Spectral Element Method

Report

June 12, 2025

1 Introduction

The Spectral Element Method (SEM) is a high-order extension of the Galerkin finite element method for solving partial differential equations. It uses high-degree polynomial basis functions within each element, allowing for greater accuracy compared to low-order methods. SEM is particularly effective for problems with smooth solutions. Its ability to achieve fast convergence with fewer degrees of freedom makes it useful in fields such as fluid dynamics and wave propagation.

The material proceeds as follows: we begin by formulating the general Galerkin approach for elliptic PDEs, then describe SEM in one and higher dimensions. Next, we discuss the SEM-NI (numerical integration) variant, its interpolation basis, and quadrature rules. This is followed by a stability and error analysis with numerical examples illustrating convergence behavior. Finally, we present the DG-SEM formulation for time-dependent problems and demonstrate how aliasing errors arise and can be mitigated through split-form techniques. Most of the text is taken from Chapter 5 of [1].

2 Spectral Element Method

2.1 Advection-Diffusion-Reaction Problem

Consider the advection-diffusion-reaction problem

$$\begin{aligned} -\nabla \cdot (\alpha \nabla u - \boldsymbol{\beta} u) + \gamma u &= f, \text{ in } \Omega \\ u &= 0, \text{ on } \partial\Omega \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is open and bounded, $f \in L^2(\Omega)$ and α is a positive scalar function satisfying $\alpha(\boldsymbol{x}) \geq \alpha_0 > 0$ for all $\boldsymbol{x} \in \Omega$, $\boldsymbol{\beta} \in [L^\infty(\Omega)]^d$ and $\gamma \in L^\infty(\Omega)$.

The weak formulation of (1) is as follows:

$$\begin{aligned} &\text{Find } u \in V \text{ such that} \\ &a(u, v) := \int_\Omega (\alpha \nabla u - \boldsymbol{\beta} u) \cdot \nabla v + \gamma u v = \int_\Omega f v =: b(v), \quad \forall v \in V \end{aligned} \tag{2}$$

where $V = H_0^1(\Omega)$, $a(.,.)$ defines a bilinear form on V and $b(.)$ defines a linear functional on V.

Following the usual finite element procedure of discretizing the domain, we divide Ω into a set of $M \geq 2$ subdomains denoted by Ω_m for $m = 1, \ldots, M$. Let h_m denote the size of Ω_m and $h := \max h_m$ and $N_m \in \mathbb{N}$ be some constants. Then, we define the SEM space as follows:

$$V_h = \{v \in V : v|_{\Omega_m} \in \mathcal{P}^{N_m}(\Omega_m) \text{ for all } m = 1, \ldots, M\} \tag{3}$$

We want to find $u_h \in V_h$ such that

$$a(u_h, v_h) = b(v_h), \quad \forall v_h \in V_h$$

$$\implies \sum_{m=1}^{M} a_{\Omega_m}(u_h, v_h) = \sum_{m=1}^{M} b_{\Omega_m}(v_h), \quad \forall v_h \in V_h \tag{4}$$

where

$$a_{\Omega_m}(u_h, v_h) = \int_{\Omega_m} ((\alpha \nabla u_h - \boldsymbol{\beta} u_h) \cdot \nabla v_h + \gamma u_h v_h)\, d\boldsymbol{x}, \qquad b_{\Omega_m}(v_h) = \int_{\Omega_m} f v_h \, d\boldsymbol{x}$$

SEM differs from FEM, in the choice of interpolation and quadrature points, which is described in the following section.

2.2 Spectral Element Method in 1D

Let $\Omega = (a, b)$ with $a < b$. Divide $\bar{\Omega} = [a, b]$ into a set of $M \geq 2$ intervals denoted by $\Omega_m = (x_{m-1}, x_m)$ for $m = 1, \ldots, M$. Each element is mapped to the reference interval $[-1, 1]$ and LGL interpolation points are considered on the reference interval.

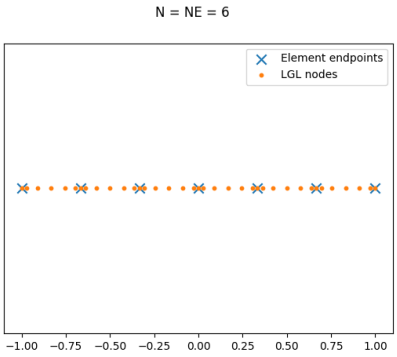

Figure 1: Disctrization of $[-1, 1]$ into $M = NE = 6$ elements where in each element $N_m = N = 6$.

Denote $\hat{\Omega} = [-1, 1]$ to be the reference interval and consider $N_m + 1$ Legendre-Gauss-Lobatto (LGL) nodes $\hat{x}_i$, $i = 0, \ldots, N_m$, given by the roots of the polynomial

$$P_{N_m}(x) = (1 - x^2) L'_{N_m}(x),$$

where L_{N_m} is the Legendre polynomial of degree N_m. Consider $N_m + 1$ characteristic Lagrange polynomials $\hat{\psi}_i$ corresponding to these nodes. Then, we have

$$\hat{\psi}_i(\hat{x}_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \qquad i, j \in \{0, \ldots, N_m\}.$$

Consider the mapping $F_m : \hat{\Omega} \to \bar{\Omega}_m$ define by

$$x = F_m(\hat{x}) := \frac{h_m}{2} \hat{x} + \frac{x_m + x_{m-1}}{2}$$

Denote by $\psi_i^{(m)}$, $i = 0, \ldots, N_m$ the basis functions in Ω_m obtained by

$$\psi_i^{(m)} = \hat{\psi}_i \circ F_m^{-1}, \quad i = 0, \ldots, N_m$$

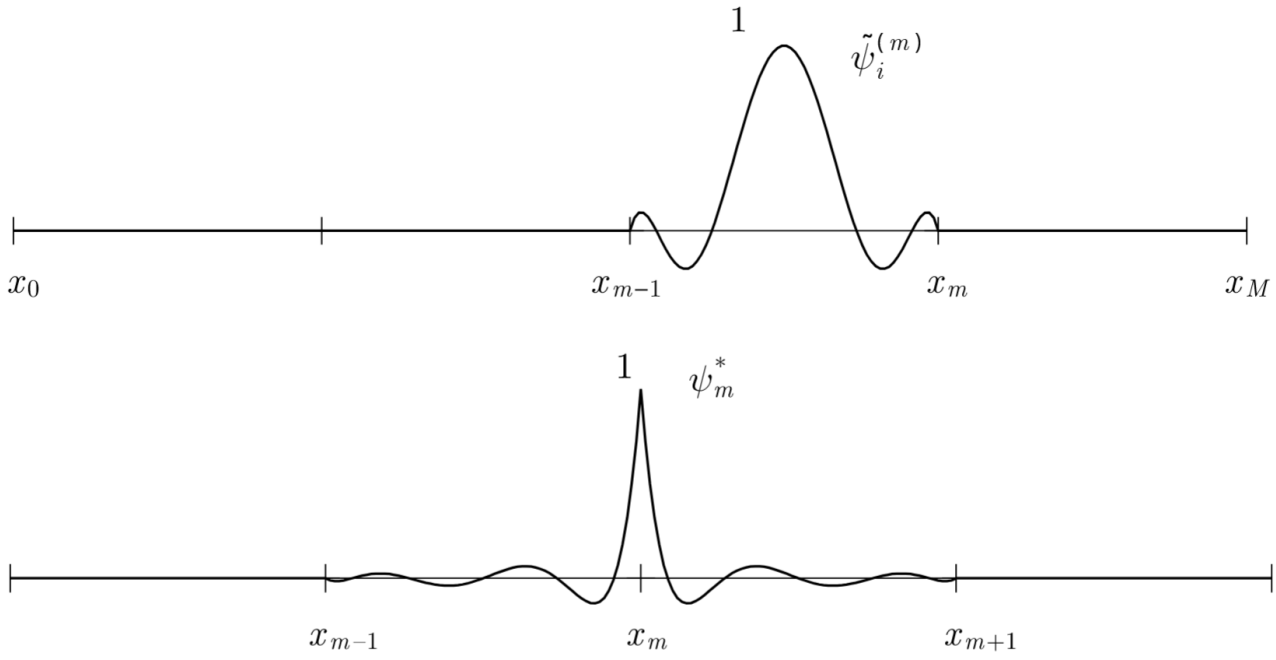

Figure 2: Internal and interface basis functions

For each $m \in \{1, \ldots, M\}$, the functions $\psi_i^{(m)}$ for $i = 0, \ldots, N_m$ form a local basis for Ω_m.
A global representation of basis for V_h is given as follows:

- Internal basis functions $\tilde{\psi}_i^{(m)}(x)$ which has support in only in Ω_m:

$$\tilde{\psi}_i^{(m)}(x) = \begin{cases} \psi_i^{(m)}(x) & x \in \Omega_m \\ 0 & \text{otherwise} \end{cases}, \quad \forall i \in \{1, \ldots, N_m - 1\}$$

- Interface basis functions $\psi_m^*(x)$ which has support in $\Omega_m \cup \Omega_{m+1}$:

$$\psi_m^*(x) = \begin{cases} \psi_{N_m}^{(m)}(x) & x \in \Omega_m \\ \psi_0^{(m+1)}(x) & x \in \Omega_{m+1} \\ 0 & \text{otherwise} \end{cases}$$

These functions form a basis for the SEM space V_h and so we get $\dim V_h = \sum_{m=1}^{M}(N_m - 1) + M - 1 = \sum_{m=1}^{M} N_m - 1$.
The solution $u_h \in V_h$ can be expressed as

$$u_h(x) = \sum_{m=1}^{M-1} u_m^* \psi_m^*(x) + \sum_{m=1}^{M} \sum_{i=1}^{N_m - 1} u_i^{(m)} \tilde{\psi}_i^{(m)}(x)$$

with u_m^* indicates the unknown value of u_h at node x_m and $u_i^{(m)}$ indicates the unknown value of u_h at internal LGL nodes $x_i^{(m)}$.

Typically, LGL quadrature formulas are used for computing the integrals. In $\hat{\Omega} = [-1, 1]$, the quadrature points are precisely the LGL nodes described earlier and the quadrature weights are given by

$$w_i = \frac{2}{N_m(N_m+1)} \frac{1}{[L_{N_m}(\hat{x}_i)]^2}, \quad i = 0, \ldots, N_m$$

Using this quadrature rule, we replace (4) by more flexible SEM-NI version:

$$\sum_{m=1}^{M} a_{\Omega_m, N_m}(u_h, v_h) = \sum_{m=1}^{M} (\alpha u_h' - \beta u_h, v_h')_{\Omega_m, N_m} + (\gamma u_h, v_h)_{\Omega_m, N_m} = \sum_{m=1}^{M} (f, v_h)_{\Omega_m, N_m} = \sum_{m=1}^{M} b_{\Omega_m, N_m}(v_h), \ \ \forall v_h \in V_h$$

where

$$(u_h, v_h)_{\Omega_m, N_m} = \sum_{j=0}^{N_m} u_h(x_j^{(m)}) v_h(x_j^{(m)}) w_j^{(m)}$$

is the analog of LGL quadrature in Ω_m, where $w_j^{(m)}$ denote the quadrature weights in Ω_m.
The nodal basis as constructed above is the natural choice in SEM-NI. The pattern of the associated stiffness matrix depends on the ordering of these basis functions. The unknown coefficients are obtained by solving the above system of linear equations.

2.3 Spectral Element Method in higher dimensions

As described previously, we consider a partition $\mathcal{T} = \{\Omega_m\}$ of Ω, where each element Ω_m is obtained by a transformation F_m from the reference element $\hat{\Omega}$ in $\mathbb{R}^d$. For simplicity, we assume that the reference element is the d-dimensional cube i.e. $\hat{\Omega} = (-1, 1)^d$.
Suppose we have $N_m = (N_m^{(1)}, \ldots, N_m^{(d)})$ and we consider the nodes $\{(\hat{x}_1^{(i)}, \ldots, \hat{x}_d^{(i)})\}$, where $\hat{x}_j^{(i)}$ are $N_m^{(i)} + 1$ LGL nodes in x_j-direction in $\hat{\Omega}$. Then, we define the polynomial space

$$\mathcal{P}^{N_m}(\hat{\Omega}) = \hat{\mathcal{P}}^{N_m^{(1)}, \ldots, N_m^{(d)}}(\hat{\Omega}) = \text{span}\{\hat{x}_1^{(i_1)} \ldots \hat{x}_d^{(i_d)} : 0 \leq i_k \leq N_m^{(k)} \text{ for all } k = 1, \ldots, d\}$$

The Lagrange basis for this space is given by the tensor product of Lagrange basis in each coordinate.

$$\begin{aligned} L_{i_1 \ldots i_d}(\hat{x}_1^{j_1}, \ldots, \hat{x}_d^{j_6}) = \delta_{i_1 j_1} \ldots \delta_{i_d j_d} = l_{i_1}(\hat{x}_1^{j_1}) \ldots l_{i_d}(\hat{x}_d^{j_d}) \\ \implies L_{i_1 \ldots i_d} = l_{i_1} \otimes \cdots \otimes l_{i_d} \end{aligned}$$

where $0 \leq i_k, j_k \leq N_m^{(k)}$ for all $k = 1, \ldots, d$ and l_i's are the characteristic Lagrange functions with respect to the LGL nodes $\hat{x}_i$.
The basis functions for $\mathcal{P}^{N_m}(\Omega_m)$ are obtained by mapping these basis function onto Ω_m via F_m^{-1}. The glueing procedure is similar as discussed in the one-dimensional case, where the interface basis functions in any element at a node that lies on any edge, is glued with the interface basis function at the same node in the neighboring element sharing that particular edge. This glueing procedure allows for different polynomial degrees in along each direction, provided the polynomial degrees along an interface between two elements agree on both sides.

Using LGL quadrature rule or any suitable numerical integration formula, we get for $u_h \in V_h$,

$$\sum_{m=1}^{M} a_{\Omega_m,N_m}(u_h, v_h) = \sum_{m=1}^{M} b_{\Omega_m,N_m}(v_h), \quad \forall v_h \in V_h \tag{5}$$

We can also choose the unit triangle to be the reference element, where constructing the basis functions is much more complicated.

2.4 Stability and Error Analysis

Assume that $\nabla \cdot \boldsymbol{\beta}$ exists and satisfies

$$\frac{1}{2}\nabla \cdot \boldsymbol{\beta} + \gamma \geq 0 \quad \text{in } \Omega \tag{6}$$

Under such assumptions, we get that the bilinear form $a(.\,,.)$ is continuous and coercive i.e. it satisfies

$$|a(u,v)| \leq A_0 ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)}, \qquad \forall u, v \in H_0^1(\Omega)$$

and

$$a(v,v) \geq \alpha_* ||v||^2_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega)$$

for suitable constants $A_0 \geq \alpha_* > 0$. Then, the SEM problem has a unique solution due to Lax-Milgram theorem which satisfies the uniform stability estimate

$$||u_h||_{H^1(\Omega)} \leq \frac{1}{\alpha_*} ||f||_{L^2(\Omega)}$$

A similar result holds for the solution of the SEM-NI problem (5) under more restrictive assumptions on the coefficients. The condition (6) is sufficient to establish stability of SEM-NI provided the advection term is treated in skew-symmetric form i.e. the term $-\sum_m (\boldsymbol{\beta} u_h, \nabla v_h)_{N_m,\Omega_m}$ in $a_{N_m,\Omega_m}(u_h, v_h)$ is replaced by

$$\begin{aligned} -\frac{1}{2}\sum_m (\boldsymbol{\beta} u_h, \nabla v_h)_{N_m,\Omega_m} + \frac{1}{2}\sum_m (\boldsymbol{\beta} \cdot \nabla u_h, v_h)_{N_m,\Omega_m} \\ + \frac{1}{2}\sum_m ((\nabla \cdot \boldsymbol{\beta}) u_h, v_h)_{N_m,\Omega_m} \end{aligned}$$

Then, the stability estimate takes the form

$$||u_h||_{H^1(\Omega)} \lesssim \frac{1}{\alpha_*} \sum_m ||f||_{C(\bar{\Omega}_m)}$$

where the term $||f||_{C(\bar{\Omega}_m)}$ bounds the discrete norm of f in the SEM-NI version.
Using Galerkin orthogonality relations together with the coercivity and continuity of bilinear form $a(.\,,.)$, we get the bound

$$||u - u_h||_{H^1(\Omega)} \leq \frac{A_0}{\alpha_*} \inf_{v_h \in V_h} ||u - v_h||_{H^1(\Omega)}$$

known as Cea's lemma. This result shows that the error $u - u_h$ in the energy norm is proportional to the best approximation error of u in V_h in the same norm.
At this point, we are left with the problem of finding a particular element $u_h \in V_h$ for which the norm of $u - u_h$

behaves asymptotically as the infimum. A natural choice is the SEM interpolant $u_h = I_h u$. $I_h u$ is a function in V_h, which interpolates the function u at $N_m + 1$ LGL points in Ω_m.
Under suitable assumptions on the partition $\mathcal{T} = \{\Omega_m\}$ and polynomial degree N_m, we get

$$||u - u_h||_{H^1(\Omega)} \leq \frac{A_0}{\alpha_*} \left\{ \sum_m C(s_m) h_m^{2(\min(N_m+1,s)-1)} N_m^{2(1-s_m)} ||u||^2_{H^{s_m}(\Omega_m)} \right\}^{\frac{1}{2}} \tag{7}$$

where $H^{s_m}(\Omega_m)$ measures the local smoothness of u in Ω_m and the constants $C(s_m) > 0$ are independent of h_m and N_m.
In particular, if $N_m = N$ for all m and if h_m satisfies $ch \leq h_m \leq c'h$ for all m, then

$$||u - u_h||_{H^1(\Omega)} \leq C_1(s) h^{\min(N+1,s)-1} N^{1-s} ||u||_{H^s(\Omega)} \tag{8}$$

where $s = \min_m s_m$.
When the exact solution u is very smooth i.e.when s is very large, it is advantageous to improve the quality of the approximation by keeping h fixed and increasing N. In such case, we achieve spectral (exponential) convergence with N. This phenomenon is illustrated in the next section via an example problem.

2.5 1D Example

Consider the following problem

$$\begin{aligned} -\Delta u + u &= (\pi^2 + 1)\sin(\pi x), \text{ in } (0,1) \\ u(0) &= u(1) = 0 \end{aligned} \tag{9}$$

The exact solution for this problem is given by $u(x) = \sin(\pi x)$. In most of the SEM-NI realizations, the polynomial degree within each element are equal. In such case, we have the accuracy estimate (8). The error table and plots obtained from the SEM implementation in cases: (a) increasing number of elements and (b) increasing polynomial degrees, are described below.

NE	**N = 2**				**N = 4**			
	L2-Error	**L2-Rate**	**H1-Error**	**H1-Rate**	**L2-Error**	**L2-Rate**	**H1-Error**	**H1-Rate**
2	1.788656×10^{-2}	–	0.206590	–	1.075629×10^{-4}	–	2.637394×10^{-3}	–
4	2.033514×10^{-3}	3.136829	0.051210	2.012278	3.374269×10^{-6}	4.994462	1.669890×10^{-4}	3.981288
8	2.482120×10^{-4}	3.034330	0.012776	2.003007	1.055485×10^{-7}	4.998597	1.047068×10^{-5}	3.995326
16	3.084226×10^{-5}	3.008592	0.003192	2.000747	3.299197×10^{-9}	4.999648	6.549478×10^{-7}	3.998832
32	3.849545×10^{-6}	3.002149	0.000798	2.000186	1.031064×10^{-10}	4.999909	4.094252×10^{-8}	3.999708

Table 1: Convergence results for fixed polynomial degree N and increasing number of elements NE

(a) In this case, the rate of convergence in H^1-norm and L^2-norm is N and $N + 1$, respectively, which is evident from Table 1. These rate of convergence are the usual logarithmic convergence rate α obtained by

$$\alpha \sim \frac{\log\left(\frac{e_1}{e_2}\right)}{\log\left(\frac{h_1}{h_2}\right)}$$

Thus, when we keep the polynomial degree N fixed and increase the number of elements, SEM behaves much like standard FEM where the rate of convergence depends on the polynomial degree.

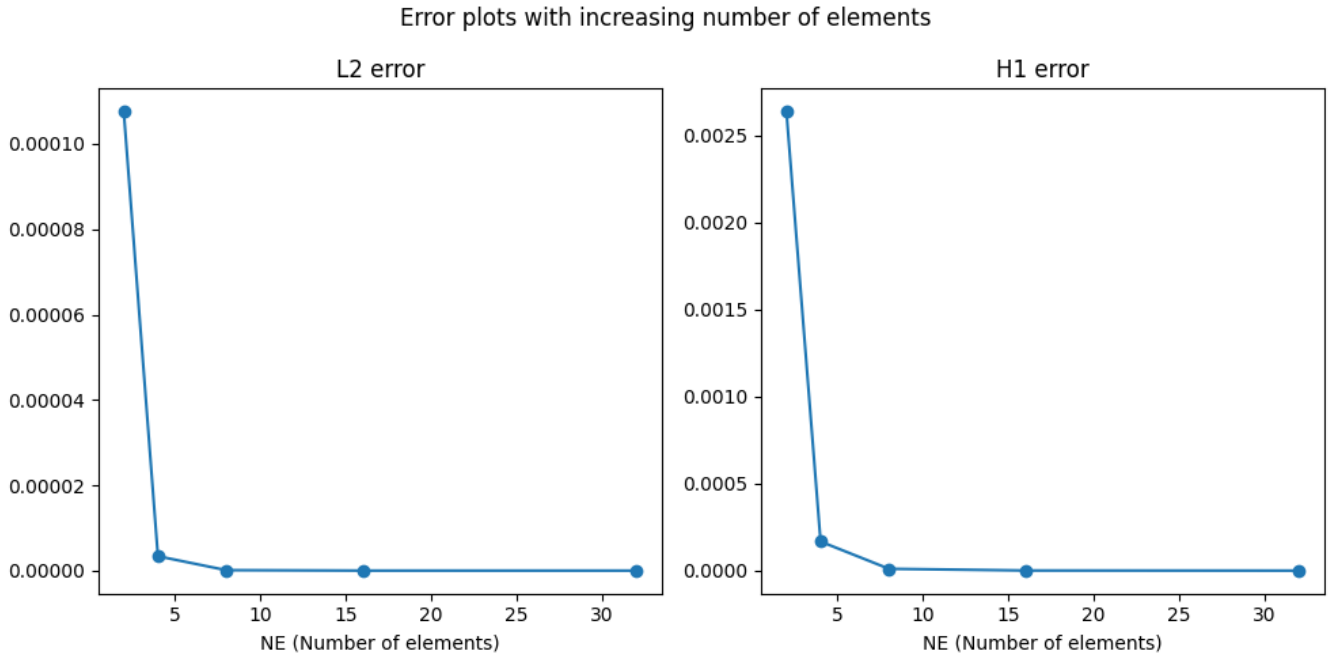

Figure 3: Error plots for fixed polynomial degree $N = 4$ and increasing number of elements.

N	$NE = 4$				$NE = 8$			
	L2-Error	**L2-Rate**	**H1-Error**	**H1-Rate**	**L2-Error**	**L2-Rate**	**H1-Error**	**H1-Rate**
1	1.204547×10^{-1}	–	1.488231×10^{0}	–	2.506198×10^{-2}	–	7.200980×10^{-1}	–
2	2.529542×10^{-2}	1.560651	2.921623×10^{-1}	1.628034	2.875822×10^{-3}	2.165013	7.242161×10^{-2}	2.296883
3	2.068955×10^{-3}	2.503580	3.782801×10^{-2}	2.044260	1.270477×10^{-4}	3.119532	4.777496×10^{-3}	2.718588
4	1.521170×10^{-4}	2.610149	3.729839×10^{-3}	2.316685	4.771937×10^{-6}	3.281810	2.361582×10^{-4}	3.007170
5	9.680631×10^{-6}	2.754523	2.944741×10^{-4}	2.538929	1.525528×10^{-7}	3.442997	9.321078×10^{-6}	3.232223
6	5.346467×10^{-7}	2.896276	1.936068×10^{-5}	2.721947	4.217076×10^{-9}	3.588369	3.061757×10^{-7}	3.415874
7	2.598321×10^{-8}	3.024155	1.090198×10^{-6}	2.876885	1.024712×10^{-10}	3.717315	8.612737×10^{-9}	3.570917
8	1.125893×10^{-9}	3.138875	5.367995×10^{-8}	3.011075	2.219447×10^{-12}	3.832324	2.118615×10^{-10}	3.705065
9	4.398907×10^{-11}	3.242390	2.348231×10^{-9}	3.129377	5.224213×10^{-14}	3.749124	4.631124×10^{-12}	3.823133
10	1.564464×10^{-12}	3.336398	9.241328×10^{-11}	3.235147	2.878261×10^{-14}	0.596118	2.795165×10^{-13}	2.807494

Table 2: Convergence results for increasing polynomial degree with $NE = 4$ and $NE = 8$

(b) Since the exact solution is smooth, we get spectral (exponential) convergence which is evident form the Table 2. Here the spectral rate of convergence α is obtained by

$$\alpha \sim \frac{\log\left(\frac{e_1}{e_2}\right)}{\Delta N}$$

Thus, if we keep doubling polynomial degree then the error decreases rapidly or exponentially, which is evident from the Table 2. The corresponding error plots in Figure 4 also describe this spectral convergence rate.

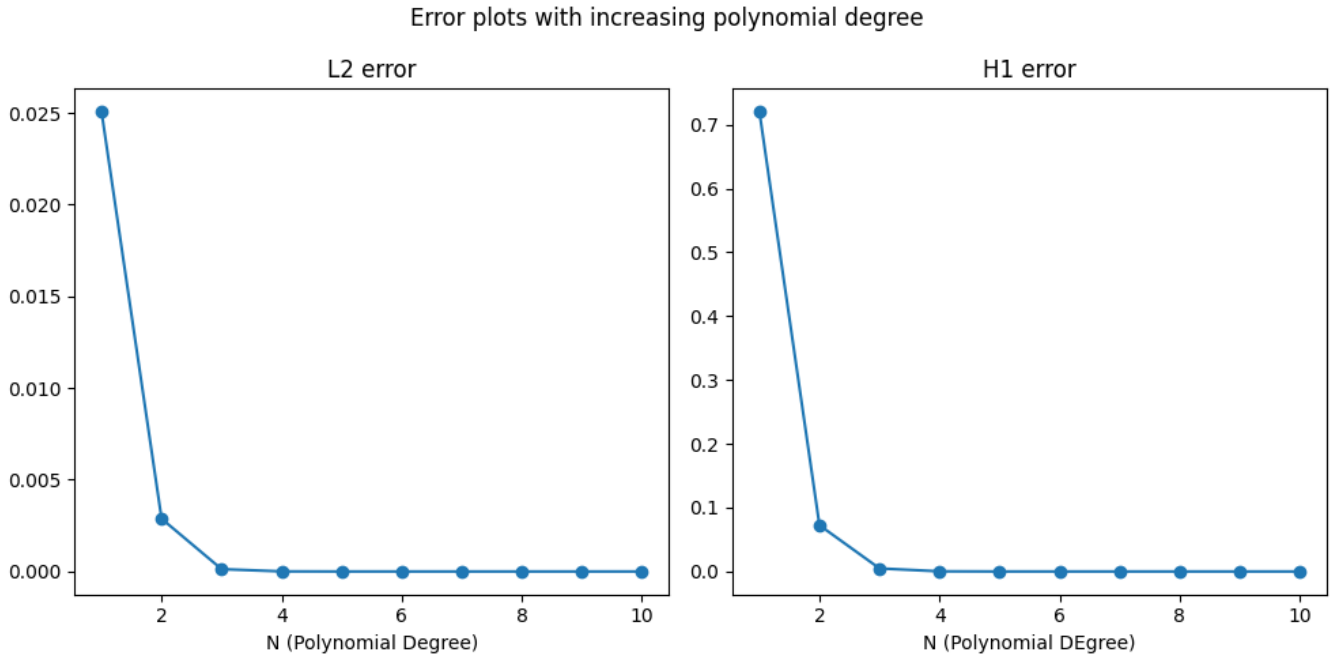

Figure 4: Error plots for fixed number of elements $NE = 8$ and increasing polynomial degree.

3 DG-SEM

3.1 Time dependent problems and DG-SEM

Consider the hyperbolic conservation laws in $\Omega \subset \mathbb{R}^d$

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{F}(u) = 0 \tag{10}$$

with the flux function $\boldsymbol{F} : \mathbb{R} \to \mathbb{R}^d$.
Suppose we have a partition $\mathcal{T} = \{\Omega_m\}$ of Ω, as described before. Consider the DG-SEM space V_h consisting of piecewise polynomials of degree N_m in each Ω_m given by

$$V_h = \{v \in L^2(\Omega) : v|_{\Omega_m} \in \mathcal{P}^{N_m}(\Omega_m) \text{ for all } m = 1, \ldots, M\}. \tag{11}$$

Note that, as in DG, here we do not enforce continuity across each element. Thus, in general, $V_h \subset H^1(\mathcal{T})$ but $V_h \not\subset H^1(\Omega)$. Multiplying each term in (10) with $v_h \in V_h$ and integrating the flux term by parts over each Ω_m, we get

$$\sum_m \left(\int_{\Omega_m} \frac{\partial u_h}{\partial t} \cdot v_h \, dx - \int_{\Omega_m} \boldsymbol{F}(u_h) \cdot \nabla v_h \, dx + \int_{\partial \Omega_m} (\boldsymbol{F}(u_h) \cdot \boldsymbol{n}) v_h \, ds \right) = 0$$

Then, we get

$$\sum_m \left(\left(\frac{\partial u_h}{\partial t}, v_h \right)_{\Omega_m} - (\boldsymbol{F}(u_h), \nabla v_h)_{\Omega_m} + \int_{\partial \Omega_m} (\boldsymbol{F}(u_h) \cdot \boldsymbol{n}) v_h \, ds \right) = 0$$

Replacing $\boldsymbol{F}(u)$ on Γ by an appropriate numerical flux function $\hat{\boldsymbol{F}}$, we get

$$\sum_m \left(\left(\frac{\partial u_h}{\partial t}, v_h \right)_{\Omega_m} - (\boldsymbol{F}(u_h), \nabla v_h)_{\Omega_m} + \int_{\partial\Omega_m} \hat{\boldsymbol{F}}(u_h) \cdot [\![v_h]\!] \, ds \right) = 0 \tag{12}$$

where $(u_h, v_h)_{\Omega_m}$ represent the integrals on Ω_m. In DG-SEM-NI version, these integrals are replaced by numerical approximation using LGL quadrature formula on Ω_m. As described previously, the collocation of interpolation points and quadrature points results in block diagonal coefficient matrix. Furthermore, since there is no requirement of continuity across the boundary, we can use Gauss-Legendre points, which are similar to LGL points but do not include any nodes on the boundary of the element.
For illustrating the above statement, consider the 1D advection case in $\Omega = (a, b)$ with inflow boundary condition, where $F(u) = \beta u$.

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial (\beta u)}{\partial x} &= 0, \qquad \text{in } \Omega \times [0, T] \\ u(x, 0) &= u_0(x), \quad \text{in } \Omega \\ u(a, t) &= g(t) \end{aligned} \tag{13}$$

The DG weak form (12) reduces to

$$\sum_m \left(\left(\frac{\partial u_h}{\partial t}, v_h \right)_{\Omega_m} - (\beta u_h, \nabla v_h)_{\Omega_m} + \int_{\partial\Omega_m} \hat{\boldsymbol{F}}(u_h) \cdot [\![v_h]\!] \, ds \right) = 0$$

The algebraic form is given by

$$M \frac{d}{dt} u_h^m + L u_h^m = 0$$

where u_h^m is the unknown vector of values of u_h at interpolation points in Ω and $\frac{d}{dt} u_h^m$ describes the approximated integrals involving time-derivative of u_h using appropriate time-integration methods. Here, the mass matrix M is block diagonal. DG-SEM method is analogous to SEM, where the entries of this mass matrix M and advection matrix L are computed using specific set of quadrature rules. However, this collocation of quadrature points and interpolation points are still not accurate enough to compute the integrals exactly in the case of non-linear problems.

3.2 Energy stability for 1D Burgers' equation in split-form

Consider the 1D Burgers' equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial u^2}{\partial x} &= 0, \qquad \text{in } \Omega \times [0, T] \\ u(x, 0) &= u_0(x), \quad \text{in } \Omega \\ u(a, t) &= g(t) \end{aligned} \tag{14}$$

where $\Omega = (0, L)$. The weak form for this equation is as follows:

$$\sum_m \left(\left(\frac{\partial u_h}{\partial t}, v_h \right)_{\Omega_m} - \frac{1}{2} (u_h^2, \nabla v_h)_{\Omega_m} + \int_{\partial\Omega_m} \hat{\boldsymbol{F}}(u_h) \cdot [\![v_h]\!] \, ds \right) = 0$$

If we consider the SEM approximation space V_h, then the numerical solution $u_h \in \mathcal{P}^N$. However, nonlinear terms such as u_h^2 lie in $\mathcal{P}^{2N}$. In the weak form, when testing against a function $v_h \in \mathcal{P}^N$, the term $\left(\frac{u_h^2}{2}, \nabla v_h\right)_{\Omega_m}$ involves a degree $2N$ integrand, while Gauss–Lobatto quadrature used in SEM integrates polynomials exactly up to degree $2N-1$. In practice, u_h^2 is typically computed pointwise at collocation nodes and projected back into $\mathbb{P}^N$. This projection introduces aliasing due to the mismatch between the quadrature degree and the integrand's polynomial degree, reuslting in potential energy instability. This instability can be handled by rewriting the flux term into split-form (or skew-symmetric form) so that each term is a polynomial of lower degree and is computed accurately using quadratures.
The split form of Burgers' equation (14) can be given by

$$\frac{\partial u}{\partial t} + \frac{2}{3}\frac{1}{2}\frac{\partial u^2}{\partial x} + \frac{1}{3}u\frac{\partial u}{\partial x} = 0 \tag{15}$$

Multiplying this by u_h to get the energy estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||u_h||^2 + \frac{1}{3}\int_0^L u_h\frac{\partial u_h^2}{\partial x} + u_h^2\frac{\partial u_h}{\partial x} = 0 \\
\implies &\frac{1}{2}\frac{d}{dt}||u_h||^2 - \frac{1}{3}\int_0^L u_h^2\frac{\partial u_H}{\partial x} + \frac{1}{3}u_h^3|_0^L + \frac{1}{3}\int_0^L u_h^2\frac{\partial u_h}{\partial x} = 0 \\
\implies &\frac{1}{2}\frac{d}{dt}||u_h||^2 + \frac{1}{3}u_h^3|_0^L = 0
\end{aligned}$$

Thus, the energy is only dependent on the boundary data, which is a key property of energy-stable formulations [2].

4 Summary

SEM is analogous to FEM, but with a specific set of quadrature rules. The collocation of quadrature and interpolation points in SEM-NI give rise to sparse matrices. SEM is a powerful method for approximating the smooth solutions, providing spectral (or exponential) convergence. However, SEM struggles with complex geometries. Mappings from the reference element to physical elements can become complicated, especially for curvilinear or distorted meshes. In a similar fashion, DG-SEM is analogous to standard DG methods. DG-SEM is much less stable than standard DG methods, especially for non-linear problems. The stability can be enhanced by replcaing the non-linear terms into skew-symmetric form as discussed in the case of 1D Burgers' equation.

References

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