On a Dynamic Input–Output Framework for Price-Feedback-Based Production Planning

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Abstract

This paper develops a dynamic extension of the classical Leontief input–output framework for large-scale production planning under bounded information and computational constraints. The model formulates planning as a viability-preserving tracking problem in which production, investment, and final demand evolve jointly through feedback based on capital coefficients, cost-based prices, and low-order autoregressive demand extrapolation. To ensure scalability, the Leontief inverse is approximated using sparse linear algebra and truncated Neumann series rather than repeated matrix inversion. We derive conditions for convergence, feasible growth, and bounded error propagation, showing that approximation error decays exponentially with iteration count. Numerical simulations calibrated to empirical input–output data demonstrate stable coordination between aggregate demand, realised output, and productive capacity under stochastic demand shocks, with deviations remaining small and bounded.

Keywords: Leontief model, input–output analysis, dynamic systems, computational economics

1. Introduction

Input–output (IO) analysis has long provided a systematic framework for describing the structural interdependencies of production in complex economies. Since Leontief's original formulation, IO models have been widely applied in empirical analysis, forecasting, and policy evaluation, owing to their transparent accounting structure and explicit representation of intersectoral flows. However, classical IO models are fundamentally static, treating production coefficients and final demand as fixed over the period of analysis. This limitation has motivated a substantial literature on dynamic extensions of the IO framework.

Existing dynamic input–output models typically incorporate time through capital accumulation, inventory adjustment, or lagged production responses. While these approaches have advanced the descriptive power of IO analysis, they often face two persistent challenges. First, many dynamic formulations become computationally demanding or unstable when scaled to large, highly disaggregated economies. Second, demand is frequently treated as an exogenous or weakly coupled process, limiting the ability of the model to capture feedback between production, capacity, and realized demand over time.

This paper proposes a dynamic input–output framework that explicitly incorporates demand feedback while remaining computationally tractable at realistic scales. Building on the classical Leontief system, we formulate production dynamics as an iterative process in which output, intermediate requirements, and final demand are updated jointly across time steps. The resulting system can be expressed in matrix form and solved using sparse linear algebra techniques and Neumann series approximations, allowing the model to scale to large economies without requiring full matrix inversion at each iteration.

A central feature of the proposed framework is the explicit treatment of feedback between realized demand and subsequent production decisions. Rather than assuming demand paths exogenously, the model allows aggregate demand to evolve endogenously as a function of output, capacity constraints, and adjustment dynamics. This feedback structure makes it possible to study convergence properties, stability conditions, and transient deviations from potential output within a unified accounting framework.

The contribution of the paper is threefold. First, we develop a dynamic extension of the Leontief input–output model that integrates demand feedback while preserving the accounting consistency of the classical system. Second, we provide an analysis of computational complexity, convergence, and error propagation associated with the iterative solution, highlighting conditions under which the model remains stable and tractable. Third, we demonstrate the feasibility of the approach through numerical simulations calibrated to aggregate economic data, illustrating how the model reproduces key macroeconomic relationships under plausible assumptions.

The framework is intended as a proof of concept rather than a complete representation of any specific economy. While the simulations demonstrate internal consistency and convergence, the model abstracts from institutional detail, behavioural heterogeneity, and explicit price formation. These limitations delineate the scope of the present analysis and point toward directions for future empirical and theoretical extensions.

The remainder of the paper is structured as follows. Section 2 presents the dynamic formulation of the input–output system and its computational implementation. Section 3 analyses convergence properties and error dynamics. Section 4 reports simulation results and discusses their interpretation. Section 5 concludes and outlines directions for further research.

2. Theoretical Foundations and Literature Review

The input-output model, originally developed by Wassily Leontief, has served as a cornerstone of economic analysis since the mid-20th century. The fundamental premise of this model lies in representing the interdependencies between different sectors of an economy through a matrix of technical coefficients, enabling economists to understand how changes in final demand propagate through the entire economic system. While the static Leontief model provides valuable insights into economic structure, its limitations in capturing dynamic adjustments and feedback mechanisms have long been recognised by economists as needing to be rectified. Leontief input-output models are empirical, linear, simultaneous representations of the interdependent economic activities within a country's economy. They are designed to organise economic data systematically into input-output tables which capture flows of goods and services between industries. These models serve as analytical tools primarily for economic planning and conditional economic projections, aiming to provide internally consistent estimates of output requirements given final demand. The theoretical foundation rests on generalised linear systems reflecting averages of economic behaviour over time rather than stochastic predictions.Leontief (1955) Later developments in input-output analysis have focused on incorporating dynamic elements to better reflect real-world economic behaviour. The dynamic input-output model addresses the temporal aspects of production, investment, and capital building that are absent from static formulations. Leontief (1986) This evolution represents a significant advancement in economic modelling, as it allows for the analysis of adjustment processes and optimisation of production schedules over time. The integration of cybernetic feedback mechanisms further enhances the model's applicability to real-world economic planning scenarios. The theoretical foundation of dynamic input-output models rests on several key assumptions about economic behaviour and production relationships. First, the model assumes that production technologies exhibit constant returns to scale- a standard assumption in I-O models. The second, assumption is that the "A" matrix remains stable over the planning horizon as the paper aims for monthly planning. However, this assumption is relaxed where the technical coefficients are recalculated over time, for example every year. The mathematical framework underlying dynamic input-output models extends the basic Leontief equation $(I-A)\vec{X} = \vec{d}$ such that it includes temporal elements and investment considerations. In this formulation, $\vec{X}$ represents the gross output vector, A denotes the technical coefficients matrix, and $\vec{d}$ signifies the final demand vector. The dynamic extension incorporates the capital coefficients matrix B, which relates investment requirements to changes in production capacity, resulting in the fundamental dynamic equation: $B\Delta\vec{X} = (I - A)\vec{X} - \vec{d}$. Existing dynamic input-output (IO) models, such as Leontief 1986 extensions and Wagner's 1957 linear programming solutions, incorporate capital coefficients B and temporal investment but rely on direct matrix inversions unsuitable for large-scale ($n < 1000$) systems due to O(n^3) complexity. Sparse matrix techniques appear in general IO analysis (e.g., Miller and Blair 2009) for real economies with low density, yet applications remain limited to static cases or small simulations without rigorous error bounds. Neumann series approximations for Leontief inverses exist in signal processing (e.g., massive MIMO) and statistics (L^∞ convergence proofs), but economic planning literature applies them rarely without integration into dynamic frameworks. Some critiques of the model suggest that the investment does not scale linearly with growth. However, for the implementation in the paper, the size of the investment is much smaller than the demand. Hence, this assumption is held to be true for the time and growth scale. While the proposed framework shares the goal of modeling economic dynamics with established paradigms such as Computable General Equilibrium (CGE), Dynamic Stochastic General Equilibrium (DSGE), and Agent-Based Models (ABMs), it diverges significantly in methodology and scope. CGE models, though detailed in sectoral disaggregation, typically rely on static or recursive-dynamic optimization under perfect foresight and are not designed for high-frequency, feedback-driven planning. DSGE models focus on aggregate fluctuations and rational expectations around a steady state, prioritizing macroeconomic consistency over detailed inter-industry coordination.
ABMs excel at capturing heterogeneous agent behavior and emergent complexity but face prohibitive computational and data requirements at the economy-wide scale envisioned here. In contrast, our approach adopts a control-theoretic and bounded-rationality perspective: it foregoes global optimization or equilibrium assumptions in favor of a viable, tracking-based rule system that operates under material balance and capacity constraints. The model is explicitly designed for large-scale applicability—leveraging sparse linear algebra and autoregressive feedback to manage hundreds of thousands of sectors—making it uniquely suited for real-time production planning in complex, coordinated economies where detailed optimization is computationally infeasible and perfect information is unavailable.

3. Methodology and Model Development

3.1. Formal Planner Problem: A Cybernetic Formulation

We model the economy as a large-scale production network with inter-sectoral dependencies. Time is discrete and indexed by $t \in \mathbb{N}$.

State variables.. Let the system state be

$$\vec{s}(t) \equiv \big(\vec{X}(t_n),\, \vec{K}(t_n)\big)$$

where

- $\vec{X}(t_n) \in \mathbb{R}^n_+$ is the gross output vector
- $\vec{K}(t_n) \in \mathbb{R}^n_+$ is the productive capital stock

Control variables. The planner selects control inputs

$$\vec{u}(t_n) \equiv \big(\vec{F}(t_n), \vec{I}(t_n)\big)$$

where

- $\vec{I}(t_n) \in \mathbb{R}^n_+$ denotes gross investment,
- $\vec{F}(t_n) \in \mathbb{R}^n_+$ denotes Final Produciton demand.

System dynamics. Material balance:

$$(I - A)\vec{X}(t_n) = \vec{F}(t_n) = \vec{C}_p(t_n) + \vec{I}(t_n) + \vec{G}(t_n)$$

Capital accumulation:

$$\vec{K}(t_{n+1}) = (I - \hat{\delta})\vec{K}(t_n) + \vec{I}(t_n),$$

where $\hat{\delta}$ is a diagonal matrix of depreciation rates.

Capacity constraint:

$$0 \leq B\vec{X}(t_n) \leq \vec{K}(t_n)$$

Demand estimate evolution:

$$\vec{C}_p(t_n) = \vec{C}_p(t_{n-1}) + \Delta\vec{C}_p(t_{n-1})$$

where $\Delta\vec{C}(t_{n-1})$ is the change in estimated demand as determined by the prices and previous demand.

Demand:

$$\vec{C}(t_n) = f(\vec{P}(t_n), Y_d(t_n))$$

here we define $Y_d(t_n)$ as the household income disposable at any point of time t_n

Viability constraints. The system must remain within the feasible set

$$\mathcal{V} = \left\{ s \;\middle|\; \begin{array}{l} \vec{F}(t_n) \geq 0, \vec{K}(t_n) \geq 0,\; \vec{I}(t_n) \geq 0 \\ 0 \leq B\vec{X}(t_n) \leq \vec{K}(t_n) \end{array} \right\} \tag{1}$$

Reference trajectory and tracking error. Let the reference signal be the expected demand path

$$\vec{r}(t_n) \equiv \vec{C}(\vec{P}_0, t_n) + \vec{I}(t_n) + \vec{G}(t_n)$$

Define the tracking error as

$$\vec{e}(t_n) \equiv (I - A)\vec{X}(t_n) - \vec{r}(t_n).$$

Planner's problem (cybernetic form). The planner's objective is to design a feedback control law

$$\vec{u}(t_n) = \pi\big(\vec{s}(t_n)\big)$$

such that the following conditions hold:

$$s(t_n) \in \mathcal{V} \quad \forall t \quad \text{(viability)},$$

$$\frac{1}{N}\sum_{i=1}^{N} \|\vec{P}_0{\cdot}\vec{e}(t_i)\| \leq \varepsilon_d \quad \forall t_i \quad \text{(bounded tracking error)},$$

$$\limsup_{t\to\infty} \|\vec{P}_0 \cdot \vec{e}(t_n)\| < \infty \quad \text{(dynamic stability)}$$

$$g(t_n) \leq g_{\max}(\Gamma) \equiv \frac{1}{\rho(H(\Gamma))} - \delta \quad \text{(growth feasibility)}.$$

Interpretation.. This formulation defines economic planning as a viability-preserving tracking problem under bounded information and computational constraints. The planner does not solve a global optimisation problem, but implements a feedback control rule that stabilizes production and investment around an evolving demand trajectory while respecting material balance and capacity constraints. Here, ε_d A secondary objective of the planning system is also to not perform investment when there exists more unused capital than the capital activation needed, this for a growing economy implies a small output gap after some point in time t_n. Mean Absolute Output Gap $\leq \varepsilon_{\mathrm{USA}}$

3.2. Static Leontief Model Foundation

The development of the dynamic input-output model begins with a comprehensive analysis of the static Leontief framework, which serves as the foundation for all subsequent extensions. The static model establishes the fundamental relationship between production, consumption, and final demand through the matrix equation $(I - A)\vec{X} = \vec{F}$, where each element represents a specific economic quantity that can be measured and analysed. This equation encapsulates the essential insight that the gross production of any industry must equal the sum of its intermediate consumption by all other industries, plus the final demand for its output.

To illustrate the mechanics of the model, we begin with a simplified two-industry system that captures the essential features of inter-industry relationships while remaining mathematically tractable. In this toy model, the production equations take the form $F_1 + Z_{1,1} + Z_{1,2} = X_1$ and $F_2 + Z_{2,1} + Z_{2,2} = X_2$ where $Z_{i,j}$ represents the consumption of industry i's output by industry j. By introducing the technical coefficients $a_{i,j} = Z_{i,j}/X_j$, which represent the amount of input i required for output j, the system can be expressed in the matrix form as $\vec{X} = \vec{F} + A\vec{X}$. Leontief (1936) The mathematical elegance of this formulation lies in its ability to capture complex economic interdependencies through relatively simple linear relationships. The matrix $(I - A)$ represents the term referred to in literature as "flow matrix" Wagner (1957) when inverted, and its elements can be interpreted as the total direct and indirect requirements of each industry's output needed to satisfy a unit of final demand. This interpretation provides powerful insights into the structure of economic systems and enables economists to trace the full implications of changes in the final demand throughout the entire production network. The gross production, hence, in the static model, will be given in the following form

$$\boxed{\vec{X} = (I - A)^{-1}\vec{F}} \tag{2}$$

This is equal to the market clearing prices determined by the identity that expenditure is equal to the income. $\vec{P}_c \cdot \vec{F} = (\hat{w}\vec{L}) \cdot \vec{X}$ This is solved in the following manner:

$$[\vec{P}_c^T(I - A) - (\hat{w}\vec{L})^T]\vec{X} = 0$$

This gives the following result for the general equilibrium prices of such an economy:

$$\vec{P}_c = (I - A^T)^{-1}\hat{w}\vec{L} \tag{3}$$

3.3. Dynamic Investment with feedback

For clarity, throughout this section:[1]

3.3.1. Price feedback mechanics and AR(1) extrapolation

The derivation of prices follows a similar pattern as before to maintain the equality in the expenditure and income system. Along with the material balance system, we introduce the term of taxes, which accounts for the fraction of the net income spent by the government on investment and government spending, we consider the case for $r = 0$ as profit based investment is replaced by the tax-based investment for the planned investment system $T(t_n)$.

$$\sum_{n=1}^{N} \hat{w}\vec{L} \cdot \vec{X}(t_n) - T(t_n) = \sum_{n=1}^{N} \vec{P}_c(t_n) \cdot \vec{C}(t_n) \tag{4}$$

This can be simplified similarly to before with a new equality. $(I - A)\vec{X}(t_n) = \vec{F}(t_n) = \vec{C}_p(t_n) + \vec{I}(t_n) + \vec{G}(t_n)$

$$\sum_{n=1}^{N} (\hat{w}\vec{L})^T (I - A)^{-1}\vec{F}(t_n) = \sum_{n=1}^{N} \vec{P}_c(t_n) \cdot \vec{C}(t_n) + T(t_n)$$

This equation implies the price is equal to $\vec{P}_c(t_n)$. The price 'oscillates' around $(I - A^T)^{-1}\hat{w}\vec{L}$. We can see a similar relationship for the taxes (or savings) in this economic model. Taxation is defined as the following where tau is a parameter to allow for lower or higher taxation changing demand:

$$T(t_n) = (1 - \tau)\vec{P}_0 \cdot [\vec{I}(t_n) + \vec{G}(t_n)]$$

Therefore it is obtained that the following relationship holds

$$\sum_{n=1}^{N} \vec{P}_c(t_n) \cdot \vec{C}(\vec{P}_c) = \vec{P}_0 \cdot \sum_{n=1}^{N} \vec{C}_p(t_n) + \tau[\vec{I}(t_n) + \vec{G}(t_n)]$$

Now, we require similar to this that all markets clear simultaneously up to some maximum error, for all points of time t_n. At each planning step, realized consumer demand is given by

$$\vec{P}_c(t_n) \cdot \vec{C}(\vec{P}_c(t_n)) = \vec{P}_0 \cdot (\vec{C}_p(t_n) + \tau[\vec{I}(t_n) + \vec{G}(t_n)])$$

The system can be broken down by a Jacobian, we define the $\vec{P}_c(t_n) = \vec{P}_0 + \Delta\vec{P}(t_n)$

$$(\vec{P}_0 + \Delta\vec{P}(t_n)) \cdot (\vec{C}(P_0, t_n) + J_d(\vec{P}_0)\Delta\vec{P}(t_n)) = \vec{P}_0 \cdot (\vec{C}_p(t_n) + \tau[\vec{I}(t_n) + \vec{G}(t_n)]) \tag{5}$$

Here, we can also see that $\vec{C}(\vec{P}_c, t_n) = \vec{C}_p(t_n)$ since the prices ($\vec{P}_c$)are defined as the market clearing prices . There is also another condition which must be imposed.

$$\Delta P(t_n) \cdot \vec{C}_p(t_n) = \tau[\vec{I}(t_n) + \vec{G}(t_n)]$$

This simplifies the expression to the following form:

$$\vec{C}_p(t_n) - \vec{C}(\vec{P}_0, t_n) = J_d(\vec{P}_0)\Delta\vec{P}(t_n)$$

The expression is the negative of the excess demand in each sector. The excess demand at $\vec{P}_0$ is the feedback signal for the planner. To form the final expression we divide and multiply with the expression of $\hat{P}_0\hat{C}_p^{-1}$ which gives elasticity matrix, for tractability we consider the matrix to be diagonal. The elasticity matrix entries are ϵ_i. Which gives the final expression [2]:

$$\frac{C_{i,p}(t_n) - C_i(\vec{P}_0, t_n)}{C_i(t_n)} = \frac{\Delta P_i(t_n)}{P_{0,i}}\epsilon_i$$

Therefore we can define the change in the estimated demand as the following:

$$\Delta\vec{C}_p(t_n) \approx -\sum_{i=1}^{n} \vec{C}_p(t_{n-1})\frac{\epsilon_i \Delta P_i(t_{n-1})}{P_{0,i}}\vec{e}_i \tag{6}$$

This makes the rule equivalent to in cybernetic terms as the following:

$$\vec{C}_p(t_n) = \vec{C}(t_{n-1})$$

We consider a simplified diagonal matrix E for this analysis. This leads the model to have a lag of one time step compared to the actual demand of the consumer. This is because in the ideal case where the

[1] $\vec{C}$ denotes realised consumer demand; $\vec{C}_p$ denotes the planner's estimated demand level; and $\Delta\vec{C}_{p,E}$ denotes the expected change in consumer demand.

[2] Elasticities may be estimated empirically without committing to a specific utility function; the framework is compatible with multiple microeconomic interpretations but does not require one.

the price changes act exactly as described by the log-linear relationship. The resulting expression leads to the difference term being the excess demand at prices $\vec{P}_0$ at time t. This approximation may hold true for the demand curve locally. At each planning step, prices are assumed to adjust such that markets clear in the narrow accounting sense that supply equals demand. This local clearing condition ensures material balance and budget feasibility but does not imply convergence to a stationary equilibrium. Preferences, demand, and productive capacity evolve endogenously over time due to investment dynamics, technological change, and behavioural shifts.

The model does not have the finite difference of the demand into the future but by using a set of AR(1) extrapolations for each good from $i \in (1, n)$ we may extrapolate possible demand paths into the future. Along with this, we take the expectation value of the system which removes the Gaussian error term leaving the following relation[3] :

$$\Delta \vec{C}_{p,E}(t_n) \equiv \Phi_v(t_n) \Delta \vec{C}_p(t_{n-1}) + \vec{c}_v(t_n)$$

We can find the second order differences similarly with the use of the first difference extrapolated. A similar protocol is applied to the second differences in the past along with the extrapolated difference, and they are used together to extrapolate the second difference.

$$\Delta^2 \vec{C}_{p,E}(t_n) \equiv \Phi_a(t_n) \Delta^2 \vec{C}_p(t_{n-1}) + \vec{c}_a(t_n)$$

This specific formulation is taken as the algorithm will be using a derivative of demand in the following section. Along with this, to minimise the percentage error in the estimation of the demand, the finite difference has been estimated instead of the demand as a whole due to the significant errors associated with the AR(1) models in predicting the demand. In the causal relationship between savings and investments we first use equation 9 to determine how much investment should occur in which sectors and this sets the tax rates on the incomes.

The demand extrapolation mechanism is intentionally restricted to low-order autoregressive processes. While multivariate methods such as VAR or state-space models may offer higher predictive accuracy in low-dimensional settings, their parameter and data requirements scale quadratically with the number of sectors. For an economy-wide planning problem with n on the order of 10^5 sectors, such approaches are computationally and statistically infeasible.

A more generalized version of this could be the following

For the implementation of the model, it is recommended that a rolling window of $10 - 12$ time steps may be used for the data of price elasticity of demand as well as demand changes input in the AR(1) system, this allows the system to look at recent data and allow for estimating the temporally dependent values of demand side data which is highly dependent on the psychological needs of the consumers. To conclude: prices in this model are signals with their time average value derived from labour and indirect input requirements. They are used to infer excess demand or surplus and to update investment decisions through the feedback rule in equation 6. The prices in this model converge to the cost prices asymptotically if demand is constant but in most applications this may not be the case hence they aren't assumed to converge to Walrasian general equilibrium in which no agent changes their behaviour but are equilibrium or market clearing prices in the accounting sense.

3.3.2. Dynamic Input–Output Model

The dynamic input–output model considered in this paper follows the formulation introduced by Wagner Wagner (1957). The core material balance condition is given in equation 7:

$$\boxed{(I - A)\vec{X}(t_n) = \vec{C}_p(t_n) + \vec{G}(t_n) + \vec{I}(t_n)} \tag{7}$$

This equation states that gross production must be sufficient to satisfy final demand and investment demand. In a dynamic setting, the investment vector represents the resources allocated toward expanding productive capacity. Investment is therefore equal to the surplus of gross output remaining after final demand has been met, scaled by the structure of inter-sectoral production. As a result, economic growth is directly proportional to surplus production, with proportionality governed by the capital coefficients matrix.

At each planning step, production must adjust to reach a level of gross output capable of meeting the forecasted demand. As shown in the previous section, demand evolves over time in response to price changes, and the planning mechanism reallocates production accordingly. The central objective of this section is to determine how the economy transitions from its current production level to the level required to satisfy consumer needs in the subsequent time period.

The investment component of the dynamic input–output model addresses one of the central problems of economic planning: the allocation of resources for future capacity expansion. Investment is determined by the capital coefficients matrix B and the required change in production:

$$\boxed{\vec{I}(t_n) = B\Delta\vec{X} + \Delta\vec{K}_u(t_n) + \hat{\delta}\vec{K}(t_n)} \tag{8}$$

Here, $\Delta\vec{X}$ can be decomposed into changes in consumption and investment demand, where consumption is defined as the sum of private consumer demand and government demand. With this decomposition, the only remaining unknown is the evolution of investment itself over time. We take the limit of the $\Delta\vec{K}_u \approx \vec{0}$ This leads the only dominant term to be the change

[3]The AR(1) process is employed as an instrumental extrapolation rule rather than as a structural model of consumer behaviour. Stability of the planning procedure does not rely on forecast optimality as feedback is used to correct errors.

in capital employed which is defined as $\vec{K}_e = B\vec{X}$ and therefore $\Delta\vec{K}_e = B\Delta\vec{X}$.
To solve for investment dynamics, we employ a Neumann series approximation applied to a differential operator. For analytical convenience, we consider a continuous-time approximation, treating the discrete change in demand $\Delta\vec{C}_p(t)$ over a planning interval as a time derivative $\partial_t\vec{C}_p(t)$, under the assumption that δt is sufficiently small. This continuous-time formulation is not intended as an alternative model of economic dynamics, but rather as an analytical device for studying the structure and stability of the discrete-time planning rule introduced later.

Defining the terms used as the following:

$$M = B(I-A)^{-1}, M\frac{d[\vec{C}_p(t)+\vec{G}(t)]}{dt} = \vec{f}(t)$$

we obtain the following expression for investment in continuous time:

$$\frac{d\vec{K}_e(t)}{dt} = M\frac{d^2\vec{K}_e(t)}{dt^2} + \vec{f}(t) \tag{9}$$

This equation can be written formally as

$$\frac{d\vec{K}_e(t)}{dt} = (I - M\partial_t)^{-1}\vec{f}(t)$$

If the operator norm of $\|M\partial_t\| \leq 1$ a Neumann series expansion exists. This condition may fail if the derivative operator is unbounded for arbitrary $\vec{f}(t)$; consequently, the approximation is valid only for economies growing below a maximum sustainable balanced growth rate $g_{\max}$.

Under this condition, investment can be expressed as the infinite series

$$\boxed{\frac{d\vec{K}_e(t)}{dt} = \sum_{k=1}^{\infty} M^k \frac{d^k[\vec{C}_p(t)+\vec{G}(t)]}{dt^k}} \tag{10}$$

The corresponding expression for final production in continuous time is

$$\vec{F}(t) = \vec{C}_p(t) + \vec{G}(t) + \sum_{k=1}^{\infty} M^k \frac{d^k[\vec{G}(t)+\vec{C}_p(t)]}{dt^k} + \hat{\delta}\vec{K}(t)$$

To simplify the analysis, we assume that the planned bundle for consumer demand in each sector grows at a constant rate g_i and the government demand grows at the same rate. Under these assumptions, each successive term in the series decays geometrically relative to the previous one, allowing the series to be approximated by truncation:

$$\vec{I}(t) = \sum_{k=1}^{\infty}(M\hat{g})^k[\vec{C}_p(t) + \vec{G}(t_n)]$$

For small short-term growth rates g_i, higher-order terms rapidly become negligible. The series therefore provides an asymptotic. To approximate the series [4]:

$$\begin{aligned}\vec{I}(t) &= M\left[\frac{d\vec{G}(t)}{dt} + \frac{d\vec{C}_p(t)}{dt}\right] + M\frac{d\vec{I}(t)}{dt} + \hat{\delta}\vec{K}(t) \\ &= \sum_{k=1}^{\infty} M^k \frac{d^k[\vec{G}(t)+\vec{C}_p(t)]}{dt^k} + \hat{\delta}\vec{K}(t)\end{aligned} \tag{11}$$

Hence, we take the following approximation taking the series to second order:

$$\frac{d\vec{I}(t)}{dt} \approx M\frac{d^2[\vec{C}_p(t)+\vec{G}(t)]}{dt^2}$$

We now discretize this, as shown before using the AR(1) extrapolations, we will be using the terms obtained from them in the actual algorithm since we do not know what the values of the future consumer bundles are due to their determination via the administered price signals. Therefore we have the solution given by the following equation: $\Delta\vec{K}_e(t) = M[\Delta\vec{G}(t_n) + \Delta\vec{C}_{p,E}(t_n)] + M^2[\Delta^2\vec{G}(t_n) + \Delta^2\vec{C}_{p,E}(t_n)]$

The only remaining part of investment which is yet to be determined is the unused capital, which is administratively set to minimize the investment needed. The evolution of unused capital reflects the difference between available capital and employed capital. Unused capital therefore acts as a buffer stock:

$$\vec{K}_u(t_n) = (1-b)\vec{K}(t_n) - \hat{K}_e(t_n)$$

with $\hat{K}_e(t_n) = \vec{K}_e(t_{n-1}) + \Delta\vec{K}_e(t_{n-1})$ and the capital stock accumulates in the following way. $\vec{K}(t_n) = (I - \hat{\delta})\vec{K}(t_{n-1}) + \vec{I}(t_{n-1})$ Now, Total investment at each planning step is therefore given by

$$\boxed{\vec{I}(t_n) \equiv \hat{\delta}\vec{K}(t_n) + \Delta\vec{K}_e(t_n) + \Delta\vec{K}_u(t_n)} \tag{12}$$

The change in unused capacity is bounded by its available stock:

$$\Delta\vec{K}_u(t_n) = -\vec{K}_u(t_n)$$

and otherwise satisfies

$$\Delta\vec{K}_u(t_n) = -\Delta\vec{K}_e(t_n)$$

implying that new productive capacity is activated from idle capital before new capital formation occurs. Equivalently,

$$\Delta\vec{K}_u(t_n) = -\min(\vec{K}_u(t_n), \Delta\vec{K}_e(t_n))$$

This rule enforces non-negativity of gross investment and ensures priority activation of idle capacity.

Finally, gross output is determined by

$$\boxed{\vec{X}(t_n) = (I-A)^{-1}\vec{F}(t_n)} \tag{13}$$

and, for a closed economy, final demand is given by

$$\vec{F}(t_n) = \vec{C}_p(t_n) + \vec{G}(t_n) + \vec{I}(t_n)$$

[4] This series is taken as investment because in the absence of unused capital, the change in employed capital is equal to the change in capital

4. The Implementation of the Theory

4.1. Approximation of the Leontief Matrix inverse:

To conclude the efficiency of approximating the vector instead of computing it directly, we simulate by generating a random A matrix of size $n \times n$ where $n \in [1; 1000]$. The data we extract from the simulation is the computation time required concerning matrix sizes, the relative error of the approximation for different matrix sizes, and with respect to the number of iterations. The approximation of the inverse Leontief matrix is as follows G.H. Golub (2013):

$$\boxed{(I - A)^{-1} = \lim_{k \to \infty} \sum_{i=0}^{k} A^i = I + A + \ldots + A^k} \tag{14}$$

The series is truncated after the K-th iteration as it is assumed that any $A^{k+1} = 0$, hence, it has minimal contribution to the series. It can be iteratively approximated by multiplying it by the input vector (d), which gives the following equation $\vec{X}^{(0)} = \vec{d}$ and $\vec{d}_k = A^k \vec{d}$:

$$\vec{X}^{(k+1)} = \vec{X}^k + A\vec{d}_k$$

We must note that the equation 14 only holds if $\rho(A) < 1$. The code can be found in the link present in the appendix.

4.2. Errors and Acceptable Limits

As can be seen in the figure 2, there are errors present in the computation of the matrix approximation. This error is characterised in the following way as seen from the result in 12:

$$\delta(A) = (I - A)^{-1} - \sum_{i=0}^{k} A^i$$

Which gives the following result:

$$\delta(A) = \sum_{i=k+1}^{\infty} A^i$$

As a matrix, it does not provide us with any relative error, but it allows the computation of the relative error by the following equation: Let the error between the two terms be δ, where delta is found by the modulus of relative deviation divided by the modulus of the complete solution:

$$\delta(n, k) = \frac{|[\vec{d} - (I - A)\vec{X}]|}{|\vec{d}|}$$

This is simplified to the following form:

$$\delta(n, k) = \frac{|(I - A)\delta(A)\vec{d}|}{|\vec{d}|} = \frac{|[A^{k+1}\vec{d}]|}{|\vec{d}|}$$

For this paper, it has been considered as a function of the iterations of the algorithm. As the error rate stays relatively constant with respect to matrix size. This relationship is observed in figure-1 It can be seen

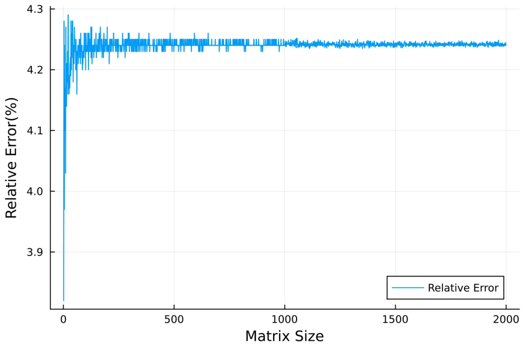

Figure 1: Relative error vs matrix size

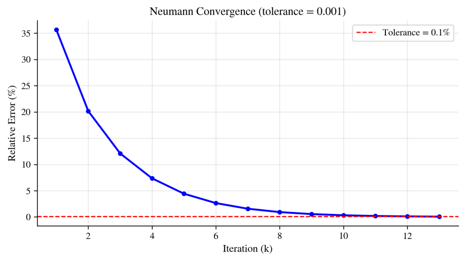

Figure 2: Relative error versus number of Iderations

that the relationship between the two is an exponentially decreasing function - as predicted by the equations presented above, and seen in figure-2.

$$\boxed{\delta(k) = \delta(0)e^{-\alpha k}} \tag{15}$$

The tolerance is set as the following Tolerance $= 0.001$ Hence, the error in the model must be less than $\delta(k) < \varepsilon$; this in the simulation is determined to be at 13 Iderations for the A-matrix provided from the Spanish economy. (See appendix)

4.3. The Space and Time Complexity of the Algorithm.

We consider the complexity of carrying out the multi-step investment computation with respect to time and memory, based on the computation algorithm provided in Section IV-A. At first, we observe the computation time and memory allocation as a function of matrix size, while keeping the time-steps constant. Afterwards, we do the opposite to see the feasibility of planning for more future periods.

The computation time with respect to matrix size appears to grow quadratically, so it is fair to postulate that we are observing a time complexity of $O(n^2)$.

The memory allocated is in a quadratic relationship with matrix size. Therefore, has a complexity of $O(n^2)$.

4.4. Number of planned industries and applications

While no official source enumerates the exact number of distinct goods produced by China's state-controlled economy, the National Bureau of Statistics' Statistical Classification of Products (2017) defines approximately 38,000 product subclasses that encompass the entire economy.National Bureau of Statistics of China (2017) The state sector dominates production in many

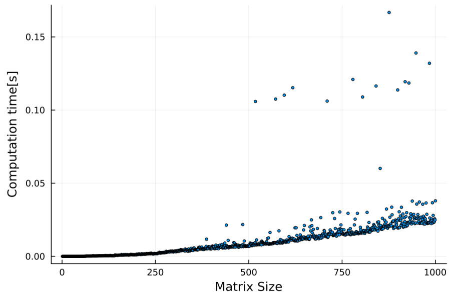

Figure 3: Matrix size vs computation time

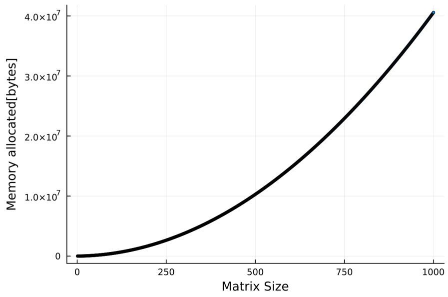

Figure 4: Matrix size vs Memory allocated

of these subclasses, particularly in strategic industries. Accounting for product variants (sizes, specifications, models) within each subclass—especially in complex manufacturing sectors—suggests the state-controlled economy produces on the order of several hundred thousand to over one million distinct goods. Hypothetically the algorithm can be used for as high as one million ($n = 10^6$) and be solved in realistic time but it may not be possible yet to obtain the data on such a high detail for the management of production and distribution, hence, it implies that for now, the model may work best for the management of the state owned enterprises in coordinating between each other on the product subclass level and/or for firms with multiple branches of production with interrelations to manage the coordination of goods between them.

4.5. Data Requirements

The Dynamic Leontief model requires comprehensive input-output tables with technical coefficients matrix A (inter-industry inputs per unit output), capital coefficients matrix B (investment needs for capacity growth), final demand vector d (consumer/government demand), wages matrix $\hat{w}$, price elasticities, and AR(1) parameters for demand forecasting across thousands of sectors. Real-time granular data demands monthly or higher-frequency updates at 5-digit SIC level or equivalent (e.g., 1,000,000+ products as in China's state sector), including gross output $\vec{X}$, capacity utilization, imports/exports, and price fluctuations for feedback loops. These must cover interdependencies for sparse matrix approximations and Neumann series inversions, with rolling windows of 10-12 periods for elasticities and AR(1) inputs.Hötte (2025)

4.6. Feasibility Challenges

Obtaining real-time granular data faces significant hurdles: traditional IO tables from sources like national statistical offices (e.g., BLS, Census Bureau) are annual or quarterly with lags, relying on surveys that cannot match monthly planning horizons proposed in the paper. Emerging alternatives like anonymized payment data show promise for high-frequency inter-industry flows correlating with GDP and IO structures, but require validation against official tables and face issues like late-arriving data, scaling for millions of sectors, bandwidth limits, and integration with legacy systems. Computational feasibility improves with sparse techniques (e.g., 6-minute runs for 100,000 sectors), but data sparsity in developing economies and error propagation (needing 2 iterations for finding investment as well as final gross production).

5. Testing and Validation

5.1. Simulation Testbed

To evaluate the internal consistency and dynamic stability of the proposed planning algorithm, we construct a numerical simulation based on a calibrated multi-sector input–output economy. The simulation operates on a monthly time step and is parametrized using an empirical input–output table for Spain consisting of 64 production sectors. These sectors are classified into three broad categories—heavy, medium, and light industry—to reflect systematic differences in capital intensity and depreciation behaviour.

Inter-industry production requirements are represented by a fixed technical coefficients matrix $A \in \mathbb{R}^{64\times 64}$. Capital requirements are captured through a diagonal capital–output matrix B, whose entries vary by industry group, with higher capital coefficients assigned to heavy industry and lower coefficients to light industry. Sectoral depreciation rates are likewise heterogeneous: heavy industries are assigned lower depreciation rates, while light industries face higher depreciation, reflecting faster capital turnover.

The following objects are held fixed over the simulation horizon: the technical coefficients matrix A, the capital coefficients matrix B, and the sectoral depreciation matrix $\hat{\delta}$. Endogenous variables include gross output $X(t)$, capital stock $K(t)$, planned consumer demand $C_p(t)$, realised consumption $C(t)$, and gross investment $I(t)$.

Initial gross output $X(0)$ is chosen to satisfy the static Leontief system for the initial planned demand vector $C_p(0)$. Initial capital stocks $K(0)$ are set such that productive capacity exceeds realised output by approximately 0.973%, introducing an exogenous buffer of idle capacity. This buffer serves both as a realism-enhancing feature and as a stress test for the planner's ability to absorb demand fluctuations without inducing instability.

Final demand consists of household consumption, investment, and government expenditure. Government spending grows exogenously at a constant real rate. Household consumption is generated using sectoral Cobb–Douglas demand functions and adapts endogenously in response to income and relative prices. To induce non-trivial dynamics, stochastic preference shocks are introduced every six periods.

At each planning step, gross output is determined by solving the Leontief production system using a truncated Neumann series approximation of $(I - A)^{-1}$. Real GDP is measured consistently as aggregate value added. The model enforces capacity feasibility by constraining output through the available capital stock at each time step.

5.2. Testing Methodology

The simulation evaluates the planning algorithm under the joint interaction of adaptive demand formation, capital accumulation, and capacity constraints. Performance is assessed along three explicit criteria:

1. **Demand–output consistency:** Realised GDP Y_t should closely track aggregate demand D_t, such that
$$\left| \frac{Y_t - AD_t}{Y_t} \right| < \varepsilon_d \quad \forall t,$$
indicating successful coordination between planned demand and feasible production.
2. **Capacity feasibility and output gap dynamics:** Realised output must remain bounded by productive capacity,
$$B\vec{X}(t) \leq \vec{K}(t) \quad \forall t,$$
with deviations from capacity exhibiting stable and predictable dynamics.
3. **Dynamic stability:** Transient oscillations induced by stochastic demand shocks and feedback-driven adjustments must decay over time rather than amplify.

Aggregate demand is computed as the real value of consumption, investment, and government expenditure at base prices. Excess demand feeds back into planned consumption through the price-based adjustment mechanism described in Section 3, combined with short-horizon AR(1) extrapolation of demand changes. Investment responds endogenously to both depreciation requirements and expected changes in capital demand.

The output gap is defined as

$$\text{Output Gap}_t = \frac{Y_t - Y_t^{\text{pot}}}{Y_t^{\text{pot}}},$$

where Y_t^{pot} denotes real potential GDP implied by the available capital stock. To avoid artefacts from demand-forecast initialization, the first 12 periods are excluded from the reported results, as they are used solely to generate sufficient data for the AR(1) estimation.

5.3. Results

Figures 5 and 6 summarize the dynamic behaviour of the simulated economy under the proposed planning algorithm.

Figure 5 displays the evolution of realised real GDP, real potential GDP, and aggregate demand over the simulation horizon. Real GDP closely tracks aggregate demand throughout the simulation. This alignment arises from the feedback mechanism linking excess demand to planned consumption and, through the truncated Neumann expansion, to investment and capacity activation. As a result, demand shocks are absorbed primarily through adjustments in capacity utilisation rather than through persistent shortages or surpluses.

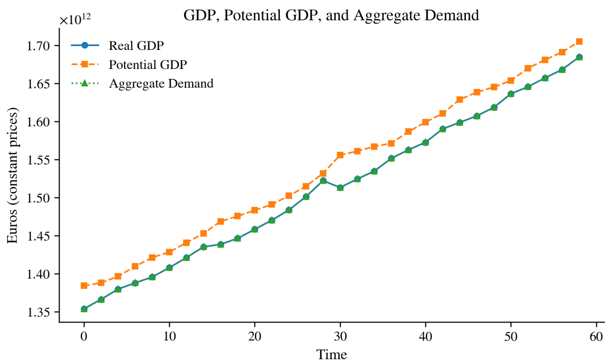

Figure 5: Real GDP, real potential GDP, and aggregate demand over time.

The deviation between realised output and aggregate demand remains small and bounded, with oscillations staying below $\varepsilon_d = 0.05\%$ of real GDP. Real potential GDP consistently exceeds realised GDP by an average of 1.56%. This persistent output gap reflects the exogenously imposed capacity buffer, augmented by stochastic demand fluctuations and forecast error from the AR(1) extrapolation. Importantly, the gap neither collapses nor grows endogenously, indicating that idle capacity is activated when needed but not systematically overbuilt.

The near-parallel growth paths of realised and potential output indicate convergence to a stable growth trajectory rather than demand-driven overheating or persistent underutilisation.

Figure 6 presents the percentage output gap over time. The gap exhibits bounded fluctuations correlated with the timing of stochastic preference shocks, but displays no evidence of divergence or instability. Oscillations decay as demand adjustments propagate through the feedback and investment rules.

Taken together, these results demonstrate that the planning algorithm generates a dynamically stable growth path. Aggregate demand, realised output, and productive capacity remain coherently aligned, while transient deviations are absorbed through damped oscillations rather than amplified feedback. Over a horizon of 60 periods, the mean absolute output gap is 1.56%, below the benchmark tolerance $\varepsilon_{\text{USA}} = 1.78\%$. While the simulation abstracts from many

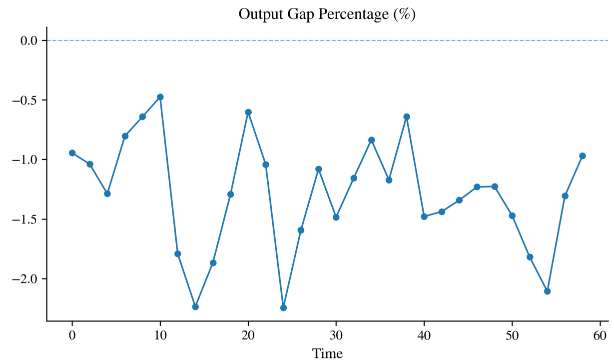

Figure 6: Percentage output gap over time.

real-world frictions, the results indicate that the proposed feedback-based planning mechanism is capable of maintaining feasibility and stability under non-trivial demand dynamics.

We similarly find the excess demand in the system is described by the following figure 7:

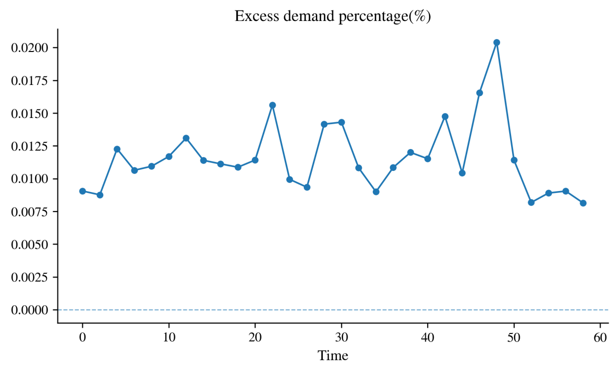

Figure 7: Excess Demand

Each of the spikes in the system correspond to the stochastic shocks added into the system each 6 time steps each with the maximum excess demand less than 0.02% which is successfully allows for the system to meet the criterion.

6. Policy Relevance and Decision-Making Applications

6.1. Core Policy Applications

The framework addresses critical decision-making gaps in production planning for large-scale coordinated economies. Traditional static Leontief models lack temporal dynamics and capacity considerations essential for policymakers directing inter-sectoral investment and resource allocation. This model operationalises monthly-frequency planning (with annual IO tables), enabling real-time production adjustments through AR(1) demand forecasting with price elasticity feedback (Eq. 4), monthly investment decisions via the reactive investment mechanism ($\vec{I}(t_n)$ derived from demand changes), and buffer-based capacity management to prevent shortages. Government demand ($\vec{G}(t_n)$) enters as an exogenous policy lever, allowing planners to simulate sector-specific output targets and assess downstream requirements across supply chains—critical for coordinating state-owned enterprises (SOEs) without relying on market signals.

6.2. Strategic Decision Domains

1. Investment Allocation and Capital Planning: The model distinguishes capital employment (mathematical construct: $\Delta\vec{K}_e$) from real investment ($\vec{I} = \Delta\vec{K}_e - min(\vec{K}_e, \vec{K}_u) + \hat{\delta}\vec{K}$), enabling policymakers to optimise where capital deployment occurs. Chinese context: applicable to coordinating 10^5 products across SOEs, with 6-minute computation enabling monthly plan adjustments.
2. Sectoral Interdependency and Supply Chain Coordination: By tracing inter-industry flows through the technical coefficients matrix A (recalculated annually), planners identify bottleneck sectors where capacity constraints propagate via the Leontief inverse $(I - A)^{-1}$, using Neumann approximations to handle large systems. Example decision: if agriculture demand rises (5 percent), the model computes required chemical fertiliser output, machinery production, and labour reallocation across linked sectors—impossible in static models. AR(1) rolling windows (10-12 periods) adapt predictions to sectoral demand volatility, improving responsiveness vs. fixed elasticity assumptions.

6.3. Decision Support Features

1. Error Tolerance and Confidence Bounds: The model calibrates acceptable error to historical US output gaps (1.78 percent mean deviation from potential GDP, 1950-2024), requiring 56 iterations for $\delta(k) < 0.5\%$ computational error. (Note: according to tests with real economic data the number of iterations needed for target deviation is approximately $k = 30$)
2. Heterogeneous Wage Planning: Sraffian pricing with heterogenous wages (Eq. 2: $\vec{P}_0 = (I - A^T)^{-1}\hat{w}\vec{L}$ allows differentiated value-added contributions by skill/sector. Policymakers can simulate wage policy (increasing w_i for strategic sectors) and observe sectoral price impacts, critical for equity and sectoral incentives.

7. Conclusion

This paper has developed a dynamic extension of the classical Leontief input–output framework aimed at large-scale production planning under bounded information and computational constraints. By integrating capital coefficients, administered price-based feedback, and low-order autoregressive demand extrapolation, the model provides a rule-based mechanism for coordinating production and investment without reliance on global optimisation, equilibrium solving, or perfect foresight.

The framework reconceptualises economic planning as a control and coordination problem rather than a welfare-maximisation exercise. Prices, derived from labour and indirect input requirements, act as accounting-based error signals indicating excess demand or surplus and guide adaptive investment responses. Under bounded growth rates and forecasting errors, the system tracks an evolving demand path while respecting material balance constraints, capacity limits, and feasibility conditions.

From a computational perspective, the use of sparse matrix representations and Neumann series approximations enables tractable solution of large input–output systems. The accompanying error analysis demonstrates exponential decay of approximation error with iteration count and shows that acceptable accuracy can be achieved within empirically observed output-gap bounds in market economies. Simulation results confirm that aggregate demand, realised output, and productive capacity remain coherently aligned over time, with deviations remaining small and dynamically stable.

The approach deliberately abstracts from non-linear behavioural responses, financial dynamics, and strategic price formation, and it depends on the availability of high-frequency, high-resolution input–output data. Demand forecasting errors propagate into investment decisions, though they remain bounded under the conditions analysed. Accordingly, the framework is intended as a production coordination mechanism rather than a comprehensive macroeconomic model.

Overall, the results demonstrate that dynamic, feedback-based economic planning grounded in classical input–output theory is mathematically coherent and computationally feasible at large scales, providing a foundation for future empirical implementation and extension.

8. Future Research Directions

1. Empirical Validation and Real-World Implementation Research Directions:

Conduct pilot implementation on a nation's manufacturing sector (e.g., China SOEs, Vietnam state enterprises, or India's priority sectors) using official IO tables combined with high-frequency proxies (business payments, customs data for trade, sensor streams for capacity utilisation). Measure forecast accuracy of AR(1) demand extrapolation against actual quarterly/monthly production, comparing to traditional econometric baselines (ARIMA, VAR models).

Validate price feedback mechanism (Eq. 4) empirically: use scanner data or PPI indices to test whether observed sectoral price deviations $\Delta\vec{P}(t_n)$ correlate with model-predicted demand changes $\Delta\vec{C}_p(t_n)$ via elasticity parameters ϵ_i. Test robustness of rolling 10-12 period windows for elasticity estimation.

2. Non-linear and Cybernetic Feedback Enhancements Gap: AR(1) feedback is linear with lag-1 structure; price adjustment assumes instant market clearing around $\vec{P}_0$.

Research Directions: Cybernetic Control Systems: Formalise the model as a feedback control problem: define target production trajectory $\vec{F}(t)$ (from policy), compute control inputs (investment $\vec{I}(t_n)$, government demand $\vec{G}(t_n)$) to minimise deviations. Apply optimal control theory (Pontryagin maximum principle, dynamic programming) to derive optimal rather than myopic allocation.

3. Multiple time scale planner system: Gap: The current planning system can't dynamically allocate goods which can be made on time scales smaller than the time step.

Research Directions: Multi-tier system: A system may be formalized in future research, such that different sub-algorithms exist for planning on different time scales such that the equation 6 holds true for the goods on those time scales. Such as factory construction may happen on a annual time scale rather than monthly, and the different planning sub-algorithms may interact via the exchange of goods between firms ($\vec{I}_{str}$).

3. International Trade Considerations

A natural extension of the dynamic Leontief planning framework involves the integration of *international trade*, allowing the planner to account for exports and imports in addition to domestic production and consumption. In real economies, trade plays a crucial role in optimizing resource allocation, exploiting comparative advantage, and stabilizing supply chains, yet its explicit incorporation into high-frequency, feedback-driven input–output models remains challenging.

The primary difficulty lies in simultaneously satisfying domestic final demand while leveraging foreign markets, all under *capital and capacity constraints.* This requires determining production plans that not only meet domestic aggregate demand ($\vec{F}(t_n)$) but also optimize the net surplus available for trade, given fluctuating world prices. The resulting problem is naturally formulated as a linear program over the feasible set defined by the Leontief material balances and capital limitations, but with an objective function weighted by *world market prices* rather than domestic value-added alone.

A promising solution is to treat the *trade surplus maximization* as a secondary planning objective. Formally, for each planning period, the planner could fix domestic production at or above $F_{\min}$ and then choose gross outputs X such that:

$$\max_X \ \vec{P}_w \cdot \big((I-A)\vec{X} - \vec{F}\big)$$

subject to:

$$0 \le B\vec{X} \le \vec{K}, \quad (I-A)\vec{X} \ge \vec{F}, \quad X \ge 0,$$

where P_w denotes the vector of world prices for tradable goods. This formulation identifies the feasible production plan that maximizes externally valued surplus without violating domestic requirements or overusing capital. Computationally, the problem can be efficiently solved using *Interior-Point Methods (IPM)*, which exploit the sparsity of A and B to provide high-accuracy solutions for large-scale economies.

9. Acknowledgements

We acknowledge Giovanni Paiela for his contributions to error calculations in this paper.

Appendix A. Relative Error with respect to the Number of Iterations

To relative error with respect to the number of iterations. Which will vary between 1 and 50. This relationship is observed in the best curve fit for the data provided by this is given in the form of a decreasing exponent of the following form: $\delta(x) = \delta(0)e^{-\alpha k}$ and the value of $\alpha = \frac{1}{k}\ln(\frac{\delta(0)}{\delta(k)}) = 0.48$

Appendix B. Notation table

This table provides all the notation details for the symbols and notation used in this paper.

Table B.1: Table of Variables

Symbol	Meaning of symbol
A	Technical coefficient matrix
B	Capital coefficient matrix
$\hat{w}$	wages matrix
Γ	consumption rate matrix
ϵ_i	Elasticity
$\vec{K}(t_n)$	Capital vector
$\vec{X}(t_n)$	Gross output vector
$\vec{C}(t_n)$	Realized Consumer demand vector
$\vec{C}_p(t_n)$	Estimated Consumer demand vector
$\Delta\vec{C}_{p,E}(t_n)$	Expected difference in consumer demand
$\vec{I}(t_n)$	Gross investment vector
$\vec{I}_{str}(t_n)$	strategic Investment vector
$\vec{P}_0(t_n)$	Cost price vector
$\vec{P}_c(t_n)$	Price vector for clearing the market
$\vec{L}$	Labour time vector
ε_{USA}	USA Mean absolute Output Gap
ε_d	Maximum Excess demand
Φ	persistence factor of AR(1)
$\vec{c}$	Intercept factor of AR(1)

Appendix C. The maximum growth rate

To obtain the maximum growth rate let us assume $B\vec{X} = \vec{K}$ and some depreciation δ.

$$\vec{X}(t_{n+1}) = (1+g)\vec{X}(t_n)$$

and the dynamic equation with depreciation is the following:

$$B[\vec{X}(t_{n+1}) - (1-\delta)\vec{X}(t_n)] = (I-A)\vec{X}(t_n) - \vec{d}(t_n)$$

By the equation for time evolution of the final production, it is determined the maximum growth rate is the following, given an initial distribution of consumption $\vec{d} = \Gamma\vec{F}$ where Γ is a diagonal matrix:

$$B[(g+\delta)\vec{X}(t_n)] = (I-\Gamma)(I-A)\vec{X}(t_n)$$

Which is found to be the following result:

$$(g+\delta)\vec{X}(t_n) = B^{-1}(I-\Gamma)(I-A)\vec{X}(t_n)$$

By the Perro-Frobenius method:

$$g_{max}(\Gamma) + \delta = \rho(B^{-1}(I-\Gamma)(I-A)) = \rho(H(\Gamma))$$

Here we define the H matrix as $H(\Gamma) = B^{-1}(I-\Gamma)(I-A)$, this implies that given a distribution Γ the maximum rate is dependent on the savings rate, perfectly in line with existing literature such as Harrow-Domar growth model. This demonstrates that there exist a planning system which may be able to find a feasible growth path if the growth rate is below the maximum growth rate.

Then we finally arrive at the following expression:

$$g_{max}(\Gamma) = \rho(H(\Gamma)) - \delta$$

Appendix D. Stability and Z-Domain Analysis of the Planning Adjustment Rule

Appendix D.1. Planning Update Rule

Let $C(t)$ denote realized final demand at time t, and $C_p(t)$ the planned final demand used as input to the Leontief production system. The planner updates planned demand according to observed excess demand:

$$\vec{C}_p(t) = \vec{C}_p(t-1) + K_p\big[\vec{C}_p(t-1) - \vec{C}(t-1)\big],$$

where $K_p > 0$ is a proportional adjustment parameter.

Define the planning error

$$\vec{e}(t) := \vec{C}_p(t) - \vec{C}(t)$$

Substituting the planner update rule, the exact error dynamics are

$$\vec{e}(t) = (1+K_p)\vec{e}(t-1) - \Delta\vec{C}(t-1),$$

where $\Delta\vec{C}(t-1) := \vec{C}(t) - \vec{C}(t-1)$

Equation for error dynamics holds without linearization or assumptions about instantaneous market clearing.

Appendix D.2. Balanced Growth Assumption

Assume final demand follows balanced growth:

$$\vec{C}(t) = (1+g)\vec{C}(t-1), \qquad g > 0$$

Then

$$\Delta\vec{C}(t) = g\vec{C}(t-1),$$

and the error recursion becomes

$$\vec{e}(t) = (1+K_p)\vec{e}(t-1) - g\vec{C}(t-1)$$

Because $C(t)$ grows over time, absolute errors need not converge even when relative errors are stable. Define the relative planning error:

$$\varepsilon_i(t) := \frac{e_i(t)}{C_i(t-1)}$$

Using $C(t) = (1+g)C(t-1)$, the normalized error dynamics are

$$\vec{\varepsilon}(t) = (1+K_p)(1+g)\vec{\varepsilon}(t-1) - g\mathbf{1}$$

Appendix D.3. Appendix X: Nyquist Stability of Relative Error Dynamics

$$\varepsilon(t) = a\,\varepsilon(t-1), \qquad a = (1+K_p)(1+g).$$

The corresponding open-loop transfer function is

$$L(z) = az^{-1}.$$

Evaluated on the unit circle $z = e^{i\omega}$,

$$L(e^{i\omega}) = ae^{-i\omega},$$

which traces a circle of radius $|a|$ centered at the origin. The discrete-time Nyquist criterion requires that this locus not encircle the critical point -1.

Hence the stability condition is

$$|a| < 1 \iff |(1+K_p)(1+g)| < 1.$$

Equivalently,

$$-1 - \frac{1}{1+g} < K_p < -1 + \frac{1}{1+g}$$

Appendix E. Mean Absolute output gap

The mean absolute output gap for a market economy $\varepsilon_{\text{USA}} = 1.781\%$ is determined by the use of the data from the Federal bank of St. Louis (more commonly known as FRED)

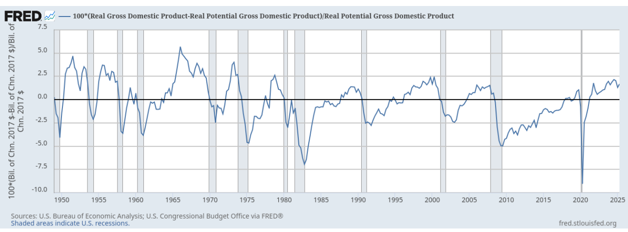

Figure E.8: US economy percentage output gap 1950-2025

Appendix F. Simulation Code and Dataset for Reproducibility

The hyperlink presented here gives direct access to an open science foundation page, The page includes data sets taken from the Spanish Input-Output table (2022) with A-matrix as provided in the data from the National Statistics Institute of Spain Instituto Nacional de Estadística (2022) along with the data on consumption, investment and government spending:
Simulation Files
To run the program, the Python file and the datasets must be in the same folder.

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