$\alpha = \dfrac{2\pi \nu m}{\hbar}$

$$\psi_v = \begin{cases} e^{-\alpha x^2/2}\left(c_0 + c_0 x^2 + \cdots c_v x^v\right) & \text{for } v \text{ even} \\ e^{-\alpha x^2/2}\left(c_1 x + c_3 x^3 + \cdots c_v x^v\right) & \text{for } v \text{ odd} \end{cases}$$

$\hbar = 1$

$\phi_1 = e^{-\pi \nu x^2}\left(c_1 x + c_3 x^3\right)$

$\phi_0 = e^{-\pi \nu m x^2}\left(c_0\right)$

normalizing

$$\int_{-\infty}^{\infty} |\phi_0|^2 \, dx = 1$$

$$\phi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha x^2}{2}}$$

$$\Rightarrow \phi_0 = \left(\frac{2\nu m \pi}{\frac{\hbar}{\pi}}\right)^{1/4} e^{-\frac{\pi \nu m x^2}{\hbar}}$$

$$\phi_1 = \left(\frac{4\alpha^3}{\pi}\right)^{1/4} e^{-\alpha x^2/2} x$$

$$\Rightarrow \phi_1 = \left(\frac{4\left(\frac{8\pi^3 \nu^3 m^3}{\hbar^3}\right)}{\pi}\right)^{1/4} e^{-\frac{\pi \nu m x^2}{\hbar}} x$$

$$\Rightarrow \phi_0 = \left(2\nu m\right)^{1/4} e^{-\pi m \nu x^2}$$

$$\phi_1 = \left(32\pi^2 \nu^3 m^3\right)^{1/4} e^{-\pi \nu m x^2} x$$

$\langle \phi_0 | H_{in} | \phi_0 \rangle =$

$$\int_0^L (2v\pi m)^{1/4} e^{-\pi m v x^2} \left(\frac{-1}{2m} \frac{d^2}{dx^2} + \alpha x \right) \left((2v\pi m)^{1/4} e^{-\pi m v x^2} \right) dx$$

$$= \int_0^L (2v\pi m)^{1/4} e^{-\pi m v x^2} \left[\frac{\pm 1}{2m} (2v\pi m)^{1/4} \left(2\pi m v e^{-\pi m v x^2} + 4\pi^2 m^2 v^2 x^2 e^{-\pi m v x^2} \right) + \alpha (2\pi m v)^{1/4} x e^{-\pi m v^2} \right] dx$$

$$= (2\pi m v)^{1/2} (\pi v) \int_0^L e^{-2\pi m v x^2} dx + (2\pi m v)^{1/2} (2\pi^2 m v^2) \int_0^L x^2 e^{-2\pi m v x^2} dx$$

$$+ \alpha (2\pi m v)^{1/2} \int_0^L x e^{-2\pi m v x^2} dx.$$

$$= (2\pi m v)^{1/2} (\pi v) \int_0^L e^{-2\pi m v x^2} dx + (2\pi m v)^{1/2} (2\pi^2 m v^2) \int x \cdot x e^{-2\pi m v x^2} dx$$

$$+ \alpha (2\pi m v)^{1/2} \int_0^L e^{-2\pi m v x^2} \cdot \frac{(-2\pi m v) x}{(-2\pi m v)} dx$$

$$\cancel{= (2\pi m v)^{1/2} (\pi v) \int_0^L e^{-2\pi m v x^2} dx + (2\pi m v)^{1/2} (2\pi^2 m v^2) \left[x \int x e^{x^2} dx - \int \right.}$$

$$\cancel{= (2\pi m\upsilon)^{1/2}(\pi\upsilon)\int_0^L e^{-2\pi m\upsilon x^2}dx + (2\pi m\upsilon)^{1/2}(2\pi^2 m\upsilon^2)}$$

let $f(x) = x$, $g'(x) = x e^{(-2\pi m\upsilon)x^2}$

$$\int x \cdot x e^{-2\pi m\upsilon x^2} dx = x\int x e^{-2\pi m x^2} dx - \int 1 \cdot \int x e^{-2\pi m x^2} dx$$

$$= x \cdot \left(\frac{e^{-2\pi m\upsilon x^2}}{-4\pi m\upsilon} \right) - \int \frac{e^{-2\pi m\upsilon x^2}}{-4\pi m\upsilon} dx$$

$$= \frac{x \left(e^{-2\pi m\upsilon x^2}\right)}{-4\pi m\upsilon} + \frac{1}{4\pi m\upsilon}\int e^{-2\pi m\upsilon x^2} dx$$

$\Rightarrow \langle \phi_0 | H_{in} | \phi_0 \rangle =$

$$(2\pi m\upsilon)^{1/2}(\pi\upsilon)\int_0^L e^{-2\pi m\upsilon x^2} dx + \alpha(2\pi m\upsilon)^{1/2}\left[\frac{e^{-2\pi m\upsilon x^2}}{-2\pi m\upsilon}\right]_0^L$$

$$+ (2\pi m\upsilon)^{1/2}(2\pi^2 m\upsilon^2)\left(\left[\frac{x\left(e^{-2\pi m\upsilon x^2}\right)}{-4\pi m\upsilon}\right]_0^L + \frac{1}{4\pi m\upsilon}\int_0^L e^{-2\pi m\upsilon x^2} dx\right)$$

$$= (2\pi m\upsilon)^{1/2}(\pi\upsilon)\int_0^L e^{-2\pi m\upsilon x^2} dx + \frac{\alpha(2\pi m\upsilon)^{1/2}\left[e^{-2\pi m\upsilon x^2}\right]_0^L}{-2\pi m\upsilon}$$

$$- (2\pi m\upsilon)^{1/2}\left(\frac{\pi\upsilon}{2}\right)\left[x e^{-2\pi m\upsilon x^2}\right]_0^L + (2\pi m\upsilon)^{1/2}\left(\frac{\pi\upsilon}{2}\right)\int_0^L e^{-2\pi m\upsilon x^2} dx$$

$$= (2\pi m\upsilon)^{1/2}\left(\frac{3\pi\upsilon}{2}\right)\int_0^L e^{-2\pi m\upsilon x^2} dx + \frac{(2\pi m\upsilon)^{1/2}\alpha}{+2\pi m\upsilon}\left[1 - e^{-2\pi m\upsilon L^2}\right]$$

$$- (2\pi m\upsilon)^{1/2}\left(\frac{\pi\upsilon}{2}\right)\left[L e^{-2\pi m\upsilon L^2}\right]$$

$\psi(x) = \langle x | \psi \rangle$

$\psi(x) \neq |\psi\rangle$

$|\psi\rangle$

$\psi_0(x) = \langle x | 0 \rangle$

$N_0 e^{-\alpha x^2}$

$\langle x | n \rangle$

$\underbrace{(a + a^\dagger)}_{\hat{x}}|n\rangle = a|n\rangle + a^\dagger|n\rangle = \sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle$

$\langle \phi_0 | H_{in} | \phi_1 \rangle =$

$$\int_0^L dx\,(2\pi m v)^{1/4} e^{-\pi m v x^2}\left(-\frac{1}{2m}\frac{d^2}{dx^2}+\alpha x\right)\left((32\pi^2 m^3 v^3)^{1/4} e^{-\pi m v x^2}\cdot x\right)$$

$$= \int_0^L (2\pi m v)^{1/4} e^{-\pi m v x^2}\left[\frac{-1}{2m}(32\pi^2 m^3 v^3)^{1/4}\left((-2\pi m v) e^{-\pi m v x^2} + (-2\pi m v)\cdot 2x\cdot e^{-\pi m v x^2} + (-2\pi m v)^2 x^3 e^{-\pi m v x^2}\right) + \alpha (32\pi^2 m^3 v^3)^{1/4} x^2 e^{-\pi m v x^2}\right] dx$$

$$= \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(2\pi m v)\int_0^L x e^{-2\pi m v x^2} dx + \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(4\pi m v)\int_0^L x e^{-2\pi m v x^2} dx$$

$$+ \frac{(64\pi^2 m^4 v^4)}{2m}(2\pi m v)^2 \int_0^L x^3 e^{-2\pi m v x^2} dx$$

$$+ (64\pi^2 m^4 v^4)^{1/4}\alpha \int_0^L x^2 e^{-2\pi m v x^2} dx$$

$$= \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(6\pi m v)\int_0^L x\, e^{-2\pi m v x^2} dx$$

$$+ (64\pi^2 m^4 v^4)^{1/4}\alpha\int_0^L x^2 e^{-2\pi m v x^2} dx$$

$$+ \frac{(64\pi^2 m^4 v^4)^{1/4}(2\pi m v)^2}{2m}\int_0^L x^3 e^{-2\pi m v x^2} dx$$

$$= \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(6\pi m v)\left[\frac{e^{-2\pi m v x^2}}{-4\pi m v}\right]_0^L +$$

$$(64\pi^2 m^4 v^4)^{1/4}\alpha\left[\frac{x(e^{-2\pi m v x^2})}{-4\pi m v} + \frac{1}{4\pi m v}\int e^{-2\pi m v x^2} dx\right]_0^L$$

$$+ \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(4\pi^2 m^2 v^2)\int_0^L x^3 e^{-2\pi m v x^2} dx$$

$$= \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}\frac{(6\pi m v)}{4\pi m v}\left(1 - e^{-2\pi m v L^2}\right) +$$

$$\frac{(64\pi^2 m^4 v^4)^{1/4}}{4\pi m v}\alpha\left[-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx\right] +$$

$$\frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(4\pi^2 m^2 v^2)\int_0^L e^{-2\pi m v x^2}\cdot x^3\, dx$$

$$\int x^3 e^{-2\pi m v x^2} dx = \int x^2 \cdot x e^{-2\pi m v x^2} dx$$

$$f(x) = x^2 \quad , \quad g'(x) = x e^{-2\pi m v x^2}$$

$$= x^2 \int x e^{-2\pi m v x^2} dx - \int 2x \left(\int x e^{-2\pi m v x^2} dx \right) dx$$

$$= x^2 \left[\frac{e^{-2\pi m v x^2}}{-4\pi m v} \right] - \int 2x \cdot \frac{e^{-2\pi m v x^2}}{(-4\pi m v)} dx$$

$$= \frac{x^2 \left(e^{-2\pi m v x^2} \right)}{-4\pi m v} - 2 \frac{e^{-2\pi m v x^2}}{(-4\pi m v)^2}$$

$$\Rightarrow \langle \phi_0 | H_{in} | \phi_1 \rangle =$$

$$\frac{(64\pi^2 m^4 v^4)^{1/4}}{2m} \frac{(6\pi m v)}{4\pi m v} \left[1 - e^{-2\pi m v L^2} \right] +$$

$$\frac{(64\pi^2 m^4 v^4)^{1/4}}{4\pi m v} \alpha \left(-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx \right) +$$

$$\frac{(64\pi^2 m^4 v^4)^{1/4}}{2m} \frac{(4\pi^2 m^2 v^2)}{4\pi m v} \left[-x^2 e^{-2\pi m v x^2} + \frac{2 e^{-2\pi m v x^2}}{+4\pi m v} \right]_0^L$$

$$= \frac{\cancel{2}\cancel{m} v (4\pi^2)^{1/4}}{\cancel{2}\cancel{m}} \frac{\cancel{(6\pi m v)}^{3}}{\cancel{2}\cancel{4\pi m v}} \left(1 - e^{-2\pi m v L^2} \right)$$

$$+ \frac{\cancel{2}\cancel{m}\cancel{v} (4\pi^2)^{1/4}}{2\cancel{4}\pi \cancel{m}\cancel{v}} \alpha \left(-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx \right)$$

$$+ \frac{\cancel{2}\cancel{m} v (4\pi^2)^{1/4}}{\cancel{2}\cancel{m}} \frac{(\cancel{4}\pi^{\cancel{2}} m^{\cancel{2}} v^{\cancel{2}})}{\cancel{4\pi m v}} \left(\frac{2 e^{-2\pi m v L^2}}{+4\pi m v} - L^2 e^{-2\pi m v L^2} - \frac{2}{+4\pi m v} \right)$$

$$= \frac{3v (4\pi^2)^{1/4}}{2} \left(1 - e^{-2\pi m v L^2} \right) + \frac{(4\pi^2)^{1/4}}{2\pi} \alpha \left(-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx \right)$$

$$+ (\pi m v^2)(4\pi^2)^{1/4} \left(\frac{2 e^{-2\pi m v L^2}}{+4\pi m v} - L^2 e^{-2\pi m v L^2} - \frac{2}{+4\pi m v} \right)$$

$\langle \phi_1 | H_{in} | \phi_0 \rangle =$

$$\int_0^L dx \cdot (32\pi^2 v^3 m^3)^{1/4} x \cdot e^{-\pi m v x^2} \cdot \left(\frac{-1}{2m}\frac{d^2}{dx^2} + \alpha x\right) \cdot \left((2 v m)^{1/4} e^{-\pi m v x^2}\right)$$

$$= \int_0^L (32\pi^2 v^3 m^3)^{1/4} \cdot x e^{-\pi m v x^2} \left[\frac{+1}{2m}(2 m v)^{1/4}\left(2\pi m v e^{-\pi m v x^2} + 4\pi^2 m^2 v^2 x^2 e^{-\pi m v x^2}\right) + \alpha (2 m v)^{1/4} x \cdot e^{-\pi m v x^2}\right] dx$$

$$= \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(2\pi m v)\int_0^L x e^{-2\pi m v x^2} dx + \frac{(64\pi^2 m^4 v^4)^{1/4}(4\pi^2 m^2 v^2)}{2m}\int_0^L x^3 e^{-2\pi m v x^2} dx$$

$$+ (64\pi^2 m^4 v^4)^{1/4} \alpha \int_0^L x^2 e^{-2\pi m v x^2} dx$$

$$= \frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}(2\pi m v)\left[\frac{e^{-2\pi m v x^2}}{-4\pi m v}\right]_0^L +$$

$$\frac{(64\pi^2 m^4 v^4)^{1/4}}{2m}\frac{(4\pi^2 m^2 v^2)}{4\pi m v}\left[-x^2 e^{-2\pi m v x^2} + \frac{2 e^{-2\pi m v x^2}}{+4\pi m v}\right]_0^L$$

$$+ (64\pi^2 m^4 v^4)^{1/4} \alpha \left[\frac{x(e^{-2\pi m v x^2})}{-4\pi m v} + \frac{1}{4\pi m v}\int e^{-2\pi m v x^2} dx\right]_0^L$$

$$= \frac{2mv(4\pi^2)^{1/4}}{2m}\frac{(2\pi m v)}{4\pi m v}\left(1 - e^{-2\pi m v L^2}\right) +$$

$$\frac{2mv(4\pi^2)^{1/4}}{2m}\frac{(4\pi^2 m^2 v^2)}{4\pi m v}\left(\frac{2 e^{-2\pi m v L^2}}{+4\pi m v} - L^2 e^{-2\pi m v L^2} - \frac{2}{+4\pi m v}\right)$$

$$+ \frac{2mv(4\pi^2)^{1/4}}{4\pi m v}\alpha\left(-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx\right)$$

$$= \frac{v(4\pi^2)^{1/4}}{2}\left(1 - e^{-2\pi m v L^2}\right) +$$

$$(4\pi^2)^{1/4}(\pi m v^2)\left(\frac{2 e^{-2\pi m v L^2}}{+4\pi m v} - L^2 e^{-2\pi m v L^2} - \frac{2}{+4\pi m v}\right)$$

$$+ \frac{(4\pi^2)^{1/4}\alpha}{2\pi}\left(-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx\right)$$

$$\langle \phi_1 | H_{in} | \phi_1 \rangle =$$

$$\int_0^L dx \cdot (32\pi^2 m^3 v^3)^{1/4} x \cdot e^{-\pi m v x^2} \left[\frac{-1}{2m} \frac{d^2}{dx^2} + \alpha x \right] \left((32\pi^2 m^3 v^3)^{1/4} x \cdot e^{-\pi m v x^2} \right)$$

$$= \int_0^L dx (32\pi^2 m^3 v^3)^{1/4} x \cdot e^{-\pi m v x^2} \cdot \left[\frac{-1}{2m} (32\pi^2 m^3 v^3)^{1/4} \left((-2\pi m v) x e^{-\pi m v x^2} + (-2\pi m v) \cdot 2x e^{-\pi m v x^2} + (-2\pi m v)^2 x^3 e^{-\pi m v x^2} \right) + \alpha (32\pi^2 m^3 v^3)^{1/4} x^2 e^{-\pi m v x^2} \right] dx$$

$$= \frac{(32\pi^2 m^3 v^3)^{1/2}}{2m} (6\pi m v) \int_0^L x^2 e^{-2\pi m v x^2} dx +$$

$$(32\pi^2 m^3 v^3)^{1/2} \frac{(2\pi m v)^2}{2m} \int_0^L x^4 e^{-2\pi m v x^2} dx +$$

$$(32\pi^2 m^3 v^3)^{1/2} \alpha \int_0^L x^3 e^{-2\pi m v x^2} dx$$

$$= \frac{(32\pi^2 m^3 v^3)^{1/2}}{2m} (6\pi m v) \left[\frac{x (e^{-2\pi m v x^2})}{-4\pi m v} + \frac{1}{4\pi m v} \int e^{-2\pi m v x^2} dx \right]_0^L$$

$$+ (32\pi^2 m^3 v^3)^{1/2} \alpha \left[\frac{x^2 (e^{-2\pi m v x^2})}{-4\pi m v} - \frac{2 e^{-2\pi m v x^2}}{(-4\pi m v)^2} \right]_0^L$$

$$+ (32\pi^2 m^3 v^3)^{1/2} \frac{(2\pi m v)^2}{2m} \int_0^L x^4 e^{-2\pi m v x^2} dx$$

$$\int x^4 e^{-2\pi m v x^2} dx = \int x^3 \cdot x e^{-2\pi m v x^2} dx$$

$$= x^3 \int x e^{-2\pi m v x^2} - \int 3x^2 \cdot \left(\int x e^{-2\pi m v x^2} dx \right) dx$$

$$= x^3 \frac{e^{-2\pi m v x^2}}{(-4\pi m v)} - \int 3x^2 \frac{e^{-2\pi m v x^2}}{(-4\pi m v)} dx$$

$$= \frac{x^3 e^{-2\pi m v x^2}}{(-4\pi m v)} - \frac{3}{(-4\pi m v)} \int x \cdot x e^{-2\pi m v x^2} dx$$

$$= \frac{x^3 e^{-2\pi m v x^2}}{(-4\pi m v)} - \frac{3}{(-4\pi m v)} \left(x \int x e^{-2\pi m v x^2} dx - \int 1 \int x e^{-2\pi m v x^2} dx \right)$$

$$= \frac{x^3 e^{-2\pi m v x^2}}{(-4\pi m v)} - \frac{3}{(-4\pi m v)} \left(\frac{x e^{-2\pi m v x^2}}{(-4\pi m v)} - \frac{e^{-2\pi m v x^2}}{(-4\pi m v)^2} \right)$$

$$= \frac{x^3 e^{-2\pi m v x^2}}{(-4\pi m v)} - \frac{3x e^{-2\pi m v x^2}}{(-4\pi m v)^2} + \frac{3 e^{-2\pi m v x^2}}{(-4\pi m v)^3}$$

$$\Rightarrow \langle \phi_1 | H_{in} | \phi_1 \rangle =$$

$$\frac{(32\pi^2 m^3 v^3)^{1/2}}{2m} \frac{(3)}{4\pi m v} \left[-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx \right]$$

$$+ (32\pi^2 m^3 v^3)^{1/2} \alpha \left(\frac{L^2 e^{-2\pi m v L^2}}{-4\pi m v} - \frac{2 e^{-2\pi m v L^2}}{(-4\pi m v)^2} + \frac{2}{(-4\pi m v)^2} \right)$$

$$+ (32\pi^2 m^3 v^3)^{1/2} \frac{(2\pi m v)^2}{2m} \left[\frac{L^3 e^{-2\pi m v L^2}}{-4\pi m v} - \frac{3L e^{-2\pi m v L^2}}{(-4\pi m v)^2} + \frac{3 e^{-2\pi m v L^2}}{(-4\pi m v)^3} - \frac{3}{(-4\pi m v)^3} \right]$$

$$= 3\pi v (2mv)^{1/2} \left(-L e^{-2\pi m v L^2} + \int_0^L e^{-2\pi m v x^2} dx \right)$$

$$+ (2mv)^{1/2} \alpha \left(-L^2 e^{-2\pi m v L^2} - \frac{2 e^{-2\pi m v L^2}}{4\pi m v} + \frac{2}{4\pi m v} \right)$$

$$+ (2mv)^{1/2} (\pi^2 m^2 v^2) \left[-L^3 e^{-2\pi m v L^2} - \frac{3L e^{-2\pi m v L^2}}{4\pi m v} - \frac{3 e^{-2\pi m v L^2}}{(4\pi m v)^2} + \frac{3}{(4\pi m v)^2} \right]$$

$$H = \left(\frac{H+H^T}{2}\right) + \left(\frac{H-H^T}{2}\right)$$

$\hookrightarrow$ assyminetric

$\hookrightarrow$ symmetric

$$= \begin{bmatrix} \langle\phi_0|H_{in}|\phi_0\rangle & \frac{\langle\phi_0|H_{in}|\phi_1\rangle + \langle\phi_1|H_{in}|\phi_0\rangle}{2} \\ \frac{\langle\phi_0|H_{in}|\phi_1\rangle + \langle\phi_1|H_{in}|\phi_0\rangle}{2} & \langle\phi_1|H_{in}|\phi_1\rangle \end{bmatrix}$$

$$\frac{\langle\phi_0|H_{in}|\phi_1\rangle + \langle\phi_1|H_{in}|\phi_0\rangle}{2} =$$

$$v(4\pi^2)^{1/4}\left(1-e^{-2\pi mL^2}\right) + \frac{(4\pi^2)^{1/4}\alpha}{2\pi}\left(-Le^{-2\pi mL^2} + \int_0^L e^{-2\pi mL^2}dx\right)$$

$$+ (\pi m v^2)(4\pi^2)^{1/4}\left(\frac{2e^{-2\pi mL^2}}{4\pi m v} - L^2 e^{-2\pi m v L^2} - \frac{2}{m\pi m v}\right)$$

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assuming symmetric limits (i.e., $-L/2$ to $+L/2$)

$$\langle\phi_0|H_{in}|\phi_0\rangle = (2mv)^{1/2}(2\pi v)\int_0^{L/2} e^{-2\pi m v x^2}dx + (2mv)^{1/2}(4\pi^2 m v^2)\int_0^{L/2} x^2 e^{-2\pi m v x^2}dx$$

$$\langle\phi_0|H_{in}|\phi_1\rangle = (64\pi^2 m^4 v^4)^{1/4}(2\alpha)\int_0^{L/2} x^2 e^{-2\pi m v x^2}dx$$

$$\langle\phi_1|H_{in}|\phi_0\rangle = (64\pi^2 m^4 v^4)^{1/4}(2\alpha)\int_0^{L/2} x^2 e^{-2\pi m v x^2}dx$$

$$\langle\phi_1|H_{in}|\phi_1\rangle = \frac{(32\pi^2 m^3 v^3)^{1/2}}{2m}(6\pi m v)\cdot 2\int_0^{L/2} x^2 e^{-2\pi m v x^2}dx + \frac{(32\pi^2 m^3 v^3)^{1/2}(2\pi m v)^2}{2m}\cdot 2\int_0^{L/2} x^4 e^{-2\pi m v x^2}dx$$

$$\langle n'|a|n\rangle = n^{1/2}\langle n'|n-1\rangle = n^{1/2}\,\delta_{n',n-1}$$

$$\langle n'|a^{+}|n\rangle = (n+1)^{1/2}\langle n'|n+1\rangle = (n+1)^{1/2}\,\delta_{n',n+1}$$

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} x + i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} p$$

$$a^{+} = \left(\frac{m\omega}{2\hbar}\right)^{1/2} x - i\left(\frac{1}{2m\omega\hbar}\right)^{1/2} p$$

$$\Rightarrow x = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(a+a^{+})$$

$$p = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^{+}-a).$$

$$\delta_{i,j} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \quad \text{(Kronecker Delta)}$$

$$\Rightarrow \langle n'|a|n\rangle = \begin{cases} n^{1/2}, & n' = (n-1) \\ 0, & n' \neq (n-1) \end{cases}$$

$$\langle n'|a^{+}|n| = \begin{cases} (n+1)^{1/2}, & n' = (n+1) \\ 0, & n' \neq (n+1) \end{cases}$$

$$\Rightarrow \quad X = \begin{cases} \left(\frac{\hbar}{2m\omega}\right)^{1/2} n^{1/2}, & n' = (n-1) \\ \left(\frac{\hbar}{2m\omega}\right)^{1/2} (n+1)^{1/2}, & n' = (n+1) \\ 0, & \text{otherwise} \end{cases}$$

$$P = \begin{cases} i\left(\frac{m\omega\hbar}{2}\right)^{1/2} (n+1)^{1/2}, & n' = (n+1) \\ -i\left(\frac{m\omega\hbar}{2}\right)^{1/2} n^{1/2}, & n' = (n-1) \\ 0, & \text{otherwise} \end{cases}$$

$$H_{in} = -\frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 + \alpha x$$

$$= \frac{-1}{2m}\left(i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^{+}-a)\right)^2 + \alpha\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a^{+}+a)$$

$$= \frac{1}{2\cancel{m}}\left(\frac{\cancel{m}\omega\hbar}{2}\right)(a^{+}-a)^2 + \alpha\left(\frac{\hbar}{2m\omega}\right)^{1/2}(a^{+}+a)$$