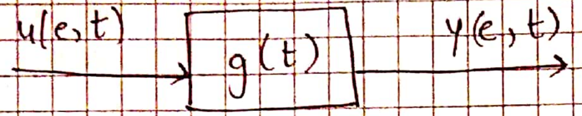
Task 5.1

A Gaussian noise signal $u(e,t)$ is the input signal of a linear system which is described by its impulse response $g(t)$



Some power spectral densities are known.

$$S_{uy} = \frac{S_1}{(1-j\omega b)(1+j\omega T_1)} \qquad S_{yy} = \frac{S_1}{1+\omega^2 T_1^2}$$

a) Determine the transfer function $G(j\omega)$ of the linear system.

Deriving Transfer function

→ If $U(j\omega)$ is input and $Y(j\omega)$ is output then transfer function of the linear system becomes:

$$Y(j\omega) = G(j\omega) \cdot U(j\omega)$$

→ We know $S_{yy}(j\omega) = E(Y^* Y)$

when Y^* is the shifted signal across Y

→ $Y = G \cdot U$

$= Y^* = U^* . G^*$ Taking conjugate of both sides

$= U^* . G^* Y = Y^* Y$ multiplying both sides by Y

$= G^* E(U^* . Y) = E(Y^* Y)$

$$\Rightarrow \boxed{G^* = \frac{S_{yy}}{S_{uy}}}$$

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$$\rightarrow G^* = \frac{S_{yy}}{S_{uy}}$$

$$\rightarrow G^* = \frac{\dfrac{S_1}{1+w^2T_1^2}}{\dfrac{S_1}{(1-jwb)(1+jwT_1)}} = \frac{(1-jwb)(1+jwT_1)}{(1+w^2T_1^2)}$$

$$= \frac{(1-jwb)(1+jwT_1)}{(1-jwT_1)(1+jwT_1)} = \left(\frac{1-jwb}{1-jwT_1}\right)$$

$\rightarrow f = \frac{a}{b}$; $f^* = \frac{a^*}{b^*}$ $\rightarrow$ for conjugating fractions

$$\boxed{G = \frac{1+jwb}{1+jwT_1}}$$

b) Is the system described by $G(jw)$ a causal system? Explain your statement.

For a causal system:

① Order of numerator of transfer function should not be greater than the order of denominator.

② Poles of the transfer function lie on left half of complex plane

Poles are roots of denominator. Either the transfer function is in Fourier domain (jw) or Laplace domain (s-domain) both scenarioes have to be fulfilled.

To find poles, equating denominator of $G(j\omega)$ to 0

$$1 + j\omega T_1 = 0$$

$$\boxed{j\omega = -\frac{1}{T_1}}$$ Constitutes to one pole of the system

To make this pole lie in left half of complex plane, T_1 must be positive in that case it will be causal. i.e <u>$T_1 > 0$ then system is causal</u>

c) Calculate the autocorrelation function $S_{uu}(\tau)$ of the input signal $u(e, t)$

We know

$$= G \cdot u = Y$$

$$= G \cdot u \cdot u^* = Y \cdot u^*$$

$$= u \cdot u^* = \frac{Y \cdot u^*}{G}$$

$$= S_{uu} = \frac{S_{uy}}{G}$$

$$= S_{uu} = \frac{\dfrac{S_1}{(1 - j\omega b)\cancel{(1 + j\omega T_1)}}}{\left(\dfrac{1 + j\omega b}{\cancel{1 + j\omega T_1}}\right)}$$

$$= S_{uu} = \frac{S_1}{(1 - j\omega b)(1 + j\omega b)}$$

$$\rightarrow S_{uu} = \frac{S_1}{(1 - j^2\omega^2 b^2)} \Rightarrow \frac{S_1}{1 + \omega^2 b^2}$$

For $S_{uu}(\tau)$ i.e converting frequency to time domain, fourier inverse table is used. / Inverse fourier transform is employed.

$$S_{uu}(\tau) = F^{-1}\left[\frac{S_1}{1+\omega^2 b^2}\right]$$

$\therefore$ Table of Fourier transforms shared

In the table for:

$$\Rightarrow \quad e^{-a|t|} = \frac{2a}{a^2+\omega^2}$$

$$\cancel{F^{-1} \times \frac{2}{2}}\left[\cancel{\frac{2a}{a^2+\omega^2}} \quad \cancel{\frac{S_1}{b^2+\omega^2 b^2}}\right]$$

$$\rightarrow \quad S_{uu}(\tau) = F^{-1}\left[\frac{S_1}{b^2\left(\frac{1}{b^2}+\omega^2\right)} \times \frac{2}{2}\right]$$

$\rightarrow$ a^2 for $e^{-a|t|} = \frac{2a}{a^2+\omega^2}$

as $a = 1/b$

$$S_{uu}(\tau) = F^{-1}\left[\frac{S_1}{2b} \frac{2(1/b)}{\left(\frac{1}{b^2}+\omega^2\right)}\right]$$

$$\boxed{S_{uu}(\tau) = \frac{S_1}{2b} e^{-\frac{1}{b}|\tau|}}$$