3/10/21 P1 Exercises in classical harmonic oscillator

1) $m = 1000g$, spring constant, $K = 65 N/m$

$x = 0$ to $x = 11 cm$.

frictionless

(a) $\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{65}{11\times10^{-2}}} = \boxed{24.30 \text{ rad/s}}$

frequency, $f = \frac{\omega}{2\pi} = \boxed{3.87 \, s^{-1}}$

time period, $T = \frac{1}{f} = \boxed{0.26 s}$ (approx.)

(b) eq. of motion is

$x = A \sin\omega t$

$\cancel{x = \left(\frac{V_0}{\omega}\right)\sin(\omega t + \phi)}$

$\cancel{v = v_0 \cos(\omega t + \phi)}$

given, $x = 0$ is equilibrium

spring stretched to $x = 11 cm$

$\Rightarrow$ amplitude $= \boxed{0.11 m}$

(c) eq. of motion is

$x = \left(\frac{V_0}{\omega}\right)\cos(\omega t)$

(started from extreme position)

$A = \frac{V_0}{\omega}$.

$v = v_0 \sin\omega t$

(our eq. is correct so far as $v = 0$ for $t = 0$ (given))

max. speed $V_m = V_0 = A\omega$.

$\Rightarrow V_m = 0.11 \times 24.30 = \boxed{2.673 m/s}$

(d) $a = v_0 \omega \sin\omega t$

$\Rightarrow a_{max.} = v_0\omega = 2.673 \times 24.30 = \boxed{64.95 m/s^2.}$

(e) from the eq. I obtained,

$\boxed{\phi = 0}$

(f) ~~x(t)~~ $x(t) = 0.11 \cos(24.03t)$

(g) $E = KE + PE = \frac{1}{2} m v^2 + \frac{1}{2} K x^2$

$x = 11 cm$, $v = 0 cm/s$.

$\Rightarrow E = 0 + \frac{1}{2} \times 65 \times \left(\frac{11}{100}\right)^2 = \boxed{39.325 J}$

(h) $x = \frac{1}{2} x_m$ (half of max.)

$x = 0.11 \cos(24.03t)$

$\Rightarrow x_m = 0.11 \Rightarrow \frac{x_m}{2} = \frac{0.11}{2}$

~~$v = v_0 \sin\omega t$~~ $\Rightarrow PE = \frac{1}{2} K x^2 = \frac{1}{2} \times 65 \times \left(\frac{0.11}{2}\right)^2$

$\Rightarrow \boxed{PE = 9.831255 J}$

total energy of the above system remains constant

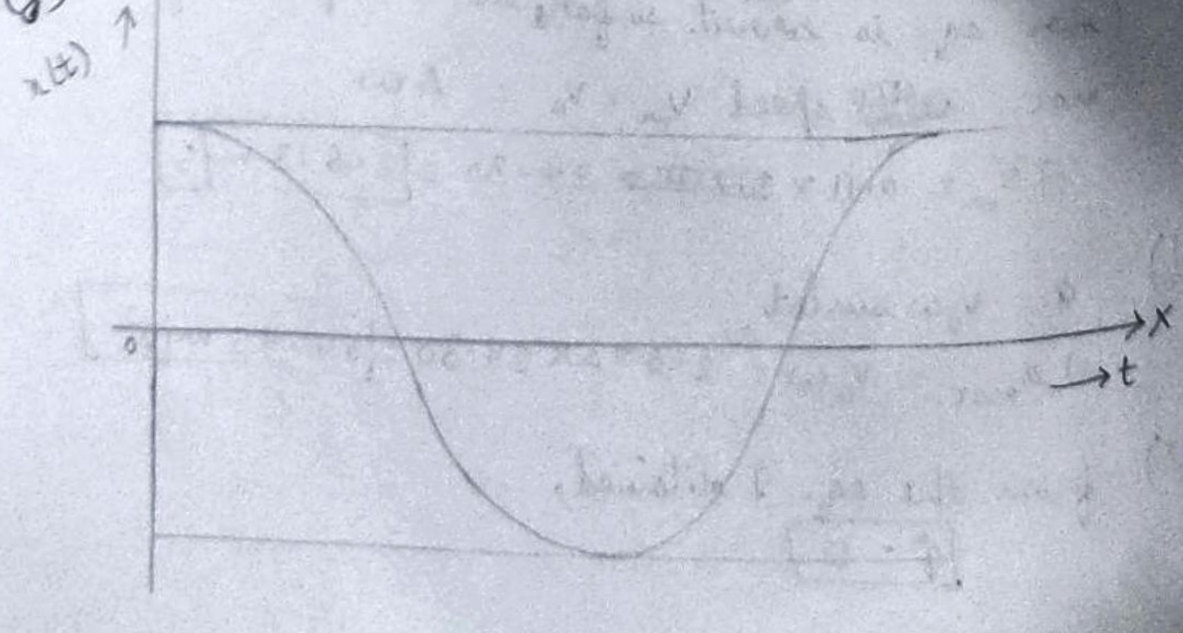
$\Rightarrow$ KE at $x = \frac{x_m}{2}$ is

$KE = 39.325 - 9.83125$

$\Rightarrow \boxed{KE = 29.49375 J}$

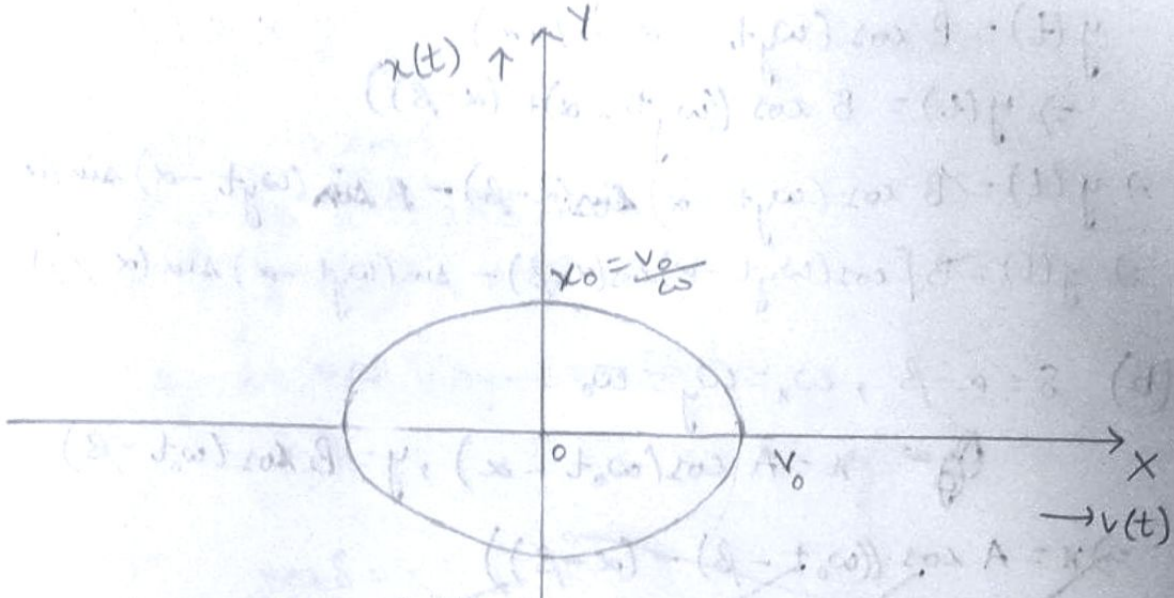
at $x = -\frac{x_m}{2}$, ~~also~~, KE and PE share the same values as that at $x = +\frac{x_m}{2}$

(j) $x = 0.11 \cos(24.03t)$



$v(t) = (0.11)(24.30)\sin(24.30t)$

$\Rightarrow v(t) = \sqrt{1 - \omega^2 x^2}$



The shape is elliptical (the above graph was supposed to look elliptical)

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2) $F = -K_x \hat{x} - K_y \hat{y}$

$\frac{d^2x}{dt^2} = -\omega_x^2 x$; $\omega_x = \sqrt{\frac{K_x}{m}}$

$\frac{d^2y}{dt^2} = -\omega_y^2 y$; $\omega_y = \sqrt{\frac{K_y}{m}}$

$v_x \cdot \frac{dv_x}{dx} = -\omega_x^2 x$

$\Rightarrow \int v_x \, dv_x = -\omega_x^2 \int x \, dx$

$\Rightarrow \frac{v_x^2}{2} = -\frac{\omega_x^2 x^2}{2} + c \Rightarrow v_x^2 = -\omega_x^2 x^2 + c_x$

(c_x = constant)

$v_x = \sqrt{c_x - \omega_x^2 x^2}$

$\frac{dx}{dt} = \sqrt{c_x - \omega_x^2 x^2}$

$\Rightarrow \int \frac{dx}{\sqrt{c_x - \omega_x^2 x^2}} = \int dt$

2) (a) $x(t) = A\cos(\omega_x t - \alpha)$

$$y(t) = B\cos(\omega_y t - \beta)$$

$$y(t) = B\cos(\omega_y t - \alpha - \beta + \alpha)$$

$$\Rightarrow y(t) = B\cos((\omega_y t - \alpha) + (\alpha - \beta))$$

$$\Rightarrow y(t) = B\cos(\omega_y t - \alpha)\cancel{\sin}\cos(\alpha-\beta) - B\cancel{\sin}\sin(\omega_y t - \alpha)\sin(\alpha-\beta)$$

$$\Rightarrow y(t) = B\left[\cos(\omega_y t - \alpha)\cos(\alpha-\beta) - \sin(\omega_y t - \alpha)\sin(\alpha - \beta)\right]$$

(b) $\delta = \alpha - \beta$, $\omega_x = \omega_y = \omega_0$

$$x = A\cos(\omega_0 t - \alpha),\ y = B\cos(\omega_0 t - \beta)$$

$$\cancel{\Rightarrow x = A\cos((\omega_0 t - \beta) - (\alpha - \beta))}$$

$$\cancel{\Rightarrow x = A\cos(\omega_0 t - \beta)\cos(\alpha - \beta) + A\sin(\omega_0 t - \beta)\sin(\alpha-\beta)}$$

$$\cancel{\Rightarrow x = \frac{A}{B}\, y\cos\delta + \sqrt{A^2 - A^2\cos^2(\omega_0 t - \beta)}\,\sin\delta}$$

$$y = B\cos((\omega_0 t - \alpha) + (\alpha - \beta))$$

$$\Rightarrow y = B\cos(\omega_0 t - \alpha)\cos(\alpha - \beta) - B\sin(\omega_0 t - \alpha)\sin(\alpha - \beta)$$

$$\Rightarrow y = \frac{B}{A}x\cos\delta - \frac{B}{A}\sqrt{A^2 - A^2\cos^2(\omega_0 t - \alpha)}\,\sin\delta$$

$$\Rightarrow y = \frac{B}{A}x\cos\delta - \frac{B}{A}\sqrt{A^2 - x^2}\,\sin\delta$$

$$\Rightarrow \boxed{Ay - Bx\cos\delta = -B\sqrt{A^2 - x^2}\,\sin\delta}$$

(c) $\Rightarrow (Ay - Bx\cos\delta)^2 = B^2(A^2 - x^2)\sin^2\delta$

$$\Rightarrow A^2y^2 + B^2x^2\cos^2\delta - 2ABxy\cos\delta = A^2B^2 - B^2x^2\sin^2\delta$$

$$\Rightarrow A^2y^2 + B^2x^2 - 2ABxy\cos\delta = A^2B^2$$

(d) $\delta = \pm\pi/2 \Rightarrow \cos\delta = 0$

$$\Rightarrow A^2y^2 + B^2x^2 = A^2B^2$$

$\Rightarrow \boxed{\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1}$ $\longrightarrow$ plot would be an ellipse

(e) $\delta = \pm\pi/2$, $A = B$ $\Rightarrow \cos\delta = 0$

$$A^2y^2 + A^2x^2 = A^4$$

$\Rightarrow \boxed{x^2 + y^2 = A^2}$ $\longrightarrow$ plot would be a circle of radius A

(f) $\delta = 0$ $\Rightarrow \cos\delta = 1$

$$\Rightarrow A^2y^2 + B^2x^2 - 2ABxy = A^2B^2$$

$$\Rightarrow (Ay - Bx)^2 = A^2B^2$$

$\Rightarrow (Ay - Bx - AB)(Ay - Bx + AB) = 0$ $\longrightarrow$ plot would be a pair of straight lines

$\delta = \pm\pi$ $\Rightarrow \cos\delta = -1$

$$\Rightarrow A^2y^2 + B^2x^2 + 2ABxy = A^2B^2$$

$$\Rightarrow (Ay + Bx)^2 = A^2B^2$$

$\Rightarrow (Ay + Bx - AB)(Ay + Bx + AB) = 0$ $\longrightarrow$ plot would be a pair of straight lines