$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ is Eigen vector}$$

Eigen vector corresponding to λ=2

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$Here\ x_1 = 0\ and\ x_3 = 0\ and\ we\ can\ take\ any\ arbitary\ value\ x_2\ i.e\ x_2 = \alpha\ (say)$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Eigen\ vector\ is\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Eigen vector corresponding to λ=3

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_3 = 0$

$-x_2 = 0$

$x_1 - x_3 = 0$

$here\ by\ solving\ we\ get\ x_1 = x_3, x_2 = 0\ say\ x_3 = \propto$

$x_1 = \propto\ ,\quad x_2 = 0\quad , x_3 = \propto$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$Eigen\ vector\ is\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Find the Eigen values and Eigen vectors of the matrix is $\begin{bmatrix} 3 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Consider characteristic equation is $|A-\lambda I|=0$ i.e. $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix}=0$

$$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda)-(16)\right]+6\left[(-6)(3-\lambda)+8\right]+2\left[24-2(7-\lambda)\right]=0$$

$$\Rightarrow (8-\lambda)\left[21-7\lambda-3\lambda+\lambda^2-16\right]+6[-18+6\lambda+8]+2[24-14+2\lambda]=0$$

$$\Rightarrow (8-\lambda)\left[\lambda^2-10\lambda-5\right]+6[6\lambda-10]+2[10+2\lambda]=0$$

$$\Rightarrow 8\lambda^2-80\lambda-40-\lambda^3+10\lambda^3+5\lambda+36\lambda-60+20+4\lambda=0$$

$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$$

$$\Rightarrow \lambda\left[-\lambda^2+18\lambda-45\right]=0$$

$$\Rightarrow \lambda=0 \quad (OR) \quad -\lambda^2+18\lambda-45=0$$

$$\Rightarrow \lambda=0, \quad \lambda=3, \quad \lambda=15$$

Eigen Values $\lambda = 0, 3, 15$

Case (i): If $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} X = 0$$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 8x_1-6x_2+2x_3=0----(1)$$

$$-6x_1+7x_2-4x_3=0----(2)$$

$$2x_1-4x_2+3x_3=0----(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{array}$$

$$\Rightarrow \frac{x_1}{21-16}=\frac{-x_2}{-18+8}=\frac{x_3}{24-14}=k$$

$$\Rightarrow \frac{x_1}{5}=\frac{-x_2}{-10}=\frac{x_3}{10}=k$$

$$\Rightarrow x_1=k, \quad x_2=2k, \quad x_3=2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} k \\ 2k \\ 2k \end{bmatrix}=k\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (ii): If $\lambda = 3$

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0 ------(1)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 ------(2)$$

$$2x_1 - 4x_2 + 0 = 0 --------(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{matrix}$$

$$\Rightarrow \frac{x_1}{0-16} = \frac{-x_2}{0+8} = \frac{x_3}{24-8} = k$$

$$\Rightarrow \frac{x_1}{-16} = \frac{-x_2}{8} = \frac{x_3}{16} = k$$

$$\Rightarrow \frac{x_1}{-2} = \frac{-x_2}{1} = \frac{x_3}{2} = k$$

$$\Rightarrow \frac{x_1}{-2} = k, \quad -x_2 = k, \quad x_3 = 2k$$

$$\Rightarrow x_1 = -2k, \quad x_2 = -k, \quad x_3 = 2k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ 2k \end{bmatrix} = k \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

Case (iii): If $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} x = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -7x_1 + (-6x_2) + 2x_3 = 0 ----(1)$$
$$-6x_1 - 8x_2 - 4x_3 = 0 -----(2)$$
$$2x_1 - 4x_2 - 12x_3 = 0 -----(3)$$

Consider (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{matrix}$$

$$\Rightarrow \frac{x_1}{96-16} = \frac{-x_2}{72+8} = \frac{x_3}{24+16} = k$$

$$\Rightarrow \frac{x_1}{80} = \frac{-x_2}{80} = \frac{x_3}{40} = k$$

$$\Rightarrow \frac{x_1}{2} = k, \quad \frac{x_2}{2} = k, \quad \frac{x_3}{1} = k$$

$$\Rightarrow x_1 = 2k, \quad x_2 = 2k, \quad x_3 = k$$

Eigen Vector is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} k$

4. Find the Eigen values and the corresponding Eigen vectors of the matrix.

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & -0 \end{bmatrix}$$

Sol: Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & -0 \end{bmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$ i.e. $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow (-2-\lambda)[-\lambda(1-\lambda)-12] - 2[-2\lambda-6] - 3[2(-2)+(1-\lambda)] = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$
$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$$
$$\Rightarrow \lambda = -3, -3, 5$$

The Eigen values are -3,-3, and 5

Case (i): If $\lambda = -3$

We get $\begin{bmatrix} -2+3 & 2 & -3 \\ 2 & 1+3 & -6 \\ -1 & -2 & 0+3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The augment matrix of the system is $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right]$

Performing $R_2 - 2R_1, R_3 + R_1$, we get $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Hence we have $x_1 + 2x_2 - 3xy = 0 \Rightarrow x_1 = -2x_2 + 3x_3$

Thus taking $x_2 = k_1$ *and* $x_3 = k_2$, we get $x_1 = -2k_1 + 3k_2; x_2 = k_1; x_3 = k_2$

Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ are the Eigen vectors corresponding to $\lambda = -3$

Case (ii): If $\lambda = 5$

We get $\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow -7x_1 + 2x_2 - 3x_3 = 0 ----(1)$$
$$2x_1 - 4x_2 - 6x_3 = 0 ----(2)$$
$$-x_1 - 2x_2 - 5x_3 = 0 ----(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{array}$$

$$\Rightarrow \frac{x_1}{20-12} = \frac{-x_2}{-10-6} = \frac{x_3}{-4-4} = k_3$$

$$\Rightarrow \frac{x_1}{8} = \frac{-x_2}{-16} = \frac{-x_3}{-8} = k_3$$

$$\Rightarrow \frac{x_1}{1} = \frac{-x_2}{-2} = \frac{-x_3}{-1} = k_3$$

$$Eigen\ vector\ is \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} k_3$$

5. Find the Eigen values and Eigen vectors of the matrix A and it's inverse where

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

The characteristic equation of "A" is given by $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 2, 3 \quad i.e.\ Eigen Values\ are\ 1, 2, 3$$

Note: In upper Δ^{le} (or) Lower Δ^{lar} of a square matrix the Eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Case (i): If $\lambda = 1$

$$\therefore (A - \lambda I) x = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_2 + 4x_3 = 0;\ x_2 + 5x_3 = 0;\ 2x_3 = 0 \quad \Rightarrow x_1 = k_1;\ x_2 = 0;\ x_3 = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k_1$$

Case (ii): If $\lambda = 2$

$$\Rightarrow \begin{bmatrix} +1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + 3x_2 + 4x_3 = 0; 5x_2 = 0; x_3 = 0$$

$$\Rightarrow -x_1 + 3k + 4(0) = 0 \Rightarrow -x_1 + 3k = 0 \Rightarrow x_1 = 3k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii): If $\lambda = 3$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 4 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + 3x_2 + 4x_3 = 0; -x_2 + 5x_3 = 0; x_3 = 0$$

$$Let \quad x_3 = k$$

$$\Rightarrow -x_2 + 5x_3 = 0 \Rightarrow x_2 = 5k$$

$$and -2x_1 + 3x_2 + 4k = 0 \Rightarrow -2x_1 + 15k + 4k = 0$$

$$\Rightarrow -2x_1 + 19x = 0 \Rightarrow x_1 = \frac{19}{2}k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{19}{2}k \\ 5k \\ k \end{bmatrix} = k\begin{bmatrix} \frac{19}{2} \\ 5 \\ 1 \end{bmatrix}$$

Note: Eigen Values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$ i.e $1, \frac{1}{2}, \frac{1}{3}$ and the Eigen vectors of A^{-1} are same as Eigen vectors of the matrix A

6. Determine the Eigen values and Eigen vectors of

$$B = 2A^2 - \frac{1}{2}A + 3I \text{ where } A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$$

Sol :– *Given that* $B = 2A - \frac{1}{2}A + 3 \quad \Rightarrow A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$

$$\text{we have } A^2 = A.A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix}$$

$$B = 2A^2 - \tfrac{1}{2}A + 3I$$

$$= 2\begin{bmatrix} 56 & -40 \\ 20 & -4 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 112 & -80 \\ 40 & -8 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 111 & -78 \\ 39 & -6 \end{bmatrix}$$

Characteristic equation of B is $|B - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{vmatrix} = 0$$

i.e. $\Rightarrow \lambda^2 + 105\lambda - 2376 = 0$

$$\Rightarrow (\lambda - 33)(\lambda - 72) = 0$$

$$\Rightarrow \lambda = 33 \quad or \quad 72$$

Eigen Values of B are 33 and 72.

Case (i): If λ = 33

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 78 & -78 \\ 39 & -39 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

Case (ii): If $\lambda = 72$

$$\Rightarrow \begin{bmatrix} 111-\lambda & -78 \\ 39 & -6-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 111-72 & -78 \\ 39 & -6-72 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} X = 0$$

$$\Rightarrow \begin{bmatrix} 39 & -78 \\ 39 & -78 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 39x_1 - 78x_2 = 0 \Rightarrow x_1 = 2x_2$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = k(say)$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} k$$

Properties of Eigen Values:

Theorem 1: The sum of the Eigen values of a square matrix is equal to its trace and product of the Eigen values is equal to its determinant.

Theorem 2: If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value A^n corresponding to the eigen vector X.

Theorem 3: A Square matrix A and its transpose A^T have the same eigen values.

Theorem 4: If A and B are n-rowed square matrices and If A is invertible show that $A^{-1}B$ and $B\,A^{-1}$ have same eigen values.

Theorem 5: If $\lambda_1, \lambda_2, \ldots\ldots\ldots\lambda_n$ are the eigen values of a matrix A then $k\,\lambda_1, k\,\lambda_2, \ldots\ldots k\,\lambda_n$ are the eigen value of the matrix KA, where K is a non-zero scalar.

Theorem 6: If λ is an Eigen values of the matrix A, then λ+k is an Eigen value of the matrix A+KI

Theorem 7: If λ_1, λ_2 … λ_n are the Eigen values of A, then $\lambda_1 - K$, $\lambda_2 - K$, … $\lambda_n - K$ are the Eigen values of (A-KI). Where K is a non-zero scalar.

Theorem 8: If $\lambda_1, \lambda_2 \dots \lambda_n$ are the Eigen values of A, find the Eigen values of the matrix $(A - \lambda I)^2$.

Theorem 9: If λ is an Eigen value of a non-singular matrix A corresponding to the Eigen vector X, then λ^{-1} *is an Eigen value of* A^{-1} *and corresponding Eigen vector X itself.*

Theorem 10: If λ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an Eigen value of the matrix AdjA.

Theorem 11: If λ is an eigen value of an orthogonal matrix A, then $\frac{1}{\lambda}$ is also an Eigen value *A*

Theorem 12: If λ is eigen value of A then prove that the eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$

Theorem 13: Suppose that A and P be square matrices of order n such that P is non singular. Then A and $P^{-1}AP$ have the same eigen values.

Corollary 1: If A and B are square matrices such that A is non-singular, then $A^{-1}B$ and BA^{-1} have the same eigen values.

Corollary 2: If A and B are non-singular matrices of the same order, then AB and BA have the same eigen values.

Theorem 14: The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Note: Similarly we can show that the Eigen values of a diagonal matrix are just the diagonal elements of the matrix.

Theorem 15: The eigen values of a real symmetric matrix are always real.

Theorem 16: For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

Note: If λ is an eigen value of A and f(A) is any polynomial in A, then the eigen value of f(A) is $f(\lambda)$.

Theorem17: The Eigen values of a Hermitian matrix are real.

Note: The Eigen values of a real symmetric are all real

Corollary: The Eigen values of a skew-Hermitian matrix are either purely imaginary (or) Zero

Theorem 18: The Eigen values of an unitary matrix have absolute value l.

Note 1: From the above theorem, we have “The characteristic root of an orthogonal matrix is of unit modulus”.

2. The only real Eigen values of unitary matrix and orthogonal matrix can be ± 1

Theorem 19: Prove that transpose of a unitary matrix is unitary.

Solved Problems:

1. For the matrix $A=\begin{bmatrix}1 & 2 & -3\\ 0 & 3 & 2\\ 0 & 0 & -2\end{bmatrix}$ **find the Eigen values of** $3A^3+5A^2-6A+2I$

Sol: The Characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}1-\lambda & 2 & -3\\ 0 & 3-\lambda & 2\\ 0 & 0 & -2-\lambda\end{vmatrix}=0$

$\Rightarrow (1-\lambda)(3-\lambda)(-2-\lambda)=0$

$\therefore$ Eigen values are 1, 3,-2.

If λ is the Eigen value of A. and F (A) is the polynomial in A then the Eigen value of f (A) is f (λ)

Let $f(A)=3A^3+5A^2-6A+2I$

$\therefore$ Eigen Value of f (A) are f (1), f (-2), f (3)

f (1) = 3+5-6+2 = 4

f (-2) = 3(-8)+5(4)-6(-2)+2 = -24+20+12+2 = 10

f (3) = 3(27)+5(9)+6(3)+2 = 81+45-18+2 = 110

The Eigen values of f (a) are f (λ) = 4, 10,110

2. Find the eigen values and eigen vectors of the matrix A and its inverse, where

A = $\begin{bmatrix}1 & 3 & 4\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix}$

Sol: Given A = $\begin{bmatrix}1 & 3 & 4\\ 0 & 2 & 5\\ 0 & 0 & 3\end{bmatrix}$

The characteristic equation of A is given by |A-λI| = 0

$\Longrightarrow \begin{vmatrix}1-\lambda & 3 & 4\\ 0 & 2-\lambda & 5\\ 0 & 0 & 3-\lambda\end{vmatrix}=0$

$\Longrightarrow (1-\lambda)[(2-\lambda)(3-\lambda)]=0$

$\Longrightarrow \lambda = 1, 2, 3$

Characteristic roots are 1, 2, 3.

Case (i): If $\lambda = 1$

$For\ \lambda = 1, becomes \begin{bmatrix} 0 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow 3x_2 + 4x_3 = 0$

$x_2 + 5x_3 = 0$

$2x_3 = 0$

$x_2 = 0, x_3 = 0$ and $x_1 = \alpha$

$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the solution where α is arbitrary constant

$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 1$

Case (ii): If $\lambda = 2$

$For\ \lambda = 2, becomes \begin{bmatrix} -1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow -x_1 + 3x_2 + 4x_3 = 0$

$5x_3 = 0 \Rightarrow x_3 = 0$

$-x_1 + 3x_2 = 0 \Rightarrow x_1 = 3x_2$

$Let\, x_2 = k$

$x_1 = 3k$

$X = \begin{bmatrix} 3k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$

is the solution where k is arbitrary constant

$\therefore X = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 2$

Case (iii): If $\lambda = 3$

$For\ \lambda = 3, becomes \begin{bmatrix} -2 & 3 & 4 \\ 0 & -1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Longrightarrow -2x_1 + 3x_2 + 4x_3 = 0$

$-x_2 + 5x_3 = 0$

$Say\ x_3 = K \implies x_2 = 5K$

$x_1 = \frac{19}{2}K$

X = $\begin{bmatrix} \frac{19}{2}K \\ 5K \\ K \end{bmatrix} = \frac{K}{2}\begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the solution, where k/2 is arbitrary constant.

$\therefore X = \begin{bmatrix} 19 \\ 10 \\ 2 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 3$

Eigen values of A $^{-1}$are 1, $\frac{1}{2}$, $\frac{1}{3}$.

We know Eigen vectors of A^{-1} are same as eigen vectors of A.

3. Find the eigen values of A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

Sol: we have A= $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

So $\overline{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ and $A^T = \begin{bmatrix} 3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\Rightarrow \overline{A} = -A^T$

Thus A is a skew-Hermitian matrix.

$\therefore$ The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow A^T = \begin{vmatrix} 3i-\lambda & -2+i \\ -2+i & -i-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 2i\lambda + 8 = 0$

$\Rightarrow \lambda = 4i, -2i$ are the Eigen values of A

4. Find the eigen values of $A = \begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

$Now\ \overline{A} = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix} and\ \left(\overline{A}\right)^T = \begin{bmatrix} -\frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}i \end{bmatrix}$

We can see that $\overline{A}^T.A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$

Thus A is a unitary matrix

$\therefore$ The characterstic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}\frac{1}{2}i-\lambda & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}i-\lambda\end{vmatrix}=0$$

Which gives $\lambda=\frac{\sqrt{3}}{2}+i\frac{1}{2} and \frac{-\sqrt{3}}{2}+\frac{1}{2}i$ and

Hence above λ values are Eigen values of A.

Diagonalization of a Matrix by similarity transformation:

Similar Matrix: A matrix A is said to be similar to the Matrix B if there Exists a non-singular matrix P such that B= $P^{-1}AP$. This transformation of A to B is known as "Similarity Transformation"

Diagonalization of a Matrix:

Let A be a square Matrix. If there exists a non-singular Matrix P and a diagonal Matrix D such that $P^{-1}AP=D$, then the Matrix A is said to be diagonalizable and D is said to be "Diagonal" form (or) canonical diagonal form of the Matrix A

Modal Matrix: The modal matrix which diagonalizes A is called the modal Matrix of A and is obtained by grouping the Eigen vectors of A into a Square Matrix.

Spectral Matrix: The resulting diagonal Matrix D is known as Spectral Matrix.

In this spectral Matrix D whose principal diagonal elements are the Eigen values of the Matrix.

Calculation of powers of a matrix:

We can obtain the power of a matrx by using diagonalization

Let A be the square matrix then a non-singular matrix P can be found such that $D = P^{-1}AP$

$D^2= (P^{-1}AP)\,(P^{-1}AP)$

$= P^{-1}A\,(PP^{-1})\,AP$

$= P^{-1}A^2P$ (since $PP^{-1}=I$)

Simlarly $D^3 = P^{-1}A^3P$

In general $D^n = P^{-1}A^nP$……..(1)

To obtain A^n, Premultiply (1) by P and post multiply by P^{-1}

Then $PD^nP^{-1} = P(P^{-1}A^nP)P^{-1} = (PP^{-1})A^n (PP^{-1}) = A^n \Rightarrow A^n = PD^nP^{-1}$

Hence $A^n = P \begin{bmatrix} \lambda_1^n & 0 & 0\cdots & 0 \\ 0 & \lambda_2^n & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \lambda_n^n \end{bmatrix} P^{-1}$

Diagonalization of a matrix:

Theorem: If a square matrix A of order n has n linearly independent eigen vectors $(X_1, X_2 \ldots X_n)$ corresponding to the n eigen values $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Note: 1. If $X_1, X_2 \ldots X_n$ are not linearly independent this result is not true.

2. Suppose A is a real symmetric matrix with n pair wise distinct eigen values $\lambda_1, \lambda_2 \cdots \lambda_n$ then the corresponding eigen vectors $X_1, X_2 \ldots X_n$ are pairwise orthogonal.

Hence if $P = (e_1, e_2 \ldots e_n)$

Where $e_1 = (X_1 / \|X_1\|)$, $e_2 = (X_2 / \|X_2\|) \ldots . e_n = (X_n / \|X_n\|)$ then P will be an orthogonal matrix.

i.e, $P^TP = PP^T = I$

Hence $P^{-1} = P^T \quad \therefore P^{-1}AP = D$

Solved Problems :

1. Determine the modal matrix P of A= $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ **. Verify that $P^{-1}AP$ is a diagonal matrix.**

Sol: The characteristic equation of A is $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$

which gives $(\lambda-5)(\lambda+3)^2 = 0$

Thus the eigen values are $\lambda=5$, $\lambda=-3$ and $\lambda=-3$

When $\lambda=5 \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By solving above we get $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Similarly, for the given eigen value $\lambda = -3$ we can have two linearly independent eigen vectors

$X_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$P = [X_1\ X_2\ X_3]$

$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ = modal matrix of A

Now det P=1(-1)-2(2)+3(0-1) = -8

$P^{-1} = \frac{adj\ P}{\det P} = -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}$

$= -\frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

$= -\frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ 6 & -12 & -18 \\ 3 & 6 & 15 \end{bmatrix}$

$P^{-1}AP = \frac{1}{8}\begin{bmatrix} -40 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ = diag [5,-3,-3].

$\therefore\ P^{-1}AP$= diag [5,-3,-3].

2. Find a matrix P which transform the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate A^4.

Sol: Characteristic equation of A is given by |A-λI| = 0 i.e. $\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$

=> (1-λ)$[(2-\lambda)(3-\lambda)-2]-0-[2-2(2-\lambda)]=0$

=> $9\,\lambda - 1)(\lambda - 209\lambda - 30 = 0$

=> $\lambda = 1, \lambda = 2, \lambda = 3$

Thus the eigen values of A are 1, 2, 3.

If x_1, x_2, x_3 be the components of an eigen vector corresponding to the eigen value λ, we have

[A-λI] X = $\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case (i): If $\lambda = 1$

$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ i.e, $0.x_1+0.x_2+0.x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$ and $x_1+x_2+x_3=0$

$x_3=0$, $x_1=-x_2$

$x_1=1$, $x_2=-1$, $x_3=0$

Eigen vector is $[1,-1,0]^T$

Also every non-zero multiple of this vector is an eigen vector corresponding to $\lambda=1$

For $\lambda=2$, $\lambda=3$ we can obtain eigen vector $[-2,1,2]^T$ and $[-1,1,2]^T$

$$P = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

The Matrix P is called modal matrix of A

$$P^{-1} = -\frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ 2 & 2 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$Now\, P^{-1}AP = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ -1 & -1 & 0 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} 0 & -1 & \frac{-1}{2} \\ -1 & 1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -49 & -50 & -40 \\ 65 & 66 & 40 \\ 130 & 130 & 81 \end{bmatrix}$$

3. Determine the modal matrix P for $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ and hence diagonalize A

Sol: Given that $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right]-1\left[(1-\lambda)-3\right]+3\left(1-3(5-\lambda)\right)=0$$

$$\Rightarrow (1-\lambda)\left(5-5\lambda-\lambda+\lambda^2-1\right)-(-2-\lambda)+3(1-15+3\lambda)=0$$

$$\Rightarrow (1-\lambda)\left(4-6\lambda+\lambda^2\right)-(-2-\lambda)+3(-14+3\lambda)=0$$

$$\Rightarrow 4-6\lambda+\lambda^2-4\lambda+6\lambda^2-\lambda^3+2+\lambda-42+9\lambda=0$$

$$\Rightarrow \lambda^3-7\lambda^2-9\lambda+9\lambda-36=0$$

$$\Rightarrow \lambda^3-7\lambda^2-36=0$$

$$\Rightarrow \lambda=-2,3,6$$

The Eigen Values are -2, 3, and 6

Case (i): If $\lambda = -2$

⇨ $[A-\lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3x_1 + x_2 + 3x_3 = 0 ----(1)$$
$$x_1 + 7x_2 + x_3 = 0 ----(2)$$
$$3x_1 + x_2 + 3x_3 = 0 ----(3)$$

From (2) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{matrix}$$

$$\Rightarrow \frac{x_1}{1-21} = \frac{-x_2}{3-3} = \frac{x_3}{21-1} = k$$

$$\Rightarrow \frac{x_1}{-20} = \frac{-x_2}{0} = \frac{x_3}{20} = k$$

$$\Rightarrow x_1 = -20k, \quad x_2 = 0, \quad x_3 = 20k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20k \\ 0k \\ 20k \end{bmatrix} = 20k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii): If $\lambda = 3$

⇨ $[A-\lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + 3x_3 = 0 ----(1)$$
$$x_1 + 2x_2 + x_3 = 0 ------(2)$$
$$3x_1 + x_2 - 2x_3 = 0 ----(3)$$

Consider (1) & (2)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -2 & 1 & 3 \\ 1 & 2 & 1 \end{array}$$

$$\Rightarrow \frac{x_1}{1-6} = \frac{-x_2}{-2-3} = \frac{x_3}{-4-1} = k$$

$$\Rightarrow \frac{x_1}{-5} = \frac{-x_2}{-5} = \frac{x_3}{-5} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5k \\ 5k \\ -5k \end{bmatrix} = -5k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii): If $\lambda = 6$

⇨ $[A- \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 1 & -11 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x_1 + x_2 + 3x_3 = 0 ----(1)$$

$$x_1 - x_2 + x_3 \quad = 0 ---(2)$$

$$3x_1 - x_2 - 5x_3 = 0 ---(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{array}$$

$$\Rightarrow \frac{x_1}{5-1} = \frac{-x_2}{-5-3} = \frac{x_3}{+1+3} = k$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{8} = \frac{x_3}{4} = k$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} k$$

$$\rho = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|\rho| = -1(-1-2)-1(0-2)+1(0+1)$$
$$|\rho| = (-1)(-3)-1(-2)+1 = 3+2+1 = 6$$

$$\rho = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$Adj(\rho) = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\rho^{-1} = \frac{Adj\,\rho}{|\rho|}$$

$$Cofactor\ of\ \ \rho = \frac{1}{6}\begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$D = \rho^{-1} A\, \rho$$

$$D = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} -1+0+3 & 1-1+3 & 1+2+3 \\ -1+0+1 & 1-5+1 & 1+10+1 \\ -3+0+1 & 3-1+1 & 3+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}\begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

4. $\boldsymbol{If\ A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ Find (a) A^8 (b) A^4

Sol: Given that A= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

The Characteristic equation of A is $|A-\lambda I| = 0$ i.e. $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)-4\right]-1\left[0+4\right]+1\left[0+4(2-\lambda)\right]=0$$
$$\Rightarrow (1-\lambda)\left[6-2\lambda-3\lambda+\lambda^2-4\right]-4+8-4x=0$$
$$\Rightarrow (1-\lambda)\left[\lambda^2-5\lambda+2\right]+4-4\lambda=0$$
$$\Rightarrow \lambda^2-5\lambda+2-\lambda^3+5\lambda^2-2\lambda+4-4\lambda=0$$
$$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$$
$$\Rightarrow \lambda = 1,2,3$$

The Eigen values are 1, 2, and 3

Case (i): If $\lambda = 1$

$$[A-\lambda I]X = 0$$

$$=>\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$=>\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1+x_2=0$, $x_1+x_2=0$, $-4x_1+4x_2+2x_3=0$

Let $x_3 = k$, $x_2 + k=0$, $x_2 = -k$

$$\Rightarrow -4x_1 + 4(-k) + 2K = 0 \Rightarrow -4x_1 - 2k = 0 \Rightarrow -4x_1 = 2k \Rightarrow x_1 = \frac{-k}{2}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{-k}{2} \\ -k \\ k \end{bmatrix} = \begin{bmatrix} +\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} - k = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \frac{-k}{2}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Case (ii): If $\lambda = 2$

$$=> \quad [A-\lambda I]X = 0$$

$$=>\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$=>\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -x_1 + x_2 + x_3 = 0 ----(1)$$

$$x_3 = 0 ---(2)$$

$$-4x_1 + 4x_2 + x_3 = 0 ---(3)$$

Consider (1) & (3)

$$\begin{matrix} x_1 & x_2 & x_3 \\ -1 & 1 & 1 \\ -4 & 4 & 1 \end{matrix}$$

$$\Rightarrow \frac{x_1}{1-4} = \frac{-x_2}{-1+4} = \frac{x_3}{-4+4} = k$$

$$\Rightarrow \frac{x_1}{-3} = \frac{-x_2}{3} = \frac{x_3}{0} = k$$

$$\Rightarrow x_1 = -k; \quad x_2 = -k$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ -k \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}(-k) \qquad \therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case (iii): If $\lambda = 3$

$$=> \quad [A - \lambda I]X = 0$$

$$=> \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} X = 0$$

$$=> \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$-4x_1 + 4x_2 = 0$$

Let $x_1 = k$ *and* $x_3 = k$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0 \Rightarrow -2k + x_2 + k = 0 \Rightarrow x_2 = k$$

$$\therefore X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} k$$

$$\therefore p = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\rho^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$D = \rho^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$D^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 3^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix}$$

$$(a).\quad A^8 = PD^8P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 256 & 0 \\ 0 & 0 & 6561 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12099 & 12355 & 6305 \\ -12100 & 12356 & 6305 \\ -13120 & 13120 & 6561 \end{bmatrix}$$

$$(b).\quad D^4 = \begin{bmatrix} 1^4 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 3^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$A^4 = PD^4P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 16 & 81 \\ 2 & 16 & 81 \\ -2 & 0 & 81 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1+64-162 & 1-48+162 & 0-16+81 \\ -2+64-162 & 2-48+162 & 0-16+81 \\ 2+0-162 & -2-0+162 & 0-0+81 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} -99 & 115 & 65 \\ -100 & 116 & 65 \\ -160 & -160 & 81 \end{bmatrix}$$

Cayley-Hamilton Theorem: Every Square Matrix satisfies its own characteristic equation

To find Inverse of matrix: If A is non-singular Matrix, then A^{-1} exists, Pre multiplying characteristic equation by A^{-1} then we have $a_0A^{n-1} + a_1A^{n-2} + ---- + a_{n-1}I + a_nA^{-1} = 0$,

$$A^{-1} = \frac{1}{a_n}\left[a_0A^{n-1} + a_1A^{n-2} + --- a_{n-1}I\right]$$

To find the powers of A: - Let K be a positive integer such that $K \geq n$

Pre multiplying (1) by A^{K-n} we get $a_0A^K + a_1A^{K-1} + ---- + a_nA^{k-n} = 0$,

$$A^K = \frac{-1}{a_0}\left[a_1A^{k-1} + a_2A^{K-1} --- + a_nA^{K-n}\right]$$

Solved Problems :

1. S.T the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ **satisfies its characteristic equation and hence find** A^{-1}

Sol: Characteristic equation of A is det $|(A - \lambda I)| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$C_2 \rightarrow C_2+C_3$ $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1-\lambda & 3 \\ 0 & 1-\lambda & 2-\lambda \end{vmatrix} = 0$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$$

By Cayley – Hamilton theorem, we have $A^3-A^2+A-I = 0$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \quad A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \quad A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^3 - A^2 + A - I = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Multiplying with A^{-1} we get $A^2 - A + I = A^{-1}$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

2. Using Cayley - Hamilton Theorem find the inverse and A^4 of the matrix

A $= \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characteristic equation is given by $|A-\lambda I|=0$ *i.e.,* $\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)^2 \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 6 & 2 & -(1+\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley – Hamilton theorem we have $A^3-5A^2+7A-3I=0$…..(1)

Multiply with A^{-1} we get

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$A^2=\begin{bmatrix}25 & 8 & -8\\ -24 & -7 & 8\\ 24 & 8 & -7\end{bmatrix}\quad A^3=\begin{bmatrix}79 & 26 & -26\\ -78 & -25 & 26\\ 78 & 26 & -25\end{bmatrix}$$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}-3 & -2 & 2\\ 6 & 5 & -2\\ -6 & -2 & 5\end{bmatrix}$$

multiplying (1) with A,we get,

A^4-5A^3+7A^2-3A= 0

A^4 = 5A^3-7A^2+3A

$$=\begin{bmatrix}395 & 130 & -130\\ -390 & -125 & 130\\ 390 & 130 & -125\end{bmatrix}-\begin{bmatrix}175 & 56 & -56\\ -168 & -49 & 56\\ 168 & 56 & -69\end{bmatrix}+\begin{bmatrix}21 & 6 & -6\\ -18 & -3 & 6\\ 18 & 6 & -3\end{bmatrix}=\begin{bmatrix}241 & 80 & -80\\ -240 & -79 & 80\\ 240 & 80 & -79\end{bmatrix}$$

3. *If* $A=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}$ **Verify Cayley-Hamilton theorem hence find A^{-1}**

Sol: - Given that $A=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}$

The characteristic equation of A is |A-λI|=0

i.e. $\begin{vmatrix}2-\lambda & 1 & 2\\ 5 & 3-\lambda & 3\\ -1 & 0 & -2-\lambda\end{vmatrix}=0$

$$\Rightarrow(2-\lambda)\left[-6-3\lambda+2\lambda+\lambda^2\right]-1\left[-10-5\lambda+3\right]+2\left[0+(3-\lambda)\right]$$

$$\Rightarrow(2-\lambda)\left[\lambda^2-\lambda-6\right]-1\left[-5\lambda-7\right]+2\left[3-x\right]=0$$

$$\Rightarrow 2\lambda^2-2\lambda-12-\lambda^3+\lambda^2+6\lambda+5\lambda+7+6-2\lambda=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+7\lambda+1=0$$

$$\Rightarrow \lambda^3-3\lambda^2-7\lambda-1=0----(1)$$

According to Cayley Hamilton theorem. Square matrix ‘A’ satisfies equation (1)

Substitute A in place of λ

Now $A=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}$

$$A^2=\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}\begin{bmatrix}2 & 1 & 2\\ 5 & 3 & 3\\ -1 & 0 & -2\end{bmatrix}=\begin{bmatrix}7 & 5 & 3\\ 22 & 14 & 13\\ 0 & -1 & 2\end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

Now $A^3 - 3A^2 - 7A - I = 0$

$$\Rightarrow \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & 3 & -7 \end{bmatrix} + \begin{bmatrix} 21 & -15 & -9 \\ -66 & -42 & -39 \\ 0 & 3 & -6 \end{bmatrix} + \begin{bmatrix} 14 & -7 & -14 \\ -35 & -21 & -21 \\ 7 & 0 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Cayley –Hamilton theorem is verified.

To find A^{-1}

$$\Rightarrow A^3 - 3A^2 - 7A - I = 0$$

Multiply A^{-1}, we get

$$A^{-1}\left(A^3 - 3A^2 - 7A - I\right) = 0$$

$$\Rightarrow A^2 - 3A - 7I - A^{-1} = 0$$

$$\Rightarrow A^{-1} = A^2 - 3A - 7I$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} -6 & -3 & -6 \\ 15 & -9 & 9 \\ 3 & 0 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

Check $A.A^{-1} = I$

$$A.A^{-1} = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

4. Using Cayley – Hamilton theorem, find A^8, if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow (1-\lambda)(-1-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - 5 = 0 - - - - (1)$$

Substitute A in place of λ

$$A^2 - 5I = 0 \Rightarrow A^2 = 5I$$

$$find\ A^8$$

$$\therefore A^8 = 5A^6 = 5(A^2)(A^2)(A^2)$$

$$= 5(5I)(5I)(5I)$$

$$= 625I$$

$$\Rightarrow A^8 = 625I$$

Quadratic Forms

A homogeneous expression of the second degree in any number of variables is called a quadratic form.

Note:A homogeneous expression of second degree means each and every term in any expression should have degree two.

Ex1:$3x^2 + 5xy - 2y^2$ is a quadratic form in two variables x and y.

Ex2: $2x^2 + 3y^2 - 4z^2 + 2xy - 3yz + 5zx$ is a quadratic form in three variables x,y and z.

Ex3: ${x_1}^2 \pm 2{x_2}^2 + 4{x_3}^2 - x_1 x_2 + x_2 x_3 + 2x_1 x_4 - 5x_3 x_4$ is a quadratic form in four variables x_1, x_2, x_3, x_4 .

The most general form of quadratic form in n variables is defined as follows

Definition: An expression of the form Q= $X^T AX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ where a_{ij}'s are constants is called a quadratic form in n variables $x_1, x_2 ....x_n$ .if the constants a_{ij}'s are real numbers it is called a real quadratic form.

i.e $X^T AX = \sum_{i=1}^{n} \sum_{j-1}^{n} a_{ij} x_i x_j$

$$= \begin{matrix} a_{11}{x_1}^2 + a_{12}x_1x_2 + ... + a_{1n}x_1x_n + \\ a_{22}{x_2}^2 + a_{2n}x_2x_n + ... + a_{n1}x_nx_1 + a_{n2}x_nx_2 + ... \\ + ann{x_n}^2 \end{matrix}$$

$$= a_{11}{x_1}^2 + (a_{12} + a_{21})x_1x_2 + ...(a_{1n} + a_{n1})(x_1x_n)$$
$$+ a_{22}x_{22} + (a_{23} + a_{32})x_2x_3 + ... + (a_{2n} + a_{n2})x_2x_n$$
$$+ ...... + a_{nn}x_n^2$$

Here A is known as the co-efficient matrix.

Theorem: Every real quadratic form in n variables $x_1, x_2 ... x_n$ can be expressed in the form $X^T BX$ where $X = [x_1, x_2 ..... x_n]^T$ a column matrix is and B is a real symmetric matrix of order n.

Quadratic form corresponding to a real symmetric matrix:

Let A=$\lfloor a_{ij} \rfloor_{nxn}$ be a real symmetric matrix and let $X = [x_1, x_2 ..... x_n]^T$ be a column matrix. Then $X^T AX$ will determine a quadratic form $X^T AX = \sum_{i=1}^{n} \sum_{j-1}^{n} a_{ij} x_i x_j$ .on expanding this we seen to be the quadratic form $\sum_{i=1}^{n} a_{ii} x_{12} + \sum_{i} \sum_{j} a_{ij} x_i x_j (i + j)$

Note: 1) $a_{11}x_1^2 + a_{22}x_{22} + a_{33}x_{32} + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$ write $a_{12} = a_{21}$ $a_{13} = a_{31}$ $a_{23} = a_{32}$

∴ Quadratic form =(x_1,x_2,x_3) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

i.e $X^T AX$, where A is a symmetric matrix Quadratic forms in more variables can similarly be written in the form $X^T AX$ by suitably defining A.

Examples:

1) Write the matrix relating to the quadratic form $ax^2 + 2hxy + by^2$ **.**

Sol: The given quadratic form can be written as

$ax^2 + 2bxy + hyx + by^2$

$\Rightarrow [x \quad y] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

∴ The corresponding matrix is $\begin{bmatrix} a & h \\ h & b \end{bmatrix}$

2) Find the symmetric matrix corresponding to the quadratic form $x_1^{\ 2} + 2x_2^{\ 2} + 4x_2x_3 + x_3x_4$

Sol: The above quadratic form can be written as

$$x_1x_1 + 0.x_1x_2 + 0.x_1x_3 + 0.x_1x_y + 0.x_2x_1 + 2x_2^{\,2} + 2x_2x_3 + 0.x_2x$$

$$+0.x_3x_1 + 2x_3x_2 + 0.x_3x_3 + \frac{1}{\alpha}x_3x_4^{2}$$

$$+0.x_4x_2 + \frac{1}{\alpha}x_y x_2^{2} + 0.x_4x_4$$

$\therefore$ The matrix relating to the above quadratic form is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$

3) Find the quadratic form relating to the symmetric matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

Sol: The quadratic form related to the given matrix is X^TAX where

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

Here $a_{11}=1, a_{22}=1, a_{33}=1, a_{12}=a_{21}=2,$

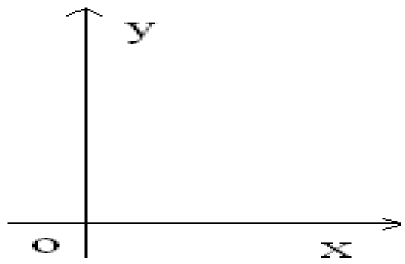
$$a_{13}=a_{31}=3.a_{23}=a_{32}=5$$

Then the corresponding quadratic form is

$$a_{11}x^2 + a_{22}y^2 + a_{33}x^2 + (a_{12}+a_{21})xy \quad +(a_{13}+a_{31})(a_{23}+a_{32})yx$$

This simplifies to $x^2 + y^2 + z^2 + 4xy + 5xz + 6yx$.

Linear Transformation of a Quadratic form:



Let the point $p(x, y)$ with

respect to a set of rectangular

axes OX and OY be transformed

to the point $p^1(x^1y^1)$,with

respect to a set of rectangular

axes OX^1 and OY^1 by the

Following relation

$$x^1 = a_1x + a_2y \qquad y^1 = b_1x + b_2y \qquad \begin{bmatrix} x^1 \\ y^1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(or) $X = AY$ Where

$$X = \begin{bmatrix} x^1 \\ y^1 \end{bmatrix} \qquad A=\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix}$$

Such a transformation is called linear transformation in two dimensions.

Note: If X=AY and Y=BZ be the two linear transformations, then X=CZ, where C=AB is called composite linear transformation.

If X^TAX is a quadratic form in n variables $x_1, x_2........x_n$ And Y^TBY is a quadratic in n Variables $y_1, y_2.....y_n$,we can see that X=PY Transform the original quadratic form into a new Quadratic form.

Note: If the matrix P is singular, the transformation Is said to be singular otherwise non-singular A non-singular transformation is also called A regular transformation.

Rank of a quadratic form:

Let X^TAX be a quadratic form over R. The rank r of A is called the rank of quadratic form.

If $r < n$ (order of A) or $|A| = 0$ or A is singular.

The quadratic form is called singular otherwise non-singular.

Canonical form (or) Normal form of a quadratic form:

Let X^TAX be a quadratic form in n variables. Then there exists a real non-singular linear transforms X=PY which transformation X^TAX to the form

$$Y^T BY = \lambda 1 y_1^{\,2} + \lambda_2 y_2^{\,2} + \dots\dots\dots\dots \lambda_n y_n^{\,2}$$

under the transformation X= PY, then $Y^T BY$ is called the canonical form of $X^T AX$.

Here B = diag$\left[\lambda_1, \lambda_2 \dots\dots\dots \lambda_n\right]$

Index of a real quadratic form:

When the quadratic form $X^T AX$ is reduced to the canonical form, it will contain Only r terms, if the rank of A is r. the terms in the canonical form may be positive, zero or negative. The number of positive terms in a normal form of quadratic form is called the index of the quadratic form.

Theorem: The number of positive terms in any two normal reductions of quadratic form is the same

(Or)

The index of a quadratic form is invariant from all normal reductions

Note: The number of negative forms in any two normal reductions of quadratic form is the same.

Signature of a Quadratic – form:

If r is the rank of a quadratic form and s is the number of positive terms in its normal form, then the excess number of positive terms over the number of negative terms

i.e., $s-(r-s)=2s-r$ is called the signature of the quadratic form

In other words, signature of the quadratic form is defined as the difference between the number of positive terms and the number of negative terms in its canonical forms

Nature of Quadratic forms:-

The quadratic form $X^T AX$ in n variables is said to be

(i) <u>Positive definite:-</u> If $r=n$ and $s=n(or)$ if all the eigen values of A are positive

(ii) <u>Negative definite:-</u> If $r=n$ and $s=0$ (or)if all the eigen values of A are negative

(iii) <u>Positive semi definite:-</u> If $r<n$ and $s=r(or)$ if all the eigen of $A\geq 0$ and atleast one eigen value is zero

(iv) <u>Negative semi definite:-</u> If $r<n$ and $s=0(or)$ if all the eigen of $A\leq 0$ and atleast one eigen value is zero

(v) Indefinite:- In all other cases (or) if A has positive as well as negative eigen values.

1. Identify the nature of the quadratic form

$x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_3x_2-4x_2x_3$

Sol. Given quadratic forms is $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_3x_2-4x_2x_3$

The matrix of the quadratic form is $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{bmatrix}$

The characteristic equation is $\begin{vmatrix}1-\lambda & -2 & 1\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=0$

Applying $R_1-R_3, we\ get \begin{vmatrix}-\lambda & 0 & \lambda\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=0$

$$\Rightarrow \lambda\begin{vmatrix}-1 & 0 & 1\\ -2 & 4-\lambda & -2\\ 1 & -2 & 1-\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda\left[-1\{(4-\lambda)(1-\lambda)-4\}+1(4-4+\lambda)\right]=0$$

$$\Rightarrow \lambda\left[\left(-4-4\lambda-\lambda+\lambda^2-4\right)+\lambda\right]=0$$

$$\Rightarrow \lambda\left[5\lambda-\lambda^2+\lambda\right]=0$$

$$\Rightarrow \lambda^2(\lambda-6)=0$$

$$\Rightarrow \lambda^2=0(or)\lambda-6=0$$

$\therefore$ Eigen values are $\lambda=0,0.6$

Which are positive and zero.

$\therefore$ The quadratic form is positive semi definite

Sylvester's Law of Inertia:

The signature of quadratic form is invariant for all normal reductions

Methods of Reduction of Q.F to Canonical Form

Method 1: Reduction to Normal form by Orthogonal Transformation:

If in the transformation $X = PY$, P is an Orthogonal matrix and if $X = PY$ transforms the quadratic form Q to the canonical form then Q is said to be reduces to canonical form by an Orthogonal transformation

Consider the quadratic form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21}) + (a_1^3 + a_1^3)x_1x_3 + (a_{23} + a_{32})x_2x_3$$

Where $a_{ij} = a_{ji}$ $\forall$ i,j and a_{ij} 's are Real

This is same as $X^T AX$, where $X^T = (x_1, x_2, x_3)$ and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ with } a_{ij} = a_{ji}$$

Note:-

1. If A has distinct eigen values $\lambda_1, \lambda_2, \lambda_3$ then the corresponding Eigen vectors X_1, X_2, X_3 are pair wise orthogonal
2. If A has eigen values $\lambda_1, \lambda_2, \lambda_3$ and X_1, X_2, X_3 are three Eigen vectors which are linearly independent, we can construct normalized Eigen vectors e_1, e_2, e_3 corresponding to $\lambda_1, \lambda_2, \lambda_3$ which are pair wise orthogonal
3. If A is an nth order square real symmetric matrix, the above results can be generalized
4. IF A is of order n and it is not possible to have n linearly independent pair wise orthogonal Eigen vectors, the above procedure does not work.

Working Rule:

1. Write the co-efficient matrix A associated with the quadratic form
2. Find the Eigen values and Eigen vector of A
3. Write the canonical form using $\lambda_1 y_1^2 + \lambda_2 y^2 + - - - - - + \lambda_n y_n^2$
4. Form the matrix P containing the normalized Eigen vectors of A. Then X = PY gives the required orthogonal transformation which reduces quadratic form to conical form

1. Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form by orthogonal transformation.

Sol: Comparing the given quadratic form with

$$a_{11}x^2 + a_{22}y^2 + a_{33}x^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$

$$\Rightarrow A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$$

$$\Rightarrow (3-\lambda)\left[(5-\lambda)(3-\lambda)-1\right]+1\left[-1(3-\lambda)+1\right]+1\left[1-(5-\lambda)\right]$$

$$\Rightarrow 3-\lambda\left[\lambda^2-8\lambda+14\right]+\left[\lambda-2\right]+\left[\lambda-4\right]=0$$

$$\Rightarrow 3\lambda^2-24+42-\lambda^3+8\lambda^2-14\lambda+\lambda-2+\lambda-4=0$$

$$\Rightarrow -\lambda^3+11\lambda^2-36\lambda+36=0$$

$$\Rightarrow \lambda^3-11\lambda^2+36-36=0$$

$$\Rightarrow (\lambda-2)\left(\lambda^2-9\lambda+18\right)=0$$

$$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-6)=6$$

$$\therefore \lambda = 2,3,6$$

The Eigen values of A are 2,3,6

The Eigen vector of A corresponding to $\lambda = 2$ is given by $(A-2I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Performing $R_2 \to R_2 + R_1$

$R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 + x_3 = 0 \,\&\, x_2 = 0 \to (1)$$

Let $x_3 = k$, then (1) given $x_1 = -k$

$$\therefore x_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$ is the Eigen vector of A corresponding to $\lambda = s$ is

$$(A-3I)X = 0 \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ applying $R_3 \to R_3 + R_2$

$\Rightarrow -x_2 + x_3 = 0 \Rightarrow x_2 = x_3 \quad and \quad -x_1 + 2x_3 - x_3 \to (2)$

Let $x_3 = k = x_2 = k$ then (1) gives

$-x_1 + 2k - k = 0 \Rightarrow x_1 = k$

$$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ is the Eigen vector of corresponding to $\lambda = 3$,

where k is the non zero the Eigen vector of A corresponding to $\lambda = 6$ is given by

$$(A-6I)X = 0 \Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} R_2 \to 3R_2 - 2R_2$$

$\Rightarrow -3x_1 - x_2 + x_3 = 0$

$-2x_2 - 4x_3 = 0$

$x_2 + 2x_3 = 0$

Let

$$x_3 = k \Rightarrow x_2 + 2k = 0 \Rightarrow x_2 = -2k$$
$$-3x_1 + 2k + k = 0$$
$$\Rightarrow x_1 = k$$
$$\therefore x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The Eigen vector of A corresponding to $\lambda = 6$, where k is non zero scalar

Modal matrix P = $[x_1 x_2 x_3]$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

Normalized modal matrix P = $\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$

This is an orthogonal matrix.

Diagonalised matrix is D = $P^{-1}AP$

$$\therefore D = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & \sqrt{3} & \sqrt{6} \\ 0 & \sqrt{3} & -2\sqrt{6} \\ \sqrt{2} & \sqrt{3} & \sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\therefore Q = Y^T D Y$$

$$Q = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$

Which is required canonical form.

2. Reduce the quadratic form $x^2+5y^2+z^2+2xy+6xz+2yz$ to the canonical form by orthogonal transformation.

Sol: Comparing the given quadratic form with

$$a_{11}x^2+a_{22}y^2+a_{33}x^2+2a_{12}xy+2a_{13}xz+2a_{23}yz$$

$$A=\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$$

The characteristic equation of A is $|A-\lambda I|=0$ i.e. $\begin{vmatrix}1-\lambda&1&3\\1&5-\lambda&1\\3&1&1-\lambda\end{vmatrix}=0$

$$\Rightarrow(1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right]-1\left[(1-\lambda)-3\right]+3\left(1-3(5-\lambda)\right)=0$$

$$\Rightarrow(1-\lambda)\left(5-5\lambda-\lambda+\lambda^2-1\right)-(-2-\lambda)+3(1-15+3\lambda)=0$$

$$\Rightarrow(1-\lambda)\left(4-6\lambda+\lambda^2\right)-(-2-\lambda)+3(-14+3\lambda)=0$$

$$\Rightarrow4-6\lambda+\lambda^2-4\lambda+6\lambda^2-\lambda^3+2+\lambda-42+9\lambda=0$$

$$\Rightarrow\lambda^3-7\lambda^2-9\lambda+9\lambda-36=0$$

$$\Rightarrow\lambda^3-7\lambda^2-36=0$$

$$\Rightarrow\lambda=-2,3,6$$

The Eigen Values are -2, 3, and 6

Case (i): If $\lambda=-2$

⇨ $[A-\lambda I]X=0$

$$\Rightarrow\begin{bmatrix}1-\lambda&1&3\\1&5-\lambda&1\\3&1&1-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}3&1&3\\1&7&1\\3&1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow3x_1+x_2+3x_3=0----(1)$$

$$x_1+7x_2+x_3=0----(2)$$

$$3x_1+x_2+3x_3=0----(3)$$

From (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{array}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -20k \\ 0k \\ 20k \end{bmatrix} = 20k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii): If $\lambda = 3$

$[A- \lambda I]X = 0$

⇨

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

⇨

$$\Rightarrow -2x_1 + x_2 + 3x_3 = 0 ----(1)$$
$$x_1 + 2x_2 + x_3 = 0 ------(2)$$
$$3x_1 + x_2 - 2x_3 = 0 ----(3)$$

Consider (1) & (2)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ -2 & 1 & 3 \\ 1 & 2 & 1 \end{array}$$

$$\Rightarrow \frac{x_1}{1-6} = \frac{-x_2}{-2-3} = \frac{x_3}{-4-1} = k$$

$$\Rightarrow \frac{x_1}{-5} = \frac{-x_2}{-5} = \frac{x_3}{-5} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5k \\ 5k \\ -5k \end{bmatrix} = -5k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii): If $\lambda = 6$

⇨ $[A- \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 1 & 3 \\ 1 & -11 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -5x_1 + x_2 + 3x_3 = 0 ----(1)$$
$$x_1 - x_2 + x_3 \quad = 0 ---(2)$$
$$3x_1 - x_2 - 5x_3 = 0 ---(3)$$

Consider (2) & (3)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{array}$$

$$\Rightarrow \frac{x_1}{5-1} = \frac{-x_2}{-5-3} = \frac{x_3}{+1+3} = k$$

$$\Rightarrow \frac{x_1}{4} = \frac{-x_2}{8} = \frac{x_3}{4} = k$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1} = k$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} k$$

Modal matrix P = $[x_1 x_2 x_3]$

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

This is an orthogonal matrix.

Diagonalised matrix is D = $P^{-1}AP$

$$D = \rho^{-1} A \rho$$

$$D = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1+0+3 & 1-1+3 & 1+2+3 \\ -1+0+1 & 1-5+1 & 1+10+1 \\ -3+0+1 & 3-1+1 & 3+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 2 & 3 & 6 \\ 0 & -3 & 12 \\ -2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Q.F Q = $y^T DY$

$$Q = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -2\, y_1^2 + 3y_2^2 + 6y_3^2$$

Which is required canonical form.

UNIT-III

MULTI VARIABLE CALCULUS (DIFFERENTIATION)

Introduction:

We know that $y = f(x)$ is a function where 'y' is dependent variable and 'x' is independent variable. We are going to expand the idea of functions to include functions for more than one independent variable. In day to day life we deal with things which depend on two or more quantity. For example, the area of the room which is a rectangle consists of two variables: length (say a) and breadth (say b) is given by $A = ab$. Similarly the volume of the rectangular parallelepiped consists of three variables a, b, c i.e., length, breadth, height is given by $V = a\,b\,c$.

In this chapter we say that z is a function of two variables x, y and write $z = f(x, y)$ where 'z' is dependent variable and 'x' & 'y' are independent variables.

Limit of a function of two variables:

A function $f(x,y)$ is said to tend to the limit l as (x,y) tends to (a,b) i.e., $x \to a \; and \; y \to b$ if corresponding to any given positive number $\in$ $\exists$ a positive number $\delta \; such \; that \; |f(x,y) - l| <\in$ for all points (x,y) whenever $|x-a| \le \delta$, $|y-b| \le \delta$.

In other words the variable value (x, y) approaches a finite fixed value l when the variable value (x,y) approaches a fixed value (a,b) i.e.,

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = l \; or \; \lim_{(x,y)\to(a,b)} f(x,y) = l$$

Continuity of a function of two variables at a point:

A function $f(x,y)$ is continuous at a point (a,b) if, corresponding to any given positive number $\in$ $\exists$ a positive number $\delta \; such \; that \; |f(x,y) - f(a,b)| <\in$ for all points (x,y) whenever $0 < (x-a)2 + (y-b)2 < \delta^2$

Note: Every differentiable function is always continuous, but converse need not be true.

Solved Problems:

1. **Evaluate** $\displaystyle\lim_{\substack{x\to 1\\ y\to 2}} \frac{2x^2y}{x^2+y^2+1}$

Sol: $\displaystyle\lim_{\substack{x\to 1\\ y\to 2}} \frac{2x^2y}{x^2+y^2+1} = \lim_{x\to 1}\left\{\lim_{y\to 2}\left[\frac{2x^2y}{x^2+y^2+1}\right]\right\}$

$$= \lim_{x\to 1}\frac{4x^2}{x^2+5}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

(or)

$$\lim_{\substack{x\to1\\y\to2}} \frac{2x^2y}{x^2+y^2+1} = \lim_{y\to2}\left\{\lim_{x\to1}\left[\frac{2x^2y}{x^2+y^2+1}\right]\right\}$$

$$= \lim_{x\to1}\frac{2y}{y^2+2}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

2. If f(x ,y) = $\frac{x-y}{2x+y}$ **show that** $\lim_{x\to0}\left\{\lim_{y\to0} f(x,y)\right\} \neq \lim_{y\to0}\left\{\lim_{x\to0} f(x,y)\right\}$

Sol: $\lim_{x\to0}\left\{\lim_{y\to0} f(x,y)\right\} = \lim_{x\to0}\left\{\lim_{y\to0}\frac{x-y}{2x+y}\right\}$

$$= \lim_{x\to0}\frac{x}{2x}$$

$$= \frac{1}{2}$$

$$\lim_{y\to0}\left\{\lim_{x\to0} f(x,y)\right\} = \lim_{x\to0}\left\{\lim_{y\to0}\frac{x-y}{2x+y}\right\}$$

$$= \lim_{y\to0}\frac{-y}{y}$$

$$= -1$$

Hence the result follows.

3. Discuss the continuity of the function

f(x,y) = $\begin{cases} \frac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Sol: Let us consider the limit of the function for testing the continuity along the line $y = mx.$

Now $\lim_{\substack{x\to o\\y\to0}} f(x,y) = \lim_{\substack{x\to o\\y\to o}} \frac{2xy}{x^2+y^2}$

$$= \lim_{x\to0}\frac{2mx^2}{x^2+m^2x^2}$$

$$= \frac{2m}{1+m^2}$$

Which is different for the different m selected.

$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist.

Consider

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0}\frac{2x(0)}{x^2+0} = \lim_{x\to 0} 0 = 0 = f(0,0)$$

$$\lim_{y\to 0} f(y,0) = \lim_{y\to 0}\frac{2y(0)}{y^2+0} = \lim_{y\to 0} 0 = 0 = f(0,0)$$

$\therefore f(x,y)$ is continuous for given values of x and y but it is not continuous at (0,0)

Partial Differentiation:

Let $z = f(x,y)$ be a function of two variables x and y. Then $\lim_{x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$, if it exists , is said to be partial derivative or partial differential coefficient of z or $f(x,y)$, w.r.t.x . It is denoted by the symbol $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x .

i.e, for the partial derivative of $z = f(x,y)$ w.r.t. 'x' , 'y' is kept constant.

Similarly, the partial derivative of $z = f(x,y)$ w.r.t. 'y' , 'x' is kept constant and is defined as $\lim_{y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$ and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

Higher order Partial Derivatives:

In general the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y and they can be differentiated repeatedly to get higher order partial derivatives,

So $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$, $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$,

$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3}$, $\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}$, $\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial x\,\partial y^2}$, and so on.

The chain rule of Partial Differentiation:

Let $z = f(u,v)$ where $u = \emptyset(x,y)$ and $= g(x,y)$. Then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Total differential coefficient:

Let $z = f(x,y)$ where $x = \emptyset(t)$ and $y = g(t)$

Substituting x and y in $= f(x,y)$, z becomes a function of a single variable t.

Then the derivative of z w.r.t. 't' i.e, $\frac{\partial z}{\partial t}$ is called the total differential coefficient or total derivative of z.

$\therefore \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$

Note: In the differential form , this result can be written as $du = \frac{\partial u}{\partial x}.\mathrm{dx} + \frac{\partial u}{\partial y}.\mathrm{dy}$

Here, du is called the total differential of u.

Solved Problems:

1.If U = log($x^3 + y^3 + z^3 - 3xyz$) , prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U = \frac{-9}{(x+y+z)^2}$

Sol: Given that U = log($x^3 + y^3 + z^3 - 3xyz$)

$\therefore \frac{\partial U}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$ (and z are constant)

$\frac{\partial U}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$ (and z are constant)

$\frac{\partial U}{\partial z} = \frac{3z^2-3yx}{x^3+y^3+z^3-3xyz}$ (and x are constant)

$$\therefore \frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}+\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}+\frac{3z^2-3yx}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$\Longrightarrow \frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z} = \frac{3}{x+y+z} \qquad \ldots\ldots..(1)$$

Now $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 U = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z}\right)$

$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$ [from (1)]

$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$

$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$

$= -\frac{9}{(x+y+z)^2}$

2.If $x^x y^y z^z = e$ show that at $x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$

Sol: Given that $x^x y^y z^z = e$

Taking logarithm on both sides, we get

$x \log x + y \log y + z \log z = \log e$

$\Longrightarrow z \log z = 1 - x \log x - y \log y$

Differentiating partially w.r.t ,'x', we get

$\left(z.\frac{1}{z} + 1.\log z\right)\frac{\partial z}{\partial x} = -\left(x.\frac{1}{x} + 1.\log x\right)$

$\Rightarrow \frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)} \qquad \ldots\ldots..(1)$

Similarly $\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{(1+\log z)}$ ....(2)

When $x = y = z$, we have

$\frac{\partial z}{\partial x} = -1$ and $\frac{\partial z}{\partial y} = -1$

Now differentiating (2) partially w.r.t, 'x', we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)}\right]$$

$$= -(1+\log y)\left[-(1+\log z)^{-2}\frac{1}{z}\frac{\partial z}{\partial x}\right] = \frac{1+\log y}{z(1+\log z)^2}\frac{\partial z}{\partial x} \quad ....(3)$$

When $x = y = z$ from (3), we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1+\log x}{x(1+\log x)^2}(-1) \quad \left(\text{since } \frac{\partial z}{\partial x} = -1\right)$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e+\log x)} \quad (\text{since } \log e = 1)$$

$$= -\frac{1}{x\log ex} = -(x\log ex)^{-1}$$

3.If $u = f(y-z, z-x, x-y)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol: Let $r = y - z, s = z - x, t = x - y$ .Then $u = f(r, s, t)$

Now $\frac{\partial r}{\partial x} = 0, \frac{\partial r}{\partial y} = 1, \frac{\partial r}{\partial z} = -1$

$\frac{\partial s}{\partial x} = -1, \frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial x} = 1$

and $\frac{\partial t}{\partial x} = 1, \frac{\partial t}{\partial y} = -1, \frac{\partial t}{\partial z} = 0$

∴ By chain rule of partial differentiation, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1) = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(-0) + \frac{\partial u}{\partial t}(-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad ...(2)$$

and

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0) = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad ...(3)$$

(1) + (2) + (3) gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right) + \left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right) + \left(-\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right) = 0$$

Jacobian :

Let $u = u(x,y)$, $v = v(x,y)$ are two functions of the independent variables x, y.

The jacobian of (u,v) w .r .t (x,y) or the jacobian transformation is given by the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ (or) $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

The determinant value is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Similarly if $u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$,then the Jacobian of u, v, w w.r.to x, y, z is given by

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Properties of Jacobians

1. If $J = \frac{\partial(u,v)}{\partial(x,y)}$ *and* $J^1 = \frac{\partial(x,y)}{\partial(u,v)}$ *then* $JJ^1 = 1$
2. If u, v are functions of r, s and r, s are functions of x, y, then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

Solved Problems:

1. **If $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$**

Sol : Given $x + y^2 = u$, $y + z^2 = v$, $z + x^2 = w$

We have $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}$

$= 1(1-0) - 2y(0 - 4xz) + 0$

$= 1 - 2y(-4xz)$

$= 1 + 8xyz$

$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\left[\frac{\partial(u,v,w)}{\partial(x,y,z)}\right]} = \frac{1}{1 + 8xyz}$

2. **Show that the functions $u = x + y + z$, $v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ and $w = x^3 + y^3 + z^3 - 3xyz$ are functionally related.**

Sol: Given $u = x + y + z$

$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

$w = x^3 + y^3 + z^3 - 3xyz$

we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x-2y-2z & 2y-2x-2z & 2z-2y-2x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$=6 \begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$c_1 \to c_1 - c_2$

$c_2 \to c_2 - c_3$

$$=6 \begin{vmatrix} 0 & 0 & 1 \\ 2x-2y & 2y-2z & z-y-x \\ x^2-yz-y^2+xz & y^2-xz-z^2+xy & z^2-xy \end{vmatrix}$$

$=6[2(x - y) (y^2 + xy – xz - z^2)-2(y - z)(x^2 + xz – yz - y^2)]$

$=6[2(x - y)(y – z)(x + y + z) – 2(y – z)(x – y)(x + y + z)]$

$=0$

Hence there is a relation between u,v,w.

3. If x + y + z = u , y + z = uv , z = uvw then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: $x + y + z = u$

$y + z = uv$

$z = uvw$

$y = uv - uvw = uv(1-w)$

$x = u - uv = u(1-v)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$R_2 \to R_2 + R_3$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

= uv [u –uv +uv]

$= u^2v$

4. If $u = x^2 - y^2$, $v = 2xy$ where $x = r\cos\theta$, $y = r\sin\theta$ S.T $\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$

Sol: Given $u = x^2 - y^2$, $v = 2xy$

$= r^2\cos^2\theta - r^2\sin^2\theta$ $\quad = 2r\cos\theta \, r\sin\theta$

$= r^2(\cos^2\theta - \sin^2\theta)$ $\quad = r^2\sin 2\theta$

$= r^2\cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & r^2(-\sin 2\theta)2 \\ 2r\sin 2\theta & r^2(\cos 2\theta)2 \end{vmatrix}$$

$$= (2r)(2r)\begin{vmatrix} \cos 2\theta & -r\sin 2\theta \\ \sin 2\theta & r(\cos 2\theta) \end{vmatrix}$$

$= 4r^2[r\cos^2 2\theta + r\sin^2 2\theta]$

$= 4r^2(r)[\cos^2 2\theta + \sin^2 2\theta]$

$= 4r^3$

5. If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol: Given $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

We have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$ux = yz(-1/x2) = \frac{-yz}{x^2}$, $uy = \frac{z}{x}$, $uz = \frac{y}{x}$

$v_x = \frac{z}{y}$, $v_y = xz(-1/y2) = \frac{-xz}{y^2}$, $v_z = \frac{x}{y}$

$w_x = \frac{y}{z}$, $w_y = \frac{x}{z}$, $w_z = xy(-1/z^2) = \frac{-xy}{z^2}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2}\cdot\frac{1}{y^2}\cdot\frac{1}{z^2}\begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 1[-1(1-1) -1(-1-1) + (1+1)]$

$= 0 -1(-2) + (2)$

$= 2 + 2$

=4

6. $If\ x = e^r sec\theta, y = e^r tan\theta\ P.T \frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol: Given x = $e^r \sec\theta$, y = $e^r \tan\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix}, \quad \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} r_x & r_y \\ \theta_x & \theta_y \end{vmatrix}$$

$x_r = e^r \sec\theta = x$, $\quad x_\theta = e^r \sec\theta \tan\theta$

$y_r = e^r \tan\theta = y$, $\quad y_\theta = e^r \sec^2\theta$

$x^2 - y^2 = e^{2r}(sec2\theta - tan2\theta)$

⇨ $2r = log(x^2 - y^2)$

⇨ $r = ½ log(x^2 - y^2)$

$$r_x = \frac{1}{2}\frac{1}{x^2-y^2}(2x) = \frac{x}{(x^2-y^2)}$$

$$r_y = \frac{1}{2}\frac{1}{x^2-y^2}(-2y) = \frac{-y}{(x^2-y^2)}$$

$$\frac{x}{y} = \frac{\sec\theta}{\tan\theta} = \frac{1/\cos\theta}{\sin\theta/\cos\theta} = \frac{1}{\sin\theta}$$

⇨ $\sin\theta = \frac{y}{x}$, $\theta = \sin^{-1}(\frac{y}{x})$

$$\theta_x = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}\, y\left(-\frac{1}{x^2}\right) = \frac{-y}{x\sqrt{x^2-y^2}}$$

$$\theta_y = \frac{1}{\sqrt{1-\frac{y^2}{x^2}}}(1/x) = \frac{1}{\sqrt{x^2-y^2}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} e^r \sec\theta \tan\theta \\ e^r \sec 2\theta \end{vmatrix} = e^{2r}\sec^2\theta - y\, e^r \sec\theta \tan\theta$$

$= e^{2r} \sec\theta[\sec^2\theta - \tan^2\theta] = e^{2r}\sec\theta$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{(x^2-y^2)} & \frac{-y}{(x^2-y^2)} \\ \frac{-y}{x\sqrt{x^2-y^2}} & \frac{1}{\sqrt{x^2-y^2}} \end{vmatrix}$$

$$=\left[\frac{x}{(x^2-y^2)\sqrt{x^2-y^2}} - \frac{y^2}{x(x^2-y^2)\sqrt{x^2-y^2}}\right]$$

$$= \frac{x^2-y^2}{x(x^2-y^2)\sqrt{x^2-y^2}} = \frac{1}{x\sqrt{x^2-y^2}} = \frac{1}{e^{2r}\sec\theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

7. If $= \boldsymbol{rcos\theta}, \boldsymbol{y = rsin\theta}$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$ .Also Show that $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$.

Sol : Given that $\boldsymbol{x = rcos\theta, y = rsin\theta}$ **-------(1)**

we have $r^2 = x^2 + y^2; \theta = tan^{-1}\left(\frac{y}{x}\right)$ -------(2)

$\frac{\partial x}{\partial r} = cos\theta, \frac{\partial y}{\partial r} = sin\theta, \frac{\partial x}{\partial \theta} = -rsin\theta, \frac{\partial y}{\partial \theta} = rcos\theta$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{cos\theta}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = cos\theta, \frac{\partial r}{\partial y} = \frac{y}{r} = sin\theta, \frac{\partial x}{\partial \theta}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} cos\theta & sin\theta \\ -\frac{sin\theta}{r} & \frac{cos\theta}{r} \end{vmatrix} = \frac{1}{r}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = r.\frac{1}{r} = 1.$$

8. if $\boldsymbol{x = \frac{u^2}{v}, y = \frac{v^2}{u}}$ then find $\frac{\partial(u,v)}{\partial(x,y)}$.

Sol: Given $\boldsymbol{x = \frac{u^2}{v}, y = \frac{v^2}{u}}$

$$\therefore \frac{\partial x}{\partial u} = \frac{2u}{v}, \frac{\partial x}{\partial v} = -\frac{u^2}{v^2}$$

$$\frac{\partial y}{\partial u} = -\frac{v^2}{u^2}, \frac{\partial y}{\partial v} = \frac{2v}{u}$$

$$\frac{\boldsymbol{\partial(x,y)}}{\boldsymbol{\partial(u,v)}} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix} = \frac{2u}{v} \cdot \frac{2v}{u} - \frac{u^2}{v^2}\frac{v^2}{u^2} = 4 - 1 = 3$$

Hence $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\partial(x,y)/\partial(u,v)} = 1/3$

Functional Dependence:

Two functions u and v are functionally dependent if their Jacobian i.e,

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

If the Jacobian of u, v is not equal to zero then those functions u, v are functionally independent.

Solved Problems :

1. **If $= \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. Hence prove that u and v are functionally dependent. Find the relation between them.**

Sol : Given $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

$$\therefore \frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}, \frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}, \frac{\partial v}{\partial x} = \frac{1}{1+x^2} \text{ and } \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$\therefore u$ and v are functionally dependent .

Now $v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}u$

$\therefore v = \tan^{-1}u$ is the functional relation between u and v.

2. **Determine whether the following functions are functionally dependent or not. If they are functionally dependent, find a relation between them.**

i) $u = e^x \sin y$, $v = e^x \cos y$ **ii)** $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

Sol: i) Jacobian $= \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$

$= e^x(-\sin^2 y - \cos^2 y) = -e^x \neq 0$

$\therefore u, v$ are functionally independent .

ii) $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

$$\therefore J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ \frac{-2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix} = \frac{2x}{y(x-y)^2} - \frac{2x}{y(x-y)^2} = 0$$

$\therefore$ u and v are functionally dependent , Now $v = \frac{x+y}{x-y} = \frac{y\,(\frac{x}{y}+1)}{y\,(\frac{x}{y}-1)} = \frac{u+1}{u-1}$

$\therefore$ $v = \frac{u+1}{u-1}$ is the functional relation between u and v .

3. Show that the functions $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$ are functionally related .find the relation between them.

Sol: We have

$u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \quad (\text{ Applying } R_1 \to R_1 + R_2)$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$= 2(x+y+z)\ (0)$ (since R_1andR_1 are identical)

$= 0$

Hence u, v and w are functionally dependent . That is , the functional relationship exists between them.

Now $w^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = v + 2u$

$\therefore$ $w^2 = v + 2u$ is the functional relation between u ,v and w.

4. Verify if $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$ are functionally dependent and if so , find the relation between them.

Sol: Given $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$

The functions u,v,w are functionally dependent if and only if $J(\frac{u,v,w}{xy,z}) = 0$

$$\text{Now } J(\frac{u,v,w}{xy,z}) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2(-1)\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (-2)(0) = 0$$

$\therefore$ u, v, w are functionally dependent

Now $u + v - 2w = (2x - y + 3z) + (2x - y - z) - 2\,(2x - y + z)$

$= (4x - 2y + 2z) - (4x - 2y + 2z) = 0$

Hence $u + v - 2w = 0$ is the functional relationship between u, v and w.

5. Show that the functions $u = x+y+z, v = x^2+y^2+z^2-2xy-2yz-2zx$ and $w = x^3+y^3+z^3-3xyz$ are functionally related.

Sol: Given $u = x+y+z$, $v = x^2+y^2+z^2-2xy-2yz-2zx$ and $w = x^3+y^3+z^3-3xyz$

$$\text{Now } \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(z-y-x) \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ (x-y-z) & (y-x-z) & (z-y-x) \\ (x^2-yz) & (y^2-xz) & (z^2-xy) \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & (z-y-x) \\ (x-y)(x+y+z) & (y-z)(x+y+z) & (z^2-xy) \end{vmatrix}$$

$C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 12\begin{vmatrix} x-y & y-z \\ (x-y)(x+y+z) & (y-z)(x+y+z) \end{vmatrix}$$

$$= 12\,(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ (x+y+z) & (x+y+z) \end{vmatrix}$$

$$= 12\,(x-y)(y-z)\,(0) \quad [C_1 \text{ and } C_2 \text{ are identical}]$$

$$= 0$$

Hence the functional relationship exists between u, v, w.

6. Prove that $u = \frac{x^2-y^2}{x^2+y^2}$, $v = \frac{2xy}{x^2+y^2}$ are functionally dependent and find the relation between them.

Sol: We are given $u = \frac{x^2-y^2}{x^2+y^2}$, $v = \frac{2xy}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2).2x-(x^2-y^2).2x}{(x^2+y^2)^2} = \frac{2x(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2).(-2y)-(x^2-y^2).2y}{(x^2+y^2)^2} = \frac{(-2y)(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2} = \frac{-4yx^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = 2\,y\left[\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}\right] = \frac{2y(y^2-x^2)}{(x^2+y^2)^2} \text{ and}$$

$$\frac{\partial v}{\partial y} = 2\,x\left[\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}\right] = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}$$

Thus $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4xy^2}{(x^2+y^2)^2} \\ \frac{2y(y^2-x^2)}{(x^2+y^2)^2} & \frac{2x(x^2-y^2)}{(x^2+y^2)^2} \end{vmatrix}$

$$= \frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^4} + \frac{8x^2y^2(y^2-x^2)}{(x^2+y^2)^4}$$

$$= \frac{8x^2y^2(x^2-y^2) - 8x^2y^2(y^2-x^2)}{(x^2+y^2)^4} = 0$$

$\therefore u, v$ are functionally dependent.

$u^2 + v^2 = \frac{(x^2-y^2)}{(x^2+y^2)^2} + \frac{4x^2y^2}{(x^2+y^2)^2} = \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1$

Hence $u^2 + v^2 = 1$ is the functional relation between u and v.

Maxima & Minima for functions of two Variables:

Definition : Let $f(x, y)$ be a function of two variables x and y.

At $= a$; $y = b$, $f(x, y)$ is said to have maximum or minimum value , if $f(a, b) > f(a+h, b+k)$ or $f(a, b) < f(a+h, b+k)$ respectively where h and k are small values.

Extremum: A function which have a maximum or minimum or both is called 'extremum'

Extreme value :- The maximum value or minimum value or both of a function is Extreme value.

Stationary points: - To get stationary points we solve the equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ i.e the pairs (a_1, b_1) (a_2, b_2) (a_3, b_3) ... ...are called stationary.

Working procedure:

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ Equate each to zero. Solve these equations for x & y we get the pair of values (a_1, b_1) (a_2, b_2) (a_3, b_3) ... ...

2. Find $l = \frac{\partial^2 f}{\partial x^2}, m = \frac{\partial^2 f}{\partial x \partial y}$, n = $\frac{\partial^2 f}{\partial y^2}$

3 i) If $ln - m^2 > 0$ and $l < 0$ at (a_1, b_1) then $f(x, y)$ is maximum at (a_1, b_1) and maximum value is $f(a_1, b_1)$

ii) If $ln - m^2 > 0$ and $l > 0$ at (a_1, b_1) then $f(x, y)$ is minimum at (a_1, b_1) and minimum value is $f(a_1, b_1)$.

iii) If $ln - m^2 < 0$ and at (a_1, b_1) then $f(x, y)$ is neither maximum nor minimum at (a_1, b_1).

In this case (a_1, b_1) is saddle point.

iii) If $ln - m^2 = 0$ and at (a_1, b_1), no conclusion can be drawn about maximum or minimum and needs further investigation. Similarly we do this for other stationary points.

Solved Problems:

1. Locate the stationary points & examine their nature of the following functions.

$\boldsymbol{u = x^4 + y^4 - 2x^2 + 4xy - 2y^2}$, $(x > 0, y > 0)$

Sol: Given $u(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

For maxima & minima $\frac{\partial u}{dx} = 0, \frac{\partial u}{\partial y} = 0$

$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ -------------------> (1)

$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y = 0 \Rightarrow y^3 + x - y = 0$ -------------------> (2)

Adding (1) & (2),

$x^3 + y^3 = 0$

$\Rightarrow x = -y$ -------------------> (3)

(1) $\Rightarrow x^3 - 2x \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

Hence (3) $\Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$

$l = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4$, $m = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = 4$ & $n = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 16$

At $(-\sqrt{2}, \sqrt{2})$, $ln - m^2 = (24 - 4)(24 - 4) - 16 = (20)(20) - 16 > 0$ and $l = 20 > 0$

The function has minimum value at $(-\sqrt{2}, \sqrt{2})$

At $(0,0)$, $ln - m^2 = (0-4)(0-4) - 16 = 0$

$(0,0)$ is not a extreme value.

2. Investigate the maxima & minima, if any, of the function $\boldsymbol{f(x) = x^3y^2(1 - x - y)}$.

Sol: Given $f(x) = x^3y^2(1 - x - y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$ $\quad \frac{\partial f}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2$

For maxima & minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

⇨ $3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \qquad \Longrightarrow x^2y^2(3 - 4x - 3y) = 0$ ---------------> (1)

⇨ $2x^3y - 2x^4y - 3x^3y^2 = 0$ => $x^3y(2 - 2x - 3y) = 0$ ----------------> (2)

From (1) & (2) $\quad 4x + 3y - 3 = 0$

$$2x + 3y - 2 = 0$$

$$2x = 1 \Rightarrow x = ½$$

$$4(½) + 3y - 3 = 0 \Rightarrow 3y = 3 - 2, y = (1/3)$$

$$l = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(1/2,1/3)} = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3 = 1/3 - 1/3 - 1/9 = -1/9$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 6x^2y - 8x^3y - 9x^2y^2$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1/2,1/3)} = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^3 = \frac{6-4-3}{12} = \frac{-1}{12}$$

$$n = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$$

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_{(1/2,1/3)} = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

$$ln - m^2 = (-1/9)(-1/8) - (-1/12)2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0 \text{ and } l = \frac{-1}{9} < 0$$

The function has a maximum value at $(1/2, 1/3)$

$$\therefore \text{Maximum value is } f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{8} \times \frac{1}{9}\right)\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{432}$$

3. Find three positive numbers whose sum is 100 and whose product is maximum.

Sol: Let x, y, z be three positive numbers.

Then $x + y + z = 100$

⇨ $z = 100 - x - y$

Let $f(x, y) = xyz = xy(100 - x - y) = 100xy - x^2y - xy^2$

For maxima or minima $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = 100y - 2xy - y^2 = 0 \Longrightarrow y(100 - 2x - y) = 0$ ----------------> (1)

$\frac{\partial f}{\partial y} = 100x - x^2 - 2xy = 0 \Longrightarrow x(100 - x - 2y) = 0$ ---------------> (2)

From (1) & (2)

$$100 - 2x - y = 0$$
$$200 - 2x - 4y = 0$$

-100 + 3y = 0 => 3y = 100 => y = 100/3

100 – x –(200/3) = 0 => x = 100/3

$l = \frac{\partial^2 f}{\partial x^2}$ =- 2y

$\left(\frac{\partial^2 f}{\partial x^2}\right)$ (100/3 , 100/3) = - 200/3

$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ = 100 -2x -2y

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)$ (100/3 , 100/3) = 100 –(200/3) –(200/3) = -(100/3)

$n = \frac{\partial^2 f}{\partial y^2}$ = -2x

$\left(\frac{\partial^2 f}{\partial y^2}\right)$ (100/3 , 100/3) = - 200/3

$ln - m^2$ = (-200/3) (-200/3) - $(-100/3)^2$ = $(100)^2$ /3

The function has a maximum value at (100/3 , 100/3)

i.e. at x = 100/3, y = 100/3 $\quad z = 100 - \frac{100}{3} - \frac{100}{3} = \frac{100}{3}$

The required numbers are $x = 100/3, y = 100/3, z = 100/3$

4. Find the maxima & minima of the function $f(x) = 2(x^2 - y^2) - x^4 + y^4$

Sol: Given $f(x) = 2(x^2 - y^2) - x^4 + y^4 = 2x^2 - 2y^2 - x^4 + y^4$

For maxima & minima $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0

$\frac{\partial f}{\partial x}$ = 4x - $4x^3$ = 0 => $4x(1-x^2)$ = 0 => x = 0 , x = ± 1

$\frac{\partial f}{\partial y}$ = -4y + $4y^3$ = 0 => $-4y(1-y^2)$ = 0 =>y = 0, y = ± 1

$l = \left(\frac{\partial^2 f}{\partial x^2}\right) = 4-12x^2$

$$m = \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$n = \left(\frac{\partial^2 f}{\partial y^2}\right) = -4 + 12y^2$$

we have $ln - m^2 = (4 - 12x^2)(-4 + 12y^2) - 0$

$= -16 + 48x^2 + 48y^2 - 144x^2y^2$

$= 48x^2 + 48y^2 - 144x^2y^2 - 16$

i) At (0 , ± 1)

$ln - m^2 = 0 + 48 - 0 - 16 = 32 > 0$

$l = 4-0 = 4 > 0$

f has minimum value at (0 , ± 1)

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(0, \pm 1) = 0 - 2 - 0 + 1 = -1$

The minimum value is '-1 '.

ii) At (± 1 ,0)

$ln - m^2 = 48 + 0 - 0 - 16 = 32 > 0$

$l = 4-12 = -8 < 0$

f has maximum value at (± 1 ,0)

$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$

$f(\pm 1, 0) = 2 - 0 - 1 + 0 = 1$

The maximum value is '1 '.

iii) At (0,0) , (± 1 , ± 1)

$ln - m^2 < 0$

$l = 4 - 12x^2$

(0 , 0) & (± 1 , ± 1) are saddle points.

f has no max & min values at (0 , 0) , (± 1 , ± 1).

5. Find the maximum and minimum values of $x^3 + y^3 - 3axy$.

Sol : let $z = x^3 + y^3 - 3axy$

Here $\frac{\partial z}{dx} = 3x^2 - 3ay = 0$ ------ (1)

$\frac{\partial z}{dy} = 3y^2 - 3ax = 0$ ----------- (2)

Solving (1) and (2) then

From (1) $3x^2 = 3ay$

$x^2 = ay$

$y = \frac{x^2}{a}$

Substitute the above value in equation (2) then we get

$3\frac{x^4}{a^2} - 3ax = 0$

$\Rightarrow 3x^4 - 3a^3x = 0$

$\Rightarrow 3x(x^3 - a^3) = 0$

$\Rightarrow x = 0, x = a.$

Corresponding values of y are $y = 0, y = a$.

Now $= \frac{\partial^2 z}{\partial x^2} = 6x$, $m = \frac{\partial^2 z}{\partial x dy} = -3a, n = \frac{\partial^2 z}{\partial y^2} = 6y$

(i) At the point $(0,0), ln - m^2 = 36xy - 9a^2 < 0$

Therefore z does not have any extreme value.

(ii) At the point $(a, a), ln - m^2 = 36a^2 - 9a^2 = 27a^2 > 0$ and

$l = 6a > 0 \; if \; a > 0$

In this case z attains minimum value and minimum value$=-a^3$.

$l = 6a < 0 \; if \; a < 0$

In this case z attains maximum value and maximum value$=-a^3$.

6. Examine for minimum and maximum values of $sinx + siny + \sin(x + y)$.

Sol : let $u(x, y) = sinx + siny + \sin(x + y)$ ---------(1)

$\frac{\partial u}{\partial x} = cosx + \cos(x + y) = 0$-----------(2)

$\frac{\partial u}{\partial y} = cosy + \cos(x + y) = 0$-----------(3)

On solving (2) and (3) we get $cosx = cosy$

$\Longrightarrow x = y$

From (2) $cosx + cos2x = 0$

$\Longrightarrow 2cos\frac{3x}{2}cos\frac{x}{2} = 0$

$\Longrightarrow \frac{3x}{2} = \pm\frac{\pi}{2}$ or $\frac{x}{2} = \pm\frac{\pi}{2}$

$\Longrightarrow x = \pm\frac{\pi}{3} \; or \; x = \pm\pi$

When $x = \pm\frac{\pi}{3}$ then $y = \pm\frac{\pi}{3}$ i.e $(\pm\frac{\pi}{3}, \pm\frac{\pi}{3})$

When $x = \pm\pi$ then $y = \pm\pi$ i.e $(\pm\pi, \pm\pi)$

$l = \frac{\partial^2 u}{\partial x^2} = -sinx - \sin(x + y)$; $m = \frac{\partial^2 u}{\partial x \partial y} = -\sin(x + y)$

$n = \frac{\partial^2 u}{\partial y^2} = -siny - \sin(x + y)$

(i) At $(\frac{\pi}{3}, \frac{\pi}{3})$ then $l = -\sqrt{3}, m = \frac{-\sqrt{3}}{2}$ $and\ n = -\sqrt{3}$

$ln - m^2 = \frac{9}{4} > 0\ and\ l < 0$

$\Longrightarrow u$ has maximum at $(\frac{\pi}{3}, \frac{\pi}{3})$

At $(\frac{\pi}{3}, \frac{\pi}{3})$, $u = \frac{3\sqrt{3}}{2}$

∴ Maximum value of $u = \frac{3\sqrt{3}}{2}$

(ii)At $(-\frac{\pi}{3}, -\frac{\pi}{3})$ we have $ln - m^2 > 0$ and $l > 0$

∴ u has a minimum at $(-\frac{\pi}{3}, -\frac{\pi}{3})$

∴ Minimum value of $u = \frac{-3\sqrt{3}}{2}$

At $(\pm\pi, \pm\pi)$, $ln - m^2 = 0$

∴ there is a need for further investigation.

7. A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

Sol : Let $x\ ft$, $y\ ft$ and $z\ ft$ be the dimensions of the box and

Let S be the surface of the box. Then we have

$S = xy + 2yz + 2zx$ (since open at the top) ------ (1)

Given that its volume $xyz = 32$ ------ (2)

From (2) , $z = \frac{32}{xy}$

Substitute the value of z in (1) , we get

$S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x = xy + \frac{64}{x} + \frac{64}{y}$

$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0$, $\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0$

By solving these equations we get $x = 4$, $y = 4$

$l = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$, $m = \frac{\partial^2 S}{\partial x dy} = 1$, $n = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$

At $(x, y) = (4,4)$, $ln - m^2 = \frac{128}{x^3} . \frac{128}{y^3} - 1 = 2.2 - 1 = 3 > 0$ and $l = \frac{128}{x^3} = 2 > 0$

Thus, S is minimum when $x = 4, y = 4$.

From (2) we get $z = 2$.

∴ The dimensions of the box for least material for its construction are 4,4,2.

8. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol : Let length, breadth and height be $2x, 2y, 2z$ and volume is V

$$\therefore V=(2x)(2y)(2z)=8xyz$$

Let $f=V^2=64x^2y^2z^2=64x^2y^2c^2(1-\frac{x^2}{a^2}-\frac{y^2}{b^2})$ (from (1) z^2 value substituted)

$$\Rightarrow f=64c^2(x^2y^2-\frac{x^4y^2}{a^2}-\frac{x^2y^4}{b^2}$$

$$\frac{\partial f}{\partial x}=64c^2\left(2xy^2-\frac{4x^3y^2}{a^2}-\frac{2xy^4}{b^2}\right)$$

$$\frac{\partial f}{\partial y}=64c^2\left(2x^2y-\frac{2x^4y}{a^2}-\frac{4x^2y^3}{b^2}\right)$$

$\frac{\partial f}{\partial x}=0 \Rightarrow \frac{2x^2}{a^2}+\frac{y^2}{b^2}=1$ and

$\frac{\partial f}{\partial y}=0 \Rightarrow \frac{x^2}{a^2}+\frac{2y^2}{b^2}=1$

Solving these equations , we get $x=\frac{a}{\sqrt{3}}$ and $y=\frac{b}{\sqrt{3}}$

Again $l=\frac{\partial^2 f}{\partial x^2}=64c^2\left(2y^2-\frac{2x^3y}{a^2}-\frac{2y^4}{b^2}\right)$

$$m=\frac{\partial^2 f}{\partial x\partial y}=64c^2\left(4xy-\frac{8x^3y}{a^2}-\frac{8xy^3}{b^2}\right)$$

$$n=\frac{\partial^2 f}{\partial y^2}=64c^2\left(2x^2-\frac{2x^4}{a^2}-\frac{12x^2y^2}{b^2}\right)$$

At $(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}})$, $ln-m^2>0$ and $l<0$

Therefore f is maximum at At $(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}})$

Maximum value of $f=64c^2\left(\frac{a^2}{3}\frac{b^2}{3}-\frac{a^4b^2}{9.3.a^2}-\frac{a^2b^4}{3.9.b^2}\right)=\frac{64c^2}{27}(3a^2b^2-2a^2b^2)$

i.e $V^2=\frac{64}{27}a^2b^2c^2$

$\Rightarrow V=\frac{8abc}{\sqrt{27}}$.

9. Find the points on the surface $z^2=xy+1$ that are nearest to the origin.

Sol : Let $P(x,y,z)$ be any point on the surface

Let the surface be $\emptyset(x,y,z)=z^2-xy-1=0$ ------- **(1)**

Let $O(0,0,0,)$ be the origin then $OP=f=\sqrt{x^2+y^2+z^2}$ ------- (2)

We have to minimize OP (i.e) subject to the condition (1)

From (2) $f^2=\boldsymbol{x^2+y^2+z^2=x^2+y^2+xy+1}$ **[** from (1) **]**

Now differentiating f partially on both sides w.r.to x and y then we get

$2f\frac{\partial f}{\partial x} = 2x + y \implies \frac{\partial f}{\partial x} = \frac{2x+y}{2f}$ ------- (3)

$2f\frac{\partial f}{\partial y} = 2y + x \implies \frac{\partial f}{\partial y} = \frac{2y+x}{2f}$ --------- (4)

Now the critical points are given by $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$$\implies 2x + y = 0 \; and \; 2y + x = 0$$

$$\implies x = 0 , y = 0$$

Now from equation (1) we get $z = \pm 1$

Therefore $P(0,0,1)$ and $Q(0,0,-1)$ are the critical points of f

Differentiate (3) partially w.r.to x and y then we get

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{2(2f) - (2x+y)2\frac{\partial f}{\partial x}}{(2f)^2}$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{(2f) - (2x+y)2\frac{\partial f}{\partial y}}{(2f)^2}$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{2(2f) - (2y+x)2\frac{\partial f}{\partial y}}{(2f)^2}$$

Now at $P(0,0,1)$, $ln - m^2 > 0$ and $l < 0$

Therefore f has minimum at $P(0,0,1)$

Now at $Q(0,0,1)$, $ln - m^2 > 0$ and $l < 0$

Therefore f has minimum at $Q(0,0,-1)$

Hence required points are $P(0,0,1)$ and $Q(0,0,-1)$.

Lagrange's method of undetermined multipliers

Suppose it is required to find the extremum for the function f(x , y , z) subject to condition

$\emptyset$(x , y , z) = 0 ------------- (1)

Step 1 : Form Lagrangean function $F(x , y , z)$ = f(x , y , z) + γ $\emptyset$(x , y , z) where γ is called Lagrange's constant, which is determined by the following conditions.

Step 2: Obtain the equations

$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 0$ --------------- (2)

$$\frac{\partial F}{\partial y} = 0 \quad => \frac{\partial f}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 0 \text{ --------------- } (3)$$

$$\frac{\partial F}{\partial z} = 0 \quad => \frac{\partial f}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 0 \text{ --------------- } (4)$$

Step 3: Solving the equations (1) (2) (3) & (4) we get the stationary point (x, y, z).

Step 4 : Substitute the value of x , y , z so obtained in equation (1) we get the extremum.

Solved Problems:

1. **Find the minimum value of $x^2 + y^2 + z^2$, given $x + y + z = 3a$**

Sol: $u = x^2 + y^2 + z^2$

$\emptyset = x + y + z - 3a = 0$

Using Lagrange's function

$$F(x, y, z) = u(x, y, z) + \gamma \emptyset(x, y, z)$$

For maxima or minima

$$\frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + \gamma \frac{\partial \emptyset}{\partial x} = 2x + \gamma = 0 \text{ ------------ } (1)$$

$$\frac{\partial F}{\partial y} = \frac{\partial u}{\partial y} + \gamma \frac{\partial \emptyset}{\partial y} = 2y + \gamma = 0 \text{ ------------ } (2)$$

$$\frac{\partial F}{\partial z} = \frac{\partial u}{\partial z} + \gamma \frac{\partial \emptyset}{\partial z} = 2z + \gamma = 0 \text{ ------------ } (3)$$

From (1) , (2) & (3)

$\gamma = -2x = -2y = -2z$

$x = y = z$

$\emptyset = x + x + x - 3a = 0$

$x = a$

$x = y = z = a$

Minimum value of $u = a^2 + a^2 + a^2 = 3a^2$

2. **Find the minimum value of $x^2 + y^2 + z^2$, given that $xyz = a^3$**

Sol: Let $u = x^2 + y^2 + z^2$ …… (1)

And $\emptyset = xyz - a^3 = 0$ …… (2)

Consider the lagrangean function $F(x,y,z) = u(x,y,z) + \lambda\, \phi(x, y, z)$

i.e , $F(x,y,z) = x^2 + y^2 + z^2 + \lambda\,(xyz - a^3)$ ……(3)

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} + \lambda \frac{\partial \emptyset}{\partial x} = 2x + \lambda yz = 0$ ……(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} + \lambda \frac{\partial \emptyset}{\partial y} = 2y + \lambda xz = 0$ ……(5)

$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} + \lambda \frac{\partial \emptyset}{\partial z} = 2z + \lambda yx = 0$ ……(6)

From (4) , (5) and (6) ,we have $\frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy} = -\frac{\lambda}{2}$ ……(7)

From the first two members , we have $\frac{x}{yz} = \frac{y}{xz} \Rightarrow x^2 = y^2$ …(8)

From the last members , we have $\frac{y}{xz} = \frac{z}{xy} \Rightarrow y^2 = z^2$ ……(9)

From (8) and (9) , we have $x^2 = y^2 = z^2 \Rightarrow x = y = z$ ……(10)

on solving (2) and (10) , we get , x = y = z = a

∴ Minimum value of $u = a^2 + a^2 = 3a^2$.

3. Find the maximum value of $u = x^2y^3z^4$ if 2x + 3y + 4z = a

Sol: Given $u = x^2y^3z^4$ …… (1)

Let $\phi(x, y, z) = 2x + 3y + 4z - a = 0$ ……(2)

Consider the lagrangean function F(x,y,z) = u(x,y,z) + λ ϕ(x, y, z)

i.e, , $F(x,y,z) = x^2y^3z^4 + \lambda(2x + 3y + 4z - a)$ ……(3)

for maxima or minima $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow 2xy^3z^4 + 2\lambda = 0 \Rightarrow xy^3z^4 = -\lambda$ ……(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 3x^2y^2z^4 + 3\lambda = 0 \Rightarrow x^2y^2z^4 = -\lambda$ …….(5)

and $\frac{\partial F}{\partial z} = 0 \Rightarrow 4x^2y^3z^3 + 4\lambda = 0 \Rightarrow x^2y^3z^3 = -\lambda$ …….(6)

From (4) and (5) , we have x = y …….(7)

From (5) and (6) , we have y = z …….(8)

Hence from (7) and (8) , we get x = y = z …….(9)

On solving (2) and (9) ,we get $x = y = z = \frac{a}{9}$

$$\therefore \text{Maximum value of } u = \left(\frac{a}{9}\right)^2 \left(\frac{a}{9}\right)^3 \left(\frac{a}{9}\right)^4 = \left(\frac{a}{9}\right)^9$$

4. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Sol: Let 2x ,2y, 2z are the length , breadth and height of rectangular solid

Then its volume V = 8 xyz ……..(1)

Let the sphere have a radius of 'r' so that $x^2 + y^2 + z^2 = r^2$ ……(2)

Consider the lagrangean function F(x,y,z) = u(x,y,z) + λ ϕ(x, y, z) i.e, F(x,y,z) = V + λ $(x^2 + y^2 + z^2 - r^2)$

$= 8xyz + \lambda\,(x^2 + y^2 + z^2 - r^2)$ …….(3)

For maxima or minima $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 8yz + 2\lambda x = 0$ ……(4)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 8zx + 2\lambda y = 0$ ……(5)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 8xz + 2\lambda z = 0$ ……(6)

From (4) ,(5) and (6) we have $2x^2\lambda = -8xyz = -2y^2\lambda = -2z^2\lambda$

$\Rightarrow x = y = z$

Thus for a maximum value x = y = z which shows that the rectangular solid is a cube.

5. Find the maximum and minimum values of x + y + z subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Sol: Let the function be $f(x, y, z) = \mathbf{x + y + z}$ subject to the constraints $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

The Auxiliary function $F(x, y, z) = x + y + z + \lambda(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1)$ ------- **(1)**

Differentiating (1) partially w.r.to x, y, z and equating to zero, we get

$\frac{\partial F}{\partial x} = 1 - \frac{\lambda}{x^2} = 0$ ----- (2)

$\frac{\partial F}{\partial y} = 1 - \frac{\lambda}{y^2} = 0$ ---- (3)

$\frac{\partial F}{\partial z} = 1 - \frac{\lambda}{z^2} = 0$ ----- (4)

On solving (2) ,(3) and (4) for x, y, z then we get $x = \pm\sqrt{\lambda}$, $y = \pm\sqrt{\lambda}, z = \pm\sqrt{\lambda}$

Now substitute these x, y, z in the given constraint then we get $\frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} = 1$

$\Rightarrow \lambda = 9 \quad \Rightarrow x = \pm 3, y = \pm 3, z = \pm 3$

Thus the maximum and minimum values are 9 and -9.

Taylor's series for a function of two variables:

Consider a function $f(x, y)$ defined in a region enclosing (a, b) and having successive partial derivatives ,then taylor's series gives an expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$ and partial derivatives of f at (a, b) and is expressed in ascending powers of $(x - a)$ and $(y - b)$.

f(x,y) = f(a,b) + (x-a) $f_x(a, b) + (y - b)f_y(a, b) + \frac{1}{2!}[\,(x - a)^2 f_{xx}(a, b) + 2(x - a)$

$(y-b)f_{xy}(a,b) + \big(y-b)^2 f_{yy}(a.b)\big] + \frac{1}{3!}[\,(x-a)^3 f_{xxx}(a,b) + 3\,(x-a)^2(y-$
$b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + \big(y-b)^3 f_{yyy}(a,b)\big]$ + ...

Note: The above expansion is called the expansion of f(x,y) at (a,b) or in the neighborhood of (a,b) or in powers of (x-a) and (y-b) .

Solved Problems:

1. **Expand $e^x \cos y$ near $(1, \frac{\pi}{4})$**

Sol: Let f(x,y) = $e^x \cos y \Rightarrow f(1, \frac{\pi}{4}) = \frac{e}{\sqrt{2}}$. Then

$f_x(x,y) = e^x \cos y \Rightarrow f(1, \frac{\pi}{4}) = \frac{e}{\sqrt{2}}$

$f_y(x,y) = -e^x \sin y \Rightarrow f(1, \frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$f_{xx}(x,y) = -e^x \cos y \Rightarrow f(1, \frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$f_{xy}(x,y) = -e^x \sin y \Rightarrow f(1, \frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

$f_{yy}(x,y) = -e^x \cos y \Rightarrow f(1, \frac{\pi}{4}) = \frac{-e}{\sqrt{2}}$

By substituting above values in taylor's series , we get

$$e^x \cos y = \frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{(x-1)^2}{2!} - (x-1)\left(y - \frac{\pi}{4}\right) - \frac{\left(y-\frac{\pi}{4}\right)^2}{2!} + \cdots\right]$$

2. **Expand $x^2y + 3y - 2$ in powers of $(x-1)$and $(y+2)$ using Taylor's thorem.**

Sol: Let f(x,y) = $x^2y + 3y - 2$, a = 1 , b = -2

Now f(a,b) = f(1,-2) = -10

$f_x(a,b) = 2xy \Rightarrow f_x(1,-2) = -10$

$f_y(a,b) = x^2 + 3 \Rightarrow f_y(1,-2) = 4$

$f_{xx}(a,b) = 2y \Rightarrow f_{xx}(1,-2) = -4$

$f_{xy}(a,b) = 2x \Rightarrow f_{xy}(1,-2) = 4$

$f_{yy}(a,b) = 0 \Rightarrow f_{yy}(1,-2) = 0$

$f_{xxx}(a,b) = 0 \Rightarrow f_{xxx}(1,-2) = 0$

$f_{xxy}(a,b) = 2 \Rightarrow f_{xxy}(1,-2) = 2$

$f_{xyy}(a,b) = 0 \Rightarrow f_{xyy}(1,-2) = 0$

$f_{yyy}(a,b) = 0 \Rightarrow f_{yyy}(1,-2) = 0$

All other partial derivatives of higher order will vanish

By substituting above values in taylor's series , we get

$x^2y + 3y - 2 = 10 + [(x-1)(-4) + (y+2)4] + \frac{1}{2}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0) + \frac{1}{6}[(x-1)^3(0) + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + (y+2)^3(0)]$

$= 10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$.

3. Expand the function $f(x,y) = e^y\log(1+y)$ in terms of x and y up to the terms of third degree using Taylor's theorem.

Sol: Given that $f(x,y) = e^y\log(1+y)$; $f(0,0) = 0$

$f_x(x,y) = e^x\log(1+y)$; $f_x(0,0) = 0$

$f_y(x,y) = e^x\frac{1}{1+y}$; $f_y(0,0) = 1$

$f_{xy}(x,y) = e^x\frac{1}{1+y}$; $f_{xy}(0,0) = 1$

$f_{xx}(x,y) = e^x\log(1+y)$; $f_{xx}(0,0) = 0$

$f_{yy}(x,y) = -e^x\frac{1}{(1+y)^2}$; $f_{yy}(0,0) = -1$

$f_{xxy}(x,y) = e^x.\frac{1}{1+y}$; $f_{xxy}(0,0) = 1$

$f_{xyy}(x,y) = -e^x.\frac{1}{(1+y)^2}$; $f_{xyy}(0,0) = -1$

$f_{xxx}(x,y) = e^x\log(1+y)$; $f_{xxx}(0,0) = 0$

$f_{yyy}(x,y) = 2e^x.\frac{1}{(1+y)^3}$; $f_{yyy}(0,0) = 2$

Therefore by Taylor's theorem

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0)$$
$$+\frac{1}{2!}\left[x^2f_{xx}(0,0) + 2xy\,f_{xy}(0,0) + y^2f_{yy}(0,0)\right]$$
$$+\frac{1}{3!}\left[x^3f_{xxx}(0,0) + 3x^2yf_{xxy}(0,0) + 3xy^2f_{xyy}(0,0) + y^3f_{yyy}(0,0)\right]$$
$$+\ ------$$

$$= 0 + 0 + y + \frac{1}{2!}[0 + 2xy - y^2] + \frac{1}{3!}[0 + 3x^2y - 3xy^2 + 2y^3]$$

$$f(x,y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3}$$

UNIT – IV

FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

Introduction:

Differential equations have wide applications in various engineering and science disciplines. In general, modelling variations of a physical quantity, such as temperature, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. Similarly, studying the variation of a physical quantity or other physical quantities would also lead to differential equations. In fact, many engineering subjects, such as mechanical vibration or structural dynamics, heat transfer, or theory of electric circuits, are founded on the theory of differential equations. It is practically important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behaviour of the systems concerned can be studied. For example, the change of strain or stress for some viscoelastic materials follows a differential equation. It is important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behavior of the systems concerned can be studied. Some of the real-world applications of first-order equations are terminal velocity of a falling mass, and the resistor-capacitor electrical circuit, Newton's law of cooling, law of natural growth and decay i.e. Population growth and decay, Radio-active decay and carbon dating (used for calculating age of fossils), orthogonal trajectories (used in satellites orbit alignment),concentration and dilution problems, problems involving variable acceleration can be solved using separable equations or linear 1st order equation (when taking drag into account) and many more.

Definition: An equation involving derivatives of one dependent variable with respect to one or more independent variables called a Differential equation.

There are two types' differential equations

1. Ordinary differential equations
2. Partial differential equations

Ordinary differential equation: A differential equation which involves only ordinary derivatives (derivatives with respect to only one independent variable) is said to be ordinary differential equation.

Ex . (1) $\frac{dy}{dx}+7xy=x^2$ **(2)** $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^x$

Partial Differential equation: A Differential equation is said to be partial if the derivatives in the equation have reference to two or more independent variables.

E. g: **1.** $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4z$

2. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

Order of a Differential equation: It is the order of the highest derivative occurring in the Differential equation. Differential equation is said to be of order 'n' if the n^{th} derivative is the highest derivative in that equation.

E. g : (1) $(x^2+1)\frac{dy}{dx} + 2xy = 4x2$

Order of this Differential equation is 1.

(2) $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$

Order of this Differential equation is 2.

(3) $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 2y = 0$.

Order=2

(4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Order is 2.

Degree of a Differential equation: Degree of a differential Equation is the highest degree of the highest derivative in the equation, after the equation is made free from radicals and fractions in its derivations.

E.g : 1) $y = x.\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ on solving we get

$(1-x^2)\left(\frac{dy}{dx}\right)^2 + 2xy.\frac{dy}{dx} + (1-y^2) = 0$. Degree = 2

2) a. $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$ on solving . we get

$a^2.\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$. Degree = 2

First order and First degree Differential Equation :

The general form of first order, first degree differential equation is $\frac{dy}{dx} = f(x, y)$ or $f(x, y, y') = 0$ [i.e $Mdx + Ndy = 0$ Where M and N are functions of x and y]. There is no general method to solve any first order differential equation The equation which belong to one of the following types can be easily solved.

In general the first order first degree differential equation can be classified as:

(1) Exact equations

(2) Non exact equations (reducible to exact equations).

Exact Differential Equations

Definition: Let $M(x, y)\, dx + N(x, y)\, dy = 0$ be a first order and first degree Differential Equationwhere M & N are real valued functions of $, y$ .Then the equation $M\, dx + N\, dy = 0$ is said to be an exact Differential equation if ∃ a function f ∍ $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

Condition for Exactness: If $M(x, y)$ & $N\,(x, y)$ are two real functions which have continuous partial derivatives then the necessary and sufficient condition for the Differential equation $M\, dx + N\, dy = 0$ is to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence solution of the exact equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is

$$\int M dx \quad + \quad \int N dy = c.$$

(y is taken as constant) (terms free from x are taken).

Solved Problems :

1. Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.

Sol : Given equation can be written as

$$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0 \ldots(1)$$

It is of the form $Mdx + Ndy = 0$.

Here

$M = y\cos x + \sin y + y$

$N = \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Clearly $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$Equation is exact.

The general solution is given by $\int Mdx + \int Ndy = c$

(y constant) (terms independent of x)

$$\Rightarrow \int (y\cos x + \sin y + y)dx + \int (0)dy = c$$

$$\Rightarrow y\sin x + (\sin y + y)x = c\,.$$

2. Solve $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$

Sol : Here $M = 1+e^{\frac{x}{y}}$ & $N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \;\&\; \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-1}{y}\right) + \left(1-\frac{x}{y}\right)e^{\frac{x}{y}}\left(\frac{1}{y}\right)$$

$$\Rightarrow \frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) \;\&\; \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right)$$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Equation is exact

General solution is

$$\int Mdx \quad + \quad \int Ndy \quad = c.$$

(y constant) (terms free from x)

$$\Rightarrow \int \left(1+e^{\frac{x}{y}}\right)dx + \int 0\ dy = c.$$

$$\Rightarrow x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$\Rightarrow x + ye^{\frac{x}{y}} = c$$

3.Solve the D.E $(x+y-1)\,dy-(x-y+2)\,dx=0$

Sol : Here $M=-(x-y+2);$

$N=(x+y-1)$

$\frac{\partial M}{\partial y}=1;\frac{\partial N}{\partial x}=1$

Clearly $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Thus the equation is exact.

General solution is $\int Mdx \quad + \quad \int Ndy \quad = c$.

(y constant) (terms free from x)

$\Rightarrow \int -(x-y+2)dx+\int (y-1)dy=c$

$\Rightarrow -\frac{x^2}{2}+xy-2x+\frac{y^2}{2}-y=c$

$\Rightarrow x^2-y^2-2xy+4x+2y=c_1$

4. Solve $(e^y+1).\cos x dx+e^y\sin x dx=0$ **.**

Sol. $M=(e^y+1)\cos, N=e^y\sin x$

$\frac{\partial M}{\partial y}=e^y\cos x;\frac{\partial N}{\partial y}=e^y\cos x$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=e^y\cos x$

Equation is exact.

Gen Sol. is $\int Mdx \quad + \quad \int Ndy \quad = c$.

(y constant) (terms free from x)

$\int (e^y+1)\cos x\,dx+\int 0\,dy=c$

$\Rightarrow e^y\sin x=c$.

5. Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left[x+\log x-x\sin y\right]dy=0$ **.**

Sol : Here $M=y\left(1+\frac{1}{x}\right)+\cos y, N=x+\log x-x\sin y$

$$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y \quad \frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$ so the equation is exact

General sol $\int Mdx \quad + \quad \int Ndy \quad = c$.

(y constant) (terms free from x)

$$\Rightarrow \int\left[y+\frac{y}{x}+\cos y\right]dx+\int 0.dy=c.$$

$$\Rightarrow y(x+\log x)+x\cos y=c.$$

6. Solve $(\boldsymbol{cosx - xcosy})\boldsymbol{dy} - (\boldsymbol{siny + ysinx})\boldsymbol{dx} = \mathbf{0}$

Sol : $N=\cos x-x\cos y \,\&\, M=-\sin y-y\sin x$

$$\frac{\partial N}{\partial x}=-\sin x-\cos y \qquad \frac{\partial M}{\partial y}=-\cos y-\sin x$$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\Rightarrow$$ the equation is exact.

General sol $\int Mdx \quad + \quad \int Ndy \quad = c$.

(y constant) (terms free from x)

$$\Rightarrow \int(-\sin y-y\sin x).dx \quad + \int 0.dy=c$$

$$\Rightarrow -x\sin y+y\cos x=c$$

$$\Rightarrow \text{ycosx - xsiny} = c$$

7. Solve $(r+\sin\theta-\cos\theta)dr+r(\sin\theta+\cos\theta)d\theta=0$

Sol : Given equation is $(r+\sin\theta-\cos\theta)dr+r(\sin\theta+\cos\theta)d\theta=0$............(1)

This is of the form $Md\theta+Ndr=0$

Where $M=r(\sin\theta+\cos\theta); N=r+\sin\theta-\cos\theta$

We have $\frac{\partial M}{\partial r}=\sin\theta+\cos\theta; \frac{\partial N}{\partial\theta}=\cos\theta+\sin\theta$

Clearly $\frac{\partial M}{\partial r}=\frac{\partial N}{\partial\theta}$

$\therefore$ The given equation is exact.

The general solution is given by $\int Md\theta+\int Ndr=c$

(r constant) (terms independent of θ)

$$\Rightarrow \int r(\sin\theta+\cos\theta)d\theta+\int rdr=c$$

$$\Rightarrow r(\sin\theta-\cos\theta)+\frac{r^2}{2}=c$$

The general solution is $r^2+2r\left(\sin\theta-\cos\right)=c_1$.

Reduction Of Non-Exact Differential Equations To Exact Form Using Integrating Factors

Definition: If the Differential Equation $M(x,y)\,dx + N\,(x,y\,)\,dy = 0$ be not an exact differential equation it can be made exact by multiplying with a suitable function which is called an Integrating factor(I.F).

Note: There may exits several integrating factors.

<u>Some methods to find an I.F to a non-exact Differential Equation $Mdx + N\,dy = 0$</u>

Case -1: Integrating factor by inspection/ (Grouping of terms).

Some useful exact differentials

1 . d (xy) = xdy +y dx

2. $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$

3. $d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{x^2}$

4. $d\left(\frac{x^2+y^2}{2}\right)$ = x dx + y dy

5. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{xdy-ydx}{xy}$

6. $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{xy}$

7. $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{x^2+y^2}$

8. $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{xdy-ydx}{x^2+y^2}$

9. $d\left(\log(xy)\right) = \frac{xdy+ydx}{xy}$

10. $d\left(\log\left(x^2+y^2\right)\right) \quad = \quad \dfrac{2\left(xdx+ydy\right)}{x^2+y^2}$

11. $d\left(\dfrac{e^x}{y}\right) \quad = \quad \dfrac{ye^x dx - e^x dy}{y^2}$

Solved Problems

1. Solve $\dfrac{y(xy+e^x)dx - e^x dy}{y^2} = 0$.

Sol : It can be written as $\dfrac{(xy^2+ye^x)dx - e^x dy}{y^2} = 0$.

$$\Rightarrow \frac{xy^2}{y^2}dx + \frac{e^x ydx - e^x dy}{y^2} = 0$$

$$\Rightarrow xdx + \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$\Rightarrow xdx + d\left(\frac{e^x}{y}\right) = 0$$

On integrating, we get $\dfrac{x^2}{2} + \dfrac{e^x}{y} = c$

This is the required solution.

2. Solve the differential equation $\left(y-x^2\right)dx + \left(x^2\cot y - x\right)dy = 0$

Sol : Given equation can be written as $ydx - xdy = x^2dx - x^2\cot y\, dy$

Dividing with x^2 , we get

$$\frac{ydx-xdy}{x^2} = dx - \cot y\, dy\,(or) - \left(\frac{xdy-ydx}{x^2}\right) = dx - \cot y\, dy$$

i.e , $-d\left(\dfrac{y}{x}\right) = dx - \cot y\, dy$

Integrating, we get $-\dfrac{y}{x} = x + \cos ec^2 y + \cos y = -x\left(x + \cos ec^2 y\right) + cx$

which is the required solution.

3. Solve $xdx + ydy + \dfrac{xdy - ydx}{x^2 + y^2} = 0$

Sol : Given equation is $xdx + ydy + \dfrac{xdy - ydx}{x^2 + y^2} = 0$

$d\left(\dfrac{x^2 + y^2}{2}\right) + d\left(\tan^{-1}\left(\dfrac{y}{x}\right)\right) = 0$ on Integrating we get

$\dfrac{x^2 + y^2}{2} + \tan^{-1}\left(\dfrac{y}{x}\right) = c$.

4. Solve $y\left(x^3.e^{xy} - y\right)dx + x\left(y + x^3.e^{xy}\right)dy = 0$.

Sol : Given equation is on regrouping

We get $yx^3e^{xy}dx - y^2dx + xydy + x^4e^{xy}dy = 0$

$x^3e^{xy}\left(ydx + xdy\right) + y\left(xdx - ydx\right) = 0$

Dividing by x^3

$e^{xy}\left(ydx + xdy\right) + \left(\dfrac{y}{x}\right).\left(\dfrac{xdy - ydx}{x^2}\right) = 0$

$d\left(e^{xy}\right) + \left(\dfrac{y}{x}\right).d\left(\dfrac{y}{x}\right) = 0$

on Integrating

$e^{xy} + \frac{1}{2}\left(\dfrac{y}{x}\right)^2 = c$ is required G.S.

5) Solve $(1 + xy)\,x\,dy + (1 - yx)\,y\,dx = 0$

Sol: Given equation is (1+xy) x dy + (1-yx) y dx =0.

(xdy + ydx) + xy (xdy – ydx) = 0.

Divided by $x^2y^2 \Rightarrow \left(\dfrac{xdy + ydx}{x^2y^2}\right) + \left(\dfrac{xdy - ydx}{xy}\right) = 0$

$\Rightarrow d\left(-\dfrac{1}{xy}\right) + \dfrac{1}{y}dy - \dfrac{1}{x}dx = 0$.

On integrating we get $-\dfrac{1}{xy} + \log y - \log x = \log c$

$-\dfrac{1}{xy} - \log x + \log y = \log c$.

6) Solve $ydx - xdy = a\left(x^2 + y^2\right)dx$

Sol : Given equation is ydx $ydx - xdy = a\left(x^2 + y^2\right)dx$

$$\Rightarrow \frac{ydx - xdy}{\left(x^2 + y^2\right)} = a\,dx$$

$$\Rightarrow d\left(\tan^{-1}\frac{x}{y} = a\,dx\right)$$

On integrating $\tan^{-1}\frac{x}{y} = ax + c$ where c is an arbitrary constant.

Method -2: If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous differential equation and $Mx + Ny \neq 0$ then $\frac{1}{Mx + Ny}$ is an integrating factor of $M\,dx + N\,dy = 0$.

Solved Problems:

1. Solve $x^2ydx - \left(x^3 + y^3\right)dy = 0$

Sol : Given equation is $x^2ydx - \left(x^3 + y^3\right)dy = 0$ ----------------(1)

Where $M = x^2y$ & $N = \left(-x^3 - y^3\right)$

Consider $\frac{\partial M}{\partial y} = x^2$ & $\frac{\partial N}{\partial x} = -3x^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Equation is not exact .

But given equation (1) is homogeneous differential equation then

$Mx + Ny = x\left(x^2y\right) - y\left(x^3 + y^3\right) = -y^4 \neq 0$.

$$\text{I.F} = \frac{1}{Mx + Ny} = -\frac{1}{y^4}$$

Multiplying equation (1) by $-\frac{1}{y^4}$

$$\Rightarrow \frac{x^2y}{-y^4}dx - \frac{x^3 + y^3}{-y^4}dy = 0 \text{---------------------- (2)}$$

$$\Rightarrow -\frac{x^2}{y^3}dx - \frac{x^3 + y^3}{-y^4}dy = 0$$

This is of the form $M_1dx + N_1dy = 0$

For $M_1 = \frac{-x^2}{y^3}$ & $N_1 = \frac{x^3+y^3}{y^4}$

$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{3x^2}{y^4}$ & $\frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4}$

$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ equation (2) is an exact D.E.

General solution is $\int M_1 dx + \int N_1 dy = c$

(y constant) (terms free from x in N_1)

$$\Rightarrow \int \frac{-x^2}{y^3} dx + \int \frac{1}{y} dy = c.$$

$$\Rightarrow \frac{-x^3}{3y^3} + \log|y| = c$$

2. Solve $y\left(y^2-2x^2\right)dx + x\left(2y^2-x^2\right)dy = 0$

Sol : Given equation is $y\left(y^2-2x^2\right)dx + x\left(2y^2-x^2\right)dy = 0$ -----------(1)

It is the form Mdx + Ndy = 0

Where $M = y\left(y^2-2x^2\right), N = x\left(2y^2-x^2\right)$

Consider $\frac{\partial M}{\partial y} = 3y^2 - 2x^2$ & $\frac{\partial N}{\partial x} = 2y^2 - 3x^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ equation is not exact .

Since equation (1) is Homogeneous differential equation then

Consider $Mx + Ny = x\left[y\left(y^2-2x^2\right)\right] + y\left[x\left(2y^2-x^2\right)\right]$

$= 3xy\left(y^2-x^2\right) \neq 0$.

$$\Rightarrow \text{I.F.} = \frac{1}{3xy\left(y^2-x^2\right)}$$

Multiplying equation (1) by $\frac{1}{3xy\left(y^2-x^2\right)}$ we get

$$\Rightarrow \frac{y\left(y^2-2x^2\right)}{3xy\left(y^2-x^2\right)}dx + \frac{x\left(2y^2-x^2\right)}{3xy\left(y^2-x^2\right)}dy = 0$$

$$M_1 = \frac{y(y^2-2x^2)}{3xy(y^2-x^2)}; N_1 = \frac{x(2y^2-x^2)}{3xy(y^2-x^2)}$$

$$\frac{\partial M_1}{\partial y} = \frac{2xy}{3(y^2 - x^2)^2} = \frac{\partial N_1}{\partial x}$$

Now it is exact

$$\frac{\left(y^2 - x^2\right) - x^2}{3x\left(y^2 - x^2\right)}dx + \frac{y^2 + x\left(y^2 - x^2\right)}{3y\left(y^2 - x^2\right)}dy = 0$$

$$\frac{dx}{x} - \frac{xdx}{y^2 - x^2} + \frac{ydy}{y^2 - x^2} + \frac{dy}{y} = 0.$$

$$\left(\frac{dx}{x} + \frac{dy}{y}\right) + \frac{2ydy}{2\left(y^2 - x^2\right)} - \frac{2xdx}{2\left(y^2 - x^2\right)} = 0$$

On integrating we get

$$\log x + \log y + \frac{1}{2}\log\left(y^2 - x^2\right) - \frac{1}{2}\log\left(y^2 - x^2\right) = \log c \Rightarrow xy = c$$

Method- 3: If the equation $Mdx + Ndy = 0$ **is of the form** $y.f(xy)dx + x.g(xy)dy = 0$ **&** $Mx - Ny \neq 0$ **then** $\frac{1}{Mx - Ny}$ **is an integrating factor of** $\boldsymbol{Mdx + Ndy = 0}$**.**

Solved Problems:

1. Solve $\left(xy\sin xy + \cos xy\right)ydx + \left(xy\sin xy - \cos xy\right)xdy = 0$**.**

Sol : Given equation $\left(xy\sin xy + \cos xy\right)ydx + \left(xy\sin xy - \cos xy\right)xdy = 0$ -------(1).

Equation (1) is of the form $y.\ f\left(xy\right)dx + x.g\left(xy\right)dy = 0$.

Where $M = \left(xy\sin xy + \cos xy\right)y$

$N = \left(xy\sin xy - \cos xy\right)x$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Longrightarrow$ equation (1) is not exact

Now consider $Mx - Ny$

$Mx - Ny = 2xycosxy \neq 0$

Integrating factor $= \dfrac{1}{2xy\cos xy}$

So equation (1) x I.F gives

$$\Rightarrow \frac{(xy\sin xy)y}{2xy\cos xy}dx + \frac{(xy\sin xy - \cos xy)x}{2xy\cos xy}dy = 0$$

$$\Rightarrow \left(y\tan xy + \frac{1}{x}\right)dx + \left(y\tan xy - \frac{1}{y}\right)dy = 0$$

$$\Rightarrow M_1dx + N_1dy = 0$$

$$\frac{\partial M_1}{\partial y} = \tan xy + xy\sec^2 xy = \frac{\partial N_1}{\partial x}$$

Now the equation is exact.

$\therefore$ General solution is $\int M_1dx + \int N_1dy = c.$.

(y constant) (terms free from x in N_1)

$$\Rightarrow \int\left(y\tan xy + \frac{1}{x}\right)dx + \int \frac{-1}{y}dy = c.$$

$$\Rightarrow \frac{y.\log|\sec xy|}{y} + \log x + (-\log y) = \log c$$

$$\Rightarrow \log|\sec(xy)| + \log\frac{x}{y} = \log c.$$

$$\Rightarrow \frac{x}{y}.\sec xy = c.$$

2. Solve $(1+xy)ydx + (1-xy)xdy = 0$

Sol : Here $M = (1+xy)y : N = (1-xy)x$

$$\frac{\partial M}{\partial y} = 1 + 2xy; \frac{\partial N}{\partial x} = 1\text{-}2xy$$

Hence, the equation is not exact

Also $Mx - Ny = 2x^2y^2 \neq 0$

$$\text{I.F} = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2} \neq 0$$

Multiply the given equation by I.F, we get

$$\Rightarrow \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \frac{-1}{2y}dy = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{2x^2y^2} = \frac{\partial N_1}{\partial x}$$

$\Rightarrow$Equation is exact.

On integrating, we get

$$\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int\frac{-1}{2y}dy=c$$

$$\Rightarrow\frac{-1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c\,.$$

$$\Rightarrow\frac{-1}{xy}+\log\left(\frac{x}{y}\right)=c_1 \quad \text{where } c_1 = 2c.$$

Method 4: If there exists a continuous single variable function $f(x)$ such that $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$, then I.F. of $Mdx+Ndy=0$ is $e^{\int f(x)dx}$

Solved Problems:

1. Solve $\left(3xy-2ay^2\right)dx+\left(x^2-2axy\right)dy=0$

Sol : Given equation is $\left(3xy-2ay^2\right)dx+\left(x^2-2axy\right)dy=0$

This is of the form $Mdx + Ndy = 0$

$$\Rightarrow M=3xy-2ay^2 \quad \& \quad N=x^2-2axy$$

$$\frac{\partial M}{\partial y}=3x-4ay \quad \& \quad \frac{\partial N}{\partial x}=2x-2ay$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \quad \text{Equation is not exact.}$$

Now consider $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\dfrac{(3x-4ay)-(2x-2ay)}{x(x-2ay)}$

$$\Rightarrow\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=\frac{1}{x}=f(x).$$

$$\Rightarrow e^{\int\frac{1}{x}dx}=x \text{ is an Integrating factor of (1)}$$

Multiplying equation (1) with I.F we get

$$\Rightarrow\frac{\left(3xy-2ay^2\right)}{1}xdx+\frac{\left(x^2-2axy\right)}{1}xdy=0$$

$$\left(3x^2y-2ay^2x\right)dx+\left(x^3-2ax^2y\right)dy=0$$

It is the form $M_1dx + N_1dy = 0$

$M_1 = 3x^2y - 2ay^2x, \quad N_1 = x3 - 2ax^2y$

$\frac{\partial M_1}{\partial y} = 3x^2 - 4axy, \quad \frac{\partial N_1}{\partial x} = 3x^2 - 4axy$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\therefore$ Equation is exact

General sol $\int M_1dx + \int N_1dy = c.$

(y constant) (terms free from x in N_1)

$\Rightarrow \int \left(3x^2y - 2ay^2x\right)dx + \int 0dy = c$

$\Rightarrow x^3y - ax^2y^2 = c.$

2. Solve $ydx + xdy + \left(1 + x^2\right)dx + x^2 \sin ydy = 0$

Sol: Given equation is $\left(y + 1 + x^2\right)dx + \left(x^2 \sin y - x\right)dy = 0.$

$M = y + 1 + x^2 \quad \& \quad N = x^2 \sin y - x$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 2x \sin y - 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ the equation is not exact.

So consider $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\left(1 - 2x \sin y + 1\right)}{x^2 \sin y - x} = \frac{-2x \sin y + 2}{x^2 \sin y - x} = \frac{-2\left(x \sin y - 1\right)}{x\left(x \sin y - 1\right)} = \frac{-2}{x}$

$I.F = e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$

Equation (1) x I.F gives $\Rightarrow \frac{y + 1 + x^2}{x^2}dx + \frac{x^2 \sin y - x}{x^2}dy = 0$

It is the form of $M_1dx + N_1dy = 0.$

$\frac{\partial M_1}{\partial y} = \frac{1}{x^2} = \frac{\partial N_1}{\partial x}$

$\therefore$ Equation is exact

General solution is thus

$$\Rightarrow \int \left(\frac{y}{x^2} + \frac{1}{x^2} + 1\right)dx + \int \sin ydy = 0$$

$$\Rightarrow \frac{-y}{x} - \frac{1}{x} + x - \cos y = c.$$

$$\Rightarrow x^2 - y - 1 - x\cos y = cx.$$

3. Solve $2xy\,dy - (x^2+y^2+1)\,dx = 0$

Sol: Given equation is $2xy\,dy - (x^2+y^2+1)\,dx = 0$ .... (1)

This is of the form $M\,dx + N\,dy = 0$, where $N = 2xy$, $M = -x^2 - y^2 - 1$

We have $\frac{\partial M}{\partial y} = -2y$ and $\frac{\partial N}{\partial x} = 2y$, so that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ The given equation is not exact.

We have $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-2y - 2y) = \frac{-2}{x} = f(x)$

$\therefore I.F. = e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = \frac{1}{x^2}$

Multiplying (1) with $\frac{1}{x^2}$, we get $\frac{2y}{x}\,dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx$ .... (2)

This is of the form $M_1\,dx + N_1\,dy = 0$, where $M_1 = -1 - \frac{y^2}{x^2} - \frac{1}{x^2}$ and $N_1 = \frac{2y}{x}$

We have $\frac{\partial M1}{\partial y} = \frac{-2y}{x^2}$ and $\frac{\partial N1}{\partial x} = \frac{-2y}{x^2}$

Since $\frac{\partial M1}{\partial y} = \frac{\partial N1}{\partial x}$, $\therefore$ (2) is exact.

General solution is given by

$$\int M_1 dx + \int N_1 dy = c$$

(y constant) (terms free from x in N_1)

$\Rightarrow \int(-1 - \frac{y^2}{x^2} - \frac{1}{x^2})\,dx + \int 0\,dy = c$

(y constant)

$\Rightarrow -x + \frac{y^2}{x} + \frac{1}{x} = c$

This is the general solution o (2) and hence of (1)

Method -5: For the equation $Mdx + Ndy = 0$ if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$ (is a function of y alone) then $e^{\int -g(y)dy}$ is an integrating factor of $M\,dx + N\,dy = 0$.

Solved Problems :

1 . Solve $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\,dy = 0$

Sol : Given equation $(3x^2y^4+2xy)dx + (2x^3y^3-x^2)\,dy = 0$ -----------------(1).

Equation is of the form $Mdx + Ndy = 0$.

where $M = (3x^2y^4+2xy)$ and $N = (2x^3y^3-x^2)$

$$\frac{\partial M}{\partial y}=12x^2y^3+2x \;;\; \frac{\partial N}{\partial x}=6x^2y^3-2x$$

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\Rightarrow \text{equation(1) is not exact.}$$

So consider $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=g(y)=\dfrac{2}{y}=g(y)$

$$I.F=e^{\int -g(y)dy}=e^{-2\int\frac{1}{y}dy}=e^{-2\log y}=\frac{1}{y^2}.$$

Equation (1) x I.F $\left(\dfrac{3x^2y^4+2xy}{y^2}\right)dx+\left(\dfrac{2x^3y^3-x^2}{y^2}\right)dy=0$

$$\Rightarrow\left(3x^2y^2+\frac{2x}{y}\right)dx+\left(2x^3y-\frac{x^2}{y^2}\right)dy=0$$

It is the form $M_1dx+N_1dy=0$

$$\frac{\partial M_1}{\partial y}=6x^2y-2\frac{x}{y^2}=\frac{\partial N_1}{\partial x}$$

$\therefore$ Equation is exact

General sol. is $\int M_1dx+\int N_1dy=c$

(y constant) (terms free from x in N_1)

$$\Rightarrow\int\left(3x^2y^2+\frac{2x}{y}\right)dx+\int 0dy=c.$$

$$\Rightarrow\frac{3x^3y^2}{3}+\frac{2x^2}{2y}=c.$$

$$\Rightarrow x^3y^2+\frac{x^2}{y}=c.$$

2. Solve $(xy^3+y)\,dx+2(x^2y^2+x+y^4)\,dy=0$

Sol: Here $M=xy^3\;y$; $N=2\left(x^2y^2+x+y^4\right)$

$$\frac{\partial M}{\partial y}=3xy^2+1; \qquad \frac{\partial N}{\partial x}=4xy^2+2$$

We see equation is not exact.

$$\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$

Also $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=g(y)$

$$\frac{-xy^2-1}{y\left(xy^2+1\right)}=-\frac{1}{y}=g(y)$$

Thus $I.F=e^{\int -g(y)dy}=e^{\int \frac{1}{y}dy}=y.$

$$\frac{\partial M_1}{\partial y}=4xy^3+2y=\frac{\partial N_1}{\partial x} \quad \text{where } M_1=xy^4+y^2; N_1=2x^2y^3+2xy+2y^5$$

Gen Sol: $\int\left(xy^4+y^2\right)dx+\int\left(2y^5\right)dy=c$

$$\frac{x^2y^4}{2}+y^2x+\frac{2y^6}{6}=c.$$

3. Solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$

Sol: The given equation is not exact.

Also $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=g(y)=\left(\dfrac{\left(y^3-4\right)-\left(4y^3+2\right)}{y^4+2y}\right)=\dfrac{-3}{y}=g(y).$

$$I.F=e^{\int g(y)dy}=e^{-3\int \frac{1}{y}dy}=\frac{1}{y^3}$$

Here $M_1=y+\dfrac{2}{y^2}; N_1=x+2y-\dfrac{4x}{y^3}$

$$\frac{\partial M_1}{\partial y}=1-\frac{2}{y^3}=\frac{\partial N_1}{\partial x}$$

∴Equation is exact.

Gen sol is $\int\left(y+\dfrac{2}{y^2}\right)dx+\int 2ydy=c.$

$$\left(y+\frac{2}{y^2}\right)x+y^2=c.$$

Applications of Differential Equations Of First Order First Degree

ORTHOGONAL TRAJECTORIES (O.T)

Definition: A curve which cuts every member of a given family of curves at a right angle is an orthogonal trajectory of the given family.

Orthogonal trajectories in Cartesian co-ordinates:

Working rule: To find the family of O.T in Cartesian form .

Let f(x,y,c) =0 ……(1) be the given equation of family of curves in Cartesian form.

Step: (1) Differentiate with respect to ' x ' and obtain F(x, y, $\frac{dy}{dx}$) = 0 ----------(2) of the given family of curves.

(2) Replace $\frac{dy}{dx}$ by- $\frac{dx}{dy}$ in (2)

Then the Differential Equation of family of O.T is F(x, y,- $\frac{dx}{dy}$) =0 ---------(3).

(3) Solve equation (3) to get the equation of family of O.T's of equation (1)

PROBLEMS:

1 . Find the O.T's of family of semi-cubical parabolas $ay^2 = x^3$. where a is a parameters.

Sol : The given family of semi-cubical parabola is $ay^2 = x^3$. --------------- (1) $a = \frac{x^3}{y^2}$

Differentiating (1) with respect to x $\Rightarrow 2ay\frac{dy}{dx} = 3x^2$

Eliminating a $\Rightarrow 2\frac{x^3}{y^2}y\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3y}{2x}$ ---------- (2)

Replace $\frac{dy}{dx}$ $by - \frac{dx}{dy}$ in (2) $\Rightarrow$ $-\frac{dx}{dy} = \frac{3y}{2x}$ $\Rightarrow$ $2xdx = -3ydy$

$\int 2xdx = \int 3ydy + c$ $\Rightarrow$ $x^2 + \frac{3y^2}{2} = c.$

2. Find the O.T of the family of circles $x^2 + y^2 + 2gx + c = 0$, where g is the parameter.

Sol: Given $x^2 + y^2 + 2gx + c = 0$ --------- (1)

(1) is represents a system of co- axial circles with g as parameter

Differentiating with respect to 'x '=> $2x + 2y\frac{dy}{dx} + 2g = 0$-------------- (2)

Substituting equation from (2) in (1) , Eliminating g

$x^2 + y^2 - \left(2x + 2y\frac{dy}{dx}\right)x + c = 0$ -------- (3)

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (3)

$$x^2 + y^2 - \left(2x - 2y\frac{dx}{dy}\right)x + c = 0$$

$$\Rightarrow\ y^2 - x^2 + 2xy\frac{dx}{dy} + c = 0$$

$$\Rightarrow 2x\frac{dx}{dy} - \frac{x^2}{y} = \left(\frac{-C-y^2}{y}\right) \text{-------- (4)}$$

Put $x^2 = u \Rightarrow 2x\frac{dx}{dy} = \frac{du}{dy}$

(4) $\Rightarrow \frac{du}{dy} - \frac{u}{y} = \left(\frac{-C-y^2}{y}\right)$ ------ (5)

I.F $= e^{\int Pdy} = e^{\int \frac{-1}{y}dy} = e^{-log y} = \frac{1}{y}$

General solution of (5) is $u(I.F) = \int Q(I.F)dy + c$

$$x^2\left(\frac{1}{y}\right) = \int\left(\frac{-C-y^2}{y}\right)\frac{1}{y}dy + c$$

$$x^2\left(\frac{1}{y}\right) = \frac{2C}{y^3} - y + c.$$

$$x^2\left(\frac{1}{y}\right) = \int\left(\frac{-C-y^2}{y}\right)\frac{1}{y}dy + c$$

$$x^2\left(\frac{1}{y}\right) = \frac{2C}{y^3} - y + c.$$

3. Prove that the system of confocal and coaxial parabolas $y^2 = 4a(x+a)$ is self orthogonal.

Sol. The equation of the family of given parabolas is $y^2 = 4a(x+a)$ ----- (1)

Differentiating (1) with respect to x

$$2y\frac{dy}{dx} = 4a \ \Rightarrow\ y\frac{dy}{dx} = 2a \text{ ------- (2)}$$

Eliminating a between (1) and (2) , we have

$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right) = 2xyy_1 + y^2{y_1}^2$ ---------- (3) Where $y_1 = \frac{dy}{dx}$

Replace y_1 by $\frac{-1}{y_1}\left(-\frac{dx}{dy}\right)$ in (3)

$$y^2 = 2xy\left(\frac{-1}{y_1}\right) + y^2\left(\frac{1}{{y_1}^2}\right)$$

$$y^2 = 2xyy_1 + y^2{y_1}^2 \text{ ------ (4)}$$

(3) and (4) are same

$\therefore$ (1) is self-orthogonal.

4. Find the O.T of family of curves $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ where a is a parameter.

5.Prove that the system of $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self-orthogonal. where λ is a parameter.

6. Find the orthogonal trajectories of the family of parabolas $y = ax^2$.

Newton's Law Of Cooling

Statement: The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let $'\theta'$ be the temperature of the body at time 't' and θ_0 be the temperature of its surrounding medium (usually air). By the Newton's law of cooling, we have

$$\frac{d\theta}{dt} \alpha (\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k(\theta - \theta_0) \quad k \text{ is positive constant}$$

$$\Rightarrow \int \frac{d\theta}{(\theta - \theta_0)} = -k \int dt$$

$$\Rightarrow log(\theta - \theta_0) = -kt + c.$$

If initially $\theta = \theta_1$ is the temperature of the body at time t=0 then

$$c = \log(\theta_1 - \theta_0)$$

$$\Rightarrow log(\theta - \theta_0) = -kt + \log(\theta_1 - \theta_0)$$

$$\Rightarrow \log\left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right) = -kt.$$

$$\Rightarrow \left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right) = e^{-kt}$$

$$\theta = \theta_0 + (\theta_1 - \theta_0) = e^{-kt}$$

Which gives the temperature of the body at time't'.

Solved Problems

1. A pot of boiling water 100^0C is removed from the fire and allowed to cool at 30^0C room temperature. 2 minutes later, the temperature of the water in the pot is 90^0C .What will be the temperature of water after 5 minutess?

Sol : We have $\theta - 30 = ce^{-kt}$ …(1)

When $t = 0, \theta = 100$

from (1), we get $c = 70$

$\therefore \theta - 30^0 = 70e^{-kt}$ …(2)

When $t = 2, \theta = 90$

From (2), $90 - 30 = 70e^{-2k}$

$$\Rightarrow 60 = 70e^{-2k}$$

$$\Rightarrow -2k = \log\left(\frac{6}{7}\right)$$

$$= 0.1542$$

$\Rightarrow k = 0.0771$

When $t = 5, \theta - 30 = 70e^{-5k}$

$\Rightarrow \theta = 77.46^0$

2.A body is originally at 80^{0}C and cools down to 60^0 C in 20 min . If the temperature of the air is 40^0 C find the temperature of body after 40 min.

Sol : By Newton's law of cooling we have

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$ where θ_0 is the temperature of the air.

$$\Rightarrow \int \frac{d\theta}{(\theta - \theta_0)} = -k\int dt \Rightarrow \log(\theta - \theta_0) = -kt + \log c$$

Here $\theta_0 = 40^0C$

$$\Rightarrow \log(\theta - 40) = -kt + \log c$$

$$\Rightarrow \log\left(\frac{\theta - 40}{c}\right) = -kt$$

$$\Rightarrow \left(\frac{\theta - 40}{c}\right) = e^{-kt}$$

$\Rightarrow \theta = 40 + ce^{-kt}$ --------------(1)

When $t = 0, \theta = 80^0C$

$\Rightarrow 80 = 40 + c \Rightarrow c = 40$ ------------(2).

When $t = 20, \theta = 60^0C \Rightarrow 60 = 40 + ce^{-20k}$ ------------(3).

Solving (2) & (3) $\Rightarrow ce^{-20k} = 20$

$$\Rightarrow 40e^{-2k} = 20$$

$$\Rightarrow k = -20\log 2$$

When $t = 40^0C$ then equation (1) is $\theta = 40 + 40e^{-\left(\frac{1}{20}\log 2\right)40}$

$$= 40 + 40e^{-2\log 2}$$

$$= 40 + \left(40X\frac{1}{4}\right)$$

$$\Rightarrow \theta = 50^0 C$$

3. An object whose temperature is 75^{0}C cools in an atmosphere of constant temperature C, at the rate of kθ, being the excess temperature of the body over that of the temperature. If after 10min, the temperature of the object falls to 65^0 C , find its temperature after 20 min. Also find the time required to cool down to 55^{0}C.

Sol : We will take one minute as unit of time.

It is given that $\frac{d\theta}{dt} = -kt$

$$\Rightarrow \theta = ce^{-kt} \text{ ------------ } (1)$$

Initially when $t = 0 \Rightarrow \theta = 75^0 - 25^0 = 50^0$

$$\Rightarrow c = 50^0$$

Hence $c = 50 \Rightarrow \theta = 50e^{-kt}$ ----------------- (2)

When $t = 10$ min $\Rightarrow \theta = 65^0 - 25^0 = 40^0$

$$\Rightarrow 40 = 50\,e^{-10k}$$

$$\Rightarrow e^{-10k} = \frac{4}{5} \text{ --------------------- } (3)$$

The value of θ when t=20 $\Rightarrow \theta = ce^{-kt}$

$$\theta = 50e^{-k}$$

$$\theta = 50(e^{-10k})^2$$

$$\theta = 50\left(\frac{4}{5}\right)^2$$

When t=20 $\Rightarrow \theta = 32^0C$.

Hence the temperature after 20min $= 32^0C + 25^0C = 57^0C$

When the temperature of the object = 55^{0}C

$\theta = 55^0C - 25^0C = 30^0C$

Let t, be the corresponding time from equation (2)

$30 = 50e^{-kt}$ ----------- (4)

From equation (3) $e^{(-k)^{10}} = \frac{4}{5}$ i.e $e^{-k} = \left(\frac{4}{5}\right)^{\frac{1}{10}}$

From equation (4), we get $30 = 50\left(\frac{4}{5}\right)^{\frac{t_1}{10}} \Rightarrow \frac{t_1}{10}\log\frac{4}{5} = \log\frac{3}{5}$

$$\Rightarrow t_1 = 10\left[\frac{\log\left(\frac{3}{5}\right)}{\log\left(\frac{4}{5}\right)}\right] = 22.9\text{min}$$

4. A body kept in air with temperature 25^0C cools from 140^0C to 80^0C in 20 min. Find when the body cools down in 35^0C.

Sol : By Newton's law of cooling $\frac{d\theta}{dt} = -k(\theta - \theta_0) \Rightarrow \frac{d\theta}{\theta - \theta_0} = -kdt$

$\Rightarrow \log(\theta - \theta_0) = kt + c$. Here, $\theta_0 = 25^0C$

$\Rightarrow \log(\theta - 25) = kt + c$ -------------(1).

When t=0, $\theta = 140^0C$

$\Rightarrow log\ (115)\ = c$

$\Rightarrow c\ = log\ (115)$.

$\Rightarrow kt\ = -log(\theta - 25) + log\ 115$--------(2)

When $t = 20, \theta = 80^0C$

$\Rightarrow\ log\ (80 - 25) = -20k + log\ 115$

$\Rightarrow\ 20\ k\ = log\ (115)\ - log(55)$ ---------(3)

Divide equation (2) by (3), we get $\frac{kt}{20k} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$

$$\Rightarrow \frac{t}{20} = \frac{\log 115 - \log(\theta - 25)}{\log 115 - \log 55}$$

When $\theta = 35^0C$ $\Rightarrow \frac{t}{20} = \frac{\log 115 - \log(10)}{\log 115 - \log 55}$

$$\Rightarrow \frac{t}{20} = \frac{\log(11.5)}{\log\left(\frac{28}{11}\right)} = 3.31$$

$\Rightarrow$ temperature $= 20 \times 3.31 = 66.2$

The temp will be 35^0C after 66.2 min.

5.The temperature of the body drops from 100^0C to 75^0C in 10 min. When the surrounding air is at 20^0C temperature. What will be its temp after half an hour. When will the temperature be 25^0C .

Sol : $\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$\log(\theta - 20) = -kt + \log c$

when $t = 0, \theta = 100^0 \Rightarrow c = 80$

when $t = 10, \theta = 75^0 \Rightarrow e^{-10k} = \frac{11}{16}$.

when $t = 30min \Rightarrow \theta = 20 + 80\left(\frac{1331}{4096}\right) = 46^0C$

when $\theta = 25^0C \Rightarrow t = 10\frac{\log 5 - \log 80}{(\log 11 - \log 18)} = 74.86min$

6. If the air is maintained at 15^0C and the temperature of the body drops from 70^0C to 40^0C in 10 minutes. What will be its temperature after 30 minutes?

Sol : If θ be the temperature of the body at time t, then $\frac{d\theta}{dt} = -k(\theta - 15)$, where k is constant

Integrating, $\int \frac{d\theta}{\theta - 15} = -k\int dt + \log c$

i.e $\log(\theta - 15) = -kt + \log c$ i.e , $\theta - 15 = ce^{-kt}$ …(1)

When $t = 0, \theta = 70^0C$ and when $t = 10, \theta = 40^0C$

$\therefore 70 - 15 = ce^{-0} \Rightarrow c = 55$ and $40 - 15 = ce^{-10k}$

$\Rightarrow \frac{25}{55} = e^{-10k}$ or $e^{-10k} = \frac{5}{11}$ …(2)

Then (1) becomes $\theta - 15 = 55e^{-kt}$

When t=30 min, $\theta = 15 + 55e^{-30k}$

$\therefore \theta = 15 + 55(e^{-10k})^3 = 15 + 55\left(\frac{5}{11}\right)^3$ using (2)

$= 15 + \frac{625}{121} = \frac{2441}{121} = 20.16^0C$.

7. In a pot of boiling water 100^0C is removed from the fire and allowed to cool at 30^0C room temperature. Two minutes later, the temperature of the water in the pot is 90^0C. What will be the temperature of the water after 5 minutes?

Sol : We have $\theta - 30^0 C = ce^{-kt}$ …(1)

when $t = 0, \theta = 100 \Rightarrow c = 70$

$\therefore \theta - 30^0 C = 70e^{-kt}$ …(2)

when, $t = 2, \theta = 90^0$

from (2), $60 = 70e^{-2k}$

$$\Rightarrow -2k = \log\left(\frac{6}{7}\right) = -0.1542$$

$$\Rightarrow k = 0.0771$$

when $t = 5, \theta - 30^0 = 70e^{-5k}$

$\Rightarrow \theta = 77.46^0$

8. The temperature of a cup of coffee is $92^0 C$ when freshly poured, the room temperature being $24^0 C$. In one min it was cooled to $80^0 C$. How long a period must elapse, before the temperature of the cup becomes $65^0 C$.

Sol : By Newton's Law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0) \text{ ;k>0}$$

$\theta_0 = 24^0 C \Rightarrow \log(\theta - 24) = -kt + \log c$ --------------(1).

When $t = 0;\ \theta = 92 \Rightarrow c = 68$

When $t = 1;\ \theta = 80^0 C \Rightarrow e^{-k} = \frac{68}{56}$

$$\Rightarrow k = \log\frac{56}{68}.$$

When $\theta = 65^0 C, t = \frac{65x41}{68^2} = 0.576\text{min}$

Law Of Natural Growth Or Decay

Statement : Let $x(t)$ or x be the amount of a substance at time 't' and let the substance be getting converted chemically . A law of chemical conversion states that the rate of change of amount $x(t)$ of a chemically changed substance is proportional to the amount of the substance available at that time

$$\frac{dx}{dt} \alpha x$$

Note: a) In case of Natural growth we take

$$\frac{dx}{dt} = kx \quad (k > 0)$$

b) In case of Natural decay, we take $\frac{dx}{dt} = -kx\ (k>0)$

Where k is a constant of proportionality

Rate Of Decay Of Radio Active Materials

Statement: The disintegration at any instant is proportional to the amount of material present in it.

If u is the amount of the material at any time 't' , then $\frac{du}{dt} = -ku$, where k is any constant ($k > 0$). i.e Law of Natural Decay is applied.

Solved Problems

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What was the value of N after $1\frac{1}{2}$ hrs.

Sol : The differential equation to be solved is $\frac{dN}{dt} = kN$

$$\Rightarrow \frac{dN}{N} = kdt$$

$$\Rightarrow \int \frac{dN}{N} = \int kdt$$

$$\Rightarrow \log N = kt + \log c$$

$$\Rightarrow N = c e^{-kt} \text{ ------------ (1).}$$

When t= 0sec, N =100 $\Rightarrow$ 100 = c $\Rightarrow$ c =100

When t =3600sec, N =332 $\Rightarrow$ 332 =100 e^{3600k}

$$\Rightarrow e^{3600k} = \frac{332}{100}$$

Now when $t = \frac{3}{2}$ hours = 5400 sec then N=100 e^{5400k}

$$\Rightarrow N = 100\left[e^{3600k}\right]^{\frac{3}{2}}$$

$$\Rightarrow N = 100\left[\frac{332}{100}\right]^{\frac{3}{2}} = 605.$$

$$\Rightarrow N = 605.$$

2.A bacterial culture, growing exponentially, increases from 100 to 400 gms in 10 hrs. How much was present after 3 hrs, from the initial instant?

Sol : Let N be the weight of bacteria culture at any $t > 0$.

Then $N = ce^{-kt}$ …(1)

By data, when $t = 0, N = 100g$

$\therefore 100 = c$

Substituting in (1), we get $N = 100e^{-kt}$ …(2)

When $t = 100, N = 400g$

from (2), $400 = 100e^{-10k}$

$\Rightarrow 4 = e^{-10k}$

$\Rightarrow -10k = \log 4$

$\Rightarrow k = -\frac{1}{10}\log 2^2 = -\frac{1}{5}\log(2)$ …(3)

When $t = 3, N = 100e^{-3k}$

$$= 100e^{-3\left(-\frac{1}{5}\log 2\right)} = 100e^{(\log 2)^{\frac{3}{5}}}$$

$$= 100X(2)^{\frac{3}{5}} = 100X8^{\frac{1}{5}} = 100X1.414$$

$$= 141.4gms$$

3.If a radioactive Carbon-14 has a half life of 5750 years, what will remain of one gram after 3000years?

Sol : Let mass of radioactive Carbon-14 at any time be denoted by $x(t)$.

Then it is known that $\frac{dx}{dt} = -kt$ where k is a constant

$\Rightarrow x = Ae^{-kt}$ where A is also a constant.

It is known that at t=0, we have 1gm of Carbon-14

$\therefore 1 = Ae^0 \Rightarrow A = 1$

$\therefore x = e^{-kt}$

However when t=5750 years, we have 1/2gm of Carbon-14.

$\therefore \frac{1}{2} = e^{-k(5750)} \Rightarrow k = \frac{1}{5750}\log 2$

Suppose t=3000years, we have to find x.

$$\therefore x = e^{-kt} = e^{-3000k} = e^{-\frac{3000}{5750}\log 2}$$

$$\Rightarrow x = (2)^{-\frac{3000}{5750}} \text{gms}$$

4. If 30% of a radioactive substance disappears in 10 days, how long will it take for 90% of it to disappear?

Sol : The differential equation of the diffusing radioactive material is,

$$\frac{dm}{dt} = -km \qquad \dots (1)$$

Separating the variables and integrating, we get

$$m = ce^{-kt} \qquad \dots (2)$$

When $t = 0$, let $m = m_1$

$$\Rightarrow m_1 = c \qquad \dots (3)$$

By data, when $t = 10, m = \frac{70m}{100}$

$$\Rightarrow \frac{70m_1}{100} = ce^{-10k} = m_1e^{-10k}$$

$$\Rightarrow e^{-10k} = \frac{7}{10} \Rightarrow k = -\frac{1}{10}\log\left(\frac{7}{10}\right)$$

$$\therefore k = \frac{1}{10}\log\left(\frac{10}{7}\right) \qquad \dots (4)$$

Required time at t is

$$\frac{10m_1}{100} = ce^{-kt} = m_1e^{-kt} \Rightarrow \frac{1}{10} = e^{-kt}$$

$$\Rightarrow t = \frac{1}{k}\log(10)$$

$$= \frac{10\log(10)}{\log 10 - \log 7} = 64.5 \text{ days.}$$

UNIT – V

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Introduction :

Differential Equations are extremely helpful to solve complex mathematical problems in almost every domain of Engineering, Science and Mathematics. Engineers will be integrating and differentiating hundreds of equations throughout their career, because these equations have a hidden answer to a really complex problem. Mathematicians and Researchers like Laplace, Fourier, Hilbert etc., have developed such equations to make our life easier. Various Transforms from Time Domain to Frequency Domain or vice versa in Engineering is only possible because of Differential Equations. In real life situations, people use such equations for solving complex fluid dynamics problems, and finding the right balance of weights and measures to build stuff like a Cantilever Truss. Other applications include free vibration analysis, Simple mass-spring system, Damped mass-spring system, forced vibration analysis, Resonant vibration analysis, simple harmonic motion, simple pendulum, pressure Change with altitude, velocity profile in fluid flow, vibration of springs, Discharge of a capacitor, Newton's second law of motion and many more.

Definition: An equation of the form $\frac{d^n y}{dx^n}+P_1(x)\frac{d^{n-1}y}{dx^{n-1}}+P_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_n(x)y=Q(x)$ where $P_1(x),P_2(x),P_3(x)\ldots P_n(x)$ and $Q(x)$ (functions of x) are continuous is called a linear differential equation of order n.

Linear Differential Equations With Constant Coefficients

Def: An equation of the form $\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_n y=Q(x)$ where $P_1,P_2,P_3\ldots P_n$, are real constants and $Q(x)$ is a continuous function of x is called an linear differential equation of order 'n' with constant coefficients.

Note:

1. Operator $D=\frac{d}{dx}$; $D^2=\frac{d^2}{dx^2}$; … … … … … … … … $D^n=\frac{d^n}{dx^n}$

$$D\,y=\frac{dy}{dx};\ D^2y=\frac{d^2y}{dx^2};\ldots\ldots\ldots\ldots\ldots\ldots\ D^n\,y=\frac{d^n y}{dx^n}$$

2. Operator $\frac{1}{D}Q = \int Q dx$ i e $D^{-1}Q$ is called the integral of Q.

To find the general solution of $f(D).y = 0$:

Here $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n$ is a polynomial in D.

Now consider the auxiliary equation: $f(m) = 0$

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n = 0$

where $P_1, P_2, P_3 \ldots P_n$ are real constants.

Let the roots of $f(m) = 0$ be $m_1, m_2, m_3 \ldots m_n$.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E $f(m) = 0$	Complementary function(C.F)
1.	$m_1, m_2, \ldots m_n$ are real and distinct.	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}$
2.	$m_1, m_2, \ldots m_n$ and two roots are equal i.e., m_1, m_2 are equal and real (i.e repeated twice) &the rest are real and different.	$y_c = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots c_n e^{m_n x}$
3.	$m_1, m_2, \ldots m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real (i.e repeated thrice) &the rest are real and different.	$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots c_n e^{m_n x}$
4.	Two roots of A.E are complex say $\propto + i\beta$, $\alpha - i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + \ldots c_n e^{m_n x}$

5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\right] + c_5 e^{m_5 x} + \ldots c_n e^{m_n x}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2 x + c_3 x^2)\cos\beta x + (c_4 + c_5 x + c_6 x^2)\sin\beta x\right]$ $+ c_7 e^{m_7 x} + \ldots c_n e^{m_n x}$
7.	If roots of A.E. irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$

Solved Problems

1. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol : Given equation is of the form $f(D).y = 0$

Where $f(D) = (D^3 - 3D + 2)y = 0$

Now consider the auxiliary equation $f(m) = 0$

$f(m) = (m^3 - 3m + 2)y = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1, 1, -2$

Since m_1 and m_2 are equal and m_3 is -2

$y_c = (c_1 + c_2 x)e^x + c_3 e^{-2x}$

2. Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

Sol : Given $f(D) = (D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$ …(1)

Auxiliary equation is $f(m) = 0$

$\Rightarrow m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$ …(2)

By inspection $m + 1$ is its factor.

$(m+1)(m^3 - 3m^2 + 4) = 0$ …(3)

By inspection m+1 is factor of $(m^3 - 3m^2 + 4)$.

$\therefore$ (3) is $(m+1)(m+1)(m^2 - 4m + 4) = 0$

$\Rightarrow (m+1)^2(m-2)^2 = 0$

$\Rightarrow m = -1, -1, 2, 2$

Hence general solution of (1) is

$y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$

3. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Sol : Given $f(D) = (D^4 + 8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 + 8m^2 + 16) = 0$

$(m^2 + 4)^2 = 0$

$(m + 2i)^2\,(m + 2i)^2 = 0$

$m = 2i, 2i, -2i, -2i$

Here roots are complex and repeated

Hence general solution is

$y_c = [(c_1 + c_2 x)cos2x + (c3 + c4x)sin2x)]$

4. **Solve $y'' + 6y' + 9y = 0$; $y(0) = -4$, $y1(0) = 14$**

Sol : Given equation is $y'' + 6y' + 9y = 0$

Auxiliary equation $f(D)y = 0 \Rightarrow D^2 + 6D + 9)\, y = 0$

A.equation $f(m) = 0 \Rightarrow (m^2 + 6m + 9) = 0$

$\Rightarrow m = -3, -3$

$yc = (c_1 + c_2 x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to $x \Rightarrow y' = (c_1 + c_2 x)(-3e^{-3x}) + c2(e^{-3x})$

Given $y'(0) = 14 \Rightarrow c_1 = -4\ \&\ c_2 = 2$

Hence we get $y = (-4 + 2x)\,(e^{-3x})$

5. Solve $4y''' + 4y'' + y' = 0$

Sol : Given equation is $4y''' + 4y'' + y' = 0$

That is $(4D^3 + 4D^2 + D)y = 0$

Auxiliary equation $f(m) = 0$

$4m^3 + 4m^2 + m = 0$

$m(4m^2 + 4m + 1) = 0 \Rightarrow m(2m+1)^2 = 0$

$m = 0, -1/2, -1/2$

$y = c_1 + (c_2 + c_3 x)\, e^{-x/2}$

6. Solve $(D^2 - 3D + 4)\, y = 0$

Sol : Given equation $(D^2 - 3D + 4)\, y = 0$

A.E. is $f(m) = 0$

$$m^2 - 3m + 4 = 0$$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$$

$$\alpha \pm i\beta = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}}\left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right)$$

To Find General solution of $f(D)\, y = Q(x)$

It is given by $y = y_c + y_p$

i.e. $y = C.F + P.I$

Where the P.I consists of no arbitrary constants and P.I of $f(D)\, y = Q(x)$

Is evaluated as

$$P.I = \frac{1}{f(D)} Q(x)$$

Depending on the type of function of $Q(x)$, P.I is evaluated .

1. Find $\frac{1}{D}(x^2)$

Sol : $\frac{1}{D}(x^2) = \int x^2 dx = \frac{x^3}{3}$

2. **Find Particular value of** $\frac{1}{D+1}(x)$

Sol : $\frac{1}{D+1}(x) = e^{-x}\int xe^x dx$ (By definition $\frac{1}{D+\alpha}Q = e^{\alpha x}\int Q\, e^{-\alpha x} dx$

$= e^{-x}(xe^x - e^x)$

$= x - 1$

General methods of finding Particular integral :

P.I of $f(D)y = Q(x)$, when $\frac{1}{f(D)}$ is expressed as partial fractions**.**

1. Solve $(D^2 + a^2)y = \sec ax$

Sol : Given equation is …(1)

Let $f(D) = D^2 + a^2$

The AE is $f(m) = 0$ i.e $m^2 + a^2 = 0$ …(2)

The roots are $m = -ai, -ai$

$y_c = c_1\cos ax + c_2\sin ax$

$$y_p = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{2ai}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right]\sec ax \quad \ldots(3)$$

$$\frac{1}{D - ai}\sec ax = e^{iax}\int \sec ax\,dx = e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}dx$$

$$= e^{iax}\int(1 - i\tan ax)dx = e^{iax}\left[x + \frac{i}{a}\log\cos ax\right] \ldots(4)$$

Similarly we get $\frac{1}{D + ai}\sec ax = e^{-iax}\left[x - \frac{i}{a}\log\cos ax\right]$ …(5)

From (3),(4) and (5), we get

$$y_p = \frac{1}{2ai}\left[e^{iax}\left\{x + \frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}\right]$$

$$= \frac{x(e^{iax} - e^{-iax})}{2ai} + \frac{1}{a^2}(\log\cos ax)\frac{(e^{iax} + e^{-iax})}{2}$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

$\therefore$ The general solution of (1) is

$$y = y_c + y_p = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

Rules For Finding P.I In Some Special Cases

Type 1: P.I of $f\ (D)y = Q(x)$ where $(x)\ = e^{ax}$, where 'a' is constant.

$$\text{Case1.P.I} = \frac{1}{f(D)}.Q(x) = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$$

When $f(a)\ \neq\ 0$

i.e In $f(D)$, put $D = a$ and Particular integral will be calculated.

Case 2: If $f(a) = 0$ then the above method fails. Then if $f(D) = (D-a)^k\,\phi(D)$ (i.e 'a' is repeated root k times).

$$\text{Then } P.I\ = \frac{1}{\phi(a)}e^{ax}.\frac{1}{k!}x^k \text{ provided } \phi(a) \neq 0$$

Type 2: P.I of $f(D)y = Q(x)$ where $Q(x)\ = sinax$ or $Q(x)\ = cosax$ where 'a' is constant then $P.I = \frac{1}{f(D)}Q(x)$.

Working Rule :

Case 1: In $f(D)$ put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then $P.I\ = \frac{\sin ax}{f(-a)^2}$

Case 2: If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $\phi(D^2)$ and hence it is a factor of $f(D)$.

Then let $f(D) = (D^2 + a^2)\phi(D^2)$.

$$\text{Then } \frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2+a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\sin ax}{D^2+a^2} = \frac{1}{\phi\left(-a^2\right)}\frac{-x\cos ax}{2a}$$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2+a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\cos ax}{D^2+a^2} = \frac{1}{\phi\left(-a^2\right)}\frac{x\sin ax}{2a}$$

Type 3: P.I for $f(D)y = Q(x)$ where $Q(x) = x^k$ where k is a positive integer, $f(D)$ can be expressed as $f(D) = [1 \pm \phi(D)]$

$$\text{Express } \frac{1}{f(D)} = \frac{1}{[1\pm\phi(D)]} = [1\pm\phi(D)]^{-1}$$

$$\text{Hence } P.I\ = \frac{1}{[1\pm\phi(D)]}Q(x)$$

$$= [1\pm\phi(D)]^{-1}x^k$$

Type 4: P.I of $f(D)y = Q(x)$ when $Q(x) = e^{ax}\,V$ where 'a' is a constant and V is function of x. where $V\ = sinax$ or $cosax$ or x^k

Then $P.I = \frac{1}{f(D)}Q(x)$

$= \frac{1}{f(D)}e^{ax}V$

$= e^{ax}\left[\frac{1}{f(D+a)}V\right]$ & $\frac{1}{f(D+a)}V$ is evaluated depending on V.

Type 5: P.I of $f(D)y = Q(x)$ when $Q(x) = xV$ where V is a function of x.

Then $P.I = \frac{1}{f(D)}Q(x)$

$= \frac{1}{f(D)}V$

$= \left[x - \frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V$

Type 6: P.I. of $f(D)y = Q(x)$ where $Q(x) = x^m v$ where v is a function of x.

When P.I. $= \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)}x^m v$, where v = cosax or sinax

i. P.I. $= \frac{1}{f(D)}x^m \sin ax = \text{I.P.of } \frac{1}{f(D)}x^m e^{iax}$

ii. P.I. $= \frac{1}{f(D)}x^m \cos ax = \text{R.P.of } \frac{1}{f(D)}x^m e^{iax}$

<u>Formulae</u>

1. $\frac{1}{1-D} = (1-D)^{-1} = 1 + D + D^2 + D^3 + --------------$
2. $\frac{1}{1+D} = (1+D)^{-1} = 1 - D + D^2 - D^3 + -----------$
3. $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + ----------$
4. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + ---------$
5. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + -----$
6. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + ---------$

Solved Problems

1. Solve $(4D^2 - 4D + 1)y = 100$

Sol : A.E is $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{-1}{2}$

$$C.F = (c_1 + c_2 x)e^{\frac{x}{2}}$$

Now $P.I = \dfrac{100}{4D^2 - 4D + 1} = \dfrac{100e^{0x}}{(2D-1)^2} = \dfrac{100}{(0-1)^2} = 100$ { since $100e^{0x} = 100$ }

Hence the general solution is $y = C.F + P.F = (c_1 + c_2 x)e^{\frac{x}{2}} + 100$

2. Solve the differential equation $(D^2 + 4)y = \sinh 2x + 7$.

Sol : Auxillary equation is $m^2 + 4 = 0$

$$\Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$$

$\therefore$ C.F is $y_c = c_1\cos 2x + c_2\sin 2x$ … (1)

To find P.I :

$$y_p = \frac{1}{D^2+4}(\sinh 2x + 7)$$

$$= \frac{1}{D^2+4}\left(\frac{e^{2x}+e^{-2x}}{2} + 7e^0\right)$$

$$= \frac{1}{2}\cdot\frac{e^{2x}}{D^2+4} + \frac{1}{2}\frac{e^{-2x}}{D^2+4} + 7\frac{e^0}{(D^2+4)}$$

$$= \frac{e^{2x}}{2(4+4)} + \frac{e^{-2x}}{2(4+4)} + \frac{7}{(0+4)}$$

$$= \frac{e^{2x}+e^{-2x}}{16} + \frac{7}{4} = \frac{1}{8}\sinh 2x + \frac{7}{4} \qquad \ldots(2)$$

$$y = y_c + y_p$$

$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{8}\sinh 2x + \frac{7}{4}$$

2. Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol : The given equation is

$(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$ …(1)

This is of the form $f(D)y = e^{-2x} + 2\sinh x$

A.E is $f(m) = 0 \Rightarrow (m+2)(m-1)^2 = 0 \therefore m = -2,1,1$

The roots are real and one root is repeated twice.

$\therefore$ C.F is $y_c = c_1e^{-2x} + (c_2 + c_3x)e^x$.

$$P.I=\frac{e^{-2x}+2\sinh x}{(D+2)(D-1)^2}=\frac{e^{-2x}+e^{x}-e^{-x}}{(D+2)(D-1)^2}=y_{p_1}+y_{p_2}+y_{p_3}$$

Now $y_{p_1}=\dfrac{e^{-2x}}{(D+2)(D-1)^2}$

Hence $f(-2)=0$. Let $f(D)=(D-1)^2$. Then $\phi(2)\neq 0$ and $m=1$

$$\therefore y_{p_1}=\frac{e^{-2x}x}{9}=\frac{xe^{-2x}}{9}$$

and $y_{p_2}=\dfrac{e^{x}}{(D+2)(D-1)^2}$. Here f(1)=0

$$=\frac{e^{x}x^2}{(3)2!}=\frac{x^2e^{x}}{6}$$

and $y_{p_3}=\dfrac{e^{-x}}{(D+2)(D-1)^2}$

Putting $D=-1$, we get $y_{p_3}=\dfrac{e^{-x}}{(1)(-2)^2}=\dfrac{e^{-x}}{4}$

$\therefore$ The general solution is $y=y_c+y_{p_1}+y_{p_2}+y_{p_3}$

i.e $y=c_1e^{-2x}+(c_c+c_3x)e^{x}+\dfrac{xe^{-2x}}{9}+\dfrac{x^2e^{x}}{6}-\dfrac{e^{-x}}{4}$

3. Solve the differential equation $(D^2+D+1)y=\sin 2x$.

Sol : A.E is $m^2+m+1=0$

$$\Rightarrow m=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}i}{2}$$

$$\therefore y_c=e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2}+c_2\sin\frac{x\sqrt{3}}{2}\right) \quad \dots (1)$$

To find P.I :

$$y_p=\frac{\sin 2x}{D^2+D+1}=\frac{\sin 2x}{-4+D+1}$$
$$=\frac{\sin 2x}{D-3}=\frac{(D+3)\sin 2x}{D^2-9}=\frac{(D+3)\sin 2x}{-4-9}$$
$$=\frac{D\sin 2x+3\sin 2x}{-13}=\frac{2\cos 2x+3\sin 2x}{-13}$$

$$\therefore y=y_c+y_p=e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2}+c_2\sin\frac{x\sqrt{3}}{2}\right)-\frac{1}{13}(2\cos 2x+3\sin 2x)$$

4. Solve $(D^2-4)y = 2\cos^2 x$

Sol : Given equation is $(D^2-4)y = 2\cos^2 x$ …(1)

Let $f(D) = D^2 - 4$ A.E is $f(m) = 0$ i.e $m^2 - 4 = 0$

The roots are $m = 2, -2$. The roots are real and different.

$\therefore$ C.F $= y_c = c_1e^{2x} + c_2e^{-2x}$

$$P.I = y_p = \frac{1}{D^2-4}(2\cos^2 x) = \frac{1}{D^2-4}(1+\cos 2x)$$

$$= \frac{e^{0x}}{D^2-4} + \frac{\cos 2x}{D^2-4} = P.I_1 + P.I_2$$

$$P.I_1 = y_{p_1} = \frac{e^{0x}}{D^2-4} \text{ [Put D=0] } = \frac{e^{0x}}{-4} = -\frac{1}{4}$$

$$P.I_2 = y_{p_2} = \frac{\cos 2x}{D^2-4} = \frac{\cos 2x}{-8} \text{ [Put } D^2 = -2^2 = -4\text{]}$$

$\therefore$ The general solution of (1) is $y = y_c + y_{p_1} + y_{p_2}$

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{\cos 2x}{8}$$

5. Solve $(D^2+1)y = \sin x \sin 2x$

Sol : Given D.E is $(D^2+1)y = \sin x\sin 2x$

A.E is $m^2+1=0 \Rightarrow m = \pm i$

The roots are complex conjugate numbers.

C.F is $y_c = c_1\cos x + c_2\sin x$

w.k.t 2sinAsinB=cos(A-B)–cos(A+B)

$$P.I = \frac{\sin x\sin 2x}{(D^2+1)} = \frac{1}{2}\frac{\cos x - \cos 3x}{(D^2+1)} = P.I_1 + P.I_2$$

Now $P.I_1 = \dfrac{1}{2}\dfrac{\cos x}{D^2+1}$

Put $D^2 = -1$ we get $D^2+1 = 0$

$$\therefore P.I_1 = \frac{1}{2}\frac{x\sin x}{2} = \frac{x\sin x}{4} \quad \left[\because \text{Case of failure} : \frac{\cos ax}{D^2+a} = \frac{x}{2a}\sin ax\right]$$

and $P.I_2 = -\dfrac{1}{2}\dfrac{\cos 3x}{D^2+1}$

Put $D^2 = -9$, we get

$$P.I_2 = -\frac{1}{2}\frac{\cos 3x}{-9+1} = \frac{\cos 3x}{16}$$

General solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1\cos x + c_2\sin x + \frac{x\sin x}{4} + \frac{\cos 3x}{16}$$

6. Solve the differential equation $(D^3 - 3D^2 - 10D + 24)y = x + 3$.

Sol : The given D.E is $(D^3 - 3D^2 - 10D + 24)y = x + 3$

A.E is $m^3 - 3m^2 - 10m + 24 = 0$

$\Rightarrow$m=2 is a root.

The other two roots are given by $m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow$m=2 (or) m = -1

One root is real and repeated, other root is real.

C.F is $y_c = e^{2x}(c_1 + c_2x) + c_3e^{-x}$

$$y_p = \frac{x+3}{(D^3 - 3D^2 - 10D + 24)} = \frac{1}{24}\frac{x^3+3}{1+\left(\frac{D^3 - 3D^2 - 10D}{24}\right)}$$

$$= \frac{1}{24}\left[\frac{1 + D^3 - 3D^2 - 10D}{24}\right]^{-1}(x+3)$$

$$= \frac{1}{24}\left[1 - \left(\frac{D^3 - 3D^2 - 10D}{24}\right)\right](x+3)$$

$$= \frac{1}{24}\left[x + 3 + \frac{10}{24}\right] = \frac{24x + 82}{576}$$

General solution is $y = y_c + y_p$

$$\Rightarrow y = e^{2x}(c_1 + c_2x) + c_3e^{-x} + \frac{24x + 82}{576}$$

7. Solve the differential equation $(D^2 - 4D + 4)y = e^{2x} + x^2 + \sin 3x$.

Sol : The A.E is $(m^2 - 4m + 4) = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore y_c = (c_1 + c_2x)e^{2x}$ … (1)

To find y_p : $y_p = \frac{1}{D^2 - 4D + 4}(e^{2x} + x^2 + \sin 3x)$

$$= \frac{e^{2x}}{(D-2)^2} + \frac{x^2}{(D-2)^2} + \frac{\sin 3x}{D^2 - 4D + 4}$$

$$= \frac{x^2}{2!}e^{2x} + \frac{x^2}{4\left(1 - \frac{D}{2}\right)^2} + \frac{\sin 3x}{-9 - 4D + 4}$$

$$= \frac{x^2}{2}e^{2x} + \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2 - \frac{(4D-5)\sin 3x}{(5+4D)}$$

$$= \frac{x^2}{2}e^{2x} + \frac{1}{4}\left(1 + \frac{2D}{2} + \frac{3D^2}{4}\right)x^2 - \frac{(4D-5)\sin 3x}{16D^2 - 25}$$

$$= \frac{x^2}{2}e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} - \frac{(12\cos 3x - 5\sin 3x)}{-144 - 25}$$

$$= \frac{x^2}{2}e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} + \frac{(12\cos 3x - 5\sin 3x)}{169} \qquad \ldots(2)$$

$$y = y_c + y_p = (c_1 + c_2 x)e^{2x} + = \frac{x^2}{2}e^{2x} + \frac{x^2}{4} + \frac{x}{2} + \frac{3}{8} + \frac{(12\cos 3x - 5\sin 3x)}{169}$$

8.Solve the differential equation $(D^2 + 4)y = x\sin x$.

Sol : Auxiliary equation is $m^2 + 4 = 0 \Rightarrow m^2 = (\mp 2i)^2$

$\therefore m = \pm 2i$. The roots are complex and conjugate.

Hence Complementary Function, $y_c = c_1\cos 2x + c_2\sin 2x$

Particular integral, $y_p = \frac{1}{D^2+4} x\sin x$

$= \text{I.P of } \frac{1}{D^2+4} xe^{ix}$

$= \text{I.P of} e^{ix} \frac{1}{(D+i)^2+4} x = \text{I.P of } e^{ix} \frac{1}{D^2+2Di+3} x$

$= \text{I.P of } \frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}\right)^{-1} x$

$= \text{I.P of } \frac{e^{ix}}{3}\left(1-\frac{D^2+2Di}{3}+...\right)x$

$= \text{I.P of } \frac{e^{ix}}{3}\left(1-\frac{2}{3}Di\right)x\left[D^2(x)=0, \text{etc}\right]$

$= \text{I.P of } \frac{1}{3}(\cos x + i\sin x)\left(x - i\frac{2}{3}\right)$

$= \frac{1}{3}\left(-\frac{2}{3}\cos x + x\sin x\right)$

Hence the general solution is

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\left(x\sin x - \frac{2}{3}\cos x\right)$$

Where c_1 and c_2 are constants.

Other Method (using type 5): $y_p = \frac{1}{D^2+4} x\sin x$

$$= \left\{x - \frac{2D}{D^2+4}\right\}\frac{\sin x}{D^2+4}$$

$$= \frac{x\sin x}{3} - \frac{2(D\sin x)}{3(D^2+4)}$$

$$= \frac{x\sin x}{3} - \frac{2\cos x}{9}$$

Hence the general solution is

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\left(x\sin x - \frac{2}{3}\cos x\right)$$

9.Solve the Differential equation$(D^2 + 5D + 6)y = e^x$

Sol : Given equation is $(D^2 + 5D + 6)y = e^x$

Here $Q(x) = e^x$

Auxiliary equation is $f(m) = m^2 + 5m + 6 = 0$

$m^2 + 3m + 2m + 6 = 0$

$m(m+3)+2(m+3)=0$

$m=-2 \text{ or } m=-3$

The roots are real and distinct

$C.F = y_c = c_1e^{-2x}+c_2e^{-3x}$

Particular Integral$= y_p = \frac{1}{f(D)}Q(x)$

$$=\frac{1}{D^2+5D+6}e^x=\frac{1}{(D+2)(D+3)}e^x$$

Put $D = 1$ in $f(D)$

$P.I = \frac{1}{(3)(4)}e^x$

Particular Integral $= y_p = \frac{1}{12}e^x$

General solution is $y = y_c + y_p$

$$y = c_1e^{-2x}+c_2e^{-3x}+\frac{e^x}{12}$$

10.Solve $y''-4y'+3y=4e^{3x}, y(0)=-1, y'(0)=3$

Sol : Given equation is $y''-4y'+3y=4e^{3x}$

i.e $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=4e^{3x}$ it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x) = 4e^{3x}$; $f(D) = D^2-4D+3$

Auxiliary equation is $f(m)=m^2-4m+3 = 0$

$m^2-3m - m+3=0$

$$m(m-3) -1(m-3)=0 \Rightarrow m=3 \text{ or } 1$$

The roots are real and distinct.

$$C.F = y_c = c_1e^{3x}+c_2e^x$$

P.I$= y_p = \frac{1}{f(D)}Q(x)$

$$=\frac{1}{D^2-4D+3}4e^{3x}$$

$$=\frac{1}{(D-1)(D-3)}4e^{3x}$$

Put $D = 3$

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2}\frac{e^{3x}}{(D-3)} = 2\frac{x'}{1!}e^{3x} = 2xe^{3x}$$

General solution is $y = y_c + y_p$

$$y = c_1e^{3x} + c_2e^{x} + 2xe^{3x} \qquad \ldots(3)$$

Equation (3) differentiating with respect to 'x'

$$y' = 3c_1e^{3x} + c_2e^{x} + 2e^{3x} + 6xe^{3x} \qquad \ldots(4)$$

By data, $y(0) = -1$, $y'(0) = 3$

From (3), $\quad -1 = c_1 + c_2 \qquad \ldots(5)$

From (4), $\quad 3 = 3c_1 + c_2 + 2$

$3c_1 + c_2 = 1 \qquad \ldots(6)$

Solving (5) and (6) we get $c_1 = 1$ and $c_2 = -2$

$y = -2e^{x} + (1 + 2x)\, e^{3x}$

11.Solve $y'' + 4y' + 4y = 4\cos x + 3\sin x, y(0) = 0, y'(0) = 0$

Sol : Given differential equation in operator for

$$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$$

A.E is $m^2+4m + 4 = 0$

$(m + 2)^2 = 0 \qquad$ then $m = -2, -2$

$\therefore$ C.F is $y_c = (c_1 + c_2x)e^{-2x}$

P.I is= $y_p = \dfrac{4\cos x + 3\sin x}{(D^2 + 4D + 4)}$ put $D^2 = -1$

$$y_p = \frac{4\cos x + 3\sin x}{(4D + 3)} = \frac{(4D-3)(4\cos x + 3\sin x)}{(4D-3)(4D + 3)}$$

$$= \frac{(4D-3)(4\cos x + 3\sin x)}{16D^2 - 9}$$

$$y_p = \frac{(4D-3)(4\cos x + 3\sin x)}{-16-9}$$

$$= \frac{-16\sin x + 12\cos x - 12\cos x - 9\sin x}{-25} = \frac{-25\sin x}{-25} = \sin x$$

$\therefore$ General equation is $y = y_c + y_p$

$$y = (c_1 + c_2x)e^{-2x} + \sin x \qquad \ldots(1)$$

By given data $y(0) = 0, c_1 = 0$ and

Differentiating (1) w.r.t 'x', $y' = (c_1 + c_2 x)(-2)e^{-2x} + e^{-2x}(c_2) + \cos x$ … (2)

given $y'(0) = 0$

Substitute in (2) $\Rightarrow -2c1 + c2 + 1 = 0$ $\therefore c2 = -1$

$\therefore$ Required solution is $y = -xe^{-2x} + \sin x$

12.Solve $(D^2+9)y = cos3x$

Sol : Given equation is $(D^2+9)y = cos3x$

A.E is $m^2+9 = 0$

$\therefore m = \pm 3i$

$y_c = C.F = c_1\cos 3x + c_2 \sin 3x$

$$y_p = P.I = \frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$$

$$= \frac{x}{2(3)}\sin 3x = \frac{x}{6}\sin 3x$$

General equation is $y = y_c + y_p$

$$y = c_1\cos 3x + c_2\sin 3x + \frac{x}{6}\sin 3x$$

13.Solve $y''' + 2y'' - y' - 2y = 1 - 4x^3$

Sol : Given equation can be written as

$(D^3 + 2D^2 - D - 2)y = 1 - 4x^3$

A.E is $m^3 + 2m^2 - m - 2 = 0$

$(m^2 - 1)(m + 2) = 0$

$m^2 = 1$ or $m = -2$

$m = 1, -1, -2$

C.F$= c_1e^x + c_2e^{-x} + c_3e^{-2x}$

$$P.I = \frac{1}{(D^3 + 2D^2 - D - 2)}(1 - 4x^3) = \frac{-1}{2\left[1 - \frac{(D^3 + 2D^2 - D)}{2}\right]}(1 - 4x^3)$$

$$= \frac{-1}{2}\left[1 - \frac{(D^3 + 2D^2 - D)}{2}\right]^{-1}(1 - 4x^3)$$

$$=\frac{-1}{2}\left[1+\frac{(D^3+2D^2-D)}{2}+\frac{(D^3+2D^2-D)^2}{4}+\frac{(D^3+2D^2-D)^3}{8}+\ldots\right](1-4x^3)$$

$$=\frac{-1}{2}\left[1+\frac{1}{2}\left(D^3+2D^2-D\right)+\frac{1}{4}\left(D^2-4D^3\right)+\frac{1}{8}\left(-D^3\right)\right]\left(1-4x^3\right)$$

$$=\frac{-1}{2}\left[1-\frac{5}{8}D^3+\frac{5}{4}D^2-\frac{1}{2}D\right](1-4x^3)$$

$$=\frac{-1}{2}\left[(1-4x^3)-\frac{5}{8}(-24)+\frac{5}{4}(-24x)-\frac{1}{2}(-12x^2)\right]$$

$$=\frac{-1}{2}\left[-4x^3+6x^2-30x+16\right]$$

$$=[2x^3-3x^2+15x-8]$$

The general solution is

$$y=C.F+P.I$$

$$y=c_1e^x+c_2e^{-x}+c_3e^{-2x}+[2x^3-3x^2+15x-8]$$

14.Solve $(D^3-7D^2+14D-8)y=e^x\cos 2x$

Sol : Given equation is

$$(D^3-7D^2+14D-8)y=e^x\cos 2x$$

A.E is $(m^3-7m^2+14m-8)=0$

$(m-1)(m-2)(m-4)=0$

Then $m=1,2,4$

C.F$=c_1e^x+c_2e^{2x}+c_3e^{4x}$

P.I$=\dfrac{e^x\cos 2x}{(D^3-7D^2+14D-8)}$

$$=e^x\frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8}\cos 2x \quad \left[\because P.I=\frac{1}{f(D)}e^{ax}v=e^{ax}\frac{1}{f\left(D+a\right)}v\right]$$

$$=e^x\frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$=e^x\frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$=e^x\frac{1}{(-4D+3D+16)}\cos 2x \text{ (Replacing } D^2 \text{ with } -2^2)$$

$$= e^x \frac{1}{(16 - D)} \cos 2x$$

$$= e^x \frac{16 + D}{(16 - D)(16 + D)} \cos 2x$$

$$= e^x \frac{16 + D}{256 - D^2} \cos 2x$$

$$= e^x \frac{16 + D}{256 - (-4)^2} \cos 2x$$

$$= \frac{e^x}{260}(16\cos 2x - 2\sin 2x)$$

$$= \frac{2e^x}{260}\left(8\cos 2x - \sin 2x\right)$$

$$= \frac{e^x}{130}\left(8\cos 2x - \sin 2x\right)$$

General solution is $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{4x} + \frac{e^x}{130}\left(8\cos 2x - \sin 2x\right)$$

15.Solve $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$

Sol : Given $(D^2 - 4D + 4)y = x^2 \sin x + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m - 2)^2 = 0$ then m=2,2

C.F= $(c_1 + c_2 x)e^{2x}$

$$\text{P.I} = \frac{x^2 \sin x + e^{2x} + 3}{(D - 2)^2} = \frac{1}{(D - 2)^2}(x^2 \sin x) + \frac{1}{(D - 2)^2} e^{2x} + \frac{1}{(D - 2)^2}(3)$$

Now $\frac{1}{(D - 2)^2}(x^2 \sin x) = \frac{1}{(D - 2)^2}(x^2)$ (I.P of e^{ix})

$$= \text{I.P of } \frac{1}{(D - 2)^2}(x^2)e^{ix}$$

$$= \text{I.P of } (e^{ix})\frac{1}{(D + i - 2)^2}(x^2)$$

$$\text{I.P of } (e^{ix})\frac{1}{(D + i - 2)^2}(x^2)$$

On simplification, we get

$$\frac{1}{(D+i-2)^2}(x^2\sin x)=\frac{1}{625}\left[(220x+244)\cos x+(40x+33)\sin x\right]$$

and $\frac{1}{(D-2)^2}e^{2x}=\frac{x^2}{2}e^{2x}$,

$$\frac{1}{(D-2)^2}(3)=\frac{3}{4}$$

P.I= $\frac{1}{625}\left[(220x+244)\cos x+(40x+33)\sin x\right]+\frac{x^2}{2}e^{2x}+\frac{3}{4}$

$y=y_c+y_p$

$$y=(c_1+c_2x)e^{2x}+\frac{1}{625}\left[(220x+244)\cos x+(40x+33)\sin x\right]+\frac{x^2}{2}e^{2x}+\frac{3}{4}$$

16.Solve the differential equation $(D^3+1)y=\cos(2x-1)$.

Sol : Given D.E is $(D^3+1)y=\cos(2x-1)$

The A.E is $m^3+1=0$

$\Rightarrow (m+1)(m^2-m+1)=0 \quad [a^3+b^3=(a+b)(a^2-ab+b^2)]$

$$\Rightarrow m=-1,\frac{1\pm i\sqrt{3}}{2}$$

$$C.F=c_1e^{-x}+e^{\frac{x}{2}}[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x]$$

$$P.I=\frac{1}{D^3+1}\cos(2x-1)$$

Putting $D^2=a^2=-4$ then we have

P.I= $\frac{1}{1-4D}\cos(2x-1)=\frac{1+4D}{1-16D^2}[\cos(2x-1)]$

Again putting $D^2=a^2=-4$ then we have

P.I= $\frac{1}{65}[\cos(2x-1)-8\sin(2x-1)]$

∴ General solution is

$$y=C.F+P.I$$

$$y=c_1e^{-x}+e^{\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right]+\frac{1}{65}[\cos(2x-1)-8\sin(2x-1)].$$

Linear equations of second order with variable coefficients

An equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, where $P(x), Q(x), R(x)$ are real valued functions of 'x' is called linear equation of second order with variable coefficients.

Variation of Parameters:

This method is applied when P, Q in above equation are either functions of 'x' or real constants but R is a function of 'x'.

Working Rule:

1. Find C.F. Let C.F= $y_c = c_1u(x)+c_2u(x)$
2. Take $P.I = y_p = Au + Bv$ where $A = -\int\frac{vRdx}{uv'-vu'}$ and $B=\int\frac{uRdx}{uv'-vu'}$
3. Write the G.S. of the given equation $y = y_c + y_p$

1. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+y=\text{cosec}x$

Sol : Given equation in the operator form is $(D^2+1)y=\text{cosec}x$ …(1)

A.E is $(m^2+1)=0$

$\therefore m=\pm i$

The roots are complex conjugate numbers.

C.F is $y_c = c_1\cos x + c_2\sin x$

Let $y_p = A\cos x + B\sin x$ be P.I. of (1)

$$u\frac{dv}{dx}-v\frac{du}{dx}=\cos^2x+\sin^2x=1$$

A and B are given by

$$A=-\int\frac{vRdx}{uv'-vu'}=-\int\frac{\sin x\,\text{cosec}x}{1}dx=-\int dx=-x$$

$$B=\int\frac{uRdx}{uv^1-vu^1}=\int\cos x.\text{cosec}x dx=\int\cot x dx=\log(\sin x)$$

$\therefore y_p = -x\cos x+\sin x.\log(\sin x)$

$\therefore$ General solution is $y = y_c + y_p$.

$$y = c_1\cos x + c_2\sin x - x\cos x + \sin x.\log(\sin x)$$

2. Solve $(D^2 - 2D + 2)y = e^x\tan x$ by method of variation of parameters.

Sol : A.E is $m^2 - 2m + 2 = 0$

$$\therefore m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$$

We have $y_c = e^x(c_1\cos x + c_2\sin x) = c_1e^x\cos x + c_2e^x\sin x$

$$= c_1(u) + c_2(u)$$

where $u = e^x\cos x, v = e^x\sin x$

$$\frac{du}{dx} = e^x(-\sin x) + e^x\cos x, \frac{dv}{dx} = e^x\cos x + e^x\sin x$$

$$u\frac{dv}{dx} - v\frac{du}{dx} = e^x\cos x(e^x\cos x + e^x\sin x) - e^x\sin x(e^x\cos x - e^x\sin x)$$

$$= e^{2x}(\cos^2 x + \cos x\sin x - \sin x\cos x + \sin^2 x) = e^{2x}$$

Using variation of parameters,

$$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} = -\int \frac{e^x\tan x}{e^{2x}}(e^x\sin x)dx$$

$$= -\int \tan x\sin x dx = \int\left(\frac{\sin^2 x}{\cos x}dx\right) = \int \frac{(1-\cos^2 x)}{\cos x}dx$$

$$= \int(\sec x - \cos x)dx = \log(\sec x + \tan x) - \sin x$$

$$B = \int \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$$

$$= \int \frac{e^x\cos x.e^x\tan x}{e^{2x}}dx = \int \sin x dx = -\cos x$$

General solution is given by $y = y_c + Au + Bv$

$$\Rightarrow y = c_1e^x\cos x + c_2e^x\sin x + [\log(\sec x + \tan x) - \sin x]e^x\cos x - e^x\cos x\sin x$$

$$\Rightarrow y = c_1e^x\cos x + c_2e^x\sin x + [\log(\sec x + \tan x) - 2\sin x]e^x\cos x.$$

3. Solve the differential equation $(D^2 + 4)\, y = sec2x$ by the method of variation of parameters.

Sol : Given equation is $(D^2 + 4)\, y = sec2x$ …..(1)

$\therefore$ A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

The roots are complex conjugate numbers.

$\therefore yc = C.F = c_1 \cos 2x + c_2 \sin 2x$

Let $yp = P.I = A\cos 2x + B\sin 2x$

Here $u = \cos 2x, v = \sin 2x$ and $R = \sec 2x$.

$\therefore \frac{du}{dx}$ = -2 $\sin 2x$ and $\frac{dv}{dx}$ = 2 $\cos 2x$

$\therefore$ u $\frac{dv}{dx} - v\frac{du}{dx} = (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x = 2(\cos^2 2x + \sin^2 2x) = 2$

A and B are given by :

$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} dx = -\int \frac{\sin 2x \sec 2x}{2} dx = -\frac{1}{2}\int \tan 2x\, dx = \frac{1}{2}\frac{\log|\cos 2x|}{2}$

$\Rightarrow A = \frac{\log|\cos 2x|}{4}$

$B = \int \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}} dx = \int \frac{\cos 2x \sec 2x}{2} dx = \frac{1}{2}\int dx = \frac{x}{2}$

$\therefore y_p = P.I = \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$

$\therefore$ The general solution is given by :

$y = y_c + y_p =$ C.F. + P.I

i.e., $y = c_1\cos 2x + c_2\sin 2x + \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$

4. Solve $(D^2 + a^2)y = \tan ax$ by the method variation of parameters.

Sol: Given $(D^2 + a^2)y = \tan ax$ i.e $\frac{d^2y}{dx^2} + a^2 y = \tan ax$ -------- (1)

Now compare equation (1) with $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + Q(x)y = R(x)$ then

$P = 0, Q(x) = a^2$ and $R(x) = \tan ax$

The solution of (1) is $y = C.F + P.I$

Finding C.F :

The A.E of (1) is $m^2 + a^2 = 0 \implies m = \pm ai$

$\therefore C.F = c_1\cos ax + c_2\sin ax = c_1 u + c_2 v$

Here $u = \cos ax$ and $v = \sin ax$

Finding P.I : $P.I = Au + Bv$

Where $A = \int \frac{vR}{uv' - vu'} dx = -\int \frac{\sin ax \tan ax}{a} = \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax} dx = \frac{-1}{a}\int \frac{1 - \cos^2 ax}{\cos ax} dx$

$$= \frac{-1}{a}\left[\int \sec ax\, dx - \int \cos ax\, dx\right]$$

$$A = \frac{-1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax$$

$$B=\int\frac{uR}{uv'-vu'}dx=\int\frac{cosaxtanax}{a}=\frac{1}{a}\int sinax\,dx=\frac{-1}{a^2}cosax$$

Therefore $P.I=Au+Bv$

P.I $=\left(\frac{-1}{a^2}log|secax+tanax|+\frac{1}{a^2}sinax\right)cosax+\left(\frac{-1}{a^2}cosax\right)sinax$

$$=\left(\frac{-1}{a^2}log|secax+tanax|\right)cosax$$

Therefore the general solution is $y=C.F+P.I$

$y=c_1cosax+c_2sinax+\left(\frac{-1}{a^2}log|secax+tanax|\right)cosax.$