5.2

Given $S_{xx}(\tau) = e^{-\alpha|\tau|}$ for $\alpha > 0$

$$y(\eta,t) = \begin{cases} \int_0^t x(\eta,\lambda)\,d\lambda & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Calculate the mean $m_y^{(1)}(t)$?

We know that $m_y^{(1)}(t) = E\{y(\eta,t)\}$

$$m_y^{(1)}(t) = E\left\{\int_0^t x(\eta,\lambda)\,d\lambda\right\}$$

$$\Rightarrow \int_0^t \{\underbrace{E\{x(\eta,\lambda)}_{\text{mean of } x}\}\}\,d\lambda \quad \text{— (1)}$$

We know that $(m_x^{(1)})^2 = \lim_{\tau\to\infty} S_{xx} \Rightarrow \lim_{\tau\to\infty} e^{-\alpha|\tau|} \Rightarrow 0$

∴ Putting value of mean in Eq. 1 we get

$$m_y^{(1)}(t) = \int_0^t 0\,d\lambda = 0.$$

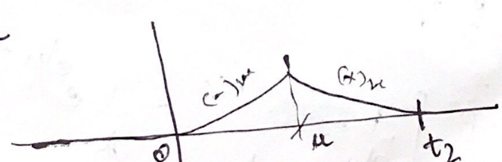
(b) Find autocorrelation function of the random process $y(\eta,t)$.

$$S_{yy} = E\{y(\eta,t_1)\}\cdot E\{y(\eta,t_2)\}.$$

$$\Rightarrow E\left\{\int_0^{t_1} x(\eta,u)\,du\right\}\cdot E\left\{\int_0^{t_2} x(\eta,v)\,dv\right\}.$$

$$\Rightarrow \int_0^{t_1}\int_0^{t_2} \underbrace{E\{x(\eta,u)\}\cdot E\{x(\eta,v)\}}\cdot dv\cdot du$$

$$\Rightarrow \underbrace{\int_0^{t_1}\int_0^{t_2} e^{-\alpha|v-u|}\,dv\,du}$$



Case I → $0 < t_1 < t_2$

$$\Rightarrow \int_0^{t_1}\left[\int_0^{u} e^{d(v-u)}\,dv + \int_u^{t_2} e^{-d(v-u)}\,dv\right]du$$

$$\Rightarrow \int_0^{t_1}\left[\frac{e^{d(v-u)}}{d}\Big|_0^{u} + \frac{e^{-d(v-u)}}{-d}\Big|_u^{t_2}\right]du$$

$$\Rightarrow \int_0^{t_1}\left[\frac{1-e^{-du}}{d} - \left[\frac{e^{-d(t_2-u)}-1}{d}\right]\right]du$$

$$\Rightarrow \int_0^{t_1}\left[\frac{1-e^{-du}+1-e^{-d(t_2-u)}}{d}\right]du$$

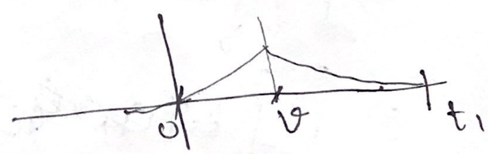
$$\Rightarrow \int_0^{t_1}\left[\frac{2-e^{-du}-e^{-d(t_2-u)}}{d}\right]du$$

$$\Rightarrow \frac{1}{d}\left[2u - \frac{e^{-du}}{-d} - \frac{e^{-d(t_2-u)}}{d}\right]_0^{t_1}$$

$$\Rightarrow \frac{1}{d^2}\left[2ud + e^{-du} - e^{-d(t_2-u)}\right]_0^{t_1}$$

$$\Rightarrow \frac{1}{d^2}\left[2t_1 d + e^{-t_1 d} - e^{-d(t_2-t_1)} - 1 + e^{-t_2 d}\right]$$

Case II $\rightarrow$ $0 < t_2 < t_1$



$$\Rightarrow \int_0^{t_2}\int_0^{t_1} e^{-d|u-v|}\,du\,dv$$

$$\Rightarrow \int_0^{t_2}\left[\frac{1}{d} - \frac{e^{-dv}}{d} - \frac{1}{d}e^{(t_1-v)} + \frac{1}{d}\right]du.$$

$$\frac{1}{d^2}\left[2dt_2 + e^{-dt_2} - 1 - e^{-d(t_1-t_2)} + e^{-dt_1}\right].$$

(c) Variance of $y(n,t)$:

$$\text{variance} = m_y^{(2)} - \underbrace{(m_y^{(1)})^2}_{}$$

↳ 0 already Calculated above

now $m_y^{(2)} = \lim_{\tau \to 0} \phi_{yy}$

Since $\tau \to 0$, $t_1 = t_2$

$$\therefore \lim_{\tau\to 0} \frac{1}{d^2}\left[2dt + e^{-dt} - 1 - 1 + e^{-dt}\right]$$

$$\Rightarrow \frac{1}{d^2}\left[2dt + 2e^{-dt} - 2\right] \text{ Ans.}$$

(d) No it is not a stionary random process as it is depending upon time. It is a non stationary random process.

For more information

Stationary Random process → donot depend upon t & τ

Weakly stationary → dependent upon τ

Non-Stationary → depends upon t.

5.3

$$S_{xx}(\tau) = a e^{-\alpha|\tau|} + b$$

$$y(\eta, t) = \begin{cases} 0 & \text{for } t \le t_0 \\ \int_{t_0}^{t} x(\eta, \lambda)\, d\lambda & \text{for } t > t_0 \end{cases}$$

(a) Determine Constants a & b

We know that $(m_x^{(1)})^2 = \lim_{\tau \to \infty} S_{xx}(\tau)$

$$\Rightarrow \lim_{\tau \to \infty} a e^{-\alpha|\tau|} + b = 0$$

$$\Rightarrow \boxed{b = 0}$$

also $m_x^{(2)} = \lim_{\tau \to 0} S_{xx}(\tau) \Rightarrow \lim_{\tau \to 0} a e^{-\alpha|\tau|} + b = 1$

$$\boxed{a = 1}$$

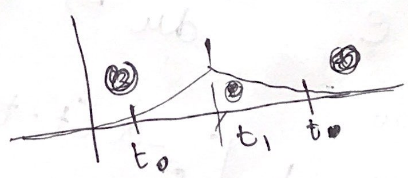
(b) Find Cross-Correlation:

$\underline{S_{xy}(t_1; t_2)} = E\{x(\eta, t_1) \cdot y(\eta, t_2)\} \rightarrow$

$$\Rightarrow E\left\{ x(\eta, t_1) \cdot \int_{t_0}^{t} x(\eta, u)\, du \right\}$$

$$\Rightarrow \int_{t_0}^{t} \underline{E\{x(\eta, t_1) \cdot x(\eta, u)} \} \, du$$

$$\Rightarrow \int_{t_0}^{t} e^{-\alpha|u - t_1|}\, du$$



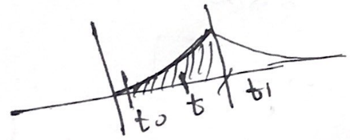
$\Rightarrow$ Case I $\rightarrow$ $t_0 < \underline{t_1} < t$

$$\int_{t_0}^{t_1} e^{d(u-t_1)}\,du + \int_{t_1}^{t} e^{-d(u-t_1)}\,du$$

$$\Rightarrow \frac{e^{d(u-t_1)}}{d}\Bigg|_{t_0}^{t_1} + \frac{e^{-d(u-t_1)}}{-d}\Bigg|_{t_1}^{t}$$

$$\Rightarrow \frac{1}{d}\left[1 - e^{d(t_0-t_1)} - \left[e^{-d(t-t_1)} - 1\right]\right]$$

$$\Rightarrow \frac{1}{d}\left[2 - e^{d(t_0-t_1)} - e^{-d(t-t_1)}\right].$$

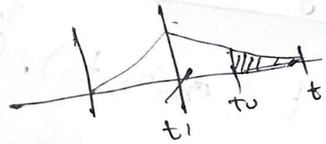


Case II → $t_0 < t < t_1$

now ~~graph~~ integral always open(+)ve

$$\therefore \int_{t_0}^{t} e^{d(u-t_1)}\,du$$

$$\Rightarrow \frac{1}{d}\left[e^{d(t-t_1)} - e^{d(t_0-t_1)}\right].$$



Case III → $t_1 < t_0 < t$

~~graph~~ integral always open (+)ve

$$\int_{t_0}^{t} e^{-d(u-t_1)}\,du \Rightarrow -\frac{1}{d}\left|e^{-d(u-t_1)}\right|_{t_0}^{t}$$

$$\Rightarrow -\frac{1}{d}\left[e^{-d(t-t_1)} - e^{-d(t_0-t_1)}\right].$$

Case IV → 0 elsewhere.

The process is non stationary.