



MODULE-1

LAPLACE TRANSFORM & INVERSE LAPLCE TRANSFORM

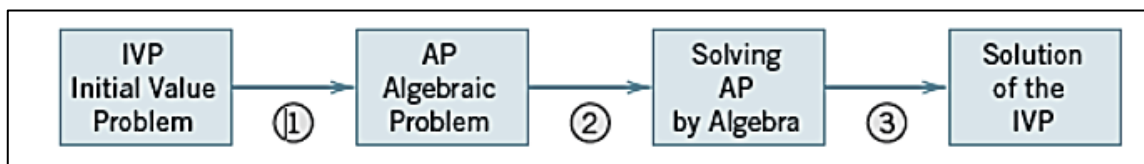


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- Laplace Transforms of Elementary functions (without proof).
- Laplace Transforms of $e^{at} \cdot f(t)$, $t \cdot f(t)$, $t^n \cdot f(t)$, $\frac{f(t)}{t}$.
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- Inverse Laplace Transforms- By the method of Partial Fractions.
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INTRODUCTION – LAPLACE TRANSFORM

Think of Laplace transforms like a super useful tool for engineers. They help solve problems involving things that change over time, like how things move or behave in the world. For example, they're great at figuring out how electrical systems, springs, mixing things, and signals change. This makes it way easier to solve tricky math problems and understand how things work in fields like engineering and physics. Solving an ODE (which stands for Ordinary Differential Equation) using the Laplace transform method involves three main steps. These steps are illustrated in Figure below:



Step 1: We start by changing the given ODE into a different kind of math problem called an algebraic equation. This new equation is called the "subsidiary equation."

Step 2: Next, we work with this subsidiary equation using only algebra, no trickier calculus stuff.

Step 3: After solving the subsidiary equation, we change the solution back to the original type of problem we were dealing with (an ODE). This gives us the final solution to the original problem we were trying to solve. The main reason to learn about Laplace transforms is that they make solving Ordinary Differential Equations (ODEs) much easier. They turn the process of solving an ODE into a simpler algebraic problem, along with some transformations. This kind of math, which changes calculus problems into algebraic ones, is called "operational calculus." So, Laplace transforms and operational calculus help us handle tricky calculus problems by turning them into more manageable algebraic tasks. In this chapter, we will explore how the Laplace transform takes a certain group of complex functions and turns them into a different set of simpler functions. We will also learn how to utilize Laplace transforms to solve differential equations.

Definition of Laplace Transform:

Let $f(t)$ be a function of t and the Laplace transform of $f(t)$ denoted by $L\{f(t)\} = F(s)$, defined as

$$F(s) = L(f) = \int_0^{\infty} e^{-st} f(t) dt \text{ ----(1).}$$

It's important for us to assume that the function $f(t)$ satisfies a certain condition: the integral of $f(t)$ must exist, which means it has a finite value. The outcome $F(s)$ is referred to as the

$$f(t) = L^{-1}(F) \text{ ----(1*)}. \text{ Note that (1) and (1*) together imply } L^{-1}(L(f)) = f \text{ and } L(L^{-1}(F)) = F$$

Laplace transform, but the procedure itself that we've just described turning $F(s)$ from a given $f(t)$ is also known as the Laplace transform. This process is a type of integral transform" represented as $\int_0^{\infty} k(s,t)f(t)dt$ with Kernel $k(s,t) = e^{-st}$. Note: It's important to understand that the Laplace transform is labelled as an "integral transform" because it performs a transformation on a function within one domain to produce a function within another domain through an integration process involving a specific function known as the kernel. This kernel, or kernel function, depends on the variables in both domains and essentially defines the characteristics of the integral transform.

Sufficient conditions for the existence of Laplace transform of given function:

The Laplace transform of given function $f(t)$ exists if the following conditions are satisfied:

1. $f(t)$ is continuous or piecewise continuous.
2. $\lim_{t \rightarrow \infty} \{e^{-at} f(t)\}$ is finite

Linearity property: If α and β are constants and $f(t)$, $g(t)$ are functions whose Laplace transforms exist then $L\{\alpha f(t) + \beta g(t)\} = \alpha L\{f(t)\} + \beta L\{g(t)\}$.

LAPLACE TRANSFORMS OF SOME ELEMENTARY FUNCTIONS

1. $f(t) = 1$

Solution: $L\{f(t)\} = \int_0^{\infty} e^{-st} 1 \cdot dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = \frac{1}{s}, \text{ If } s > 0 \therefore L\{1\} = \frac{1}{s}, s > 0$

2. $f(t) = e^{at}$

Solution: $L\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} = \frac{1}{s-a}$

$\therefore L\{e^{at}\} = \frac{1}{s-a}, s > a$

3. $f(t) = \cos at$

Solution: $L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$ We have

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx], \text{ Using this formula}$$

$$L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt = \frac{e^{-st}}{s^2 + a^2} [-s \cos at + a \sin at] \Big|_0^\infty = \frac{s}{s^2 + a^2}$$

$$\therefore L\{\cos at\} = \frac{s}{s^2 + a^2}$$

4. $f(t) = \sinh at$

$$\textbf{Solution: } L\{\sinh at\} = L\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\} = \frac{1}{2} \cdot \frac{1}{s-a} - \frac{1}{2} \cdot \frac{1}{s+a}$$

$$= \frac{a}{s^2 - a^2} \therefore L\{\sinh at\} = \frac{a}{s^2 - a^2} \text{ if } s > |a|$$

5. $f(t) = \cosh at$

$$\textbf{Solution: } L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\} = \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

$$= \frac{1}{2} \cdot \frac{1}{s-a} + \frac{1}{2} \cdot \frac{1}{s+a} = \frac{s}{s^2 - a^2} \therefore L\{\cosh at\} = \frac{s}{s^2 - a^2} \text{ if } s > |a|$$

6. $f(t) = t^n$ Where n is real number different from non-negative integer.

$$\textbf{Solution: } L\{t^n\} = \int_0^\infty e^{-st} t^n \, dt, \text{ put } st = u \Rightarrow sdt = du = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \cdot \frac{du}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n \, du = \frac{1}{s^{n+1}} \Gamma(n+1) \textbf{Note:} \text{ If } n \text{ is a positive integer } \Gamma(n+1) = n!$$

$$\therefore L\{t^n\} = \frac{n!}{s^{n+1}}, \text{ if } n \text{ is a positive integer.}$$

Some Functions $f(t)$ and Their Laplace Transforms $L(f)$:

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
3	t^2	$2/s^3$	9	$\cosh at$	$\frac{s}{s^2 - a^2}$
4	t^n ($n = 0, 1, \dots$)	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2 - a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$

WORKED EXAMPLES

1. Find the Laplace Transform of $3t + 12$.

Solution: $L\{3t + 12\} = 3L\{t\} + 12L\{1\}$

$$\Rightarrow F(s) = \frac{3}{s^2} + \frac{12}{s}$$

2. Find the Laplace Transform of $(a - bt)^2$, where a, b are constant.

Solution: $L\{(a - bt)^2\} = a^2L\{1\} - 2abL\{t\} + b^2L\{t^2\}$

$$\Rightarrow F(s) = a^2 \frac{1}{s} - 2ab \frac{1}{s^2} + b^2 \frac{2!}{s^3}$$

3. Find the Laplace Transforms of $\cos^3 \omega t$, where ω is Constant.

Solution: $L\{\cos^3 \omega t\}$

$$\Rightarrow \frac{1}{4}L\{\cos 3\omega t\} + \frac{3}{4}L\{\cos \omega t\}$$

$$\Rightarrow \frac{s}{4(s^2 + 9\omega^2)} + \frac{3s}{4(s^2 + \omega^2)}$$

4. Find the Laplace Transforms of $-5 \cdot \cos[(0.4t)]$

Solution: $L\{-5 \cdot \cos[(0.4t)]\} = -5L\{\cos[(0.4t)]\}$

$$\Rightarrow -5 \left[\frac{s}{4(s^2 + 0.4^2)} \right]$$

5. Find the Laplace Transforms of $\sin(\omega t + \theta)$, where ω, θ are Constant

Solution: to find $L\{\sin(\omega t + \theta)\}$, we Use $\sin(a + b) = \sin(a).\cos(b) + \cos(a).\sin(b)$

$$\Rightarrow L\{\sin(\omega t + \theta)\} = L\{\sin(\omega t).\cos(\theta)\} + L\{\cos(\omega t).\sin(\theta)\}$$

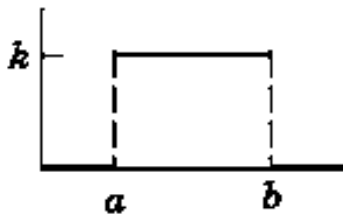
$$\Rightarrow \cos(\theta) L\{\sin(\omega t)\} + \sin(\theta) L\{\cos(\omega t)\}.$$

$$\Rightarrow F(s) = \cos(\theta) \left[\frac{\omega}{s^2 + \omega^2} \right] + \sin(\theta) \left[\frac{s}{s^2 + \omega^2} \right]$$

6. Find the Laplace Transform $L\{\sqrt{t}\}$

$$\text{Solution: } L\{\sqrt{t}\} = L\{t^{1/2}\} = \frac{\Gamma\left(\frac{1}{2} + 1\right)}{s^{\frac{1}{2} + 1}} \Rightarrow F(s) = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} \Rightarrow F(s) = \frac{\sqrt{\pi}}{2s^{3/2}}$$

7. Find the Laplace transform of



$$\text{Solution: Defining function as, } f(t) = \begin{cases} 0, & \text{if } (t < a) \\ k, & \text{if } [a \leq t \leq b] \\ 0, & \text{if } [t > b] \end{cases}$$

Apply Laplace transform to $f(t)$

$$\Rightarrow L\{f(t)\} = \int_{t=0}^a 0.e^{-st} dt + \int_a^b k.e^{-st} dt + \int_b^{\infty} 0.e^{-st} dt$$

$$\Rightarrow L\{f(t)\} = 0 + \int_a^b k.e^{-st} dt + 0$$

$$\Rightarrow L\{f(t)\} = k \int_a^b e^{-st} dt$$

$$\Rightarrow L\{f(t)\} = k \left[\frac{e^{-st}}{-s} \right]_{t=a}^{t=b} \text{ as } a \leq t \leq b$$

$$\Rightarrow L\{f(t)\} = \left[-k \frac{e^{-s(b)}}{s} + k \frac{e^{-s(a)}}{s} \right] = F(s) = \frac{k}{s} [e^{-as} - e^{-bs}].$$

8. Find the Laplace transform of



Solution: Given Graph is a straight line of the form: $\frac{x}{b} + \frac{y}{k} = 1$

$$\Rightarrow y(x) = k \left(1 - \frac{x}{b}\right)$$

\Rightarrow we need $f(t)$, replace $x \rightarrow t$

$$\Rightarrow y(t) \text{ or } f(t) = k \left(1 - \frac{t}{b}\right)$$

\Rightarrow Now apply Laplace transform on $f(t)$

$$\Rightarrow L[f(t)] = k \int_0^b e^{-st} \left(1 - \frac{t}{b}\right) dt$$

$$\Rightarrow F(s) = k \int_0^b e^{-st} (1) dt - k \int_0^b e^{-st} \left(\frac{t}{b}\right) dt$$

$$\Rightarrow F(s) = k \left[\frac{e^{-st}}{-s} \right]_{t=0}^{t=b} - \frac{k}{b} \int_0^b e^{-st} (t) dt$$

$$\Rightarrow F(s) = k \left[\frac{e^{-bs}}{-s} + \frac{1}{s} \right] - \frac{k}{b} \left[t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{s^2} \right]_{t=0}^{t=b}$$

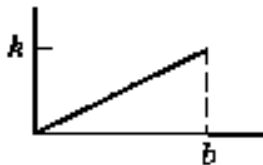
$$\Rightarrow F(s) = k \left[\frac{e^{-bs}}{-s} + \frac{1}{s} \right] - \frac{k}{b} \left[\left\{ b \frac{e^{-bs}}{-s} - (1) \frac{e^{-bs}}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right]$$

$$\Rightarrow F(s) = \left[k \frac{e^{-bs}}{-s} + \frac{k}{s} \right] - \left[\left\{ k \frac{e^{-bs}}{-s} - \frac{k}{b} \frac{e^{-bs}}{s^2} \right\} + \frac{k}{bs^2} \right]$$

$$\Rightarrow F(s) = -k \frac{e^{-bs}}{s} + \frac{k}{s} + k \frac{e^{-bs}}{s} + \frac{k}{b} \frac{e^{-bs}}{s^2} - \frac{k}{bs^2}$$

$$\Rightarrow F(s) = \frac{k}{s} + \frac{k}{b} \frac{e^{-bs}}{s^2} - \frac{k}{bs^2} \Rightarrow F(s) = \frac{k}{bs^2} [bs + e^{-bs} - 1]$$

9. Find the Laplace transform of



Solution: Given Graph is a straight line of the form $y = mx$, where $m = \frac{k}{b}$

$$\Rightarrow y(x) = \frac{k}{b}x \text{ or } f(t) = \begin{cases} \frac{k}{b}t, & \text{if } 0 \leq t \leq b \\ 0, & t \geq b \end{cases}$$

$$\Rightarrow L[f(t)] = \frac{k}{b} \int_0^b e^{-st} (t) dt$$

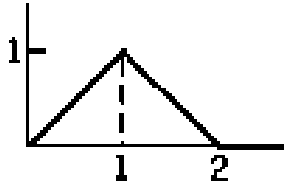
$$\Rightarrow F(s) = \frac{k}{b} \left[t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{s^2} \right]_{t=0}^{t=b}$$

$$\Rightarrow F(s) = \frac{k}{b} \left[\left\{ b \frac{e^{-bs}}{-s} - (1) \frac{e^{-bs}}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right]$$

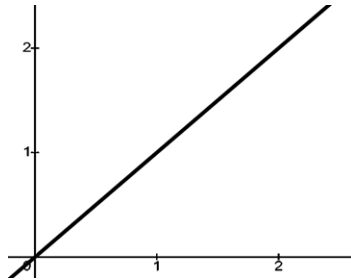
$$\Rightarrow F(s) = \left[\left\{ -k \frac{e^{-bs}}{s} - \left(\frac{k}{b} \right) \frac{e^{-bs}}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right]$$

$$\Rightarrow F(s) = \left[-k \frac{e^{-bs}}{s} - \frac{ke^{-bs}}{bs^2} + \frac{k}{bs^2} \right]$$

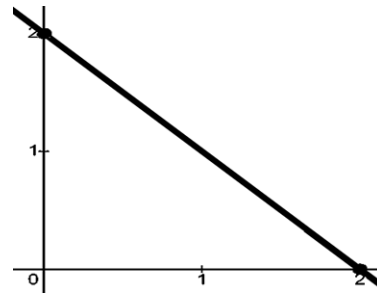
10. Find the Laplace transform of



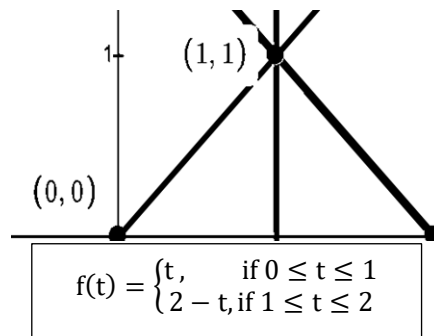
Solution: Lets Write Given graph as follows



$$y = mx, \text{ where } m = 1 \\ f(t) = t$$



$$y = 2 - x \\ f(t) = 2 - t$$



$$f(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1 \\ 2 - t, & \text{if } 1 \leq t \leq 2 \end{cases}$$

Applying Laplace transform on $f(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1 \\ 2 - t, & \text{if } 1 \leq t \leq 2 \end{cases}$

$$\Rightarrow L[f(t)] = \int_0^1 e^{-st}(t) dt + \int_1^2 e^{-st}(2 - t) dt$$

$$\Rightarrow F(s) = \left[t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{s^2} \right] \Big|_{t=0}^{t=1} + \left[(2 - t) \frac{e^{-st}}{-s} - (-1) \frac{e^{-st}}{s^2} \right] \Big|_{t=1}^{t=2}$$

$$\Rightarrow F(s) = \left[\left\{ -\frac{e^{-s}}{s} - (1) \frac{e^{-s}}{s^2} \right\} - \left\{ 0 - \frac{1}{s^2} \right\} \right] + \left[\left\{ 0 - (-1) \frac{e^{-2s}}{s^2} \right\} - \left\{ (1) \frac{e^{-s}}{-s} - (-1) \frac{e^{-s}}{s^2} \right\} \right]$$

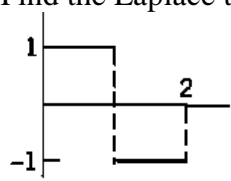
$$\Rightarrow F(s) = \left[\left\{ -\frac{e^{-s}}{s} - (1) \frac{e^{-s}}{s^2} \right\} + \frac{1}{s^2} \right] + \left[\frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - (1) \frac{e^{-s}}{s^2} \right]$$

$$\Rightarrow F(s) = -\frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$

$$\Rightarrow F(s) = \frac{1}{s^2} (1 - 2e^{-s} + e^{-2s})$$

$$\Rightarrow F(s) = \frac{1}{s^2} (1 - e^{-s})^2$$

Practice Questions

#	Find the Laplace transform of following f(t)	Answers
1	$f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$	$= \frac{e^{-(s-1)t}}{-(s-1)} \Big _0^1$
2	$f(t) = \begin{cases} t/a & 0 \leq t < a \\ 1 & t \geq a \end{cases}$	$= \frac{1}{a} \left[\frac{ae^{-as}}{-s} - \frac{1}{s^2} e^{-as} + \frac{1}{s^2} \right] + \frac{1}{s} e^{-as}$
3	$f(t) = \begin{cases} \sin 2t & 0 < t \leq \pi \\ 0 & t > \pi \end{cases}$	$= \frac{2}{s^2 + 4} (1 - e^{-\pi s})$
4	$f(t) = (5e^{3t} - 1)^2$	$\frac{25}{s-6} - \frac{10}{s-3} + \frac{1}{s}$
5	$f(t) = (2t+3)^3 + 6^t$	$= \frac{48}{s^4} + \frac{27}{s^3} + \frac{24}{s^2} + \frac{18}{s} + \frac{1}{s - \log 6}$
6	$f(t) = t^{-\frac{3}{2}} + t^{\frac{3}{2}}$	$= \frac{\frac{3}{2}\sqrt{\pi}}{s^{\frac{5}{2}}} - \frac{2\sqrt{\pi}}{s^{\frac{1}{2}}}$
7	$f(t) = \cos(2t+3) + \cos 7t \cos 3t$	$= \frac{s \cos 3}{s^2 + 4} - \frac{2 \sin 3}{s^2 + 4} + \frac{1}{2} \left\{ \frac{s}{s^2 + 100} + \frac{s}{s^2 + 16} \right\}$
8	$f(t) = 1 + \cos 2t$	$\frac{1}{s} + \frac{s}{s^2 + 4}$
9	$f(t) = t\sqrt{t} + 15t^3 + 7^t$	$= \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}} + \frac{90}{s^4} + \frac{1}{s - \log 7}$
10	Find the Laplace transform of 	$F(s) = \frac{e^{-2s} - 2e^{-s} + 1}{s}$

Laplace transform of $e^{at} f(t)$ (First Shifting property)

$$\text{If } L\{f(t)\} = F(s) \text{ then } L\{e^{at} f(t)\} = F(s-a)$$

$$\text{or, if we take the inverse on both sides } \Rightarrow e^{at} f(t) = L^{-1}F(s-a)$$

Laplace transform of $t^n f(t)$ (Multiplication by t^n)

$$\text{If } L\{f(t)\} = F(s) \text{ then } L\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Laplace transform of $\frac{f(t)}{t}$ [Division by t]

$$\text{If } L\{f(t)\} = F(s) \text{ then } L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s) ds$$

WORKED EXAMPLES

1. Find $L(-3t^4 e^{-0.5t})$

Solution: $L(-3t^4 e^{-0.5t}) = -3L(t^4 e^{-0.5t})$

$$\Rightarrow L(t^4) = \frac{4!}{s^5}$$

$$\Rightarrow L(t^4 e^{-0.5t}) = \frac{4!}{(s+0.5)^5} \text{ [using shifting property]}$$

$$\Rightarrow L(3t^4 e^{-0.5t}) = \frac{3 \cdot 4!}{(s+0.5)^5}$$

$$\Rightarrow L(3t^4 e^{-0.5t}) = \frac{3 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(s+0.5)^5}$$

$$\Rightarrow L(3t^4 e^{-0.5t}) = \frac{72}{(s+0.5)^5}$$

2. Find $L(e^{-3t} \cos \pi t)$

Solution:

$$\Rightarrow L(\cos \pi t) = \frac{s}{s^2 + \pi^2}$$

$$\Rightarrow L(e^{-3t} \cos \pi t) = \frac{s+3}{(s+3)^2 + \pi^2} \text{ [using shifting property]}$$

3. Find $L\{e^{-t} t^2\}$

Solution: We have $L\{t^2\} = \frac{2!}{s^3} \therefore L\{e^{-t} t^2\} = \frac{2!}{(s+1)^3}$ [using shifting property]

4. Find $L\{e^{2t} \sin^2 t\}$.

Solution: $L\{\sin^2 t\} = L\left\{\frac{1 - \cos 2t}{2}\right\}$

$$\Rightarrow \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos 2t\} = \frac{1}{2} \frac{1}{s} - \frac{1}{2} \frac{s}{s^2 + 4}$$

$$\Rightarrow \therefore L\{e^{2t} \sin^2 t\} = \frac{1}{2(s-2)} - \frac{1}{2} \frac{(s-2)}{(s-2)^2 + 4}, \text{ [using shifting property]}$$

5. Find $L\{t \cos 4t\}$

Solution: $L\{\cos 4t\} = \frac{s}{s^2 + 16}$

$$\Rightarrow L\{t \cos 4t\} = \frac{d}{ds} \left(\frac{s}{s^2 + 16} \right)$$

$$\Rightarrow L\{t \cos 4t\} = \frac{(s^2 + 16) \frac{d}{ds}(s) - s \frac{d}{ds}(s^2 + 16)}{(s^2 + 16)^2}$$

$$\Rightarrow L\{t \cos 4t\} = \frac{(s^2 + 16) - s(2s)}{(s^2 + 16)^2}$$

$$\Rightarrow L\{t \cos 4t\} = \frac{(s^2 + 16) - (2s^2)}{(s^2 + 16)^2}$$

$$\Rightarrow L\{t \cos 4t\} = \frac{(-s^2 + 16)}{(s^2 + 16)^2}.$$

\Rightarrow we can see only t^1 is multiplied with $\cos 4t$, so $n = 1$

\Rightarrow hence just multiply RHS of final answer with $(-1)^1$

$$\Rightarrow L\{t \cos 4t\} = (-1)^1 \frac{(-s^2 + 16)}{(s^2 + 16)^2}.$$

$$\Rightarrow L\{t \cos 4t\} = - \frac{(-s^2 + 16)}{(s^2 + 16)^2}.$$

$$\Rightarrow L\{t \cos 4t\} = \frac{(s^2 - 16)}{(s^2 + 16)^2}.$$

6. Find $L\{t e^{-at}\}$

Solution: $L\{e^{-at}\} = \frac{1}{s+a}$

$$\Rightarrow L\{t e^{-at}\} = \frac{d}{ds} \left(\frac{1}{s+a} \right)$$

$$\Rightarrow L\{t e^{-at}\} = \frac{(s+a) \frac{d}{ds}(1) - 1 \frac{d}{ds}(s+a)}{(s+a)^2}$$

$$\Rightarrow L\{t e^{-at}\} = \frac{0-1}{(s+a)^2} = \frac{-1}{(s+a)^2}$$

\Rightarrow we have t^1 so $n = 1$, hence just multiply final answer with $(-1)^1$

$$\Rightarrow L\{t e^{-at}\} = \frac{0-1}{(s+a)^2} = \frac{1}{(s+a)^2}$$

7. Find $L\{t^2 \sin at\}$.

Solution: $\{L \sin at\} = \frac{a}{s^2 + a^2} = F(s)$

$$\Rightarrow F(s) = \frac{d}{ds} \left(\frac{-a}{(s^2 + a^2)^2} \cdot 2s \right)$$

$$\Rightarrow F(s) = \frac{-2a(a^2 - 3s^2)}{(s^2 + a^2)^3}$$

8. Evaluate $L\{t e^{-2t} \sin 4t\}$.

Solution: We have $L\{\sin 4t\} = \frac{4}{s^2 + 16}$

$$\Rightarrow \therefore L\{e^{-2t} \sin 4t\} = \frac{4}{(s+2)^2 + 16} = F(s)$$

$$\Rightarrow L\{t e^{-2t} \sin 4t\} = (-1)' F'(s)$$

$$\Rightarrow = -1 \frac{d}{ds} \left(\frac{4}{(s+2)^2 + 16} \right) = \frac{4 \cdot 2(s+2)}{((s+2)^2 + 16)^2}$$

$$\Rightarrow = \frac{8(s+2)}{[(s+2)^2 + 16]^2}$$

9. Find $L\left\{\frac{\cos at - \cos bt}{t}\right\}$.

Solution: $L\left\{\frac{\cos at - \cos bt}{t}\right\}$.

$$\Rightarrow = L\left\{\frac{\cos at}{t}\right\} - L\left\{\frac{\cos bt}{t}\right\}, L\{\cos at\} = \frac{s}{s^2 + a^2}$$

$$\Rightarrow L\left\{\frac{\cos at}{t}\right\} = \int_s^\infty \frac{s}{s^2 + a^2} ds$$

$$\Rightarrow \lim_{k \rightarrow \infty} \int_s^k \frac{s}{s^2 + a^2} ds = \lim_{k \rightarrow \infty} \left[\frac{\log(s^2 + a^2)}{2} \right]_s^k = \frac{1}{2} \lim_{k \rightarrow \infty} \log \frac{k^2 + a^2}{s^2 + a^2}$$

Similarly $L\left\{\frac{\cos bt}{t}\right\} = \frac{1}{2} \lim_{k \rightarrow \infty} \log \frac{k^2 + b^2}{s^2 + b^2}$

$$\Rightarrow \therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2} \lim_{k \rightarrow \infty} \log \frac{k^2 + a^2}{k^2 + b^2} - \frac{1}{2} \log \frac{(s^2 + a^2)}{s^2 + b^2}$$

$$\Rightarrow = \frac{1}{2} \log \left(\frac{s^2 + b^2}{s^2 + a^2} \right)$$

10. Find $L\left\{\frac{1 - \cos t}{t^2}\right\}$.

Solution: $L\{1 - \cos t\} = \frac{1}{s} - \frac{s}{s^2 + 1}$

$$\Rightarrow L\left\{\frac{1 - \cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) ds = \lim_{k \rightarrow \infty} \int_s^k \frac{1}{s} - \frac{s}{s^2 + 1} ds = \lim_{k \rightarrow \infty} \left[\log s - \log \sqrt{s^2 + 1} \right]_s^k$$

$$\Rightarrow = -\log \frac{s}{\sqrt{s^2 + 1}} = \log \frac{\sqrt{s^2 + 1}}{s}$$

$$\Rightarrow \text{Now, } L\left\{\frac{1 - \cos t}{t^2}\right\} = \lim_{k \rightarrow \infty} \int_s^k \log \frac{\sqrt{s^2 + 1}}{s} ds$$

$$\Rightarrow = \lim_{k \rightarrow \infty} \int_s^k \log \sqrt{s^2 + 1} ds - \lim_{k \rightarrow \infty} \int_s^k \log s ds$$

$$\Rightarrow = \lim_{k \rightarrow \infty} \left[\log \sqrt{s^2 + 1} \cdot s - \int \frac{s}{\sqrt{s^2 + 1}} \cdot \frac{2s}{2\sqrt{s^2 + 1}} ds \right]_s^k - \lim_{k \rightarrow \infty} \left[\log s \cdot s - \int s \frac{1}{s} ds \right]_s^k$$

$$\Rightarrow = \lim_{k \rightarrow \infty} \left[s \log \sqrt{s^2 + 1} - \frac{2}{2} \int \frac{s^2}{s^2 + 1} ds \right]_s^k - \lim_{k \rightarrow \infty} [s \log s - s]_s^k$$

$$\Rightarrow = \lim_{k \rightarrow \infty} \left[s \log \sqrt{s^2 + 1} - s + \tan^{-1} s \right]_s^k - \lim_{k \rightarrow \infty} [s \log s - s]_s^k$$

$$\Rightarrow = -k + \frac{\pi}{2} - s \log \sqrt{s^2 + 1} + s - \tan^{-1} s + k + s \log s - s = \frac{\pi}{2} - s \log \sqrt{s^2 + 1} + s \log s - \tan^{-1} s$$

$$\Rightarrow = \cot^{-1} s - s \log \left\{ \frac{\sqrt{s^2 + 1}}{s} \right\}.$$

11. Evaluate $L\left\{\frac{1 - e^{-at}}{t}\right\}.$

Solution: $L\{1 - e^{-at}\} = \frac{1}{s} - \frac{1}{s + a} = F(s)$

$$\Rightarrow L\left\{\frac{1 - e^{-at}}{t}\right\} = \int_s^\infty \left\{ \frac{1}{s} - \frac{1}{s + a} \right\} ds = \lim_{k \rightarrow \infty} \int_s^k \left(\frac{1}{s} - \frac{1}{s + a} \right) ds$$

$$\Rightarrow = \lim_{k \rightarrow \infty} [\log s - \log(s + a)]_s^k$$

$$\Rightarrow = \lim_{k \rightarrow \infty} \log \frac{k}{k + a} - \log \frac{s}{s + a} = 0 - \log \frac{s}{s + a} = \log \left(\frac{s + a}{s} \right)$$

12. Evaluate $L\left[e^{-4t} \frac{\sin 3t}{t} \right].$

Solution: $L[\sin 3t] = \frac{3}{s^2 + 9}$

$$\Rightarrow L\left[\frac{\sin 3t}{t}\right] = \int_s^\infty \frac{3}{s^2 + 9} ds = \frac{3}{3} \tan^{-1} \frac{s}{3} \Big|_s^\infty = \frac{\pi}{2} - \tan^{-1} \frac{s}{3} = \cot^{-1} \frac{s}{3}$$

$$\Rightarrow L\left[e^{-4t} \frac{\sin 3t}{t}\right] = \cot^{-1} \frac{s+4}{3}.$$

13. Find $L\{e^{-t} \sin t \cos 2t\}$.

Solution: $L\{\sin t \cos 2t\} = L\left\{\frac{\sin 3t - \sin t}{2}\right\}$

$$\Rightarrow = \frac{1}{2} L\{\sin 3t\} - \frac{1}{2} L\{\sin t\} = \frac{1}{2} \frac{3}{s^2 + 9} - \frac{1}{2} \frac{1}{s^2 + 1}$$

$$\Rightarrow L\{e^{-t} \sin t \cos 2t\} = \frac{3}{2} \frac{1}{(s+1)^2 + 9} - \frac{1}{2} \frac{1}{(s+1)^2 + 1} \quad \text{using shifting property}$$

Practice Questions

#	Questions	Answers
1	Find $L\{e^{-t}(a_0 + a_1 t + a_2 t^2 + \dots a_n t^n)\}$	$\frac{a_0}{s+1} + a_1 \frac{2}{(s+1)^2} + a_2 \frac{3!}{(s+1)^3} + \dots$ $+ a_n \frac{n!}{(s+1)^{n+1}}$
2	Evaluate $L[\cos at \sinh at]$	$\frac{1}{2} \left\{ \frac{s-a}{(s-a)^2 + a^2} - \frac{s+a}{(s+a)^2 + a^2} \right\}$
3	Find $L\{t^2 \cos t\}$.	$\frac{2s(s^2 - 3)}{(s^2 + 1)^3}$
4	Evaluate $L\{t^2 e^t \sin t\}$.	$\frac{6(s-1)^2 - 2}{((s-1)^2 + 1)^3}$
5	Find $L\left\{\frac{e^{-at} - e^{-bt}}{t}\right\}$.	$\log\left(\frac{s+b}{s+a}\right)$
6	Evaluate $L\left[\frac{\sin at}{t}\right]$	$\tan^{-1}\left(\frac{a}{s}\right)$
7	Find $L\{t.e^{-2t} \sin 4t\}$	$F(s) = \frac{8(s+2)}{[(s+2)^2 + 16]^2}$

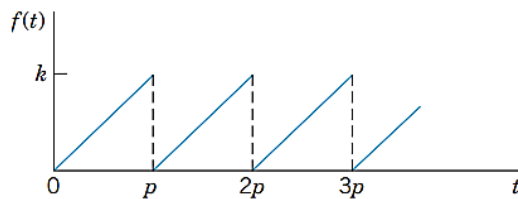
Laplace Transform of Periodic Functions:

The Laplace transform of a piecewise continuous function $f(t)$ with period T is

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt \quad (s>0)$$

WORKED EXAMPLES

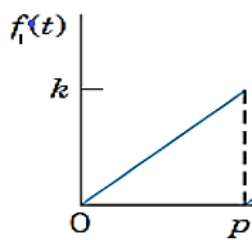
1. Find the Laplace transform of following Periodic Signals:



Solution: Saw-tooth wave with amplitude k and time period P (T=P):

Hence, Laplace transform of the periodic function is $L\{f(t)\} = \frac{1}{1-e^{-sT}} F_1(s)$, where $F_1(s)$ is

Laplace transform of One cycle. Let us consider the signal



Let us find the equation of line from point $(0,0)$ to (P, k) . Slope $(m) = \frac{k-0}{P-0} = \frac{k}{P}$,

the equation of a line passing through the origin with slope m is: $y - y_1 = m(x - x_1)$

$f_1(t) - 0 = \frac{k}{P}(t - 0)$, $f_1(t) = \frac{k}{P}(t)$, $0 \leq t \leq P$, now we apply Laplace transform on $f_1(t)$

By def. of Laplace transform $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

$$\Rightarrow L\{f_1(t)\} = \int_0^P e^{-st} k/P(t) dt$$

$$\Rightarrow L\{f_1(t)\} = \frac{k}{P} \int_0^P t e^{-st} dt$$

$$\Rightarrow F_1(s) = \frac{k}{P} \left[t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{s^2} \right] \Big|_{t=0}^{t=P}$$

$$\Rightarrow F_1(s) = \frac{k}{P} \left[\left(P \frac{e^{-sP}}{-s} - (1) \frac{e^{-sP}}{s^2} \right) - \left(0 - \frac{1}{s^2} \right) \right]$$

$$\Rightarrow F_1(s) = \frac{k}{P} \left[-P \frac{e^{-sP}}{s} - \frac{e^{-sP}}{s^2} + \frac{1}{s^2} \right] \Rightarrow F_1(s) = \frac{k}{P} \left[-P \frac{e^{-sP}}{s} - \frac{e^{-sP}}{s^2} + \frac{1}{s^2} \right]$$

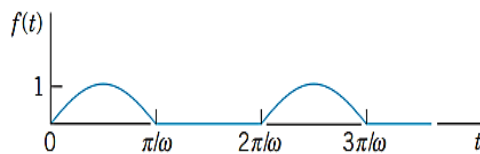
⇒ Laplace transform of the periodic function is $L\{f(t)\} = \frac{1}{1-e^{-Ps}} F_1(s)$

$$\Rightarrow L\{f(t)\} = \frac{1}{1-e^{-Ps}} \times \frac{k}{P} \left[-P \frac{e^{-sP}}{s} - \frac{e^{-sP}}{s^2} + \frac{1}{s^2} \right]$$

$$\Rightarrow L\{f(t)\} = \frac{1}{1-e^{-Ps}} \times \frac{k}{Ps^2} [-Pse^{-sP} - e^{-sP} + 1]$$

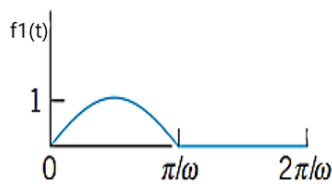
$$\Rightarrow L\{f(t)\} = \frac{1}{1-e^{-Ps}} \times \frac{k}{Ps^2} [1 - Pse^{-sP} - e^{-sP}]$$

2. Find the Laplace transform of following Periodic Signal



Solution: Half-wave rectifier time period ($T = \frac{2\pi}{\omega}$)

Laplace transform of the periodic function is $L\{f(t)\} = \frac{1}{1-e^{-sT}} F_1(s)$, where $F_1(s)$ is Laplace transform of One cycle. Let us consider the signal.



$$\Rightarrow f_1(t) = A \sin \omega_0 t, \text{ where } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\frac{2\pi}{\omega}} = 2\pi \times \frac{\omega}{2\pi} = \omega$$

$$\Rightarrow f_1(t) = \sin \omega t, \text{ when } 0 \leq T \leq \frac{2\pi}{\omega}$$

$$\Rightarrow f_1(t) = \begin{cases} \sin \omega t, & \text{if } 0 \leq T \leq \frac{\pi}{\omega} \\ 0, & \text{if } \frac{\pi}{\omega} \leq T \leq \frac{2\pi}{\omega} \end{cases}$$

By def. of Laplace transform $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$

$$\Rightarrow L\{f_1(t)\} = \int_0^{2\pi/\omega} e^{-st} f_1(t) dt$$

$$\Rightarrow L\{f_1(t)\} = \int_0^{\pi/\omega} e^{-st} \sin \omega t dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} (0) dt$$

$$\Rightarrow \therefore L\{f_1(t)\} = \int_0^{\pi/\omega} e^{-st} \sin \omega t dt \Rightarrow \left[\frac{e^{-st}}{s^2 + \omega^2} [-s \sin \omega t - \omega \cos \omega t] \right]_0^{\pi/\omega}$$

$$\Rightarrow \frac{e^{-s\pi/\omega}}{s^2 + \omega^2} [-s \sin \omega(\pi/\omega) - \omega \cos \omega(\pi/\omega)] - \frac{1}{s^2 + \omega^2} [s \sin \omega(0) - \omega \cos \omega(0)]$$

$$\Rightarrow \frac{-e^{-s\pi/\omega}}{s^2 + \omega^2} [s \sin \omega(\pi/\omega) + \omega \cos \omega(\pi/\omega)] - \frac{1}{s^2 + \omega^2} [s \sin \omega(0) - \omega \cos \omega(0)]$$

$$\Rightarrow \frac{e^{-s\pi/\omega}}{s^2 + \omega^2} [\omega] + \frac{1}{s^2 + \omega^2} [\omega] \Rightarrow F_1(s) = \frac{\omega}{s^2 + \omega^2} [e^{-s\pi/\omega} + 1]$$

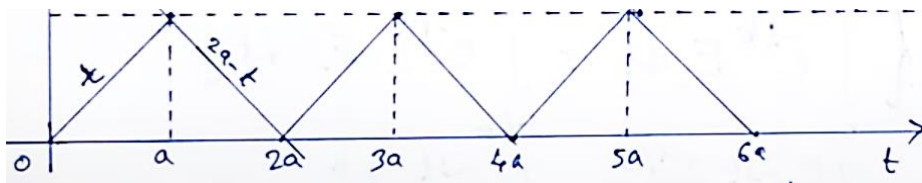
$$\Rightarrow \text{Laplace transform of the periodic function is } L\{f(t)\} = \frac{1}{1 - e^{-sT}} F_1(s),$$

$$\Rightarrow L\{f(t)\} = \frac{1}{1 - e^{-2\pi/\omega}} \left[\frac{\omega}{s^2 + \omega^2} [e^{-s\pi/\omega} + 1] \right]$$

$$\Rightarrow F(s) = \frac{1}{1 - e^{-2\pi/\omega}} \left[\frac{\omega}{s^2 + \omega^2} [e^{-s\pi/\omega} + 1] \right]$$

$$\Rightarrow F(s) = \frac{\omega(e^{-s\pi/\omega} + 1)}{(s^2 + \omega^2)(1 - e^{-2\pi/\omega})}$$

3. Find the Laplace transform of



Solution: given periodic signal can be written as $f(t) = \begin{cases} t & 0 \leq t \leq a \\ 2a - t & t > a \end{cases}$ with period $T = 2a$

$$\Rightarrow L\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$\Rightarrow = \frac{1}{1 - e^{-2as}} \left\{ \int_0^a e^{-st} t dt + \int_a^{2a} e^{-st} (2a - t) dt \right\}$$

$$\Rightarrow = \frac{1}{1 - e^{-2as}} \left\{ \left[\frac{te^{-st}}{-s} + \frac{1}{s} \int e^{-st} \right]_0^a + (2a - t) \frac{e^{-st}}{-s} + \frac{1}{s} \int e^{-st} (-1) dt \right\}_a^{2a}$$

$$\Rightarrow = \frac{1}{1 - e^{-2as}} \left\{ \left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a - \left((2a - t) \frac{e^{-st}}{s} - \frac{e^{-st}}{s^2} \right)_a^{2a} \right\}$$

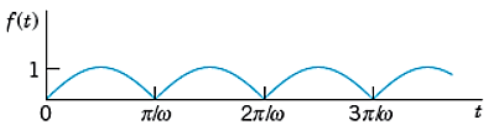
$$\Rightarrow = \frac{1}{1 - e^{-2as}} \frac{1}{s^2} \{ 1 - e^{-as} [as + 1 - e^{-as} - as^2 + 1] \}$$

$$\Rightarrow = \frac{1}{s^2(1 - e^{-2as})} (e^{-2as} - 2e^{-as} + 1)$$

$$\Rightarrow = \frac{(1 - e^{-as})^2}{s^2 (1 - e^{-as})(1 + e^{-as})}$$

$$\Rightarrow = \frac{1}{s^2} \frac{1 - e^{-as}}{1 + e^{-as}} = \frac{1}{s^2} \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$$

Practice Questions

#	Questions	Answer
1	Find the Laplace transform of Periodic Signal Full-wave rectifier 	$\frac{\omega}{s^2 + \omega^2} \coth\left(\frac{\pi s}{2\omega}\right)$
2	Find the Laplace transform of the periodic function If $f(t) = \begin{cases} E & 0 \leq t \leq a \\ -E & a < t \leq 2a \end{cases}$, where $f(t+2a) = f(t)$.	$L\{f(t)\} = \frac{E}{s} \tanh\left(\frac{as}{2}\right)$
3	$f(t) = t^2, 0 < t \leq 2$ with Period $T = 2$.	$\frac{2}{s^3(1 - e^{-2s})} [1 - (1 + 2s + 2s^2)e^{-2s}]$

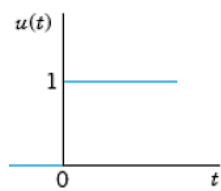
Laplace Transform of Unit Step Function (Heaviside Function):

These functions are suitable for solving ODEs with complicated right sides of considerable engineering interest, such as single waves, inputs (driving forces) that are discontinuous or act for some time only, periodic inputs more general than just cosine and sine, or impulsive

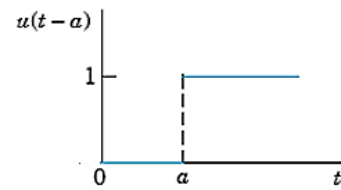
Unit Step Function (Heaviside Function) $u(t-a)$:

The **unit step function** or **Heaviside function** $u(t-a)$ is zero for $t < a$, Has a jump of Size 1 at $t = a$ (where we can leave it undefined) and is 1 for $t > a$ in a formula:

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$$



Unit step function $u(t)$



Unit step function $u(t-a)$

$$\mathcal{L} u(t-a) = \frac{e^{-as}}{s}$$

forces acting for an instant (hammer blows, for example).

Note: The unit step function is a typical “engineering function” made to measure for engineering applications, which often involve functions (mechanical or electrical driving forces) that are either “off” or “on.” Multiplying functions $f(t)$ with $u(t-a)$ we can produce all sorts of effects. The simple basic idea is illustrated as:

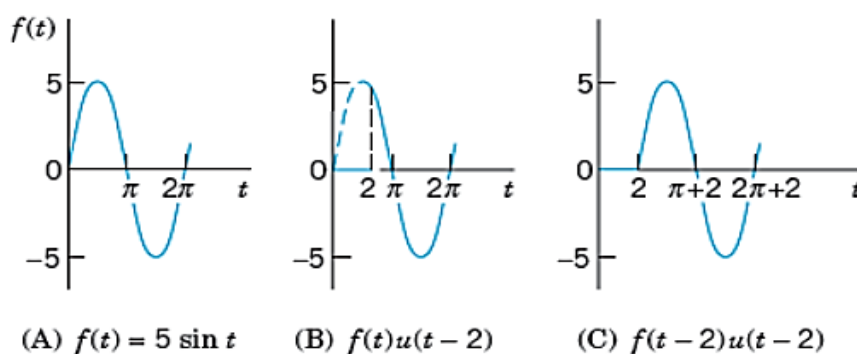


Fig.1: Effects of the unit step function: (A) Given function. (B) Switching off and on. (C) Shift.

In Fig.1 function is shown in (A). In (B) it is switched off between $t=0$ and $t=2$ [because $u(t-2) = 0$ when $t < 2$ and is switched on beginning at $t=2$]. In (C), it is shifted to the right by 2

units, say, for instance, by 2 secs, so that it begins 2 secs later in the same fashion as before.

More generally we have the following:

Let $f(t) = 0$ for all negative t . Then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ shifted (translated) to the right by the amount a .

Second Shifting Theorem; Time Shifting

If $f(t)$ has the transform $F(s)$, then the “shifted function”

$$(3) \quad f(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

has the transform $e^{-as}F(s)$. That is, if $\mathcal{L} f(t) = F(s)$, then

$$(4) \quad \mathcal{L} f(t-a)u(t-a) = e^{-as}F(s).$$

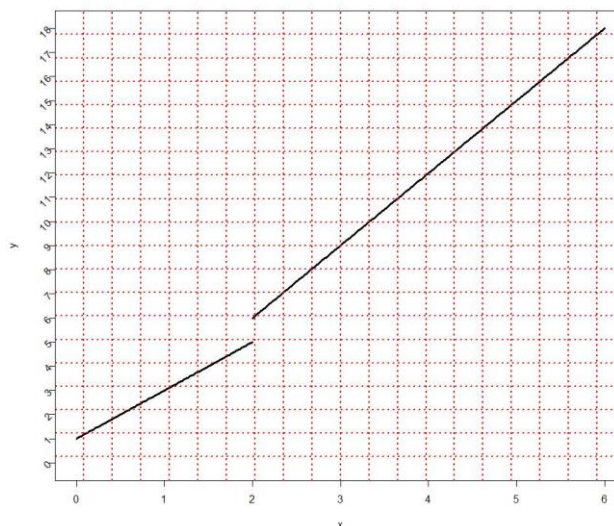
In Fig.1, the transform of $5 \sin t$ is $F(s) = \frac{5}{s^2+1}$, hence the shifted function of $5 \sin (t-2)u(t-2)$ which is shown in fig.1(C) has the transform : $\frac{5 e^{-2s}}{s^2+1}$.

Note: If $f(t) = \begin{cases} f_1(t) & 0 \leq t < a \\ f_2(t) & a \leq t < b \\ f_3(t) & t \geq b \end{cases}$ then,

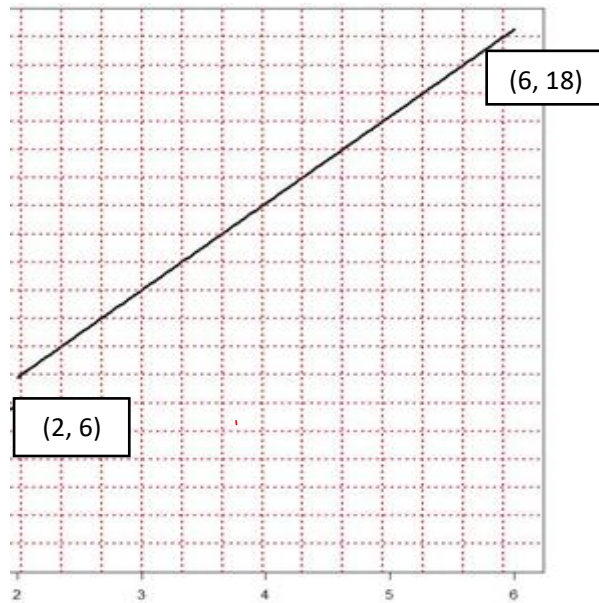
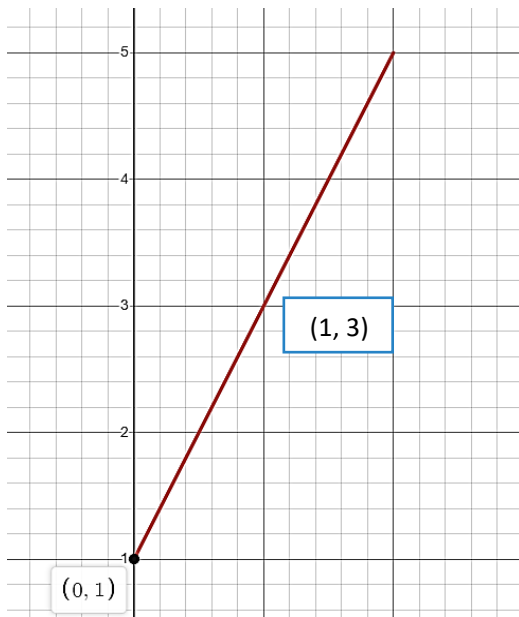
$$f(t) = f_1(t) + \{f_2(t) - f_1(t)\}u(t-a) + \{f_3(t) - f_2(t)\}u(t-b).$$

WORKED EXAMPLES

1. Identify the function $f(t)$ from the given figure and hence find its Laplace transform.



Solution:



from graph (1) it's a straight line from $(0, 1)$ to $(1, 3)$.

$$\text{slope } m = \frac{3-1}{1-0} = 2$$

$$\therefore y - y_1 = m(x - x_1) \Rightarrow (y - 1) = 2(x - 0) \Rightarrow y - 1 = 2x$$

$$\text{or } \boxed{y = 2x + 1} \text{ when } 0 \leq x \leq 2$$

it's a straight line from $(2, 6)$ to $(6, 18)$

$$\text{slope} = 3$$

$$\therefore (y - y_1) = m(x - x_1) \Rightarrow (y - 6) = 3(x - 2)$$

$$y = 3x - 6 + 6 \Rightarrow \boxed{y = 3x} \text{ when } x \geq 2$$

Now write in terms of 't'

$$f(t) = \begin{cases} 2t + 1 & \text{if } 0 \leq t < 2 \\ 3t & \text{if } t \geq 2 \end{cases}$$

$$f(t) = \begin{cases} 2t+1 & \text{if } 0 \leq t < 2 \\ 3t & \text{if } t \geq 2 \end{cases}$$

$$\therefore f(t) = (2t+1) + 3t - 2t - 1 \cdot u(t-2)$$

$$\therefore f(t) = (2t+1) + (t-1) \cdot u(t-2)$$

$$\therefore L\{f(t)\} = L\{(2t+1)\} + L\{(t-1) \cdot u(t-2)\}$$

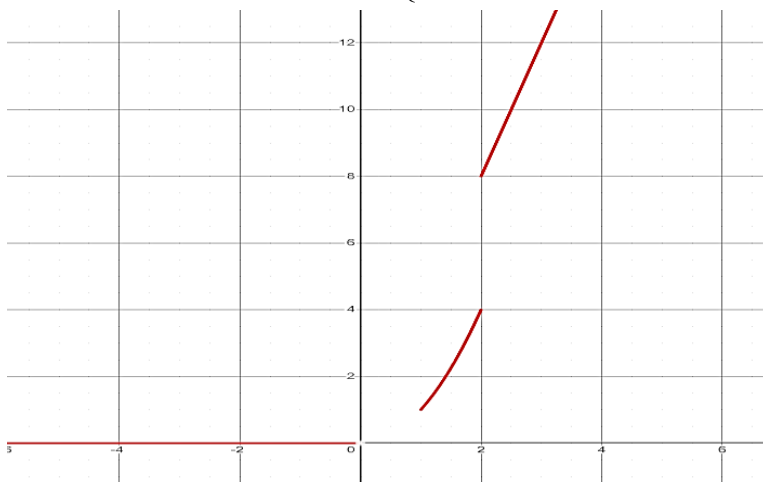
$$\begin{aligned} &L\{(2t+1)\} \\ &2L(t) + L(1) \\ &F(s) = \frac{2}{s^2} + \frac{1}{s} \end{aligned}$$

$$\begin{aligned} &L\{(t-1) \cdot u(t-2)\} \\ &f(t-2) = t-1 \\ &\text{change } t \text{ to } t+2 \text{ both side} \\ &f(t) = t+1 \\ &L\{f(t)\} = L\{t\} + L\{1\} \\ &F(s) = \frac{1}{s^2} + \frac{1}{s} \end{aligned}$$

$$\therefore F(s) = \frac{2}{s^2} + \frac{1}{s} + \left(\frac{1}{s^2} + \frac{1}{s} \right) e^{-2s}$$

2. Plot the function $f(t) = \begin{cases} t^2 & 1 \leq t < 2 \\ 4t & t \geq 2 \end{cases}$. Represent it using unit step functions. Find its Laplace transform.

Solution: We can take $f(t) = \begin{cases} 0 & 0 < t < 1 \\ t^2 & 1 \leq t < 2 \\ 4t & t \geq 2 \end{cases}$



$$\Rightarrow \therefore f(t) = 0 + (t^2 - 0)u(t-1) + (4t - t^2)u(t-2)$$

$$\Rightarrow L\{f(t)\} = L\{t^2 u(t-1)\} + L\{(4t - t^2)u(t-2)\}$$

$$\Rightarrow = L\{(t-1)^2 + 2t - 2 \cdot u(t-1)\} + L\{4 - (t-2)^2 \cdot u(t-2)\}$$

$$\Rightarrow = e^{-s} \frac{2!}{s^3} + \frac{2e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{4e^{-2s}}{s} - \frac{e^{-2s} 2!}{s^3}$$

3. If $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \cos 2t & \pi \leq t < 2\pi \\ \cos 3t & t \geq 2\pi \end{cases}$. Express $f(t)$ in terms of unit step function and hence find its Laplace transform.

Solution: $f(t) = \cos t + \{\cos 2t - \cos t\}u(t - \pi) + \{\cos 3t - \cos 2t\}u(t - 2\pi)$

$$\therefore f(t) = \cos t + \{\cos 2(t - \pi + \pi) - \cos(t - \pi + \pi)\}u(t - \pi) + \{\cos 3(t - 2\pi + 2\pi) - \cos 2(t - 2\pi + 2\pi)\}u(t - 2\pi)$$

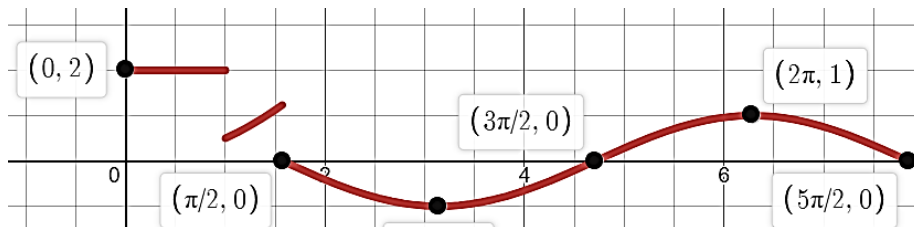
$$L\{f(t)\} = L\{\cos t\} + L\{\cos(2(t - \pi + \pi)).u(t - \pi)\} - L\{\cos(t - \pi + \pi).u(t - \pi)\} + L\{\cos(3(t - 2\pi + 2\pi)).u(t - 2\pi)\} - L\{\cos(2(t - 2\pi + 2\pi)).u(t - 2\pi)\}$$

$$\therefore L\{f(t)\} = \frac{s}{s^2+1} + e^{-\pi s} \left\{ \frac{s}{s^2+4} + \frac{s}{s^2+1} \right\} + e^{-2\pi s} \left\{ \frac{s}{s^2+9} - \frac{s}{s^2+4} \right\}$$

4. Write the following function using unit step functions and find its transform.

$$f(t) = \begin{cases} 2, & \text{if } 0 < t < 1 \\ \frac{1}{2}t^2 & \text{if } 1 \leq t < \frac{1}{2}\pi, \\ \cos(t) & t > \frac{1}{2}\pi \end{cases}$$

Solution:



$$f(t) = 2 + \left(\frac{1}{2}t^2 - 2\right).u(t - 1) + \left(\cos(t) - \frac{1}{2}t^2\right)u\left(t - \frac{1}{2}\pi\right)$$

$$\Rightarrow \therefore L\{f(t)\} = L\{2\} + L\left\{\frac{1}{2}t^2 - 2.u(t - 1)\right\} + L\left\{\cos(t) - \frac{1}{2}t^2 u\left(t - \frac{1}{2}\pi\right)\right\}$$

$$L\{2\} = \frac{2}{s}$$

$$L\left\{\frac{1}{2}t^2 - 2.u(t - 1)\right\}$$

$$f(t - 1) = \frac{1}{2}t^2 - 2$$

Change $t \rightarrow t + 1$ both side

$$f(t) = \frac{1}{2}(t + 1)^2 - 2$$

$$f(t) = \frac{1}{2}[t^2 + 2t + 1] - 2$$

$$L\{f(t)\} = \frac{1}{2}L[t^2] + L[t] + \frac{1}{2}L[1] - 2L[1]$$

$$F(s) = \frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s} - \frac{2}{s}$$

$$F(s) = \frac{1}{s^3} + \frac{1}{s^2} - \frac{3}{2s} L\left\{\cos(t) - \frac{1}{2}t^2 u\left(t - \frac{1}{2}\pi\right)\right\}$$

$$f(t - \pi/2) = \cos(t) - \frac{1}{2}t^2$$

Change $t \rightarrow t + \frac{\pi}{2}$ both side

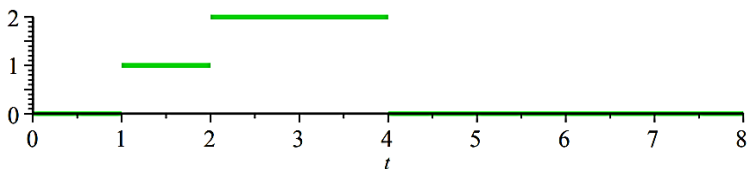
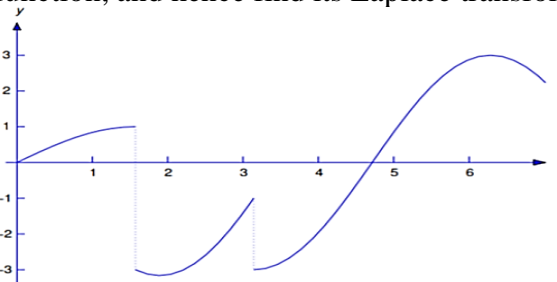
$$f(t) = \cos\left(t + \frac{\pi}{2}\right) - \frac{1}{2}\left(t + \frac{\pi}{2}\right)^2$$

$$f(t) = -\sin(t) - \frac{1}{2}[t^2 + \pi t + (\pi/2)^2]$$

$$L\{f(t)\} = -L\{\sin(t)\} - \frac{1}{2}L[t^2] - \frac{1}{2}\pi L[t] - \frac{1}{4}\pi^2 L[1]$$

$$F(s) = -\frac{1}{s^2+1} - \frac{1}{s^3} - \frac{1}{2s} - \frac{\pi^2}{4s}$$

$$\Rightarrow \therefore L\{f(t)\} = \frac{2}{s} + \left[\frac{1}{s^3} + \frac{1}{s^2} - \frac{3}{2s}\right]e^{-s} + \left[-\frac{1}{s^2+1} - \frac{1}{s^3} - \frac{\pi}{2s} - \frac{\pi^2}{4s}\right]e^{-\frac{\pi s}{2}}$$

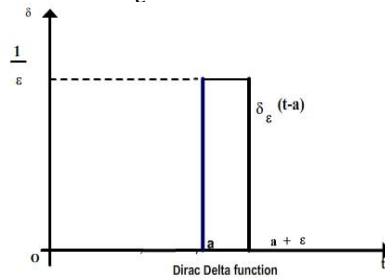
#	Practice Questions	Answer
1	<p>Express the given piecewise continuous function in terms of unit step function, and hence find its Laplace transform.</p> 	$f(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1 \\ 1 & \text{if } 1 \leq t < 2 \\ 2 & \text{if } 2 \leq t < 4 \\ 0 & \text{if } t \geq 4 \end{cases}$ $F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - 2 \frac{e^{-4s}}{s}$
2	<p>Express the given piecewise continuous function in terms of unit step function, and hence find its Laplace transform.</p> 	$f(t) = \begin{cases} \sin t, & \text{if } 0 \leq t < \frac{\pi}{2} \\ \cos t - 3 \sin t & \text{if } \frac{\pi}{2} \leq t < \pi \\ 3 \cos t & \text{if } t \geq \pi \end{cases}$ $F(s) = \frac{1}{s^2 + 1} - e^{-\frac{\pi}{2}s} \left(\frac{1 + 4s}{s^2 + 1} \right) - e^{-\pi s} \left[\frac{3 + 2s}{s^2 + 1} \right]$
3	<p>$f(t) = \begin{cases} \cos t & 0 < t < \pi \\ 1 & \pi \leq t < 2\pi \\ \sin t & t \geq 2\pi \end{cases}$. Find the Laplace Transform of $f(t)$</p>	$\therefore L\{f(t)\} = \frac{s}{s^2 + 1} + \frac{e^{-\pi s}}{s} + e^{-\pi s} \frac{s}{s^2 + 1} + \frac{e^{-2\pi s}}{s^2 + 1} - \frac{e^{-2\pi s}}{s}$

Laplace Transform of Dirac-Delta Function:

"Dirac's delta function" can be useful to model phenomena of an impulsive nature where actions of forces—mechanical, electrical, etc.—are applied over short intervals of time. We consider the function

$$f_{\varepsilon}(t - a) = \begin{cases} \frac{1}{\varepsilon} & a \leq t \leq a + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

which represents, for instance, a force of magnitude $\frac{1}{\varepsilon}$ acting from $t=a$ to $t=a + \varepsilon$, where k is positive and small.



The limit of f_{ε} as $\varepsilon \rightarrow 0$ ($\varepsilon > 0$) is denoted by $\delta(t - a)$ (i.e.,) $\delta(t - a) = \lim_{\varepsilon \rightarrow 0} f_{\varepsilon}(t - a)$.

$\delta(t - a)$ is called the Dirac delta function or the unit impulse function.

$$\delta(t - a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_0^{\infty} \delta(t - a) dt = 1,$$

$$L[\delta(t - a)] = \int_0^{\infty} e^{-st} \delta(t - a) dt = e^{-as}$$

Shifting property of $\delta(t - a)$ for a continuous function $g(t)$:

$$\int_0^{\infty} g(t) \delta(t - a) dt = g(a)$$

WORKED EXAMPLES

1. Evaluate $L\left[\frac{1}{t} \delta(t - a)\right]$

Solution: We know $L[\delta(t - a)] = e^{-as} \therefore L\left[\frac{1}{t} \delta(t - a)\right] = \int_s^{\infty} e^{-as} ds = \frac{1}{a} e^{-as}$

2. Evaluate $\int_0^{\infty} \sin 2t \delta(t - \pi/4) dt$

Solution: We know $\int_0^{\infty} f(t) \delta(t - a) dt = f(a) \therefore \int_0^{\infty} \sin 2t \delta(t - \pi/4) dt = \sin 2(\pi/4) = 1$

INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called the Inverse of Laplace transform of $F(s)$

and it is denoted by $L^{-1}\{F(s)\}$

The following methods are used to find the inverse Laplace transforms

Method 1. Use of table of Inverse Laplace Transforms.

Method 2. Use of Theorems of Inverse Laplace Transforms.

Method 3. Use of partial fractions.

From the table of Laplace transforms of elementary functions by using definition and Linearity property we can obtain a table of inverse Laplace transforms.

- | | |
|---|--|
| 1. $L\{1\} = \frac{1}{s}$ | $\therefore L^{-1}\left\{\frac{1}{s}\right\} = 1$ |
| 2. $L\{e^{at}\} = \frac{1}{s-a}$ | $\therefore L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ |
| 3. $L\{\sin at\} = \frac{a}{s^2 + a^2}$ | $\therefore L^{-1}\left\{\frac{1}{s^2 + a^2}\right\} = \frac{1}{a} \sin at$ |
| 4. $L\{\cos at\} = \frac{s}{s^2 + a^2}$ | $\therefore L^{-1}\left\{\frac{s}{s^2 + a^2}\right\} = \cos at$ |
| 5. $L\{\sinh at\} = \frac{a}{s^2 - a^2}$ | $\therefore L^{-1}\left\{\frac{1}{s^2 - a^2}\right\} = \frac{1}{a} \sinh at$ |
| 6. $L\{\cosh at\} = \frac{s}{s^2 - a^2}$ | $\therefore L^{-1}\left\{\frac{s}{s^2 - a^2}\right\} = \cosh at$ |
| 7. $L\{t^n\} = \frac{n!}{s^{n+1}}$ | $\therefore L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$ |
| 8. $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$ | $\therefore L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$ |
| 9. $L\{tf(t)\} = -F'(s)$ | $\therefore L^{-1}\{F'(s)\} = -tL^{-1}\{F(s)\} = -tf(t)$ |
| 10. $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s) ds$ | $\therefore L^{-1}\left\{\int_s^\infty F(s) ds\right\} = \frac{f(t)}{t}$ |
| 11. $L\left\{\int_0^t f(u) du\right\} = \frac{F(s)}{s}$ | $\therefore L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u) du.$ |
| 12. $L\{e^{at}f(t)\} = [L\{f(t)\}]_{s \rightarrow s-a} = F(s-a)$ | $\therefore L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\}$ |
| 13. $L\{e^{-at}f(t)\} = [L\{f(t)\}]_{s \rightarrow s+a} = F(s+a)$ | $\therefore L^{-1}\{F(s+a)\} = e^{-at}L^{-1}\{F(s)\}$ |

WORKED EXAMPLES

1. Find the inverse Laplace transform of the function $\frac{1}{2s^2 + 9}$.

Solution: $\frac{1}{2s^2 + 9} = \frac{1}{2\left(s^2 + \frac{9}{2}\right)} = \frac{1}{2\left(s^2 + \left(\frac{3}{\sqrt{2}}\right)^2\right)} \therefore L^{-1}\left\{\frac{1}{2s^2 + 9}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{s^2 + \left(\frac{3}{\sqrt{2}}\right)^2}\right\}$

$$\Rightarrow = \frac{1}{2} \sin\left(\frac{3}{\sqrt{2}}t\right) \frac{\sqrt{2}}{3}$$

2. Find the inverse Laplace transform of the function $\frac{3s + 5\sqrt{2}}{s^2 + 8}$.

Solution: $\frac{3s + 5\sqrt{2}}{s^2 + 8} = 3 \frac{s}{s^2 + (2\sqrt{2})^2} + 5\sqrt{2} \frac{1}{s^2 + (2\sqrt{2})^2}$

$$\Rightarrow \therefore L^{-1}\left\{\frac{3s + 5\sqrt{2}}{s^2 + 8}\right\} = 3 \cos 2\sqrt{2}t + \frac{5}{2} \sin (2\sqrt{2}t)$$

3. Find the inverse Laplace transform of the function $\frac{5s + 1}{s^2 - 25}$.

Solution: $\frac{5s + 1}{s^2 - 25} = \frac{5s}{s^2 - 25} + \frac{1}{s^2 - 25}$

$$5L^{-1}\left\{\frac{s}{s^2 - 5^2}\right\} + L^{-1}\left\{\frac{1}{s^2 - 5^2}\right\}$$

$$5 \cosh 5t + \frac{1}{5} \sinh 5t$$

4. Find the inverse Laplace transform of the function $\frac{s}{L^2 s^2 + n^2 \pi^2}$

Solution: $\frac{s}{L^2 s^2 + n^2 \pi^2} = \frac{1}{L^2} \left[\frac{s}{s^2 + \frac{n^2 \pi^2}{L^2}} \right]$

$$\frac{1}{L^2} L^{-1} \left[\frac{s}{(s)^2 + \left(\frac{n\pi}{L}\right)^2} \right] = \frac{1}{L^2} \cos \left[\frac{n\pi}{L} \right] t = \frac{1}{L^2} \cos \left(\frac{n\pi}{L} \right)$$

5. Find the inverse Laplace transform of the function $\frac{12}{s^4} - \frac{228}{s^6}$

Solution: we know that $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$

$$L^{-1}\left[\frac{12}{s^4} - \frac{228}{s^6}\right] = 12L^{-1}\left[\frac{1}{s^4}\right] - 228L^{-1}\left[\frac{1}{s^6}\right]$$

$$12\frac{t^3}{3!} - 228\frac{t^5}{5!}$$

$$\text{we have } \frac{t^n}{n!} = (n-1)!$$

$$12\frac{t^3}{3!} - 228\frac{t^5}{5!} = 12\frac{t^3}{6} - 228\frac{t^5}{120} = 2t^3 - 1.9t^5$$

6. Find the inverse Laplace transform of the function $\frac{s+1}{s^2+s+1}$.

$$\text{Solution: } \frac{s+1}{s^2+s+1} = \frac{s+1}{s^2+s+1+\frac{1}{4}-\frac{1}{4}} = \frac{s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{s+\frac{1}{2}+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\therefore L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = e^{-t/2} \cos \frac{\sqrt{3}}{2}t + \frac{1}{2}e^{-t/2} \sin \frac{\sqrt{3}}{2}t \frac{2}{\sqrt{3}} = e^{-t/2} \left[\cos \frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t \right]$$

7. Find the inverse Laplace transform of the function $\frac{3s+7}{s^2-2s-3}$.

$$\text{Solution: } F(s) = \frac{3s+7}{s^2-2s-3} = \frac{3s+7}{(s-1)^2-2^2} = \frac{3(s-1)+10}{(s-1)^2-4} = 3\frac{s-1}{(s-1)^2-2^2} + 10\frac{1}{(s-1)^2-2^2}$$

$$f(t) = 3 \cosh 2t e^t + 10e^t \frac{\sinh 2t}{2} = 3 \frac{e^{2t} + e^{-2t}}{2} e^t + 5e^t \frac{(e^{2t} - e^{-2t})}{2}$$

$$= \frac{3}{2}(e^{3t} + e^{-t}) + \frac{5}{2}(e^{3t} - e^{-t}) = 4e^{3t} - e^{-t}$$

8. Find inverse Laplace transform of the function $\frac{\pi}{s^2+10\pi s+24\pi^2}$

Solution: first let's complete the square in denominator

$$s^2 + 10\pi s + 24\pi^2 = s^2 + 10\pi s + 25\pi^2 + 24\pi^2 - 25\pi^2$$

$$= (s^2 + 10\pi s + 25\pi^2) - \pi^2 = (s + 5\pi)^2 - \pi^2$$

$$\Rightarrow L^{-1}\left(\frac{\pi}{s^2+10\pi s+24\pi^2}\right) = L^{-1}\left(\frac{\pi}{(s+5\pi)^2-\pi^2}\right)$$

$$L^{-1}\left(\frac{\pi}{s^2+10\pi s+24\pi^2}\right) = e^{-5\pi t} L^{-1}\left(\frac{\pi}{(s)^2-\pi^2}\right) = e^{-5\pi t} \sinh \pi t$$

Partial Fraction

9. Find the inverse Laplace transform of the function $\frac{s+10}{s^2-s-2}$

Solution:

$$s^2 - s - 2 = s^2 - 2s + s - 2$$

$$s(s-2) + 1(s-2) = (s-2)(s+1)$$

$$\text{hence } \frac{s+10}{s^2-s-2} = \frac{s+10}{(s-2)(s+1)}$$

$$\frac{s+10}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} \quad [\text{partial fraction}]$$

$$\Rightarrow s+10 = A(s+1) + B(s-2)$$

$$\Rightarrow \text{equating coefficients of } s \Rightarrow 1 = A + B \quad \text{---(1)}$$

$$\Rightarrow \text{equating coefficients of } s^0 \Rightarrow 10 = A - 2B \quad \text{---(2)}$$

$$\text{solving (1) and (2), we get } \Rightarrow A = 4, B = -3$$

$$\therefore \Rightarrow L^{-1} \left(\frac{s+10}{s^2-s-2} \right) = L^{-1} \left(\frac{s+10}{(s-2)(s+1)} \right) = L^{-1} \left(\frac{4}{(s-2)} + \frac{-3}{(s+1)} \right)$$

$$\Rightarrow L^{-1} \left(\frac{4}{(s-2)} - \frac{3}{(s+1)} \right) = L^{-1} \left[\frac{4}{(s-2)} \right] - L^{-1} \left[\frac{3}{(s+1)} \right]$$

$$\text{We know that } L\{e^{at}f(t)\} = F(s-a) \text{ or } L^{-1}\{F(s-a)\} = e^{at} \cdot f(t)$$

$$\Rightarrow L^{-1} \left[\frac{4e^{-2t}}{s} \right] \text{ changing } (s-2) \text{ to } s \text{ by multiplying } e^{2t} - L^{-1} \left[\frac{3e^t}{s} \right] \text{ changing } (s+1) \text{ to } s \text{ by multiplying } e^{-t}$$

$$\Rightarrow 4e^{2t} L^{-1} \left(\frac{1}{s} \right) - 3e^{-t} L^{-1} \left(\frac{1}{s} \right) = 4e^{2t} - 3e^{-t}$$

$$\therefore L^{-1} \left(\frac{s}{s^2-s-2} \right) = 4e^{2t} - 3e^{-t}$$

10. Find the inverse Laplace transform of the function $\frac{1+2s}{(s+2)^2(s-1)^2}$.

Solution: Let $\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2}$.

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{-1/3}{(s+2)^2} + \frac{1/3}{(s-1)^2} \therefore L^{-1} \left\{ \frac{1+2s}{(s+2)^2(s-1)^2} \right\} = -\frac{1}{3} e^{-2t} t + \frac{1}{3} e^t t$$

11. Find the inverse Laplace transform of the function $\frac{5s+3}{(s-1)(s^2+2s+5)}$.

Solution: Let $\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$.

$\Rightarrow A(s^2+2s+5) + (Bs+C)(s-1) = 5s+3$, This becomes

$$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{1}{s-1} + \frac{2-s}{s^2+2s+5}$$

$$\therefore L^{-1}\left\{\frac{5s+3}{(s-1)(s^2+2s+5)}\right\} = e^t - L^{-1}\left\{\frac{s-2}{(s+1)^2+2^2}\right\} = e^t - L^{-1}\left\{\frac{(s+1)-3}{(s+1)^2+2^2}\right\}$$

$$= e^t - L^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 3L^{-1}\left\{\frac{3}{(s+1)^2+2^2}\right\} = e^t - e^{-t} \cos 2t + 3e^{-t} \frac{\sin 2t}{2}$$

12. Find the inverse Laplace transform of the function $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$.

Solution: $\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s^2-5s+6)}$

Let $\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$

$\therefore A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) = 2s^2-6s+5$, We get

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1/2}{s-1} + \frac{-1}{s-2} + \frac{5/2}{s-3} \Rightarrow L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) = \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$$

Practice Questions

#	Questions	Answers
1	Find the inverse Laplace transform of the function $\frac{s+2}{s^2-4s+13}$.	$e^{2t} \cos 3t + 4e^{2t} \frac{\sin 3t}{3}$
2	Find the inverse Laplace transform of the function $\frac{1+2s}{(s+2)^2(s-1)^2}$	$-\frac{1}{3}e^{-2t}t + \frac{1}{3}e^t t$
3	Find the inverse Laplace transform of the function $\frac{2s-1}{s^2-6s+18}$	$e^{3t}\left(2 \cos 3t + \frac{5}{3} \sin 3t\right)$

Definition: The convolution of $f(t)$ and $g(t)$ denoted by $f(t) * g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

The Convolution Theorem:

If $L\{f(t)\} = F(s)$ and $L\{g(t)\} = G(s)$, then $L\{f(t) * g(t)\} = F(s) \cdot G(s)$

$$\Rightarrow L^{-1}[F(s)G(s)] = f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

Note: $*$ is commutative $\Rightarrow f(t) * g(t) = g(t) * f(t)$ and $f(t) * 0 = 0$.

WORKED EXAMPLES

- Find the inverse Laplace transform of the function $F(s) = \frac{1}{s(s-a)}$ using convolution theorem.

Solution: to find ILT of given $F(s)$, let $F(s) = \frac{1}{s-a}$ and $G(s) = \frac{1}{s}$

$$f(t) = e^{at} \text{ and } g(t) = 1$$

$$f(u) = e^{au} \text{ and } g(t-u) = 1$$

$$\text{by convolution we have } f(t) * g(t) = \int_{u=0}^t f(u) * g(t-u)du$$

$$f(t) * g(t) = \int_{u=0}^t e^{au} \cdot 1 du$$

$$f(t) * g(t) = \left[\frac{e^{au}}{a} \right]_{u=0}^t = \frac{e^{at}}{a} - \frac{1}{a} \Rightarrow \frac{1}{a} [e^{at} - 1].$$

- Find the inverse Laplace transform of the function $\frac{1}{(s+1)(s^2+1)}$ using convolution theorem.

Solution: Let $F(s) = \frac{1}{s^2+1}$ and $G(s) = \frac{1}{s+1} \Rightarrow g(t) = e^{-t}$ and $f(t) = \sin t$

$$L^{-1}\{F(s)G(s)\} = \int_0^t e^{-(t-u)} \sin(u) du$$

$$\text{Using } \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx)$$

$$L^{-1}\{F(s)G(s)\} = e^{-t} \int_0^t e^u \sin(u) du$$

$$= e^{-t} \left(\frac{e^u}{1+1} \right) (\sin u - \cos u) \Big|_0^t$$

$$= \frac{e^{-t} + \sin t - \cos t}{2}$$

Ques. Find the inverse Laplace transform of the function $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ using convolution theorem.

Soln. Let $F(s) = \frac{s}{(s^2+a^2)}$ and $G(s) = \frac{s}{(s^2+b^2)}$. Then,

$$f(t) = \cos(at) \text{ and } g(t) = \cos(bt)$$

$$f(u) = \cos(au) \text{ and } g(t-u) = \cos(b(t-u))$$

$$\text{Now, } L^{-1}[F(s)G(s)] = \int_0^t \cos(au)\cos(b(t-u))du$$

$$\text{Using } \cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

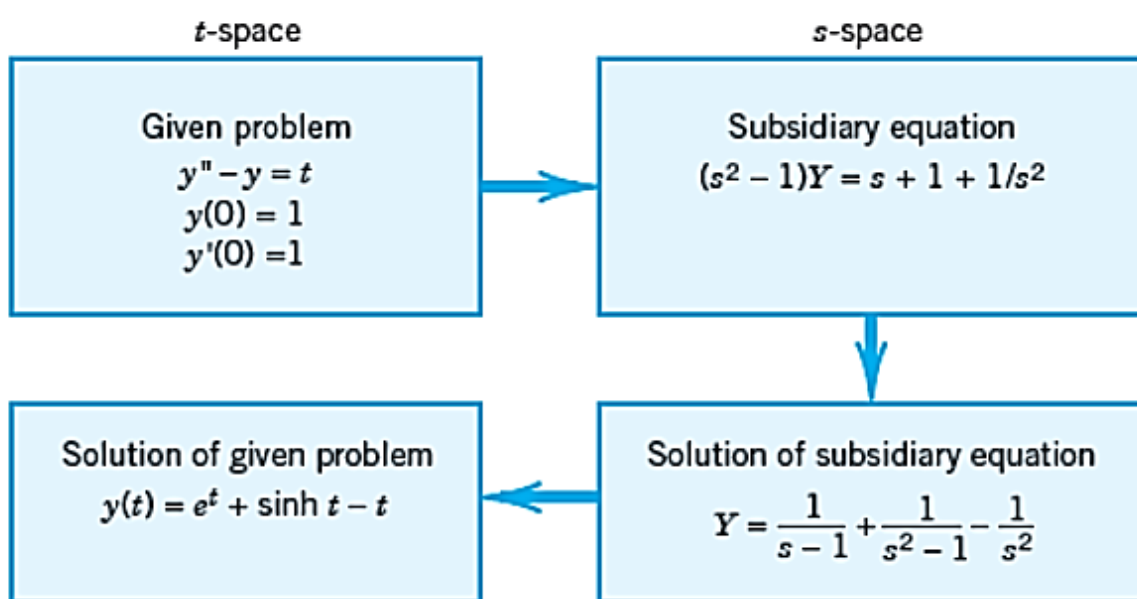
$$\begin{aligned} & \int_0^t \cos(au)\cos(b(t-u))du \\ &= \frac{1}{2} \int_0^t \cos(au + b(t-u)) + \cos(au - b(t-u))du \\ &= \frac{1}{2} \int_0^t \cos((a-b)u + bt) + \cos((a+b)u - bt)du \\ &= \frac{1}{2} \left[\frac{\sin((a-b)u + bt)}{a-b} + \frac{\sin((a+b)u - bt)}{a+b} \right] \Big|_0^t \\ &= \frac{1}{2} \left[\frac{\sin((a-b)t + bt)}{a-b} + \frac{\sin((a+b)t - bt)}{a+b} - \frac{\sin(bt)}{a-b} + \frac{\sin(bt)}{a+b} \right] \\ &= \frac{1}{2} \left[\frac{(a+b)\sin(at) + (a-b)\sin(at) - (a+b)\sin(bt) + (a-b)\sin(bt)}{(a-b)(a+b)} \right] \\ &= \frac{a\sin(at) - b\sin(bt)}{a^2 - b^2} \end{aligned}$$

Practice Questions

#	Questions	Answers
1	Find the inverse Laplace transform of the function $\frac{s^2}{(s^2 + a^2)(s^2 + b^2)}$ using convolution theorem.	$= \frac{a \sin at - b \sin bt}{a^2 - b^2}$
2	Find the inverse Laplace transform of the function $\frac{1}{s^2(s+a)^2}$ using convolution theorem	$= \frac{t e^{-at}}{a^2} + \frac{t}{a^2} + \frac{2e^{-at} - 2}{a^3}$
3	Find the inverse Laplace transform of the function $\frac{s}{(s^2 + a^2)^2}$ using convolution theorem	$= \frac{t}{2a} \sin at$

DIFFERENTIAL EQUATIONS, INITIAL VALUE PROBLEMS

Here we discuss how the Laplace transform method solves ODEs and initial value problems. We consider an initial value problem.



The general process of solution of differential equation consists of three main steps.

1. The given differential equation is transformed into a simple algebraic equation called the **subsidiary equation**.
2. The subsidiary equation is solved by pure **algebraic manipulations**.
3. Solution of the differential equation is the **inverse Laplace transform** of the solution of the **subsidiary equation**.

Note: Let $Y = L\{y(t)\}$ then we have

$$L\{y'(t)\} = sY - y(0)$$

$$L\{y''(t)\} = s^2Y - sy(0) - y'(0)$$

$$L\{y'''(t)\} = s^3Y - s^2y(0) - sy'(0) - y''(0)$$

WORKED EXAMPLES

1. Solve $y'' + y' + 9y = 0, y(0) = 0.16, y'(0) = 0$

Solution: $L[y''] + L[y'] + 9L[y] = 0$

$$\Rightarrow [s^2Y - sy(0) - y'(0)] + [sY - y(0)] + 9Y = 0$$

\Rightarrow Now apply given initial condition

$$\Rightarrow s^2Y - 0.16s + sY - 0.16 + 9Y = 0$$

$$\Rightarrow Y(s^2 + s + 9) = 0.16s + 0.16$$

$$\Rightarrow Y = \frac{0.16(s+1)}{(s^2 + s + 9)}, \text{ where } Y = L[y(t)] \text{ we need the solution for } y$$

$$\Rightarrow y = 0.16L^{-1}\left[\frac{(s+1)}{(s^2 + s + 9)}\right], \text{ we use completing the square method to solve for } y(t)$$

$$\Rightarrow s^2 + s + 9 = \left(s^2 + s + \frac{1}{4}\right) + \left(9 - \frac{1}{4}\right)$$

$$\Rightarrow = \left(s + \frac{1}{2}\right)^2 + \frac{35}{4}$$

$$\Rightarrow y = 0.16L^{-1}\left[\frac{\left(s + \frac{1}{2}\right) + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\sqrt{\frac{35}{4}}\right)^2}\right] = 0.16e^{-1/2t}L^{-1}\left[\frac{s + \frac{1}{2}}{(s)^2 + \left(\sqrt{\frac{35}{4}}\right)^2}\right] \Leftrightarrow$$

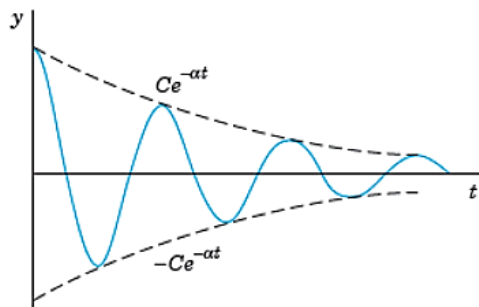
$$\Rightarrow y = 0.16e^{-1/2t}\left\{L^{-1}\left[\frac{s}{(s)^2 + \left(\sqrt{\frac{35}{4}}\right)^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s)^2 + \left(\sqrt{\frac{35}{4}}\right)^2}\right]\right\}$$

$$\Rightarrow y(t) = 0.16e^{-1/2t} \left[\cos \sqrt{\frac{35}{4}}t + \frac{1}{2} \left[\frac{1}{\sqrt{\frac{35}{4}}} \sin \sqrt{\frac{35}{4}}t \right] \right]$$

$$\Rightarrow y(t) = 0.16e^{-1/2t} \left[\cos \sqrt{\frac{35}{4}}t + 0.17 \left[\sin \sqrt{\frac{35}{4}}t \right] \right]$$

$$\Rightarrow y(t) = 0.16 \left\{ e^{-1/2t} \left[\cos 2.96t + 0.17 \left[\sin 2.96t \right] \right] \right\}$$

$$\Rightarrow y(t) = e^{-0.5t} (0.16 \cos 2.96t + 0.0272 \sin 2.96t)$$



This solution represents **damped oscillations** of the form

$$y(t) = e^{-\alpha t} (A \cos \omega t + B \sin \omega t)$$

$$= Ce^{-\alpha t} (\cos \omega t - \delta)$$

$$C = \sqrt{A^2 + B^2} \text{ \& } \delta = \tan^{-1} [B/A]$$

for Mass Spring system.

Damped oscillation: Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$

2. Solve $y''' + 2y'' - y' - 2y = 0$, $y(0) = y'(0) = 0$, $y''(0) = 6$.

Solution: Given $y''' + 2y'' - y' - 2y = 0$, $y(0) = y'(0) = 0$, $y''(0) = 6$.

Taking Laplace transforms

$$L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = 0$$

$$s^3 Y - s^2 y(0) - s y'(0) - y''(0) + 2\{s^2 Y - s y(0) - y'(0)\} - \{sY - y(0)\} - 2Y = 0$$

$$\text{Solving for } Y, \text{ We get } Y = \frac{6}{s^3 + 2s^2 - s - 2} = \frac{6}{s^2(s+2) - (s+2)} = \frac{6}{(s+1)(s-1)(s+2)}$$

$$\text{Let } Y = \frac{6}{(s+1)(s-1)(s+2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+2}$$

After resolving into partial fractions and taking inverse transforms

$$\text{We get } y(t) = -3e^{-t} + e^t + 2e^{-2t}$$

3. Solve $y'' + 4y' + 3y = e^t$, $y(0) = 1 = y'(0)$.

Solution:

$$\text{We have } y'' + 4y' + 3y = e^t \therefore L\{y''\} + 4L\{y'\} + 3L\{y\} = \frac{1}{s-1}$$

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 3Y(s) = \frac{1}{s-1}$$

$$s^2Y(s) - s - 1 + 4[sY(s) - 1] + 3Y(s) = \frac{1}{s-1}$$

$$s^2Y(s) - s - 1 + 4sY(s) - 4 + 3Y(s) = \frac{1}{s-1}$$

$$(s^2 + 4s + 3)Y(s) - s - 5 = \frac{1}{s-1}$$

$$(s^2 + 4s + 3)Y(s) = \frac{1}{s-1} + (s + 5)$$

$$= \frac{1+(s+5)(s-1)}{s-1}$$

$$= \frac{1+s^2+4s-5}{s-1}$$

Or $Y = \frac{s^2+4s-4}{(s-1)(s^2+4s+3)}$: Now, $s^2 + 4s + 3 = s^2 + s + 3s + 3 = (s + 3)(s + 1)$

$$Y = \frac{s^2+4s-4}{(s-1)(s^2+4s+3)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{C}{s+1} = \frac{1}{8(s-1)} - \frac{7}{8(s+3)} + \frac{7}{4(s+1)}$$

Taking inverse Laplace transform $y(t) = \frac{e^t}{8} - \frac{7}{8}e^{-3t} + \frac{7}{4}e^{-t}$

Practice Questions

#	Questions	Answers
1	Solve $y'' + y' - 2y = t$, $y(0) = 1$ $y'(0) = 0$.	$y(t) = -\frac{1}{4}.1 - \frac{1}{2}t + \frac{1}{4}e^{-2t} + e^t$
2	Solve $y'' + 2y' - 3y = \sin t$, $y(0) = y'(0) = 0$.	$y(t) = \frac{-1}{40}e^{-3t} + \frac{1}{8}e^t - \frac{1}{10}\cos t - \frac{1}{5}\sin t$
3	Solve $y'' + y = H(t-1)$, $y(0) = 0$ $y'(0) = 1$.	$y(t) = \sin t + [1 - \cos(t-1)]H(t-1)$

***** END OF CHAPTER*****