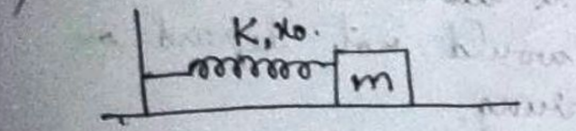
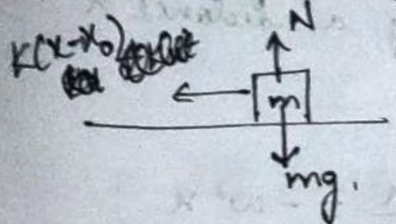
Classical Harmonic Oscillator

1)



When the spring is stretched ~~by~~ to a ~~distance~~ length x,



There would also be normal reaction between the spring and the mass at the point of contact.

2) spring force = K (displaced length)

$$a = \frac{d^2x}{dt^2} = -\frac{K(x-x_0)}{m}$$

also, $a = v \cdot \frac{dv}{dt}$

$$\Rightarrow v\frac{dv}{dx} = -\frac{K}{m}(x-x_0)$$

$$\Rightarrow \int_{v_{x_0}}^{v_x} v\,dv = -\frac{K}{m}\int_{x_0}^{x}(x-x_0)\,dx \Rightarrow \frac{(v_x^2-v_{x_0}^2)}{2} = -\frac{K}{m}\left[\frac{x^2}{2} - x_0 x\right]_{x_0}^{x}$$

$$\Rightarrow \frac{(v_x^2-v_{x_0}^2)}{2} = -\frac{K}{m}\left[\frac{x^2}{2} - xx_0 - \frac{x_0^2}{2} + x_0^2\right] \Rightarrow \frac{(v_{x}^2-v_{x_0}^2)}{2} = -\frac{K}{m}\left[\frac{x^2}{2} - xx_0 + \frac{x_0^2}{2}\right]$$

$$\Rightarrow \frac{v_x^2 - v_{x_0}^2}{2} = -\frac{K}{m}\frac{(x-x_0)^2}{2}$$

If the spring is stretched to a length x, then the length over which restoring force acts is $(x-x_0)$ as x_0 was the length of the spring initially.
If the spring is stretched by a length x, then restoring force acts over x

3) Yes, as I mentioned earlier, if we consider the spring to have stretched by a distance x, then ~~the~~ ~~deform~~ ~~x₀~~ the spring force would not depend on the initial length of the spring.

4) $a = \frac{-Kx}{m} = \frac{d^2x}{dt^2}$ (when spring is stretched by a distance x)

$$\Rightarrow \frac{d^2x}{dt^2} = -\left(\frac{K}{m}\right)x$$

This is of the form $\frac{d^2x}{dt^2} = -\omega^2 x$

where $\boxed{\omega = \sqrt{\frac{K}{m}}}$ $\Rightarrow K = m\omega^2$

Because I was stupid, and didn't get rid of x_0 before solving, I got confused and messed up. So, let's start over

5) $a = v\frac{dv}{dx} = -\frac{K}{m}x$. (from question 4)

$$\Rightarrow \int_{v_0}^{v} v\,dv = -\frac{K}{m}\int_{a_0}^{x} x\,dx$$ (~~let's assume velocity after~~

(let's assume velocity at a_0 to be v_0 and after stretching by x to be v, a_0 is an arbitrary position)

$$\Rightarrow \frac{v^2}{2} - \frac{v_0^2}{2} = -\frac{K}{m}\frac{(x^2 - a_0^2)}{2} \Rightarrow v^2 = \left(\frac{-K}{m}\right)(x^2 - a_0^2) + v_0^2$$

$$\Rightarrow v = \sqrt{-\omega^2(x^2 - a_0^2) + v_0^2}$$

for $a_0 = 0$,

$$v = \sqrt{-\omega^2 x^2 + v_0^2}$$

$$\cancel{\frac{dx}{dt} = \sqrt{-\omega^2(x^2 - x_0^2) + v_0^2}}$$

$$\cancel{\Rightarrow \int_{x_0}^{x} \frac{dx}{\sqrt{-\omega^2(x^2 - x_0^2) + v_0^2}} = \int_0^t dt}$$

$$\cancel{\Rightarrow \int_{x_0}^{x} \frac{dx}{\omega\sqrt{\left(\frac{v_0}{\omega}\right)^2 - (x^2 - x_0^2)}} = \int_0^t dt}$$

$$\cancel{\text{let } (x^2 - x_0^2) = \left(\frac{v_0}{\omega}\right)^2 \sin^2\theta}$$

$$\frac{dx}{dt} = \sqrt{-\omega^2 x^2 + v_0^2}$$

$$\Rightarrow \int_0^x \frac{dx}{\sqrt{-\omega^2 x^2 + v_0^2}} = \int_{t_0}^t dt$$

$$\Rightarrow \int_0^x \frac{dx}{\omega\sqrt{\left(\frac{v_0}{\omega}\right)^2 - x^2}} = \int_{t_0}^t dt$$

$$\cancel{\text{let } x = \left(\frac{v_0}{\omega}\right)\sin\theta}$$

$$\cancel{\Rightarrow dx = \left(\frac{v_0}{\omega}\right)\cos\theta\, d\theta}$$

$$\cancel{\Rightarrow \int_0^{\left(\frac{v_0}{\omega}\right)\sin\theta} \frac{(v_0/\omega)\cos\theta\, d\theta}{\omega\left(\frac{v_0}{\omega}\right)\cos\theta} = \int_0^t dt}$$

$$\cancel{\Rightarrow \frac{1}{\omega}\left(\frac{v_0}{\omega}\right)\sin\theta = t}$$

$$\cancel{\Rightarrow \left(\frac{v_0}{\omega^2}\right)(x) = t}$$

$$\cancel{\Rightarrow \int_0^{\left(\frac{v_0}{\omega}\right)\sin\theta} \frac{(v_0/\omega)\cos\theta\, d\theta}{\omega\left(\frac{v_0}{\omega}\right)\cos\theta} = \int_0^t dt}$$

$$\cancel{\Rightarrow \frac{1}{\omega}\int_0^{(v_0/\omega)\sin\theta} d\theta = \int_0^t dt}$$

$$\cancel{\Rightarrow \frac{1}{\omega}\int_0^x \sin^{-1}\left(\frac{\omega x}{v_0}\right) dx = \frac{1}{\omega}\left(\frac{v_0}{\omega}\right)\sin}$$

$$\Rightarrow \frac{1}{\omega}\sin^{-1}\left(\frac{\omega x}{v_0}\right) = t - t_0$$

$$\Rightarrow x = \frac{v_0}{\omega}\sin(\omega t - \omega t_0)$$

6) Since it is a second order differential equation, it should have 2 integration constants. I also seem to have 2 integration constants so, yay!

7) for $x=0$,

$$0 = \frac{v_0}{\omega} \sin(-\omega t_0)$$

(i) if $x=0$ is assumed to be the mean position,

$v_0 \neq 0$ (at mean position, velocity $\neq 0$)

$\Rightarrow t_0 = 0$ (or an integral multiple of π)

$$\Rightarrow x = \left(\frac{v_0}{\omega}\right) \sin(\omega t)$$

(ii) if $x=0$ is assumed to be the extreme position,

$v_0 = 0$ (at extreme position, velocity $=0$)

t_0 may or may not be zero

$$\Rightarrow x = \left(\frac{v_0}{\omega}\right) \sin(\omega t) \quad \text{or} \quad x = \left(\frac{v_0}{\omega}\right) \sin(\omega t - \omega t_0)$$

(iii) if $x=0$ is an arbitrary position

$$0 = \left(\frac{v_0}{\omega}\right) \sin(-\omega t_0)$$

it can be safely assumed that

$$t_0 = n\pi,\ n \in \mathbb{Z}.$$

$$x = \left(\frac{v_0}{\omega}\right) \sin(-\omega t + n\pi)$$

8) So, now we assume the particle starts at $x=0$

$\Rightarrow$ at $t=0$, $x=0$

$\Rightarrow x = \left(\frac{V_0}{\omega}\right)\sin(\omega t)$

$v(t) = \frac{dx}{dt} = V_0 \cos(\omega t)$

$a(t) = \frac{d^2x}{dt^2} = -V_0 \omega \sin(\omega t)$

it satisfies the relation $\frac{d^2x}{dt^2} = -\omega^2 x$

Also, x can be $x = \left(\frac{V_0}{\omega}\right)\sin(\omega t + n\pi)$ (a more general equation)

9) $v^2 + (x\omega)^2 = V_0^2$ $(\because \sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow v = \sqrt{V_0^2 - \omega^2 x^2}$

10) (a) $v=0$

$\Rightarrow \sqrt{V_0^2 - \omega^2 x^2} = 0 \Rightarrow \cancel{x = \pm\sqrt{\frac{V}{\omega}}}$ ~~these are your~~

$\boxed{x = \pm\left(\frac{V_0}{\omega}\right)}$

(these are your extreme positions)

(b) $x=0$

$\Rightarrow v = \sqrt{V_0^2 - 0} \Rightarrow \boxed{v = V_0}$

(this is at the mean position)

(c) $v = \frac{\max.(v)}{2}$

max. of $v = V_0$ (we can see this from the equation of $v(t)$)

$\Rightarrow \frac{V_0}{2} = \sqrt{V_0^2 - \omega^2 x^2} \Rightarrow \frac{V_0^2}{4} = V_0^2 - \omega^2 x^2$

$\Rightarrow \boxed{x = \pm\frac{\sqrt{3}\, V_0}{2\omega_0}}$

(d) $x = \frac{max.(x)}{2}$

from the equation of $x(t)$, we get

$max(x) = \frac{V_0}{\omega}$

$\Rightarrow V = \sqrt{V_0^2 - \omega^2\left(\frac{V_0}{2\omega}\right)^2} = \sqrt{V_0^2 - \frac{V_0^2}{4}}$

$\Rightarrow \boxed{V = \pm \frac{\sqrt{3}}{2}(V_0)}$

(e) I guess, it would be that there are two positions x which have the same v and these are equidistant from the position $x=0$ here (which can now be said to be the mean position).
Also, if going along +ve X-direction is considered +ve, then V is shown to be +ve when it is at a ~~position of~~ distance x along the +ve X-axis and is −ve when it is at a distance x along the −ve X-axis.

11) $KE = \frac{1}{2}mV^2$

$= \frac{1}{2}m(V_0^2 - \omega^2 x^2)$

KE is max. at $x=0$

$\Rightarrow (KE)_{max.} = \frac{1}{2}mV_0^2$

(KE) is min when when x^2 is max.

$\Rightarrow$ at $x = \pm\left(\frac{V_0}{\omega}\right)$

$\Rightarrow (KE)_{min.} = 0$

No, the kinetic energy cannot be negative.
We have seen earlier that $x_{max.} = \frac{V_0}{\omega}$

$\Rightarrow x^2 \le \frac{V_0^2}{\omega^2} \Rightarrow V_0^2 - \omega^2 x^2 \ge 0$

12) The kinetic energy is bound between two values, i.e., $KE \in \left[0, \frac{1}{2}mv_0^2\right]$

KE decreases from the mean position to the extremes

13) $TE = KE + PE$

$$= \frac{1}{2}mv^2 + \frac{1}{2}Kx^2$$

$$= \frac{1}{2}m(v_0^2 - \omega^2x^2) + \frac{1}{2}Kx^2$$

$$= \frac{1}{2}mv_0^2 - \cancel{\frac{1}{2}Kx^2} + \cancel{\frac{1}{2}Kx^2} \quad \left(\omega = \sqrt{\frac{K}{m}}\right)$$

$$\Rightarrow TE = \frac{1}{2}mv_0^2$$

The given system has its total energy constant. I think energy conservation will hold if energy is not dissipated as heat, sound, light, etc. One example of such dissipation would be when friction acts between the spring system and the floor. Of course, if you can change the system to one that can consider these forces, then I guess it can be made into a system where law of conservation of energy holds good. The example mentioned here could be modified to ~~the spring~~ spring-mass-floor system.

Back to the given mentioned system, there is no such dissipative force. So, law of conservation of energy holds true.

14) $PE = \frac{1}{2}Kx^2$

PE is max. when x^2 is max., i.e., at $x = \pm \frac{v_0}{\omega}$

PE is min. when x^2 is min., i.e, at $x=0$.

~~Qq~~ PE is min.

No, the PE of this particle can never be negative as $x^2 > 0 \Rightarrow Kx^2 > 0$

I think, the PE can be negative when the given system is oscillating in the vertical direction.

15) (a) $F = -Kx$.

(b) $W = \int_{x_A}^{x_B} F dx = \left[-\frac{1}{2}Kx^2\right]_{x_A}^{x_B} = \frac{1}{2}K(x_A^2 - x_B^2)$

(c) ~~PE at $x = x_A = \frac{1}{2}Kx_A^2$~~

~~PE at $x = x_B = \frac{1}{2}Kx_B^2$~~

~~$\Rightarrow$ W done in the above case $= PE_{x=x_A} - PE_{x=x_B}$~~

By W-E theorem,

W done = change in KE

$\Rightarrow$ W done $= KE_{x=x_B} - KE_{x=x_A}$

TE = constant $\Rightarrow PE_{x=x_A} + KE_{x=x_A} = PE_{x=x_B} + KE_{x=x_B}$

$\Rightarrow KE_{x=x_B} - KE_{x=x_A} = PE_{x=x_A} - PE_{x=x_B}$

$\Rightarrow$ W done $= (PE_{x=x_A}) - (PE_{x=x_B})$

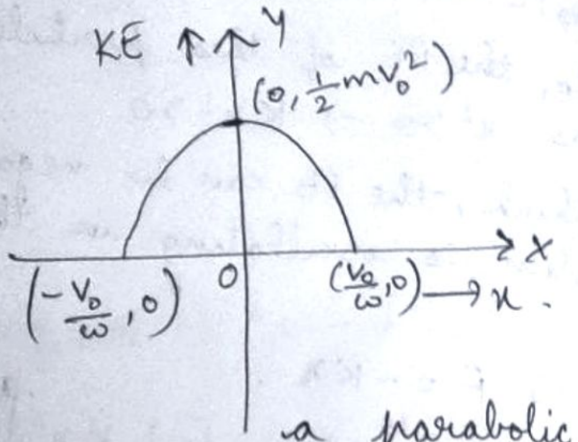
(d) $\Rightarrow PE_{x=x_A} - PE_{x=x_B} = \frac{1}{2}K.x_A^2 - \frac{1}{2}Kx_B^2$

$\Rightarrow PE = \frac{1}{2}Kx^2$

16) $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(v_0^2 - \omega^2 x^2)$

$= \frac{1}{2}\, m\omega^2\left(\left(\frac{v_0}{\omega}\right)^2 - x^2\right)$

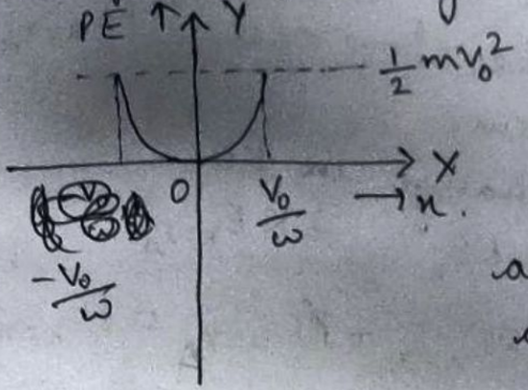
this is of the form $y = c^2 - x^2$



a parabolic curve

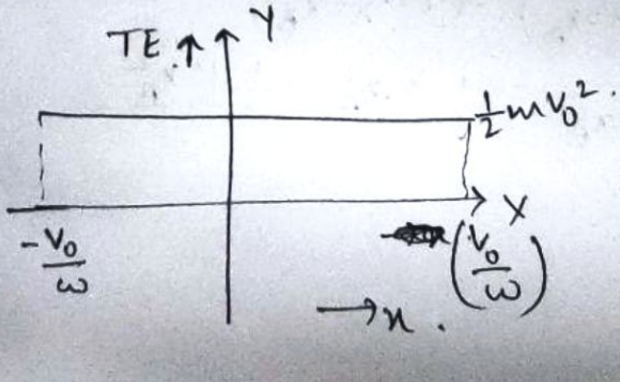
$PE = \frac{1}{2}Kx^2$

this is of the form $y = x^2$



again, a parabolic curve.

$TE = \frac{1}{2}mv^2 + \frac{1}{2}Kx^2 = \frac{1}{2}mv_0^2$ (a constant)



(a straight line parallel to X-axis)

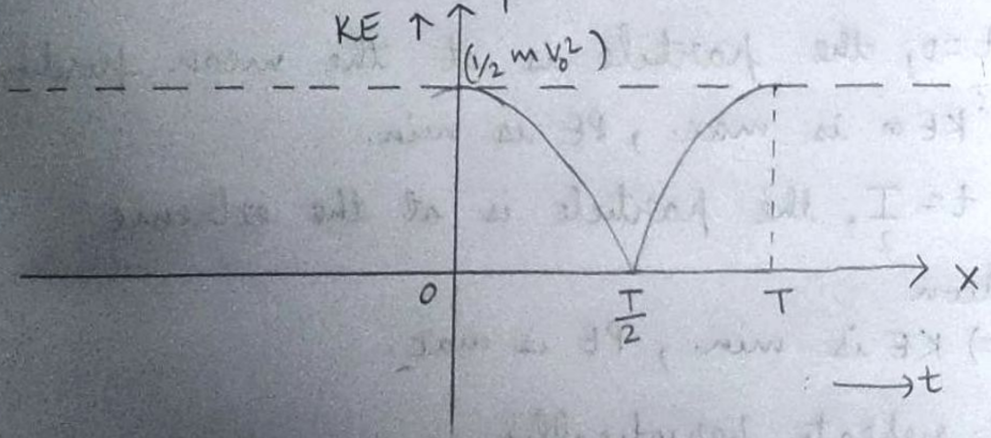
17) x, v and a as a function of time were found to be trigonometric

$$x = \frac{v_0}{\omega} \sin \omega t$$

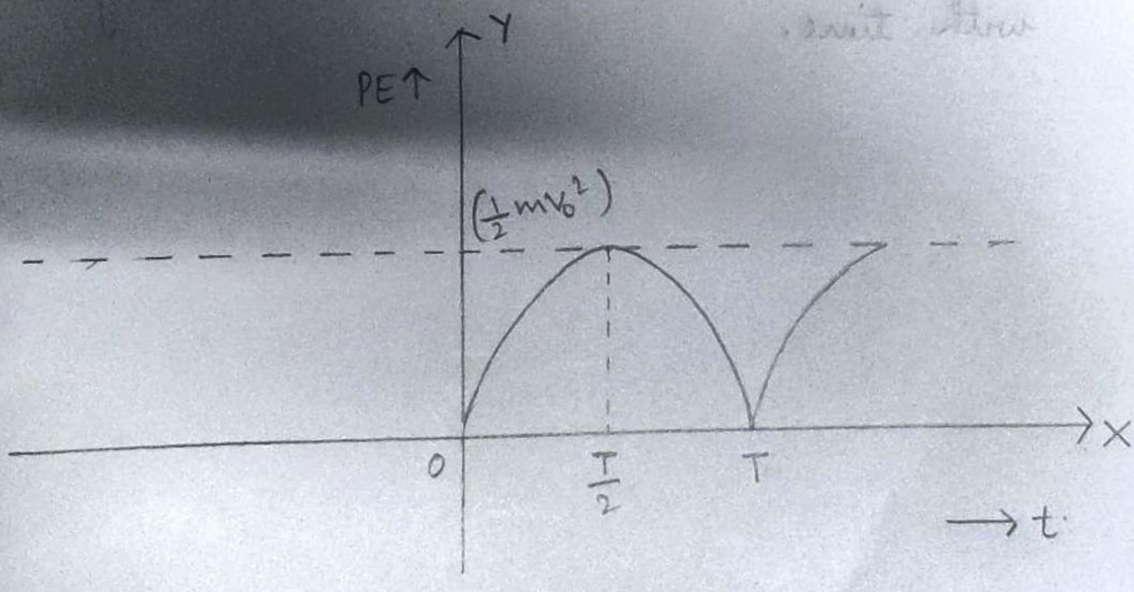
The motion is periodic with a time period of

$$\boxed{T = \frac{2\pi}{\omega}}$$

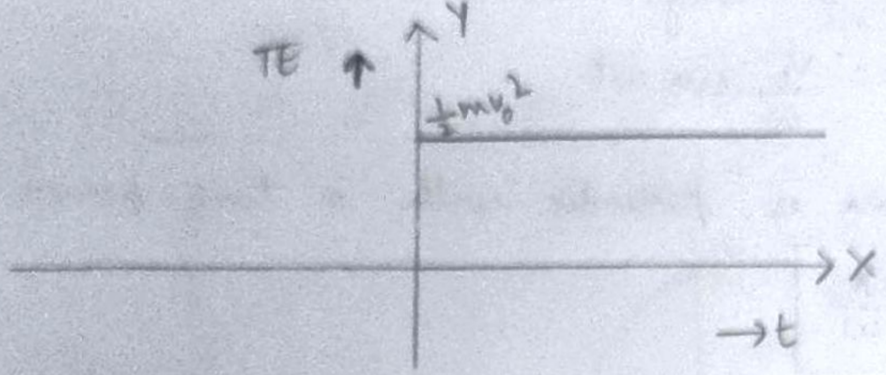
18) $KE = \frac{1}{2} m v^2 = \frac{1}{2} m v_0^2 \cos^2(\omega t)$



$PE = \frac{1}{2} k x^2 = \frac{1}{2} k \left(\frac{v_0}{\omega}\right)^2 \sin^2(\omega t) = \frac{1}{2} m v_0^2 \sin^2(\omega t)$



$TE = \frac{1}{2} m v_0^2$



KE and PE repeat values periodically with a period of $\frac{\pi}{\omega}$.

at $t=0$, the particle is at the mean position

$\Rightarrow$ KE is max., PE is min.

at $t = \frac{T}{2}$, the particle is at the extreme position

$\Rightarrow$ KE is min., PE is max.

this repeats periodically

TE is a constant, so it doesn't vary with time.