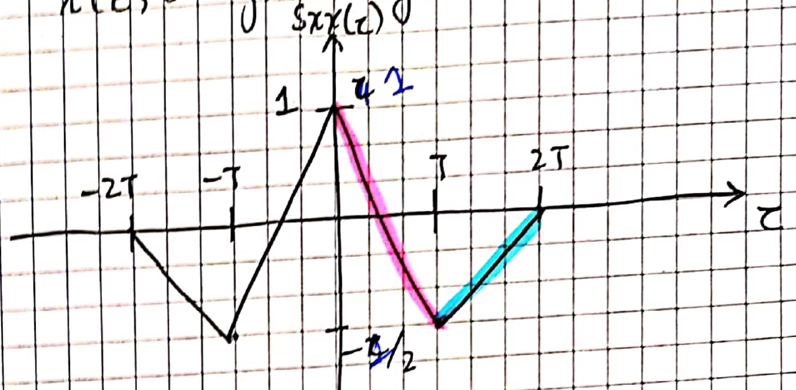
TASK 4·4

The autocorrelation function $S_{xx}(\tau)$ of random process $x(e,t)$ given by



Calculate and sketch PSD

Wiener Khinchine Theorem $\left\{ \text{simple Fourier Transform of process for } t \to f \right\}$

$$S_{xx}(w) = \int_{-\infty}^{+\infty} S_{xx}(\tau)\, e^{-jw\tau}\, d\tau$$

eq of line

$$\frac{y - y_1}{x - x_1} = m$$

Taking point ~~$(y, x) = (\tau, S_{xx})$~~

~~$(x_1, y_1) = (0, \tau)$~~

Taking 2 points on line

$(0, 1)$, $(T, -1/2)$
x, y $\quad x_1, y_1$

$$\frac{1 + 1/2}{0 - T} \Rightarrow \frac{-3}{2T} = m$$

Taking points $(\tau, S_{xx}(\tau))$
$(0, 1)$

$$S_{xx}(\tau) - 1 = \frac{-3}{2T}(\tau - 0)$$

$$S_{xx}(\tau) = -3\tau + 1 \quad \text{for interval } 0 - T$$

eq of line

using points $(T, -0.5)$ $\;x_1, y_1$
$(2T, 0)$

$$\frac{-0.5 - 0}{T - 2T} = m$$

$$m = \frac{-0.5}{-T} \Rightarrow \frac{1}{2T}$$

using points $\{\tau, S_{xx}(\tau)\}$
$(2T, 0)$

$$S_{xx}(\tau) - 0 = \frac{1}{2T}(\tau - 2T)$$

$$S_{xx}(\tau) = -1 + \frac{\tau}{2T}$$

for interval $T - 2T$

As Euler formula

$$\int_{-\infty}^{\infty} S_{xx}(\tau)\{(\cos wt + j\sin wt)\}\,dt$$

$$= \int_{-\infty}^{0} S_{xx}(\tau)\cos wt\,dt + \int_{0}^{\infty} S_{xx}(\tau)\cos wt\,dt$$
$$+ \int_{-\infty}^{0} S_{xx}(\tau)(j\sin wt)\,dt + \int_{0}^{\infty} j\sin wt\,(S_{xx}(\tau))\,dt$$

As cos is even $\Rightarrow \int_{-\infty}^{0} S_{xx}(\tau)\cos wt\,dt + \int_{0}^{\infty} S_{xx}(\tau)\cos wt\,dt$

$\Rightarrow \int_{0}^{\infty} S_{xx}(\tau)\cos wt\,dt + \int_{0}^{\infty} S_{xx}(\tau)\cos wt\,dt = 2\int_{0}^{\infty} S_{xx}(\tau)\cos wt\,dt$

As sin is odd $\Rightarrow -\int_{0}^{\infty} j\sin wt\,(S_{xx}(\tau))\,dt + \int_{0}^{\infty} j\sin wt\,(S_{xx}(\tau))\,dt$

for using and sub real parts

$$2\left[\int_{0}^{T}\left(1-\frac{3\tau}{2T}\right)\cos w\tau\,d\tau + \int_{T}^{2T}\left(-1+\frac{\tau}{2T}\right)\cos w\tau\,d\tau\right.$$

$$2\left[\int_{0}^{T}\left(\cos w\tau - \frac{3\tau\cdot\cos w\tau\,d\tau}{2T}\right) + \int_{T}^{2T}\left(-1+\frac{\tau}{2T}\right)\cos w\tau\,d\tau\right.$$

$$2\left[\frac{\sin w\tau}{w}\Big|_{0}^{T} - \frac{3}{2}\int_{0}^{T}\tau\cdot\cos w\tau\,d\tau - \frac{\sin w\tau}{w}\Big|_{T}^{2T} + \frac{1}{2}\int_{T}^{2T}\tau\cdot\cos w\tau\,d\tau\right.$$

By parts $\int f(x)\,g(x)\,dx = f(x)\int g(x)\,dx - \int \frac{df(x)}{dx}\left(\int g(x)\,dx\right)dx$

$f(x) = \tau$ $\qquad g(x) = \cos w\tau$

$\frac{df(x)}{dx} = 1$ $\qquad \int g(x)\,dx = \frac{\sin w\tau}{w}$

$$= 2\left[\frac{\sin w\tau}{w}\Big|_{0}^{T} - \frac{\sin w\tau}{w}\Big|_{T}^{2T} - \frac{3}{2}\left[\frac{\tau\sin w\tau}{w} - \int\frac{\sin w\tau}{w}\,d\tau\right] + \dots\right.$$

$$= \quad '' \quad -\frac{3}{2}\left[\frac{\tau\sin w\tau}{w} + \frac{\cos w\tau}{w^2}\right]_{0}^{T} + \dots$$

$$= \frac{2}{} \left[\frac{\sin \omega\tau}{\omega} \Big|_0^T - \frac{\sin \omega\tau}{\omega} \Big|_T^{2T} - \frac{3}{2T} \left[\frac{\tau \sin \omega\tau}{\omega} + \frac{\cos \omega\tau}{\omega^2} \right]_0^T \right.$$

$$\left. + \frac{1}{2T} \left[\frac{\tau \sin \omega\tau}{\omega} + \frac{\cos \omega\tau}{\omega^2} \right]_T^{2T} \right\}$$

$$2\left[\frac{\sin\omega T}{\omega}-\frac{\sin\omega 2T}{\omega}+\frac{\sin\omega T}{\omega}-\frac{3}{2T}\left\{\frac{T\sin\omega T}{\omega}+\frac{\cos\omega T}{\omega^2}-\frac{1}{\omega^2}\right\}\right.$$

$$\left.+\frac{1}{2T}\left[\frac{2T\sin\omega 2T}{\omega}-\frac{T\sin\omega T}{\omega}+\frac{\cos\omega 2T}{\omega^2}-\frac{\cos\omega T}{\omega^2}\right]\right]$$

$$2\left[\cancel{\frac{\sin\omega T}{\omega}}-\cancel{\frac{\sin\omega 2T}{\omega}}+\frac{\sin\omega T}{\omega}-\frac{3\sin\omega T}{2\omega}-\frac{3\cos\omega T}{2T\omega^2}+\frac{3}{2T\omega^2}\right.$$

$$\left.+\cancel{\frac{\sin\omega 2T}{\omega}}-\frac{\sin\omega T}{2\omega}+\frac{\cos\omega 2T}{2T\omega^2}-\frac{\cos\omega T}{2T\omega^2}\right]$$

$$\frac{3\sin\cancel{wT}}{\cancel{2}w} - \frac{4\cos wT}{2Tw^2} - \frac{3\sin\cancel{wT}}{\cancel{2}w} + \frac{\cos w2T}{2Tw^2} + \frac{3}{2Tw^2}$$

$$-\frac{4\cos wT}{2Tw^2} + \frac{\cos w2T}{2Tw^2} + \frac{3}{2Tw^2}$$

$$\cancel{2}\left[\frac{1}{\cancel{2}Tw^2}\left(3 - 4\cos wT + \cos w2T\right)\right]$$

$$\frac{1}{Tw^2}\left[3 - 4\cos wT + 2\cos^2 wT - 1\right]$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = \cancel{\cos^2\theta - \cos^2\theta}$ $\cancel{-\sin^2\theta}$

$\cos^2\theta - 1 + \cos^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$$\Rightarrow 2 - 4\cos wT + 2\cos^2 wT$$

$$\frac{2}{Tw^2}\left[\cos^2 wT - 2\cos wT + 1\right]$$

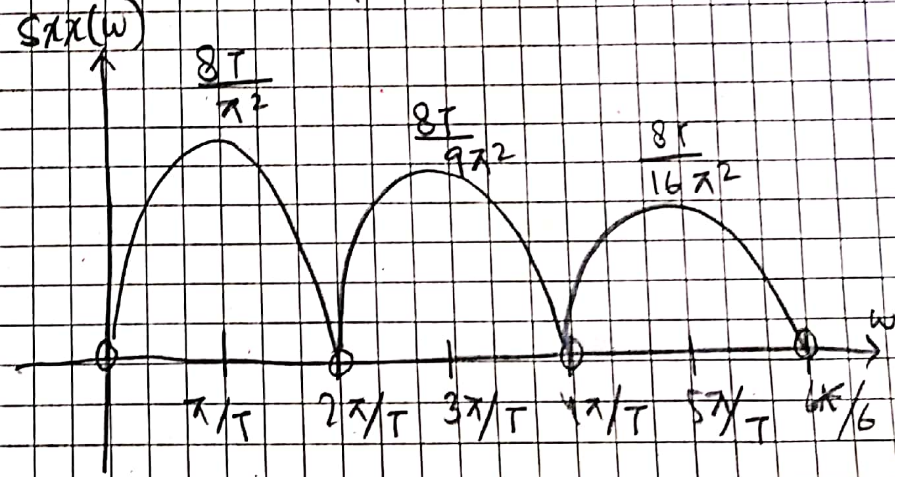
$$\frac{2(\cos wT - 1)^2}{Tw^2} \Rightarrow S_{xx}(w)$$

w	0	π/T	$2\pi/T$	$3\pi/T$	$4\pi/T$	$5\pi/T$	$6\pi/T$
$S_{xx}(w)$	0	$\frac{8T}{\pi^2}$	0	$\frac{8T}{9\pi^2}$	0	$\frac{8T}{16\pi^2}$	0

$$\frac{4\times 2}{\cancel{T}\times\frac{\pi^2}{T^{\cancel{2}}}} \Rightarrow \frac{8T}{\pi^2}$$

$$\frac{4\times 2}{\left(\frac{3\pi}{T}\right)^2\times \cancel{T}} \Rightarrow \frac{8T}{9\pi^2}$$

$$\frac{8T}{(4\pi)^2}$$



The shape of the graph (Sinc series) is due to the fact that Fourier Transform of Rectangle pulse is Sinc function. Please refer Fourier Transform Tables.pdf
Note, the acf of the case in point is a asymmetric rectangle pulse of task 3.6