SCHOOL OF ENGINEERING

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DEPARTMENT OF MATHEMATICS

MODULE 5

NUMERICAL METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

Contents

INTRODUCTION:

Partial differential equations arise in the study of many branches of applied mathematics, e.g., in fluid dynamics, heat transfer, boundary layer flow, quantum mechanics and electromagnetic theory. Only a few of these equations can be solved by analytical methods which are also complicated by requiring use of advanced mathematical techniques. In most of the cases, it is easier to develop approximate solutions by numerical methods. Of all the numerical methods available for the solution of partial differential equations, the method of **finite differences** is most commonly used. In this method, the derivatives appearing in the equation and the boundary conditions are replaced by their finite difference approximations.

CLASSIFICATION OF SECOND ORDER EQUATIONS:

The general linear partial differential equation of the second order in two independent variables is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

Such a partial differential equation is said to be

- Elliptic if , $B^2-4AC<0$
- Parabolic if $B^2-4AC=0$, and
- Hyperbolic if $B^2-4AC>0$

The below table gives the nature of the three PDE:

PDE	Nature of the PDE
Two-Dimensional Laplace Equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$	**Elliptic**
One Dimensional Heat Equation $c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}$	**Parabolic**
One Dimensional Wave Equation $c^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$	**Hyperbolic**

FINITE DIFFERENCE APPROXIMATIONS TO PARTIAL DERIVATIVES:

Let *y=y(x)* and its derivatives be single valued continuous functions of x. By Taylor's expansion we have:

$$y(x+h)=y(x)+hy'(x)+\frac{h^2}{2!}y''(x)+\frac{h^3}{3!}y'''(x)+\cdots \qquad ...(A)$$

$$y(x-h)=y(x)-hy'(x)+\frac{h^2}{2!}y''(x)-\frac{h^3}{3!}y'''(x)+\cdots \qquad ...(B)$$

Assuming h to be small and neglecting terms containing h^2, h^3,.. in (A) and (B), we have:

$\boldsymbol{y'(x)=\frac{y(x+h)-y(x)}{h}}$ (From (A)) and $\boldsymbol{y'(x)=\frac{y(x)-y(x-h)}{h}}$ (From (B))

Also (A)-(B) gives $\boldsymbol{y'(x)=\frac{y(x+h)-y(x-h)}{2h}}$

The above 3 expressions for $y'(x)$ the finite difference approximation in terms of forward, backward and central difference respectively.

Neglecting terms containing h^3, h^4,.... (A)+(B) gives the finite difference approximation for $y''(x)$

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

Let $u = u(x, y)$ be a function of two independent variables. The finite difference approximation for the first order partial derivatives and second order partial derivatives are as follows:

$\frac{\partial u}{\partial x} = \frac{u(x+h,y)-u(x,y)}{h}$, $\frac{\partial u}{\partial x} = \frac{u(x,y)-u(x-h,y)}{h}$, $\frac{\partial u}{\partial x} = \frac{u(x+h,y)-u(x-h,y)}{2h}$

$\frac{\partial u}{\partial y} = \frac{u(x,y+k)-u(x,y)}{k}$, $\frac{\partial u}{\partial y} = \frac{u(x,y)-u(x,y-k)}{k}$, $\frac{\partial u}{\partial y} = \frac{u(x,y+k)-u(x,y-k)}{2k}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h, y) - 2u(x, y) + u(x-h, y)}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x, y+k) - 2u(x, y) + u(x, y-k)}{h^2}$$

Consider a rectangular region R in the x-y plane. Divide this region into a rectangular network of sizes $\Delta x = h$ and $\Delta y = k$ as shown in Figure 1. The points of intersection of the dividing lines are called mesh points, nodal points, or grid points.

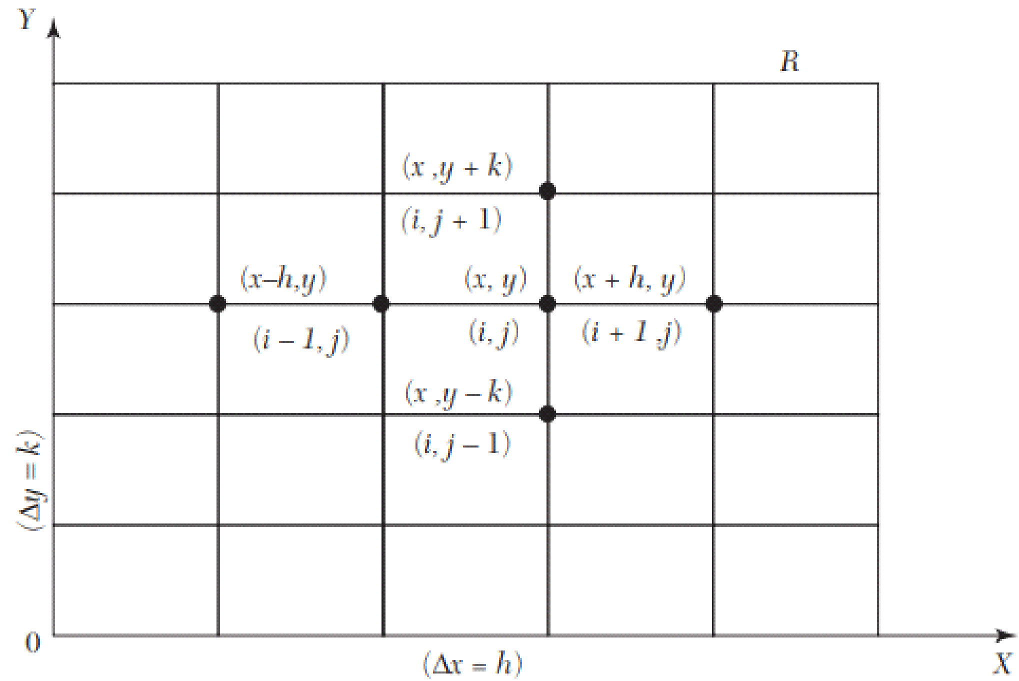

Figure 1

Writing $u(x, y) = u(ih, jk)$ as $u_{i,j}$, the above approximation become:

$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{h}$, $\quad \frac{\partial u}{\partial x} = \frac{u_{i,j} - u_{i-1,j}}{h}$, $\quad \frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$

$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j}}{k}$, $\quad \frac{\partial u}{\partial y} = \frac{u_{i,j} - u_{i,j-1}}{k}$, $\quad \frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} \qquad \ldots(\mathrm{I})$$

ELLIPTIC EQUATIONS:

The **Laplace equation** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and the

Poisson's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$

are examples of elliptic partial differential equations. The Laplace equation arises in steady-state flow and potential problems. Poisson's equation arises in fluid mechanics, electricity and magnetism and torsion problems.

SOLUTION OF LAPLACE'S EQUATION:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ldots (1)$$

Let us suppose that the region R is such that it can be divided into a network of square mesh of size *h*. Using the approximation for $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ in (1) and simplifying we get

$$\boldsymbol{u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}\right]} \quad \text{(Fig. 2)} \quad \ldots(2)$$

which is known as **STANDARD FIVE POINT FORMULA**.

Sometimes a formula similar to (2) is used which is given by

$$\boldsymbol{u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right]} \quad \text{(Fig. 3)} \quad \ldots(3)$$

which is known as **DIAGONAL FIVE POINT FORMULA.**

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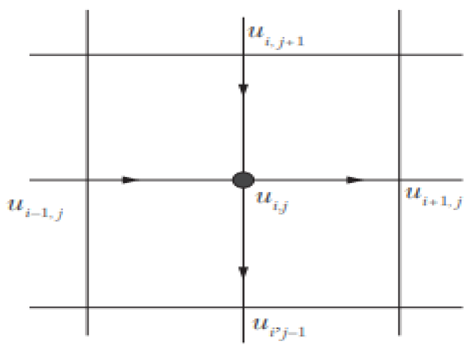


Fig. 2

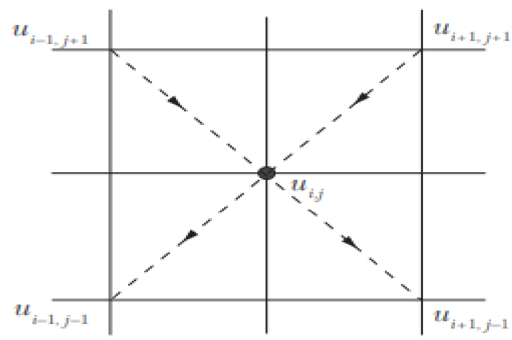
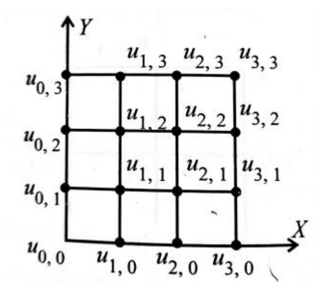


Fig. 3

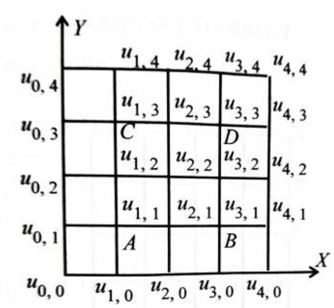
Working Procedure for Problems:

ODD number of squares



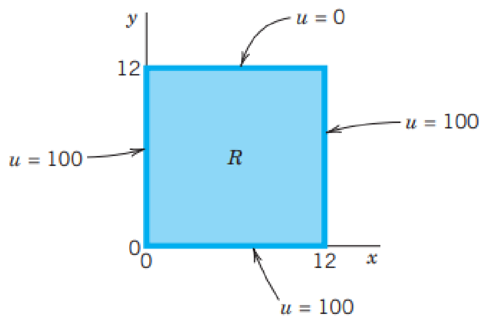
Compute $u_{11}, u_{21}, u_{12}, u_{22}$ using **STANDARD FIVE POINT FORMULA**

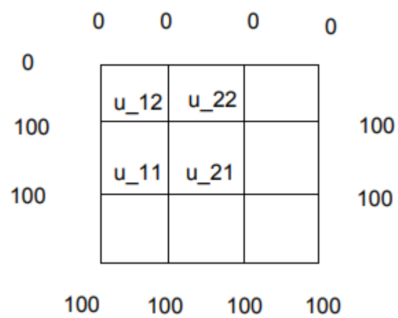
EVEN number of squares



Compute u_{22} using **STANDARD FIVE POINT FORMULA.** Compute $u_{11}, u_{31}, u_{13}, u_{33}$ using **DIAGONAL FIVE POINT FORMULA.** Compute $u_{21}, u_{23}, u_{12}, u_{32}$ using **STANDARD FIVE POINT FORMULA**

Eg. 1: The four sides of a square plate of side 12 cm, made of homogeneous material, are kept at constant temperature and as shown in Fig. Using a grid of mesh 4 cm, find the (steady-state) temperature at the mesh points.





Solution: We use the standard five point formula $u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}]$

And in each equation, taking to the right all the terms resulting from the given boundary values. Then we obtain the system:

$$u_{11} = \frac{1}{4}[100 + u_{21} + u_{12} + 100]$$

$$u_{12} = \frac{1}{4}[100 + u_{22} + 0 + u_{11}]$$

$$u_{21} = \frac{1}{4}[u_{11} + 100 + u_{22} + 100]$$

$$u_{22} = \frac{1}{4}[u_{12} + 100 + 0 + u_{21}]$$

Re writing the above equations as:

$4u_{11} - u_{12} - u_{21} \quad = 200$ ...(1)

$-u_{11} \quad + 4u_{21} - u_{22} = 200$ ...(2)

$-u_{11} + 4u_{12} \quad - u_{22} = 100$ ...(3)

$\quad -u_{12} - u_{21} + 4u_{22} = 100$ ...(4)

Eliminating u_{11} from eq (1) and (2) $[(1) + 4 * (2)]$

$4u_{11} - u_{12} - u_{21} + (-4u_{11} + 16u_{12} - 4u_{22}) = 1000$

$\Rightarrow -u_{12} + 15u_{21} - 4u_{22} = 1000$ ...(5)

Eliminating u_{11} from eq (2) and (3) $[(2) - (3)]$

$$-u_{11} \quad + 4u_{21} - u_{22} - (-u_{11} + 4u_{12} - u_{22}) = 1000$$

$\Rightarrow -4u_{12} + 4u_{21} = 100$ ...(6)

Now solving the below 3 equations in calculator

$-u_{12} - u_{21} + 4u_{22} = 100$ ..(4), $-u_{12} + 15u_{21} - 4u_{22} = 1000$ ...(5), $-4u_{12} + 4u_{21} = 100$ ...(6)

$$u_{12} = 62.5, \quad u_{21} = 87.5, \quad u_{22} = 62.5$$

Substituting above values in (1), $u_{11} = 87.5$

$\therefore u_{11} = u_{21} = 87.5$ and $u_{12} = u_{22} = 62.5$

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Note: The equations (1)-(4) can be solved using Gauss Elimination Method

$$\left(\begin{array}{cccc|c} 4 & -1 & -1 & 0 & 200 \\ -1 & 0 & 4 & -1 & 200 \\ -1 & 4 & 0 & -1 & 100 \\ 0 & -1 & -1 & 4 & 100 \end{array}\right) \sim \left(\begin{array}{cccc|c} -1 & 0 & 4 & -1 & 200 \\ 4 & -1 & -1 & 0 & 200 \\ -1 & 4 & 0 & -1 & 100 \\ 0 & -1 & -1 & 4 & 100 \end{array}\right) R_2 \leftrightarrow R_1$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -200 \\ 4 & -1 & -1 & 0 & 200 \\ -1 & 4 & 0 & -1 & 100 \\ 0 & -1 & -1 & 4 & 100 \end{array}\right) R_2 = R_2 \times (-1) \sim \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -200 \\ 0 & -1 & 15 & -4 & 1000 \\ 0 & 4 & -4 & 0 & -100 \\ 0 & -1 & -1 & 4 & 100 \end{array}\right) \begin{array}{l} R_2 = R_2 - 4R_1 \\ R_3 = R_3 + R_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -200 \\ 0 & 1 & -15 & 4 & -1000 \\ 0 & 1 & -1 & 0 & -25 \\ 0 & -1 & -1 & 4 & 100 \end{array}\right) \begin{array}{l} R_2 = R_2 \times (-1) \\ R_3 = \frac{R_3}{4} \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -200 \\ 0 & 1 & -15 & 4 & -1000 \\ 0 & 0 & 14 & -4 & 975 \\ 0 & 0 & -16 & 8 & -900 \end{array}\right) \begin{array}{l} R_3 = R_3 - R_2 \\ R_4 = R_4 + R_2 \end{array} \sim \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -200 \\ 0 & 1 & -15 & 4 & -1000 \\ 0 & 0 & 14 & -4 & 975 \\ 0 & 0 & 4 & -2 & 225 \end{array}\right) R_4 = \frac{R_4}{-4}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -200 \\ 0 & 1 & -15 & 4 & -1000 \\ 0 & 0 & 14 & -4 & 975 \\ 0 & 0 & 0 & -12 & -750 \end{array}\right) R_4 = 14R_4 - 4R_3$$

$$u_{22} = 62.5, \quad u_{21} = 87.5, \quad u_{12} = 62.5, \quad u_{11} = 87.5$$

$u_{11} = u_{21} = 87.5$ and $u_{12} = u_{22} = 62.5$

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Eg. 2: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the square region bounded by the co ordinate axes and the lines *x=4, y=4* with the boundary conditions given in the analytical expressions,

(i) $u(0, y) = 0, 0 \leq y \leq 4$

(ii) $u(4, y) = 12 + y, 0 \leq y \leq 4$

(iii) $u(x, 0) = 3x, 0 \leq x \leq 4$

(iv) $u(x, 4) = x^2, 0 \leq x \leq 4$

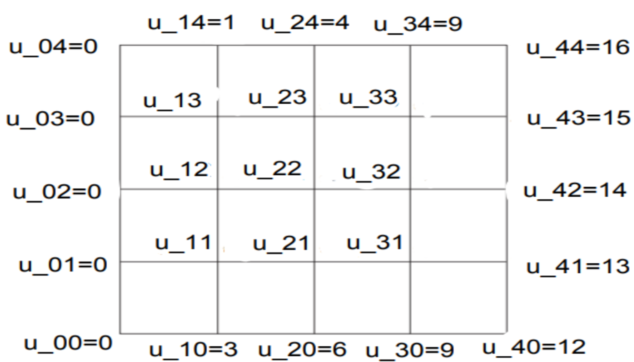
Solution: We will divide the square region into 16 squares of size 1 unit.

$$u(0,y)=0 \Longrightarrow u_{00}=u_{01}=u_{02}=u_{03}=u_{04}=0$$

$$u(4,y)=12+y \;\Longrightarrow u_{40}=12, u_{41}=13, u_{42}=14, u_{43}=15, u_{44}=16$$

$$u(x,0)=3x \;\Longrightarrow u_{00}=0, u_{10}=3, u_{20}=6, u_{30}=9, u_{40}=12$$

$$u(x,4)=x^2 \Longrightarrow u_{04}=0, u_{14}=1, u_{24}=4, u_{34}=9, u_{44}=16$$



We use the standard five point formula (2 steps)

$$\Longrightarrow u_{22}=\frac{1}{4}[0+14+4+6]=6$$

Using diagonal formula we calculate $u_{11}, u_{31}, u_{13}, u_{33}$

$$u_{i,j}=\frac{1}{4}\left[u_{i-1,j+1}+u_{i+1,j-1}+u_{i+1,j+1}+u_{i-1,j-1}\right]$$

$$\Longrightarrow u_{11}=\frac{1}{4}[0+6+6+0]=3$$

$$\Longrightarrow u_{13}=\frac{1}{4}[0+6+0+4]=2.5$$

$$\Longrightarrow u_{31}=\frac{1}{4}[6+12+14+6]=9.5$$

$$\Rightarrow u_{33} = \frac{1}{4}[6 + 16 + 14 + 4] = 10$$

Using the standard five point formula again we find $u_{12}, u_{32}, u_{21}, u_{23}$

$$\Rightarrow u_{12} = \frac{1}{4}[0 + 6 + 2.5 + 3] = 2.875$$

$$\Rightarrow u_{32} = \frac{1}{4}[6 + 14 + 10 + 9.5] = 9.875$$

$$\Rightarrow u_{21} = \frac{1}{4}[6 + 6 + 3 + 9.5] = 6.125$$

$$\Rightarrow u_{23} = \frac{1}{4}[6 + 4 + 2.5 + 10] = 5.625$$

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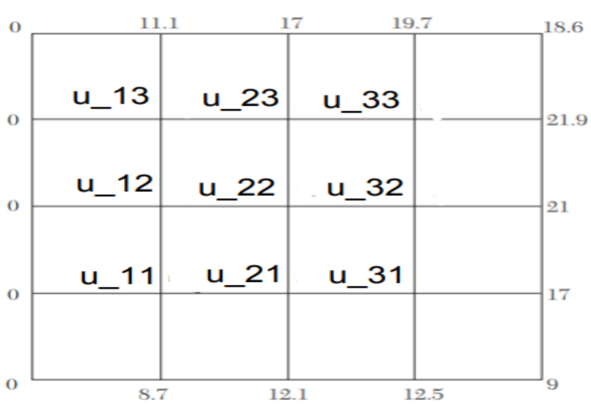
Eg. 3: For the given square mesh with boundary values as shown in the figure, find the values at the interior points.



Solution: We use the standard five point formula $(2\ steps)$

$$\Rightarrow u_{22} = \frac{1}{4}[0 + 21 + 17 + 12.1] = 12.5$$

Using diagonal formula we calculate $u_{11}, u_{31}, u_{13}, u_{33}$

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right]$$

$$\Rightarrow u_{11} = \frac{1}{4}[0 + 12.5 + 0 + 12.1] = 6.15$$

$$\Rightarrow u_{13} = \frac{1}{4}[0 + 17 + 0 + 12.5] = 7.4$$

$$\Rightarrow u_{31} = \frac{1}{4}[12.1 + 21 + 12.5 + 9] = 13.65$$

$$\Rightarrow u_{33} = \frac{1}{4}[12.5 + 18.6 + 17 + 21] = 17.28$$

Using the standard five point formula again we find $u_{12}, u_{32}, u_{21}, u_{23}$

$$\Rightarrow u_{12} = \frac{1}{4}[0 + 12.5 + 7.4 + 6.15] = 6.52$$

$$\Rightarrow u_{32} = \frac{1}{4}[12.5 + 21 + 17.28 + 13.65] = 16.12$$

$$\Rightarrow u_{21} = \frac{1}{4}[6.15 + 13.65 + 12.5 + 12.1] = 11.12$$

$$\Rightarrow u_{23} = \frac{1}{4}[7.4 + 17.28 + 17 + 12.5] = 13.55$$

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PRACTICE QUESTIONS:

Eg. 1 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for *0<x<1*, *0<y<1* given that, $u(x,0) = u(0,y) = 0$, $u(x,1) = 6x, 0 < x \leq 1$ and $u(1,y) = 3y, 0 < y < 1$. Divide the region into 9 square meshes.

Solution: Using standard five point formula we obtain the following system of equations:

$$4u_{11} - u_{12} - u_{21} \quad = 0$$

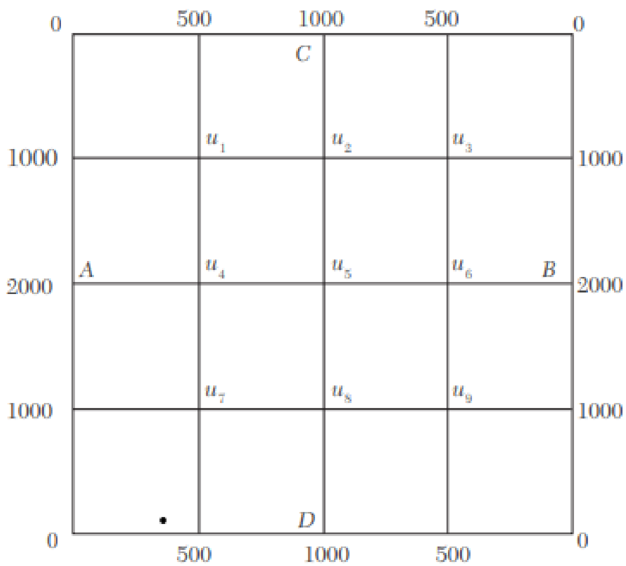
$$-u_{11} \quad + 4u_{21} - u_{22} = 1$$

$$-u_{11} + 4u_{12} \quad - u_{22} = 2$$

$$-u_{12} - u_{21} + 4u_{22} = 6$$

$u_{11} = \frac{1}{2}, u_{21} = \frac{7}{8}, u_{12} = \frac{9}{8}, u_{22} = 2.$

Eg. 2 For the given square mesh with boundary values as shown in the figure, find the values at the interior points.



Solution: Since the boundary values of u are symmetrical about AB, $\therefore$ $u_7 = u_1$, u8 = u2 , u9 = u3 . Also the values of u being symmetrical about CD. u3 = u1 , u6 = u4 , u9 = u7 . Thus it is sufficient to find the values u1 , u2 , u4 , and u5 .

$$u_1 = 1125, u_2 = 1188, u_4 = 1438, u_5 = 1500$$

SOLUTION OF POISSON'S EQUATION:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \ldots (1)$$

The standard five-point formula is:

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) \ldots (2)$$

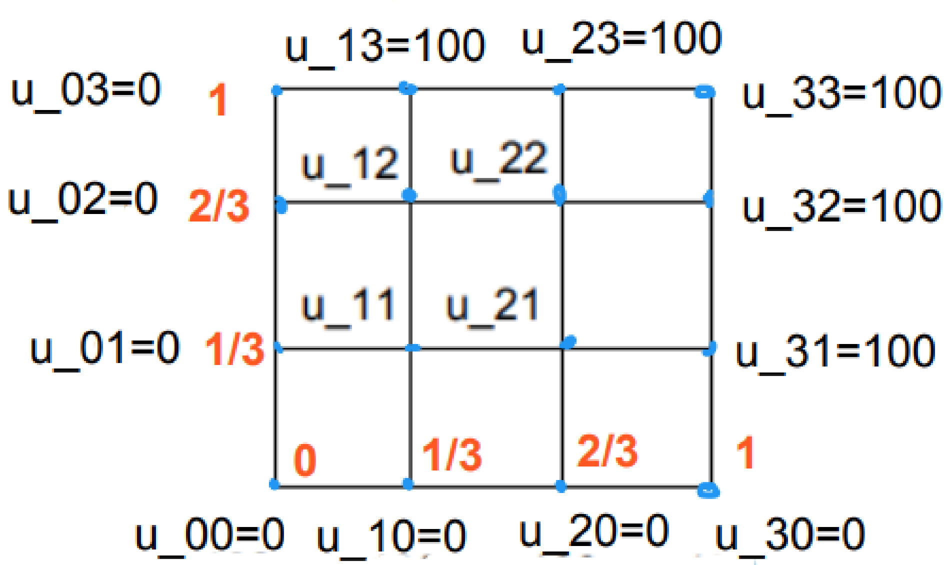
Eg. 1 Solve the Poisson equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -81xy$, 0 < x < 1, 0 < y < 1 given that *u(0, y) = 0, u(x, 0) = 0, u(1, y) = 100, u(x, 1) = 100* and *h = 1/3.*

Solution: Here $h = \frac{1}{3}$

The standard five-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2[-81(ih * jh)] = -\left(\frac{1}{3}\right)^4 (81) * ij = -ij$ ----(A)



For $i = 1, j = 1,$ (A) becomes $-4u_{11} + u_{12} + u_{21} \quad = -1$ ...(1)

For $i = 1, j = 2,$ (A) becomes $u_{11} - 4u_{12} \quad + u_{22} = -102$ ...(2)

For $i = 2, j = 1,$ (A) becomes $u_{11} \quad - 4u_{21} + u_{22} = -102$ ...(3)

For $i = 2, j = 2,$ (A) becomes $u_{12} + u_{21} - 4u_{22} = -204$ ...(4)

Eliminating u_{11} from eq (1) and (2) $[(1) + 4 * (2)]$

$-4u_{11} + u_{12} + u_{21} + (4u_{11} - 16u_{12} + 4u_{22}) = -409$

$\Longrightarrow -15u_{12} + u_{21} + 4u_{22} = -409$ ...(5)

Eliminating u_{11} from eq (2) and (3) $[(2) - (3)]$

$\Longrightarrow -4u_{12} + 4u_{21} \ = 0$ ...(6)

Now solving the below 3 equations in calculator

$u_{12} + u_{21} - 4u_{22} = -204$ ..(4), $-15u_{12} + u_{21} + 4u_{22} = -409$ ...(5), $-4u_{12} + 4u_{21} \ = 0$ ...(6)

$$u_{12} = 51.0833 = u_{21},\ u_{22} = 76.5416$$

Substituting above values in (1), $u_{11} = 25.7916$

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Eg. 2 Solve the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10)$ over the square with *sides x = 0 = y, x = 3 = y* with *u = 0* on the boundary and mesh length = 1.

Solution: Here $h = 1$

The standard five-point formula for the given equation is

$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh) = -10(i^2 + j^2 + 10)$ ...(A)

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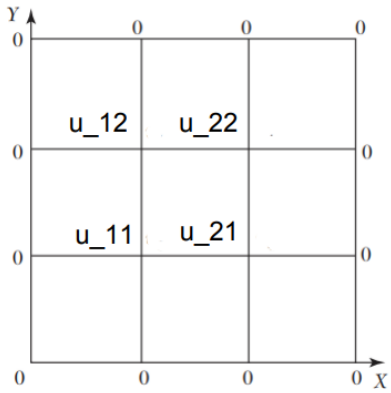

The equations from (A) are:

$$-4u_{11} + u_{12} + u_{21} = -120,\ u_{11} - 4u_{12} + u_{22} = -150,$$

$$u_{11} - 4u_{21} + u_{22} = -150, u_{12} + u_{21} - 4u_{22} = -180$$

Solving we get, $u_{11} = 67.5, u_{12} = 75$, $u_{21} = 75$, $u_{22} = 82.5$.

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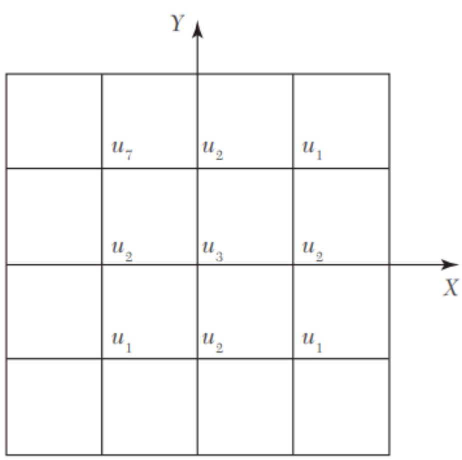
PRACTICE QUESTION:

Eg. 1 Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 8x^2y^2$ for the square mesh in the below figure with $u(x, y) = 0$ on the boundary and mesh length 1.



Solution: Hint u_3 co ordinate is (0,0). u_1 co ordinate is (-1,-1) and so on.

$$u_1 = -3, u_2 = -2, u_3 = -2$$

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