School Of Engineering, Dayananda Sagar University, 3rd Sem Module 3 : Fourier Transform

Fourier Series is used for functions that are periodic on a finite interval only. When the functions are non-periodic and are defined on the whole x-axis, this concept is extended and it leads to Fourier integrals.

1 Fourier Integral

If $f(x)$ is a piecewise continuous function with right and left hand derivative at every point and if it is absolutely integrable on the x-axis, then The fourier integral is given by

$$\boxed{f(x) = \int_0^\infty [A(\omega)cos\omega x + B(\omega)sin\omega x]d\omega} \tag{1.1}$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)cos(\omega v)dv$ and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)sin(\omega v)dv$.

Ques 1. Find the fourier integral representation of $f(x) = \begin{cases} 1, & \text{if} \quad |x| < 1 \\ 0, & \text{if} \quad |x| > 1 \end{cases}$.

Soln. $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)cos(\omega v)dv$
cos is an even function.

$$\begin{aligned} A(\omega) &= \frac{1}{\pi}\int_{-1}^{1} cos(\omega v)dv \\ &= \frac{2}{\pi}\int_{0}^{1} cos(\omega v)dv \\ &= \frac{2}{\pi}\left[\frac{sin(\omega v)}{\omega}\right]\Bigg|_0^1 \\ &= \frac{2}{\pi}\left[\frac{sin(\omega)}{\omega}\right] \end{aligned}$$

$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)sin(\omega v)dv$.

$$B(\omega) = \frac{1}{\pi}\int_{-1}^{1} f(v)sin(\omega v)dv = 0,$$

since sin is an odd function.

Thus,

$$\begin{aligned} f(x) &= \int_0^\infty [A(\omega)cos\omega x + B(\omega)sin\omega x]d\omega \\ &= \int_0^\infty \frac{2}{\pi}\left[\frac{sin(\omega)}{\omega}\right] cos(\omega x)d\omega \end{aligned}$$

2 Complex Form of Fourier Integral

The fourier integral is given by

$$\boxed{f(x) = \int_0^\infty [A(\omega)cos\omega x + B(\omega)sin\omega x]d\omega} \tag{2.1}$$

where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)cos(\omega v)dv$ and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)sin(\omega v)dv$.

Substituting A and B into the integral for f (2.1), we get,

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)[cos(\omega v)cos(\omega x) + sin(\omega v)sin(\omega x)]dvd\omega \tag{2.2}$$

Using $cos(a)cos(b) + sin(a)sin(b) = cos(a-b)$ in (2.2), we get

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)cos(\omega x - \omega v)dv\right] d\omega \tag{2.3}$$

The integral in brackets is an even function of ω, $F(\omega)$, because $cos(\omega x - \omega v)$ is an even function of ω, the function f does not depend on w, and we integrate with respect to v not ω. Hence the integral of $F(\omega)$ from $\omega = 0$ to ∞ is $\frac{1}{2}$ times the integral of $F(\omega)$ from $-\infty$ to ∞. Thus

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(v)cos(\omega x - \omega v)dv\right] d\omega \tag{2.4}$$

Since, $sin(\omega x - \omega v)$ is an odd function of ω, the integral becomes an odd function of ω, Hence

$$\frac{1}{2\pi}\int_{-\infty}^\infty \left[\int_{-\infty}^{\infty} f(v)sin(\omega x - \omega v)dv\right] d\omega = 0 \tag{2.5}$$

We now take the integrand of (2.4) plus $\iota(=\sqrt{-1})$ times the integrand of (2.5), and use the formula $e^{\iota x} = cosx + \iota sinx$, we have

$$f(v)cos(\omega x - \omega v) + \iota f(v)sin(\omega x - \omega v) = f(v)e^{\iota(\omega x - \omega v)} \tag{2.6}$$

Thus, the **Complex Fourier Integral** is given by

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^{\infty} f(v)e^{\iota\omega(x-v)}dvd\omega} \tag{2.7}$$

Ques 1. Using fourier integral representation, show that

$$\int_0^\infty \frac{cos(\omega x) + \omega sin(\omega x)}{1+\omega^2} d\omega = \begin{cases} 0, & \text{if} \quad x < 0 \\ \frac{\pi}{2}, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$$

Soln. Consider, (take $\pi = 1$, initially)

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0 \\ \frac{1}{2}, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

Consider complex form of fourier integral, (use $e^{\iota\theta} = cos\theta + \iota sin\theta$)

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) e^{-\iota\omega v} dv \right] e^{\iota\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{-v} e^{-\iota\omega v} dv \right] e^{\iota\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{0}^{\infty} e^{-v(1+\iota\omega)} dv \right] e^{\iota\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{e^{-v(1+\iota\omega)}}{-1-\iota\omega} \right] \Bigg|_0^\infty e^{\iota\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{1+\iota\omega} \right] e^{\iota\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{1+\iota\omega} \times \frac{1-\iota\omega}{1-\iota\omega} \right] (cos\omega x + \iota sin\omega x) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1-\iota\omega}{1+\omega^2} \right] (cos\omega x + \iota sin\omega x) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{1+\omega^2} \right] (cos\omega x + \iota sin\omega x - \iota\omega cos\omega x + \omega sin\omega x) d\omega \end{aligned}$$

Considering only the real part, we get

$$= \frac{1}{\pi} \int_0^\infty \frac{cos(\omega x) + \omega sin(\omega x)}{1+\omega^2} d\omega$$

Thus,

$$\frac{1}{\pi} \int_0^\infty \frac{cos(\omega x) + \omega sin(\omega x)}{1+\omega^2} d\omega = \begin{cases} 0, & \text{if} \quad x < 0 \\ \frac{1}{2}, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

Taking π to the other side, we get our desired result.

3 Fourier Transform and its Inverse

Re-writing (2.7),as a product of exponential functions, we have

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(v)e^{-\iota\omega v}dv\right]e^{\iota\omega x}d\omega \tag{3.1}$$

The expression in brackets is a function of ω, denoted by $\widehat{f}(\omega)$, and is called the **Fourier Transform** of f; Re-writing $v=x$, we have

$$\boxed{\widehat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-\iota\omega x}dx} \tag{3.2}$$

The **Inverse Fourier Transform** of $\widehat{f}(\omega)$ is given by

$$\boxed{f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{\iota\omega x}dx} \tag{3.3}$$

Notation:

$$\boxed{\widehat{f}=\mathfrak{F}(f)} \tag{3.4}$$

$$\boxed{f=\mathfrak{F}^{-1}(\widehat{f})} \tag{3.5}$$

Result: Existence of the Fourier Transform

If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $\widehat{f}(\omega)$ of $f(x)$ given by (3.2) exists.

3.1 Worked Examples

Ques 1. Find the Fourier Transform of $f(x)=1$ if $|x|<1$ and $f(x)=0$ otherwise.

Soln. Using $\widehat{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-\iota\omega x}dx$, we get

$$\begin{aligned}\widehat{f}(\omega)&=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}1.e^{-\iota\omega x}dx\\&=\frac{1}{\sqrt{2\pi}}\frac{e^{-\iota\omega x}}{-\iota\omega}\Bigg|_{-1}^{1}\\&=\frac{1}{-\iota\omega\sqrt{2\pi}}(e^{-\iota\omega}-e^{\iota\omega})\end{aligned} \tag{3.6}$$

Using $e^{\iota\omega}=cos\omega+\iota sin\omega$ and $e^{-\iota\omega}=cos\omega-\iota sin\omega$, we get,

$$e^{-\iota\omega}-e^{\iota\omega}=(cos\omega-\iota sin\omega)-(cos\omega+\iota sin\omega)=-2\iota sin\omega \tag{3.7}$$

Using (3.7) in (3.6), we get

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{-\iota\omega\sqrt{2\pi}}(-2\iota sin\omega)\\ &= \frac{2sin\omega}{\omega\sqrt{2\pi}}\\ &= \sqrt{\frac{2}{\pi}}\frac{sin\omega}{\omega}\end{aligned}$$

Ques 2. Find the Fourier Transform $\mathfrak{F}(e^{-ax})$ of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$, if $x < 0$, here $a > 0$.
Soln. Using $\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\iota\omega x}dx$, we get

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}e^{-\iota\omega x}dx\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-(a+\iota\omega)x}}{-(a+\iota\omega)}\right)\Bigg|_0^\infty\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a+\iota\omega}\right)\end{aligned}$$

Ques 3. Find the Fourier Transform of $f(x) = \begin{cases} 1, & \text{if} -b < x < b \\ 0, & \text{otherwise} \end{cases}$.
Soln. Using $\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\iota\omega x}dx$, we get

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-b}^{b} e^{-\iota\omega x}dx\\ &= \frac{1}{\sqrt{2\pi}}\frac{e^{-\iota\omega x}}{-\iota\omega}\Bigg|_{-b}^{b}\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-\iota\omega b}}{(-\iota\omega)} - \frac{e^{\iota\omega b}}{(-\iota\omega)}\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-\iota\omega b}}{(-\iota\omega)} + \frac{e^{\iota\omega b}}{(\iota\omega)}\right)\end{aligned}$$

$$\widehat{f}(\omega) = \frac{1}{\iota\omega\sqrt{2\pi}}(e^{\iota\omega b} - e^{-\iota\omega b})$$

Using $sin\theta = \frac{e^{\iota\theta}-e^{-\iota\theta}}{2\iota}$, we get

$$\begin{aligned}\widehat{f}(\omega) &= \frac{1}{\iota\omega\sqrt{2\pi}}2\iota sin\omega b\\ &= \sqrt{\frac{2}{\pi}}\frac{sinb\omega}{\omega}\end{aligned}$$

Ques 4. Find the Fourier transform of $f(x) = \begin{cases} e^{\iota ax}, & \text{if} -b < x < b \\ 0, & \text{otherwise} \end{cases}$.

Soln. Using $\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-\iota\omega x}dx$, we get

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} e^{\iota ax}e^{-\iota\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-b}^{b} e^{(\iota a-\iota\omega)x}dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{e^{(\iota a-\iota\omega)x}}{(\iota a-\iota\omega)}\Bigg|_{-b}^{b} \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{(\iota a-\iota\omega)b}}{(\iota a-\iota\omega)} - \frac{e^{(\iota a-\iota\omega)b}}{(\iota a-\iota\omega)} \right) \\
&= \frac{1}{\sqrt{2\pi}(\iota a-\iota\omega)} \left(e^{(\iota a-\iota\omega)b} - e^{-(\iota a-\iota\omega)b} \right) \\
&= \frac{1}{\iota\sqrt{2\pi}(a-\omega)} \left(e^{\iota(a-\omega)b} - e^{-\iota(a-\omega)b} \right)
\end{aligned}$$

Using $sin\theta = \frac{e^{\iota\theta}-e^{-\iota\theta}}{2\iota}$, we get

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\iota\sqrt{2\pi}(a-\omega)} 2\iota sin(a-\omega)b \\
&= \sqrt{\frac{2}{\pi}} \frac{sin(a-\omega)b}{(a-\omega)}
\end{aligned}$$

Ques 5. Find the Fourier Transform of $e^{-|x|}, -\infty < x < \infty$.

Soln. Using $\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-\iota\omega x}dx$, we get

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|}e^{-\iota\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|}(cos(\omega x) - \iota sin(\omega x)dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} e^{-|x|}cos(\omega x) - \iota \int_{-\infty}^{\infty} e^{-|x|}sin(\omega x)dx \right)
\end{aligned}$$

The integrand $e^{-|x|}cos(\omega x)$ is an even function and $e^{-|x|}sin(\omega x)$ is an odd function; Hence $\int_{-\infty}^{\infty} e^{-|x|}cos(\omega x)dx = 2\int_{0}^{\infty} e^{-|x|}cos(\omega x)dx$ and $\int_{-\infty}^{\infty} e^{-|x|}sin(\omega x)dx = 0$. Thus,

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-|x|}cos(\omega x)dx \\
&= \frac{1}{\sqrt{2\pi}} 2 \int_{0}^{\infty} e^{-x}cos(\omega x)dx
\end{aligned}$$

Using $\int e^{at}cosbt = \frac{e^{at}}{a^2+b^2}(acos(bx) + bsin(bx))$, we get

$$\begin{aligned}
\widehat{f}(\omega) &= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+\omega^2}(-cos\omega x + \omega sin\omega x)\right]\Bigg|_0^\infty \\
&= \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+\omega^2}\right]
\end{aligned}$$

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Ques 6. Find the Fourier Transform of $f(x) = xe^{-x}, 0 \leq x \leq \infty$.

Soln. Using $\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\iota\omega x}dx$, we get

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-x}e^{-\iota\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-x}e^{-\iota\omega x}dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} xe^{-x(1+\iota\omega)}dx
\end{aligned}$$

Using Bernoulli's rule of integration by parts,
$\int uv = u_0v_1 - u_1v_2 + u_2v_3 - u_3v_4 + \dots$
where $u_0, u_1, u_2, u_3, \dots$ i.e all $u's$ represent successive derivatives of u;
$v_0, v_1, v_2, v_3, \dots$ i.e all $v's$ represent successive integrals of v .

$u = x, u_1 = 1, u_2 = 0, v = e^{-x(1+\iota\omega)}, v_1 = \frac{e^{-x(1+\iota\omega)}}{-(1+\iota\omega)}, v_2 = \frac{e^{-x(1+\iota\omega)}}{(1+\iota\omega)^2}$

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\left[x\frac{e^{-x(1+\iota\omega)}}{-(1+\iota\omega)} - \frac{e^{-x(1+\iota\omega)}}{(1+\iota\omega)^2}\right]\Bigg|_0^\infty \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(1+\iota\omega)^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(1+\iota\omega)^2}\frac{(1-\iota\omega)^2}{(1-\iota\omega)^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{(1-\iota\omega)^2}{(1+\iota\omega)^2(1-\iota\omega)^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{(1-\iota\omega)^2}{[(1+\iota\omega)(1-\iota\omega)]^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{(1-\iota\omega)^2}{(1^2-(\iota\omega)^2)^2}\right] \\
&= \frac{1}{\sqrt{2\pi}}\left[\frac{(1-\iota\omega)^2}{(1+(\omega)^2)^2}\right]
\end{aligned}$$

4 Fourier Sine and Cosine Transform

The Fourier Transform of a function $f(x)$ is given by

$$\boxed{\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)cos(\omega x)dx} \tag{4.1}$$

The inverse Fourier Cosine Transform

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f_c}(\omega)cos(\omega x)d\omega} \tag{4.2}$$

The Fourier Sine Transform is given by

$$\boxed{\widehat{f_s}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)sin(\omega x)dx} \tag{4.3}$$

The inverse Fourier Sine Transform is given by

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \widehat{f_s}(\omega)sin(\omega x)d\omega} \tag{4.4}$$

4.1 Worked Examples

Ques 1. Find the Fourier Sine and Cosine transforms of the function
$f(x) = \begin{cases} k, & \text{if} \quad 0 < x < a \\ 0, & \text{if} \quad x > a \end{cases}$

Soln. Fourier sine transform of the function $f(x)$ is given by (4.3),
$\widehat{f_s}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)sin(\omega x)dx$, then

$$\begin{aligned}
\widehat{f_s}(\omega) &= \sqrt{\frac{2}{\pi}} k \int_0^a sin(\omega x)dx \\
&= \sqrt{\frac{2}{\pi}} k \left(\frac{-cos(\omega x)}{\omega}\right)\Bigg|_0^a \\
&= \sqrt{\frac{2}{\pi}} k \left(\frac{1 - cos(\omega a)}{\omega}\right)
\end{aligned}$$

Fourier cosine transform of the function $f(x)$ is given by (4.1),
$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)cos(\omega x)dx$, then

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}}k \int_0^a cos(\omega x)dx$$
$$= \sqrt{\frac{2}{\pi}}k \left(\frac{sin(\omega x)}{\omega}\right)\Big|_0^a$$
$$= \sqrt{\frac{2}{\pi}}k \left(\frac{sin(\omega a)}{\omega}\right)$$

Ques 2. Find the Fourier Sine and Cosine transforms of the function $f(x) = e^{-x}$

Soln. Fourier cosine transform of the function $f(x)$ is given by (4.1), $\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)cos(\omega x)dx$, then

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x}cos(\omega x)dx$$

Using $\int e^{ax}cos(bx)dx = \frac{e^{ax}}{a^2+b^2}\{acos(bx) + bsin(bx)\}$

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \frac{e^{-x}}{(-1)^2 + \omega^2} \left(-cos(\omega x) + \omega sin(\omega x)\right)\Big|_0^\infty$$
$$= \sqrt{\frac{2}{\pi}} \left(\frac{1}{1+\omega^2}\right)$$

Fourier sine transform of the function $f(x)$ is given by (4.3), $\widehat{f_s}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)sin(\omega x)dx$, then

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x}sin(\omega x)dx$$

Using $\int e^{ax}sin(bx)dx = \frac{e^{ax}}{a^2+b^2}\{asin(bx) - bcos(bx)\}$

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \frac{e^{-x}}{(-1)^2 + \omega^2} \left(-sin(\omega x) - \omega cos(\omega x)\right)\Big|_0^\infty$$
$$= \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2}\right)$$

Ques 3. Find the Fourier sine and cosine transform of the following function
$$f(x) = \begin{cases} x, & \text{if } \ 0 \le x \le 1 \\ 2-x, & \text{if } \ 1 \le x \le 2 \\ 0, & \text{if } \ x > 2 \end{cases}$$

Soln. Fourier cosine transform of the function $f(x)$ is given by (4.1),
$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)cos(\omega x)dx$, then

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \left(\int_0^1 xcos(\omega x)dx + \int_1^2 (2-x)cos(\omega x)dx + \int_2^\infty 0.cos(\omega x)dx \right)$$

Using Bernoulli's rule of integration by parts,
$\int uv = u_0v_1 - u_1v_2 + u_2v_3 - u_3v_4 + \dots$
where $u_0, u_1, u_2, u_3, \dots$ i.e all $u's$ represent successive derivatives of u;
$v_0, v_1, v_2, v_3, \dots$ i.e all $v's$ represent successive integrals of v .

For $\int_0^1 xcos(\omega x)dx$,
$u = u_0 = x, u_1 = 1, u_2 = 0$ and $v = cos(\omega x), v_1 = \frac{sin(\omega x)}{\omega}, v_2 = -\frac{cos(\omega x)}{s^2}$

For $\int_1^2 (2-x)cos(\omega x)dx$,
$u = u_0 = 2 - x, u_1 = -1, u_2 = 0$ and $v = cos(\omega x), v_1 = \frac{sin(\omega x)}{\omega}, v_2 = -\frac{cos(\omega x)}{s^2}$

Thus,

$$\begin{aligned}
\widehat{f_c}(\omega) &= \sqrt{\frac{2}{\pi}} \left[\left(x\frac{sin(\omega x)}{\omega} + \frac{cos(\omega x)}{\omega^2} \right) \Bigg|_0^1 + \left((2-x)\frac{sin(\omega x)}{\omega} + \frac{cos(\omega x)}{\omega^2} \right) \Bigg|_1^2 \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\left(\frac{sin(\omega)}{\omega} + \frac{cos(\omega)}{\omega^2} - \frac{1}{\omega^2} \right) + \left(-\frac{cos(2\omega)}{\omega^2} - \frac{sin(\omega)}{\omega} + \frac{cos(\omega)}{\omega^2} \right) \right] \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{2cos(\omega) - 1 - cos(2\omega)}{\omega^2} \right) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{2cos(\omega) - (1 + cos(2\omega))}{\omega^2} \right), \quad \text{Using} \quad 1 + cos(2\omega) = 2cos^2\omega \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{2cos(\omega) - 2cos^2(\omega)}{\omega^2} \right) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{2cos(\omega)(1 - cos(\omega))}{\omega^2} \right)
\end{aligned}$$

Fourier sine transform of the function $f(x)$ is given by (4.1),
$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)sin(\omega x)dx$, then

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \left(\int_0^1 xsin(\omega x)dx + \int_1^2 (2-x)sin(\omega x)dx + \int_2^\infty 0.sin(\omega x)dx \right)$$

Using Bernoulli's rule of integration by parts,
$\int uv = u_0v_1 - u_1v_2 + u_2v_3 - u_3v_4 + \dots$
where $u_0, u_1, u_2, u_3, \dots$ i.e all $u's$ represent successive derivatives of u;
$v_0, v_1, v_2, v_3, \dots$ i.e all $v's$ represent successive integrals of v .

For $\int_0^1 xsin(\omega x)dx$,
$u = u_0 = x, u_1 = 1, u_2 = 0$ and $v = sin(\omega x), v_1 = \frac{-cos(\omega x)}{\omega}, v_2 = \frac{-sin(\omega x)}{s^2}$

For $\int_1^2 (2-x)sin(\omega x)dx$,
$u = u_0 = 2-x, u_1 = -1, u_2 = 0$ and $v = sin(\omega x), v_1 = \frac{-cos(\omega x)}{\omega}, v_2 = \frac{-sin(\omega x)}{s^2}$

Thus,

$$\begin{aligned}
\widehat{f_c}(\omega) &= \sqrt{\frac{2}{\pi}} \left[\left(-x\frac{cos(\omega x)}{\omega} + \frac{sin(\omega x)}{\omega^2} \right) \Big|_0^1 + \left(-(2-x)\frac{cos(\omega x)}{\omega} - \frac{sin(\omega x)}{\omega^2} \right) \Big|_1^2 \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\left(\frac{-cos(\omega)}{\omega} + \frac{sin(\omega)}{\omega^2} \right) + \left(-\frac{sin(2\omega)}{\omega^2} + \frac{cos(\omega)}{\omega} + \frac{sin(\omega)}{\omega^2} \right) \right] \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{2sin(\omega) - sin(2\omega)}{\omega^2} \right) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{2sin(\omega) - 2sin(\omega)cos(\omega)}{\omega^2} \right), \quad \text{Using} \quad sin(2\omega) = 2sin(\omega)cos(\omega) \\
&= \sqrt{\frac{2}{\pi}} \left(\frac{2sin(\omega)(1 - cos(\omega))}{\omega^2} \right)
\end{aligned}$$

Ques 4. Find the Fourier cosine transform of the function
$f(x) = \begin{cases} cos(x), & \text{if} \quad 0 < x < a \\ 0, & \text{if} \quad x \geq a \end{cases}$

Soln. Fourier cosine transform of the function $f(x)$ is given by (4.1),
$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)cos(\omega x)dx$, then

$$\widehat{f_c}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^a cos(x)cos(\omega x)dx$$

Using $cos(a)cos(b) = \frac{1}{2}\left(cos(a+b) + cos(a-b)\right)$,

$$\begin{aligned}
\widehat{f_c}(\omega) &= \sqrt{\frac{2}{\pi}} \int_0^a \frac{1}{2} \left(cos(1+\omega)x + cos(1-\omega)x \right) dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{sin(1+\omega)x}{1+\omega} + \frac{sin(1-\omega)x}{1-\omega} \right) \Big|_0^a \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{sin(1+\omega)a}{1+\omega} + \frac{sin(1-\omega)a}{1-\omega} \right)
\end{aligned}$$