# Copyright Notice

These slides are distributed under the Creative Commons License.

DeepLearning.ai makes these slides available for educational purposes. You may not use or distribute these slides for commercial purposes. You may make copies of these slides and use or distribute them for educational purposes as long as you cite DeepLearning.AI as the source of the slides.

For the rest of the details of the license, see https://creativecommons.org/licenses/by-sa/2.0/legalcode

DeepLearning.AI

Math for Machine Learning

# Linear algebra - Week 4

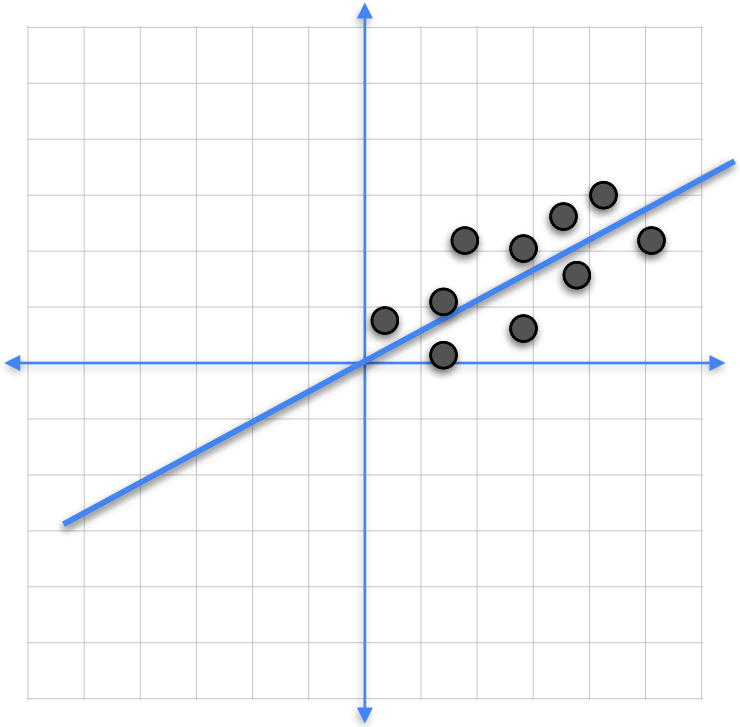
# W4 Lesson 1

DeepLearning.AI

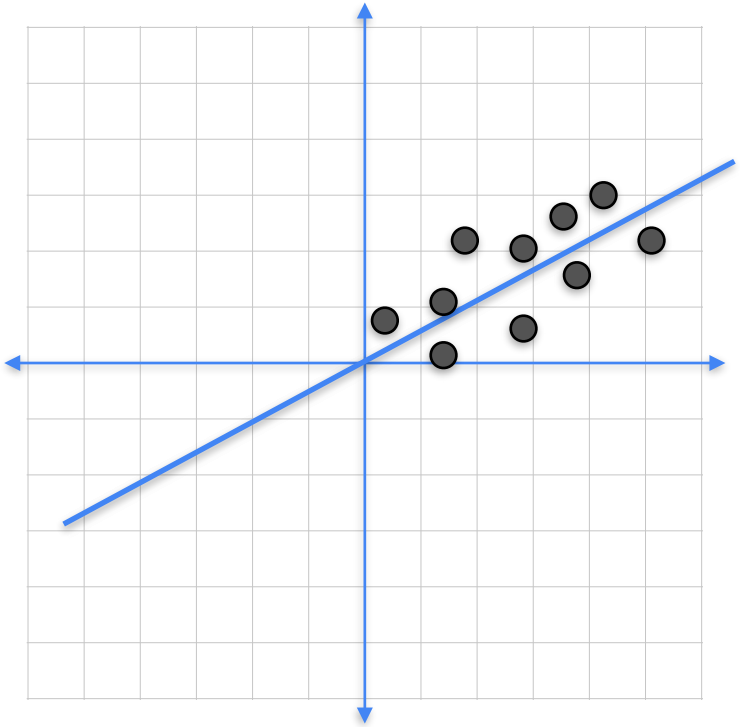
# Determinants and Eigenvectors

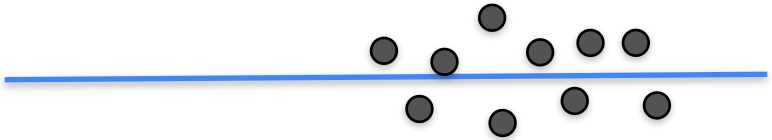
## Machine learning motivation

# Principal Component Analysis

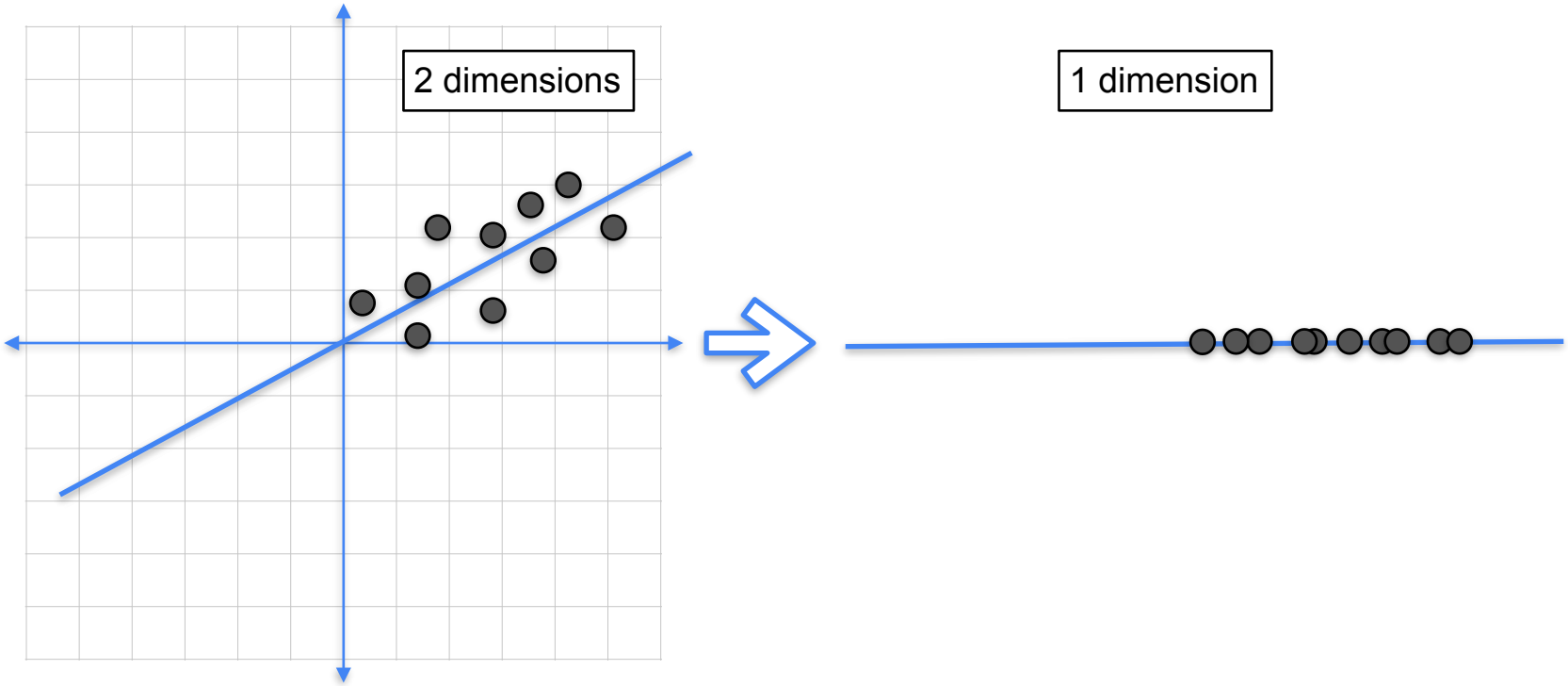


# Principal Component Analysis



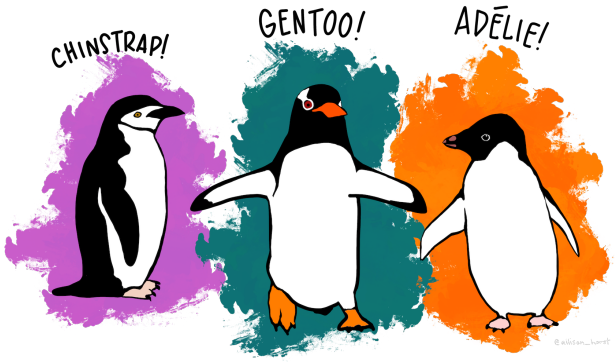


# Principal Component Analysis



# Principal Component Analysis

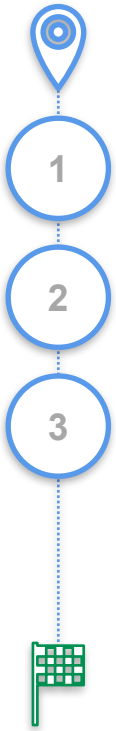
- Reduce dimensions (columns) of dataset
- Preserve as much information as possible




| species | culmen_length_mm | culmen_depth_mm | flipper_length_mm | body_mass_g |
|---|---|---|---|---|
| Adelie | 40.6 | 17.2 | 187.0 | 3475.0 |
| Adelie | 38.9 | 17.8 | 181.0 | 3625.0 |
| Adelie | 35.7 | 16.9 | 185.0 | 3150.0 |
| Gentoo | 50.0 | 15.3 | 220.0 | 5550.0 |
| Adelie | 34.5 | 18.1 | 187.0 | 2900.0 |

| PC1 | PC2 | species |
|---|---|---|
| 1.353843 | -0.422253 | Adelie |
| 1.760446 | -0.350965 | Adelie |
| 2.005766 | -1.113797 | Adelie |
| -2.585758 | 0.061768 | Gentoo |
| 2.438111 | -0.786227 | Adelie |

# What to expect?



# What to expect?

Linear transformation

1. Singular / Non-singular
2. 
3. 

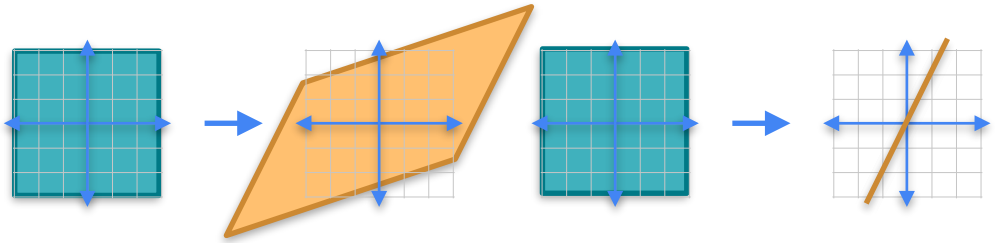
Characterize your transformation

**Non-singular**

| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

**Singular**

| | |
|---|---|
| 1 | 1 |
| 2 | 2 |



# What to expect?

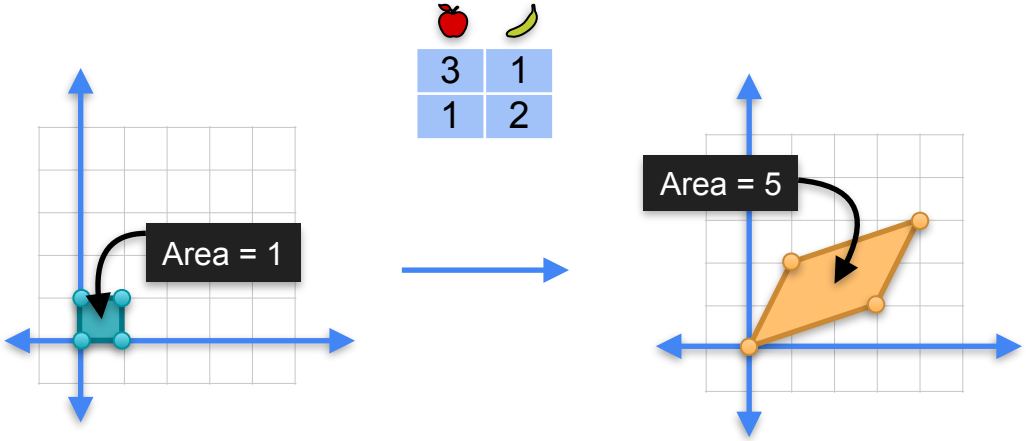
Linear transformation

1

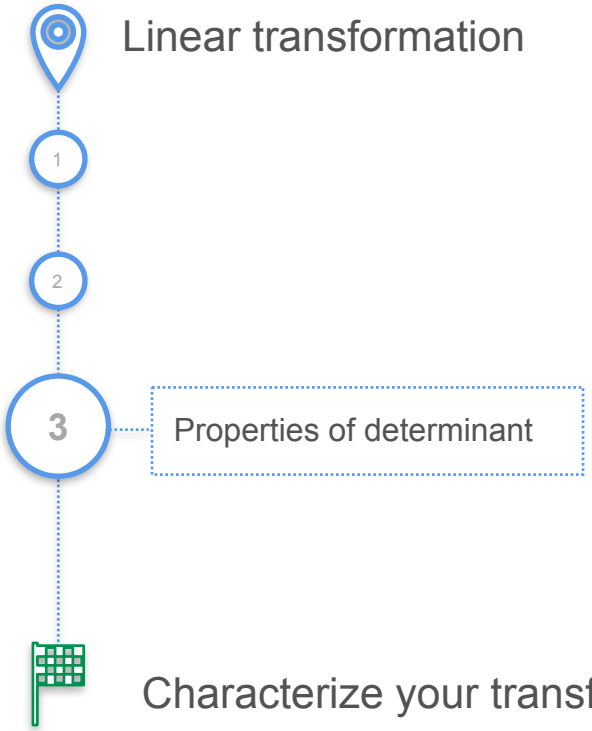
2 Determinant as area

3

Characterize your transformation



# What to expect?




$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

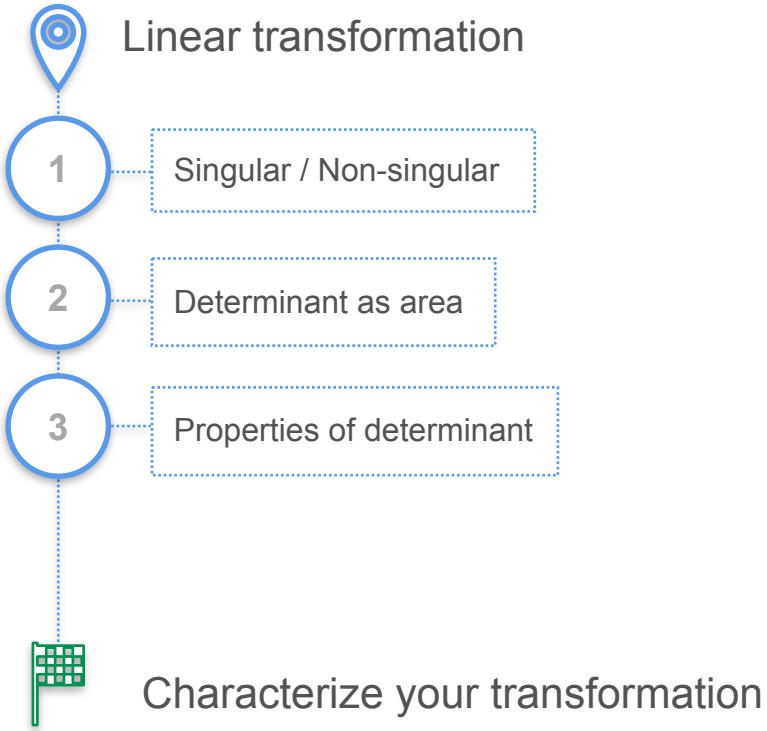
Det = 5 Det = 3 Det = 15

$= 5 \cdot 3$

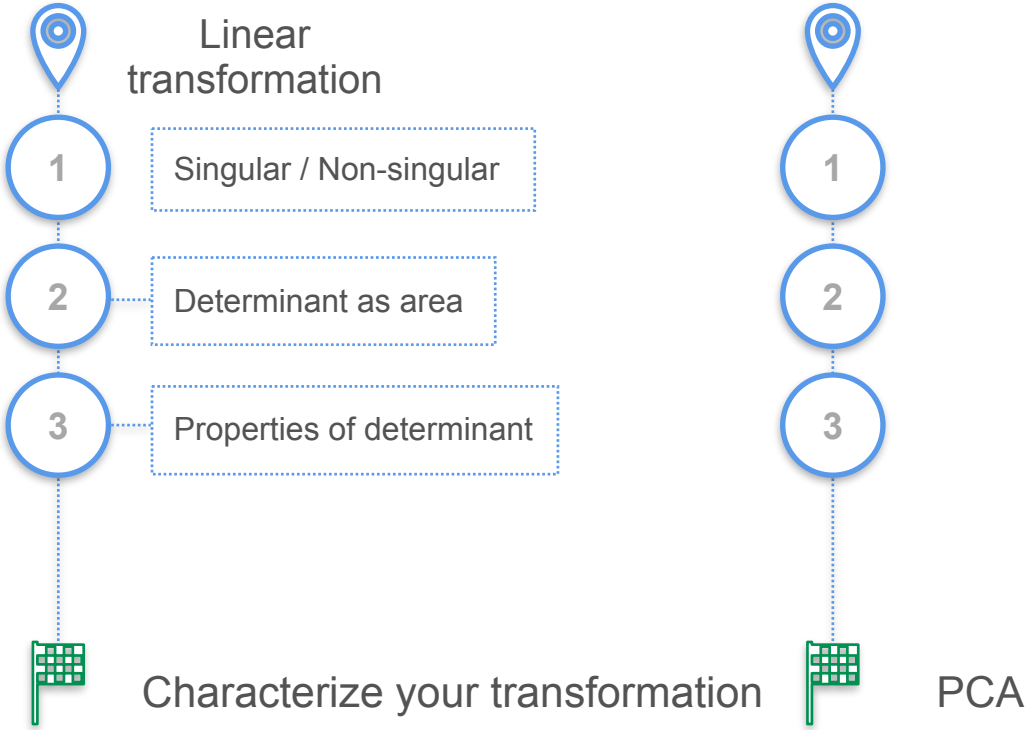
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

Det = 5 Det = 0.2 = $\frac{1}{5}$

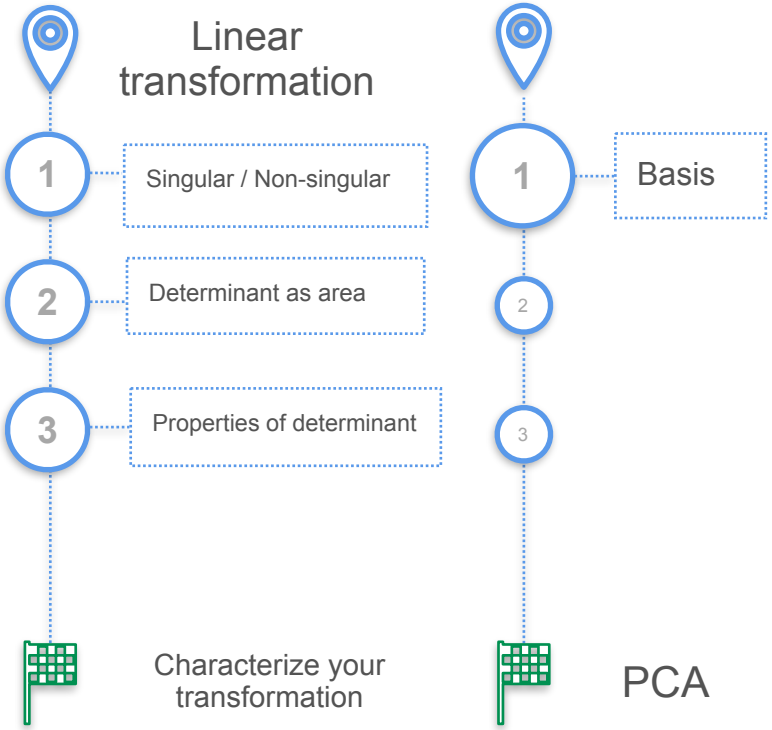
# What to expect?

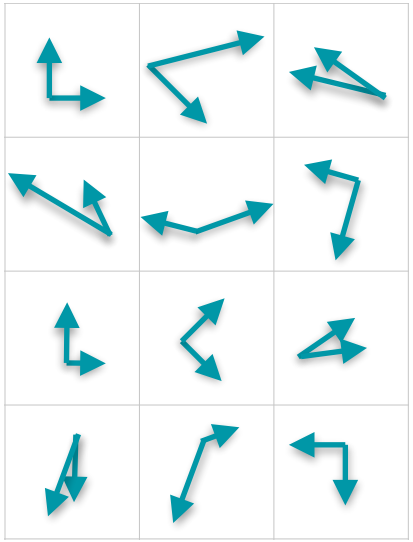


# What to expect?

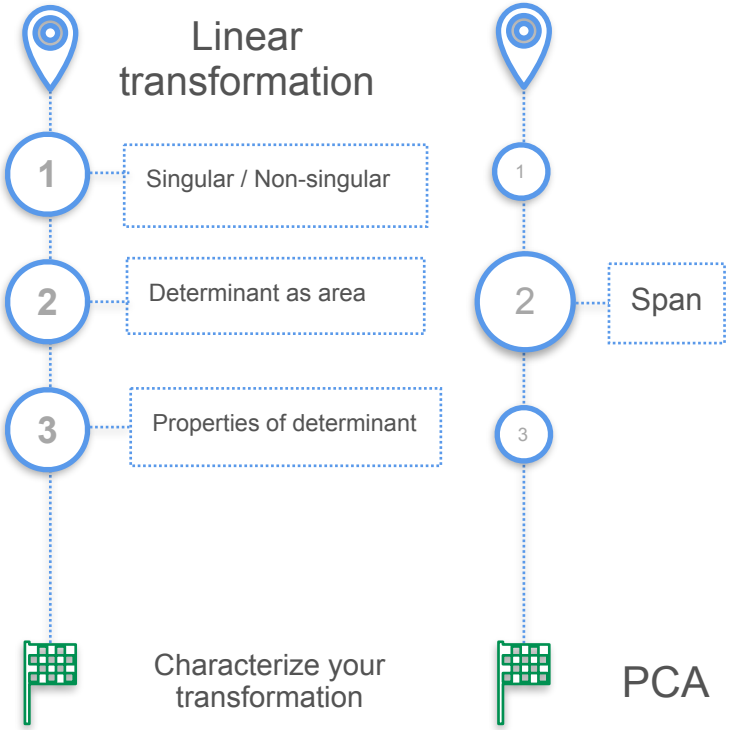


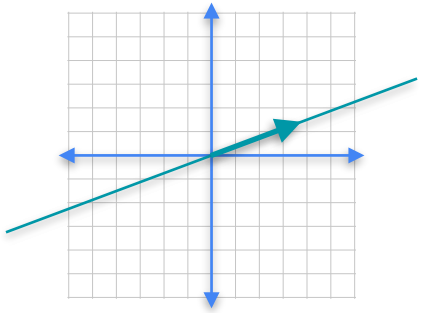
# What to expect?



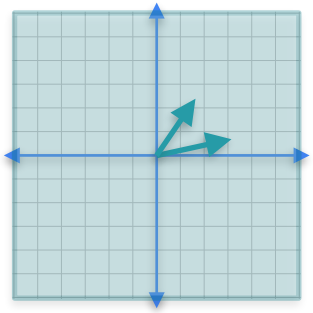



# What to expect?



1 element
Dimensions: 1



2 elements
Dimensions: 2

# What to expect?

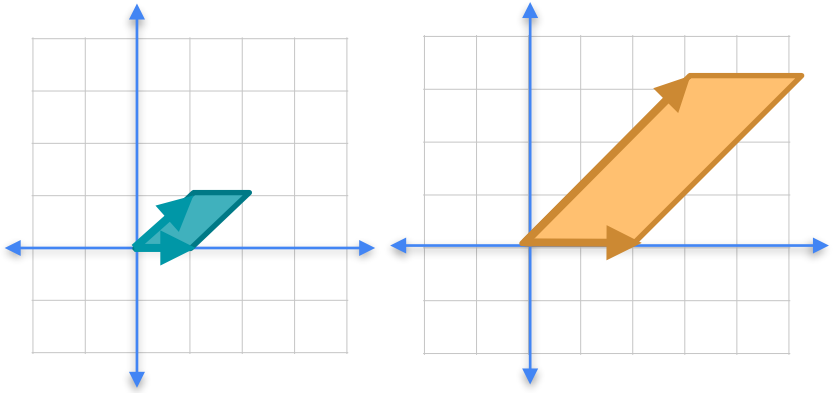
Linear transformation

1. Singular / Non-singular
2. Determinant as area
3. Properties of determinant

Characterize your transformation

1. Eigenvalues and Eigenvectors

PCA



| | | | | |
|---|---|---|---|---|
| 2 | 1 | 1 | = | 2 |
| 0 | 3 | 0 | | 0 |

(1,0) → (2,0)

$$A v_1 = \lambda_1 v_1$$

# What to expect?

Linear transformation

1. Singular / Non-singular
2. Determinant as area
3. Properties of determinant

Characterize your transformation

1. Basis
2. Span
3. Eigenvalues and Eigenvectors

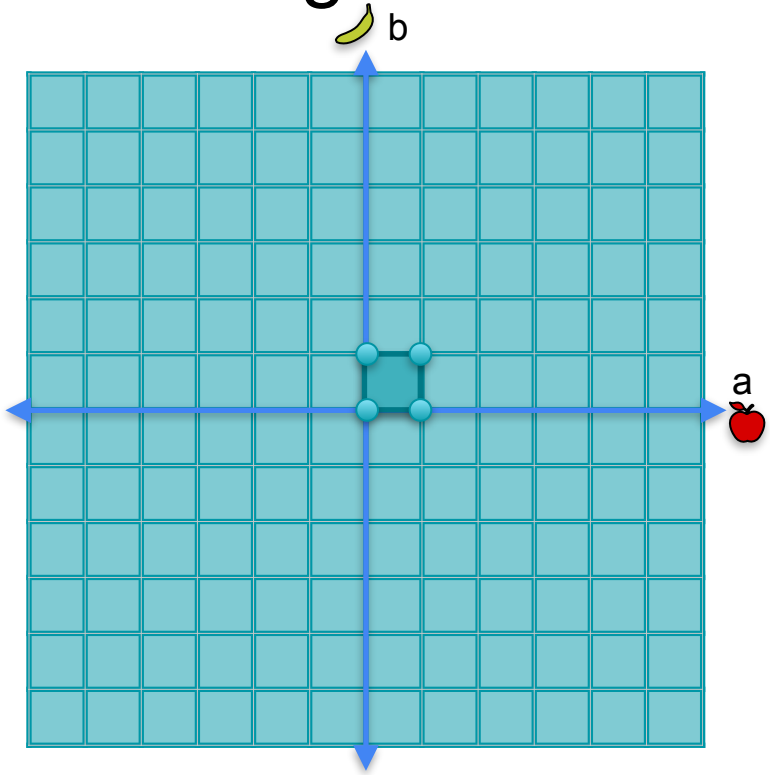
PCA

DeepLearning.AI

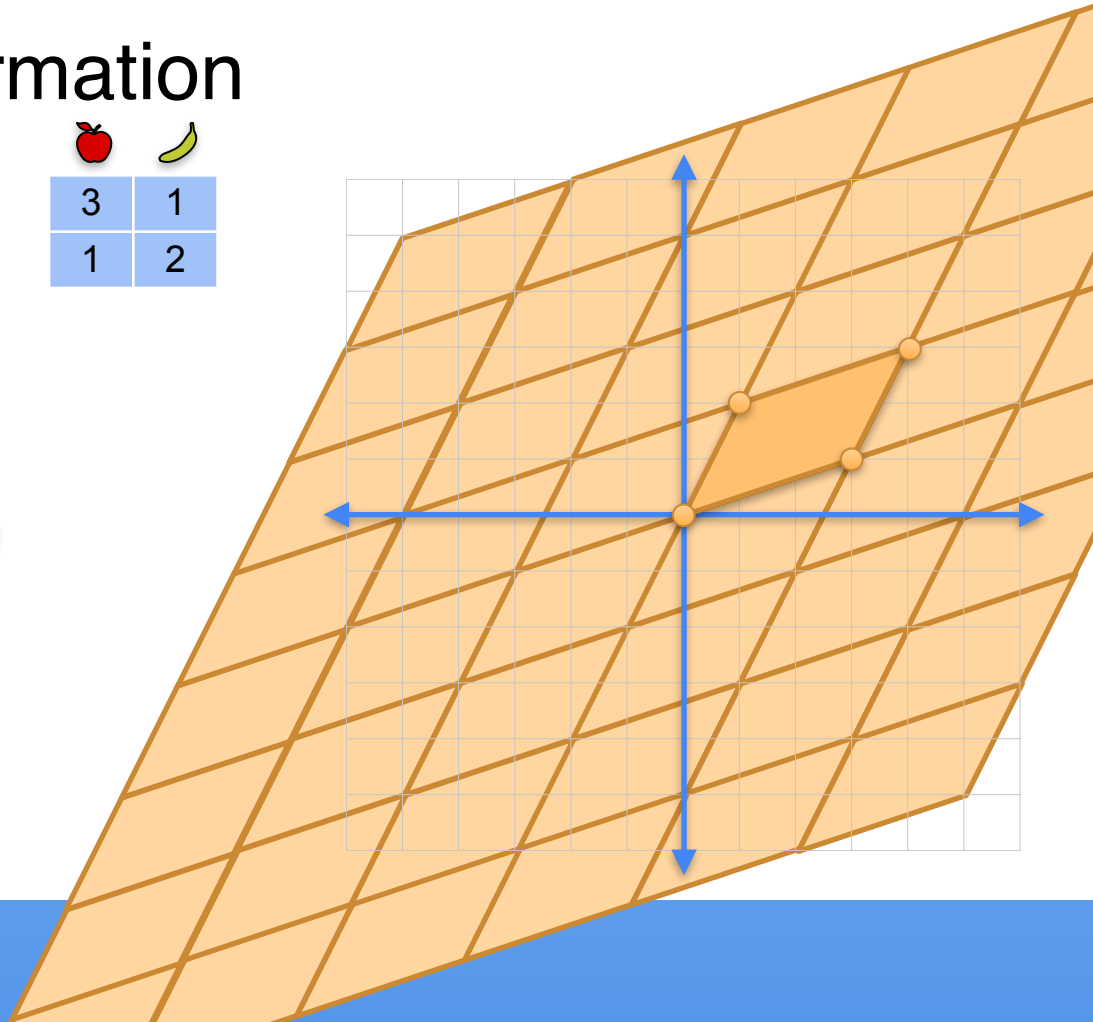
Determinants and Eigenvectors

# Singularity and rank of linear transformations

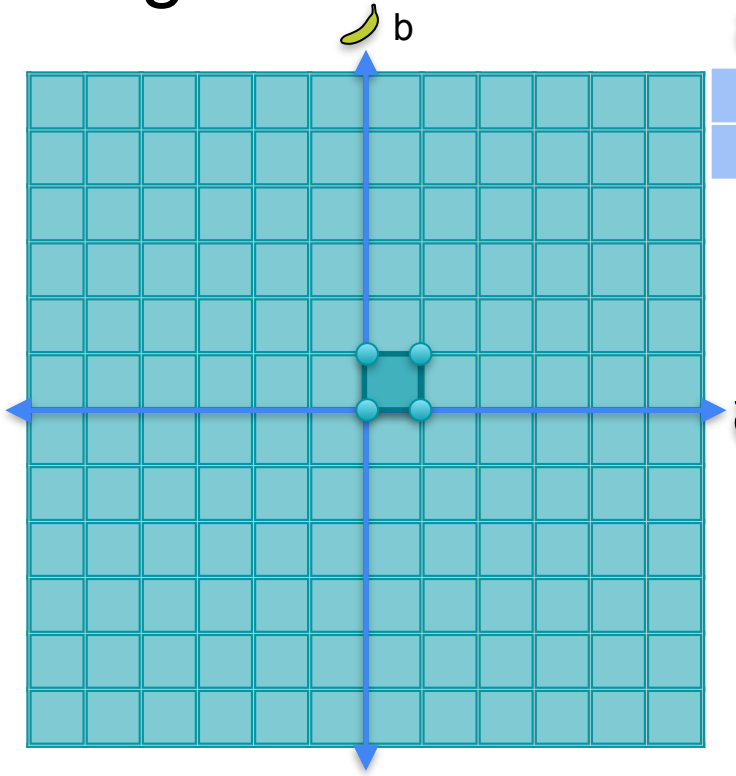
# Non-singular transformation




| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

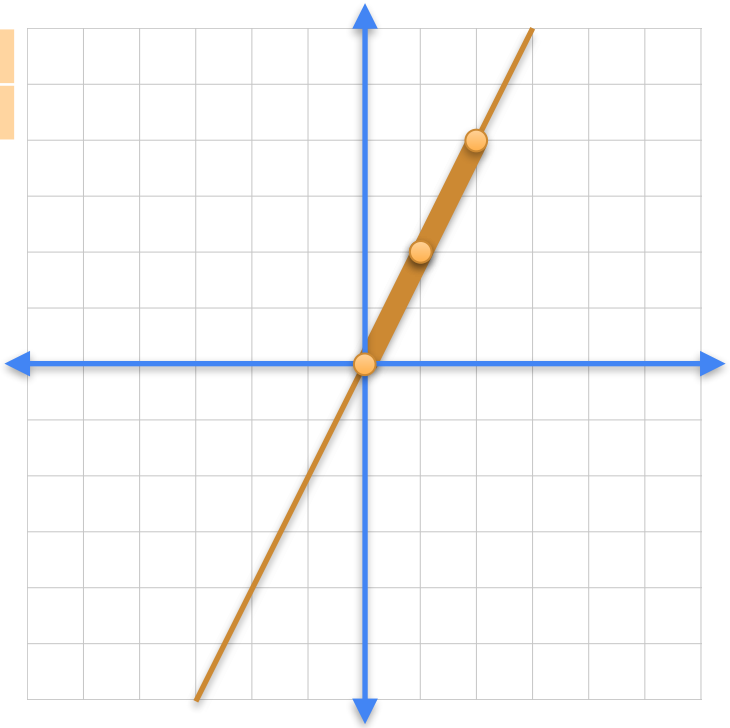


# Singular transformation

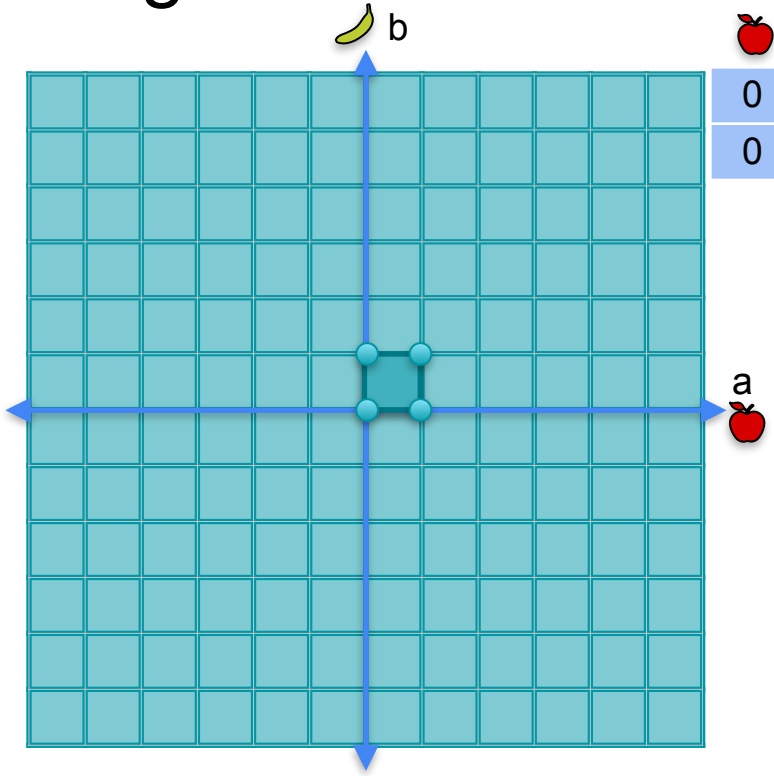



| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | = | 2 |
| 2 | 2 | 1 | | 4 |

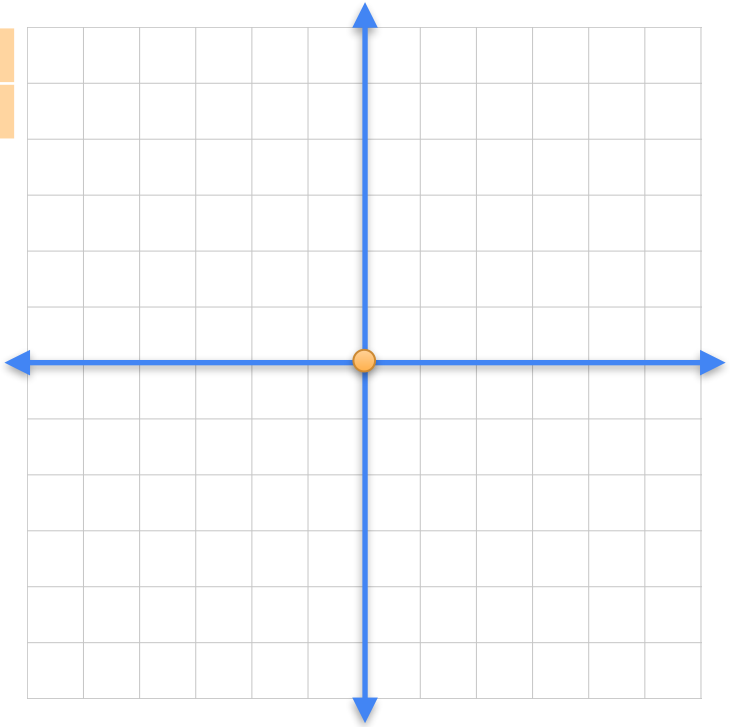
(0,0) → (0,0)
(1,0) → (1,2)
(0,1) → (1,2)
(1,1) → (2,4)



# Singular transformation




| | | | | |
|---|---|---|---|---|
| 0 | 0 | a | = | 0 |
| 0 | 0 | b | | 0 |



# Singular and non-singular transformations

**Non-singular**

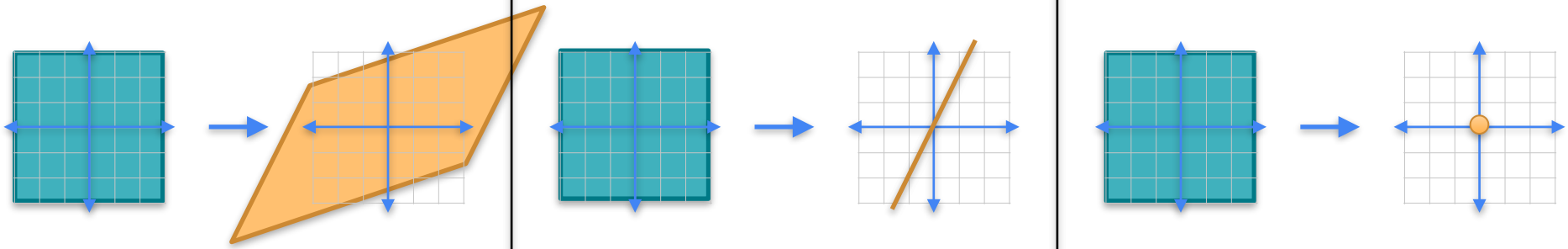
| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

**Singular**

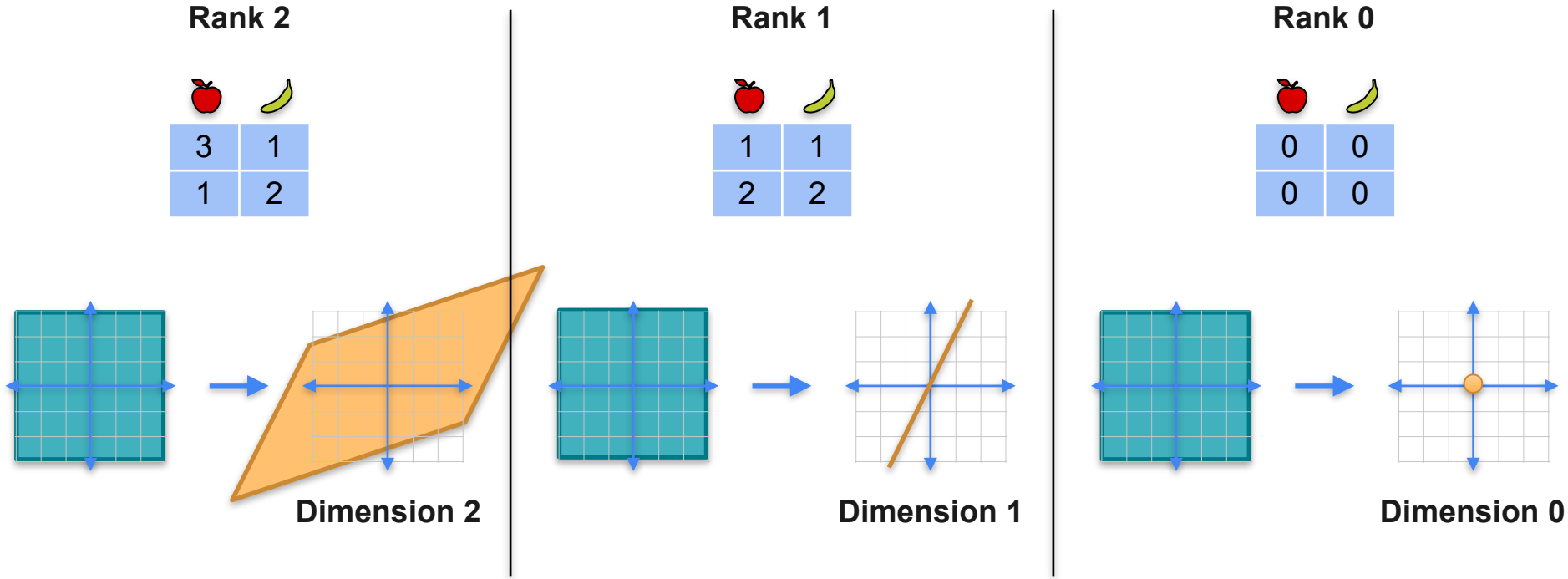
| | |
|---|---|
| 1 | 1 |
| 2 | 2 |

**Singular**

| | |
|---|---|
| 0 | 0 |
| 0 | 0 |



# Rank of linear transformations

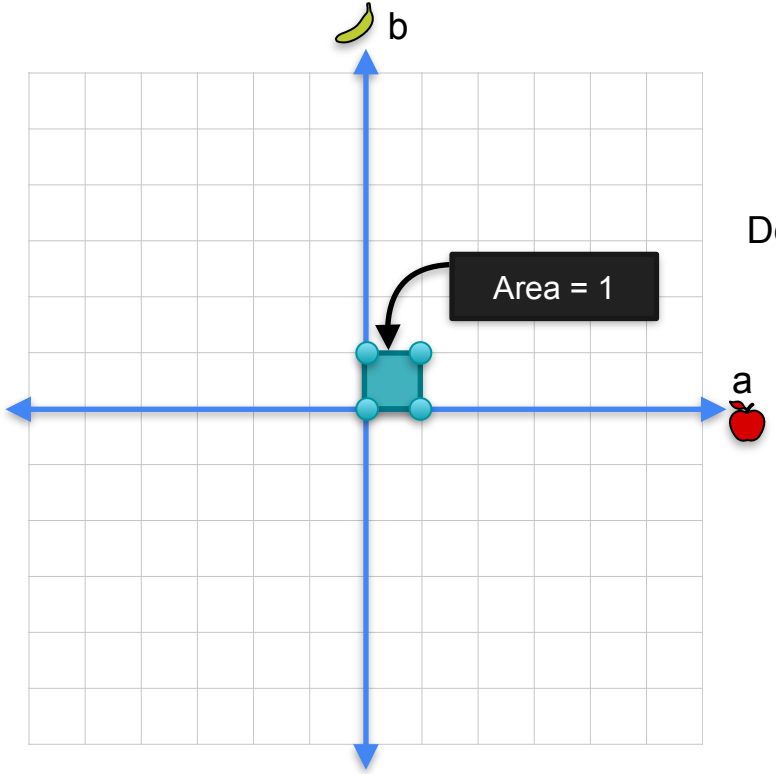


DeepLearning.AI

# Determinants and Eigenvectors

## Determinant as an area

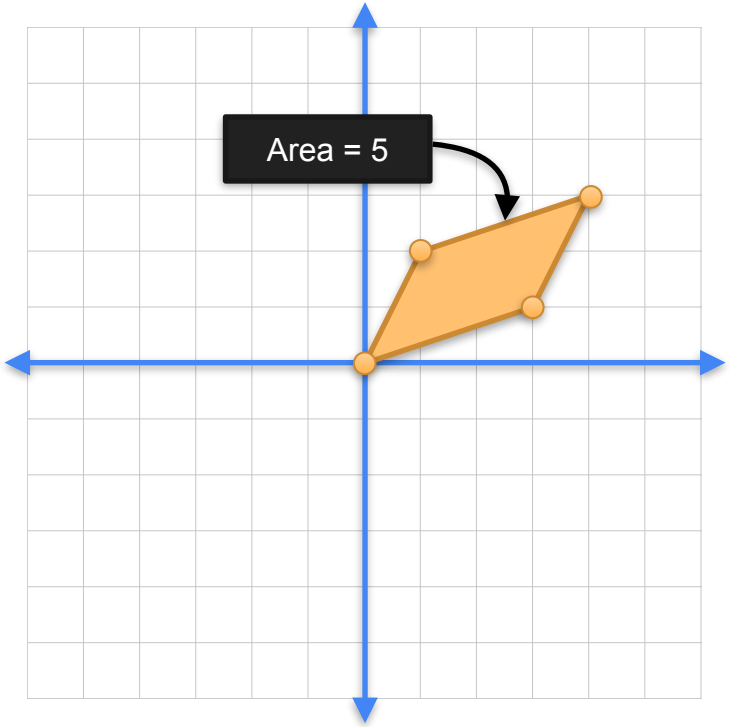
# Determinant as an area



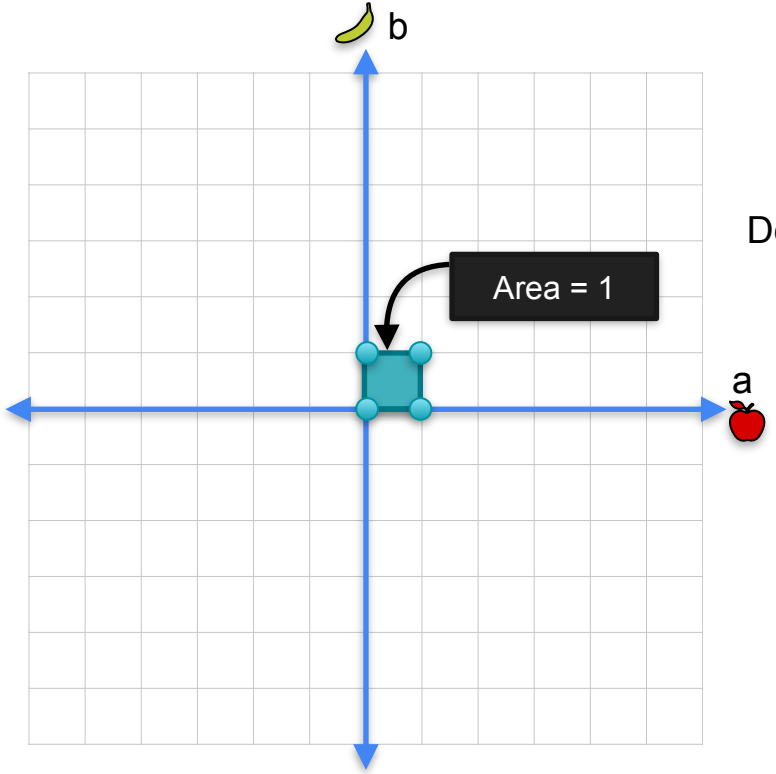

| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$



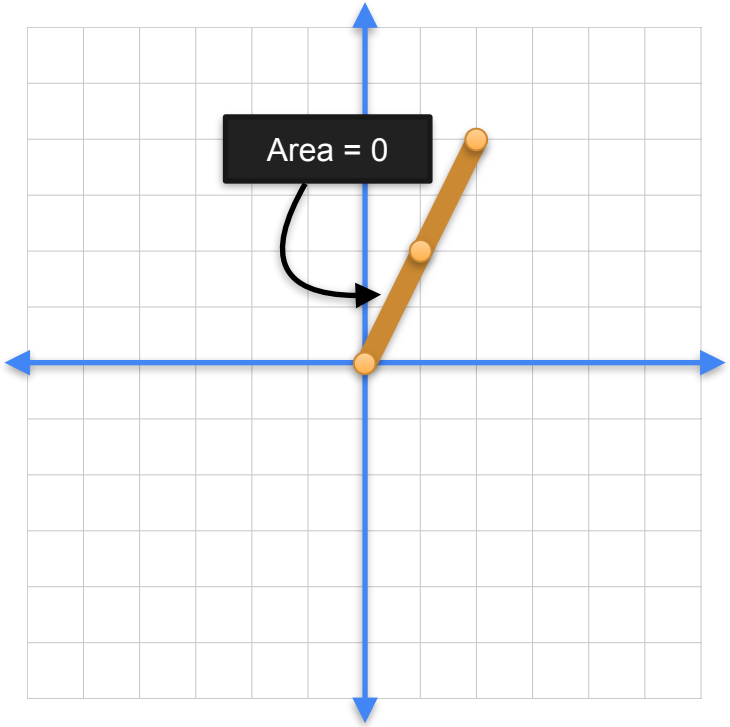
# Determinant as an area



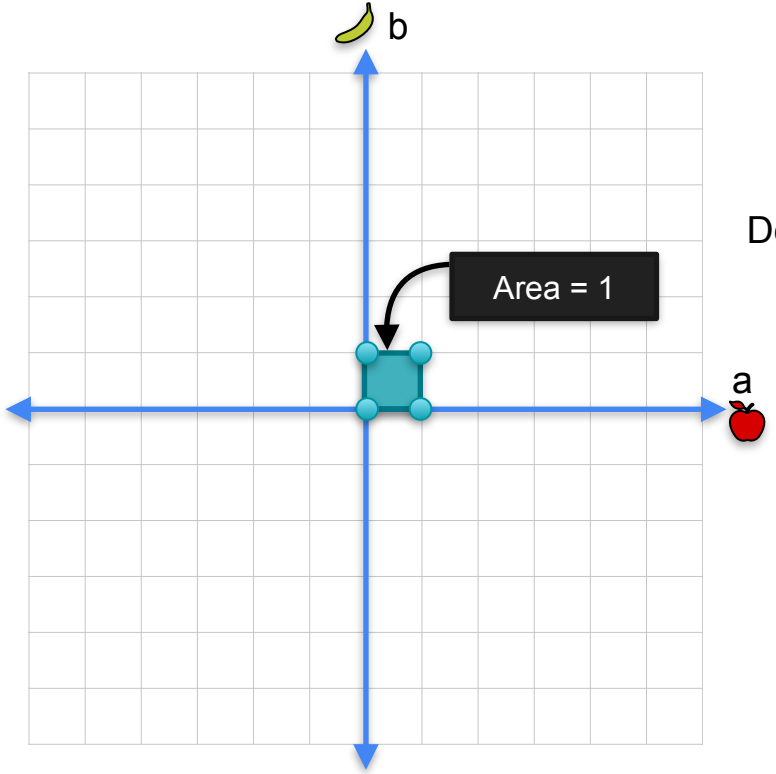

| | |
|---|---|
| 1 | 1 |
| 2 | 2 |

$$\text{Det} = 1 \cdot 2 - 1 \cdot 2$$

$$\text{Det} = 0$$



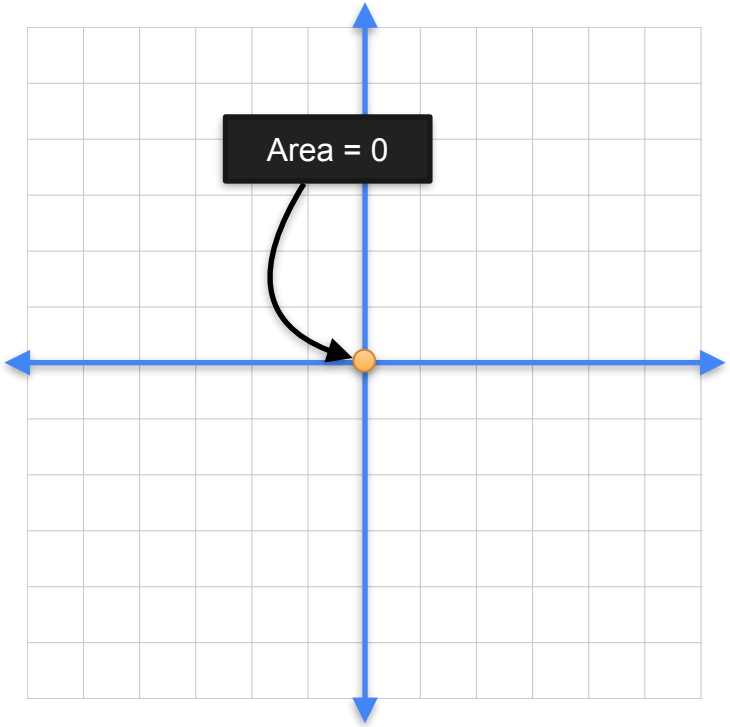
# Determinant as an area




| | |
|---|---|
| 0 | 0 |
| 0 | 0 |

$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$

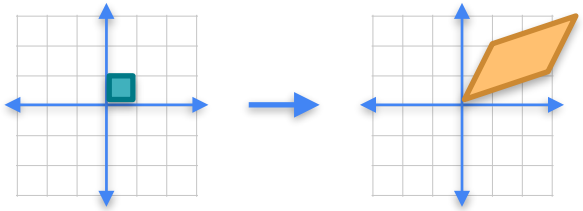


# Determinant as an area

**Non-singular**

| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

Determinant = 5

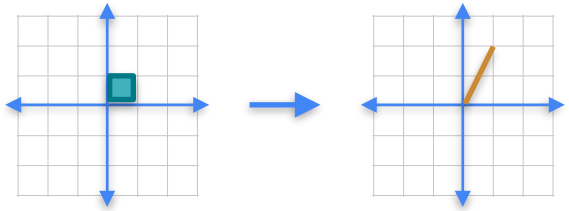


Area = 5

**Singular**

| | |
|---|---|
| 1 | 1 |
| 2 | 2 |

Determinant = 0

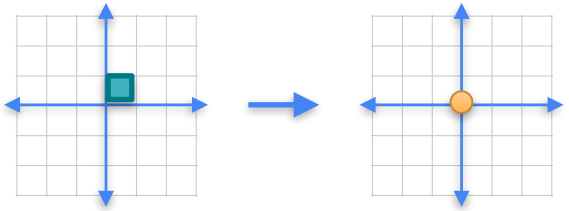


Area = 0

**Singular**

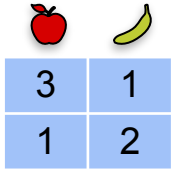
| | |
|---|---|
| 0 | 0 |
| 0 | 0 |

Determinant = 0



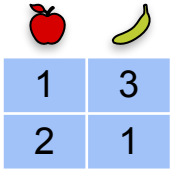
Area = 0

# Negative determinants?




Det = $3 \cdot 2 - 1 \cdot 1$

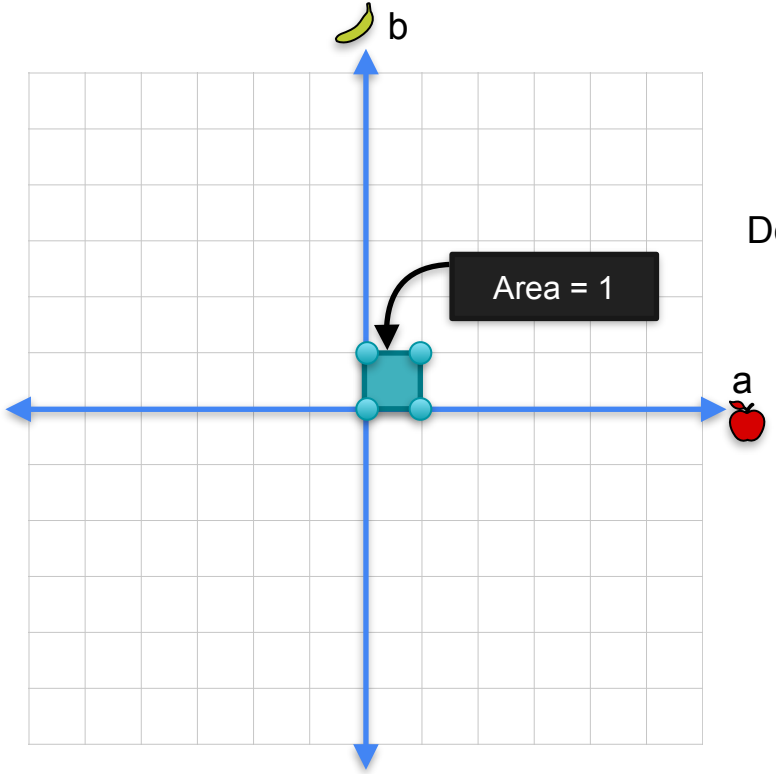
Det = $5$




Det = $1 \cdot 1 - 3 \cdot 2$

Det = $-5$

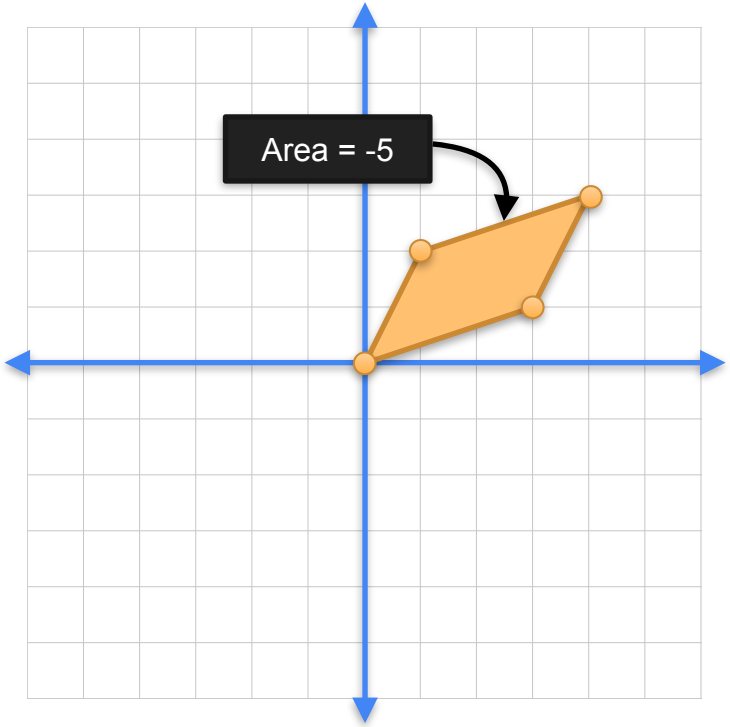
# Determinant as an area



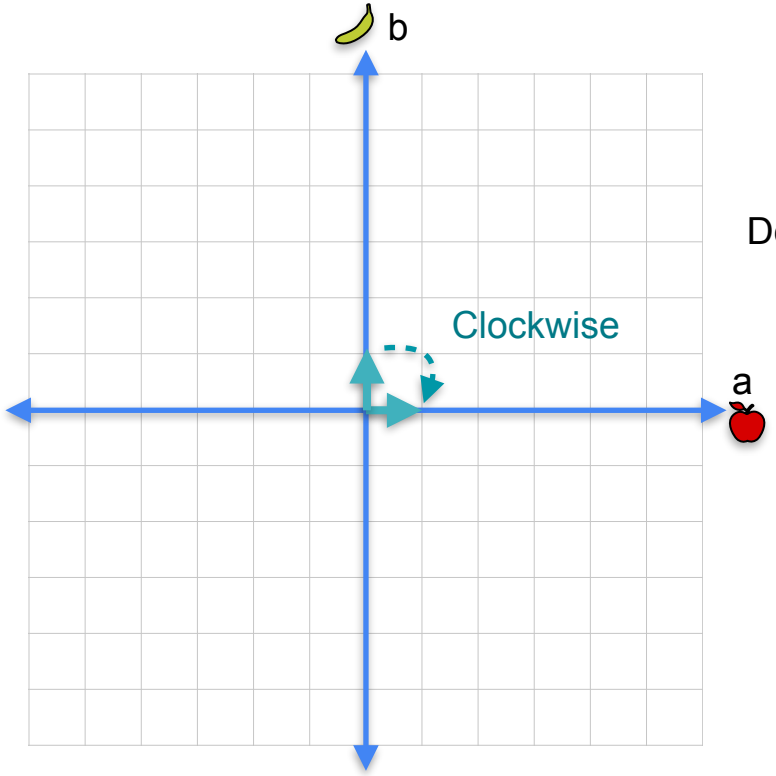

| | |
|---|---|
| 1 | 3 |
| 2 | 1 |

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



# Determinant as an area

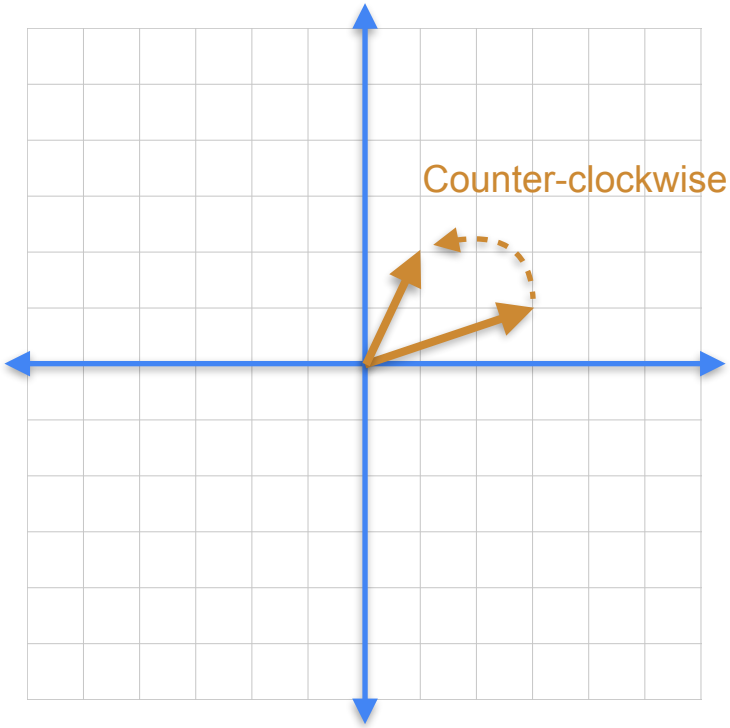



| | |
|---|---|
| 1 | 3 |
| 2 | 1 |

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

Negative

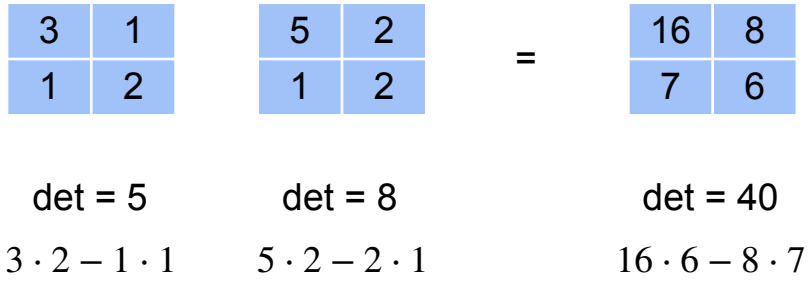


DeepLearning.AI

# Determinants and Eigenvectors

## Determinant of a product

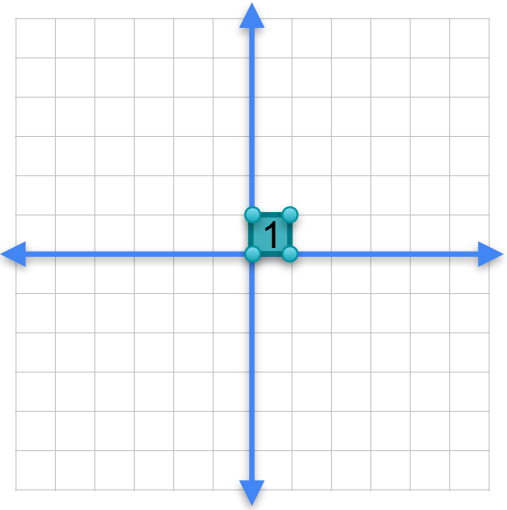
# Determinant of a product



# Determinant of a product

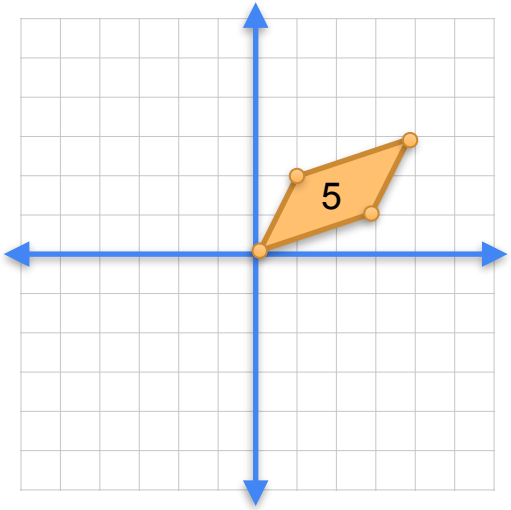
$$\det(AB) = \det(A)\det(B)$$

# Determinant of a product



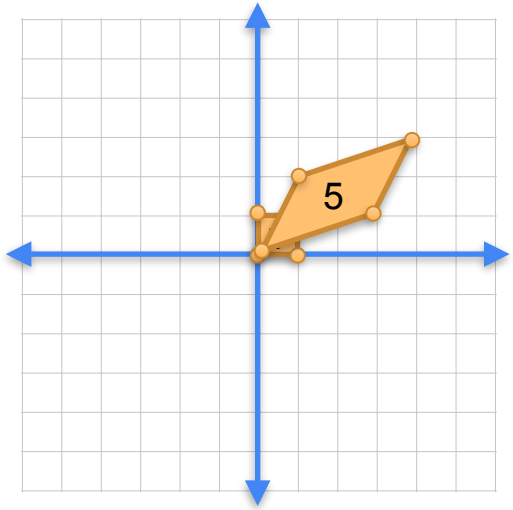

| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

Det = 5



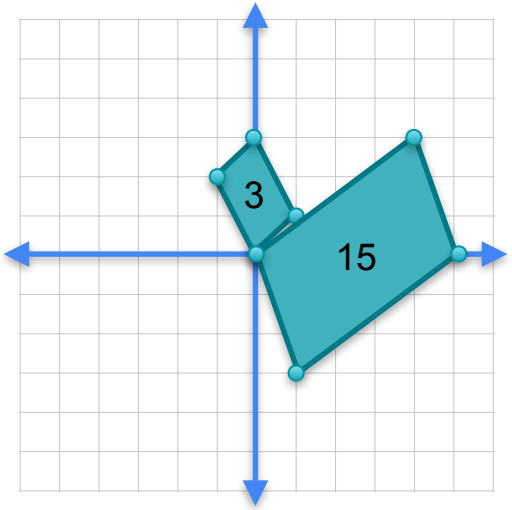

Area blows up by 5

# Determinant of a product



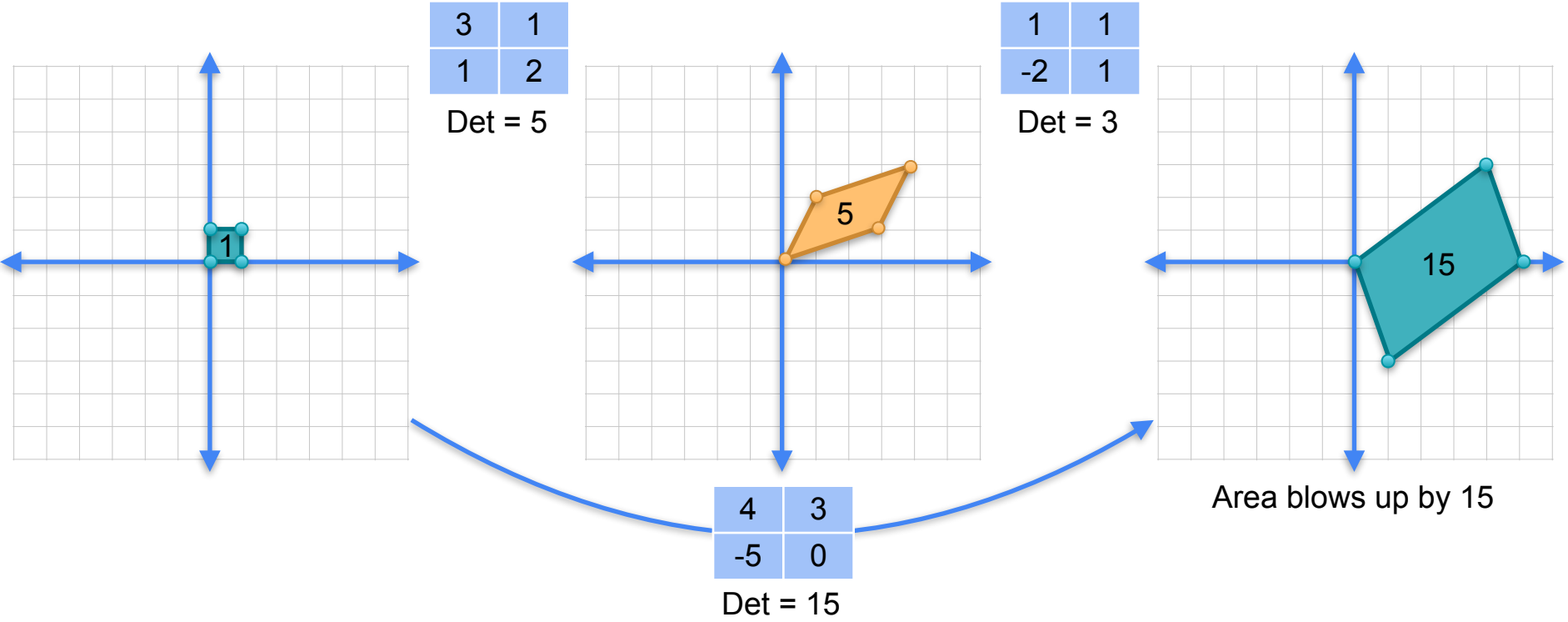

| | |
|---|---|
| 1 | 1 |
| -2 | 1 |

Det = 3




Area blows up by 3

# Determinant of a product



# Quiz

- The product of a singular and a non-singular matrix (in any order) is:
  - Singular
  - Non-singular
  - Could be either one

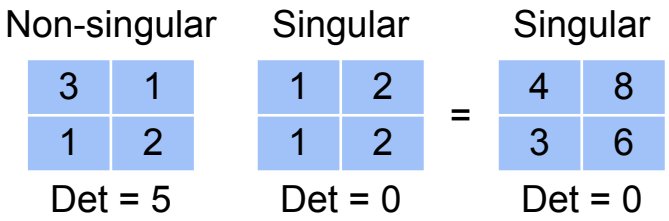
# Solution

- If A is non-singular and B is singular, then det(AB) = det(A) x det(B) = 0, since det(B) = 0. Therefore det(AB) = 0, so AB is **singular**.

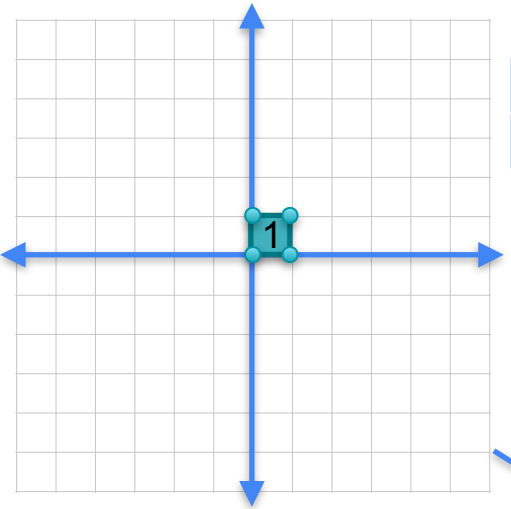
# When one factor is zero

$$5 \cdot 0 = 0$$

# When one factor is singular…

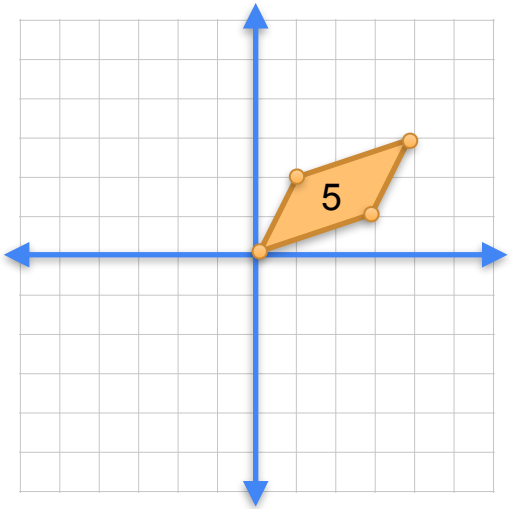


# If one factor is singular…



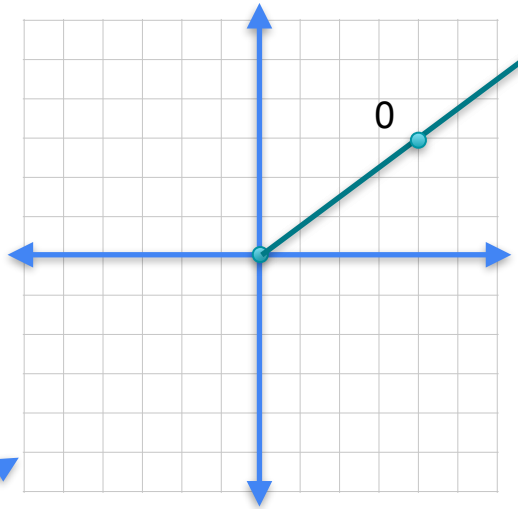

| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

Det = 5




| | |
|---|---|
| 1 | 2 |
| 1 | 2 |

Det = 0




Area blows up by 0

| | |
|---|---|
| 4 | 8 |
| 3 | 6 |

Det = 0

DeepLearning.AI

# Determinants and Eigenvectors

## Determinant of inverse

# Quiz

- Find the determinants of the following matrices

| | |
|---|---|
| 0.4 | -0.2 |
| -0.2 | 0.6 |

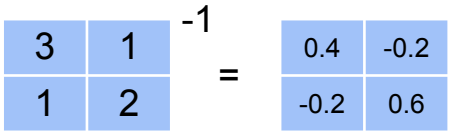
| | |
|---|---|
| 0.25 | -0.25 |
| -0.125 | 0.625 |

# Solution

Det $\begin{vmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{vmatrix}$ = (0.4)(0.6) - (-0.2)(-0.2) = 0.2

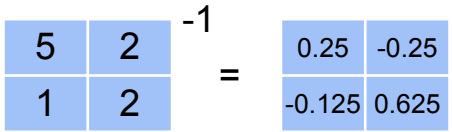
Det $\begin{vmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{vmatrix}$ = (0.25)(0.625) - (-0.125)(-0.25) = 0.125

# Determinant of an inverse



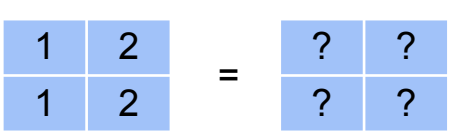

det = 5 det = 0.2

$$5^{-1} = 0.2$$




det = 8 det = 0.125

$$8^{-1} = 0.125$$




det = 0 det = ???

$$0^{-1} = ???$$

# Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

# Why?

Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(AA^{-1}) = \det(A)\det(A^{-1})$$

$$\det(I) = \det(A)\det(A^{-1})$$

$1$

$\frac{1}{\det(A)}$

# Determinant of the identity matrix

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\det(I) = 1$$

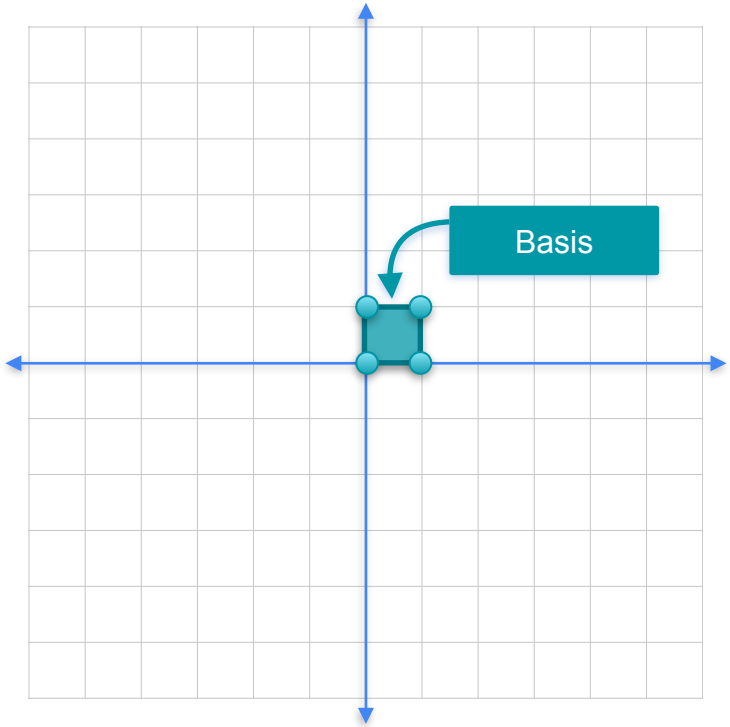
# W4 Lesson 2

DeepLearning.AI

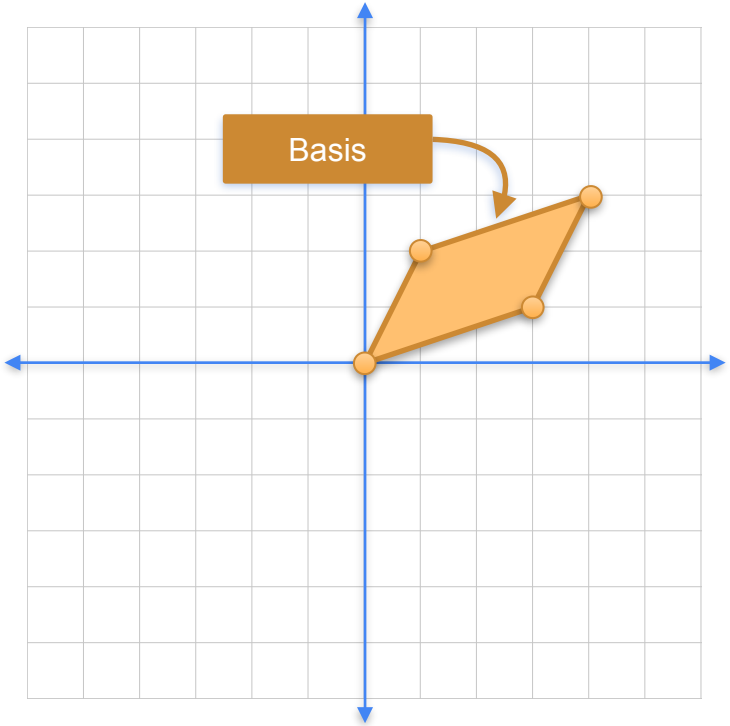
# Determinants and Eigenvectors

## Bases

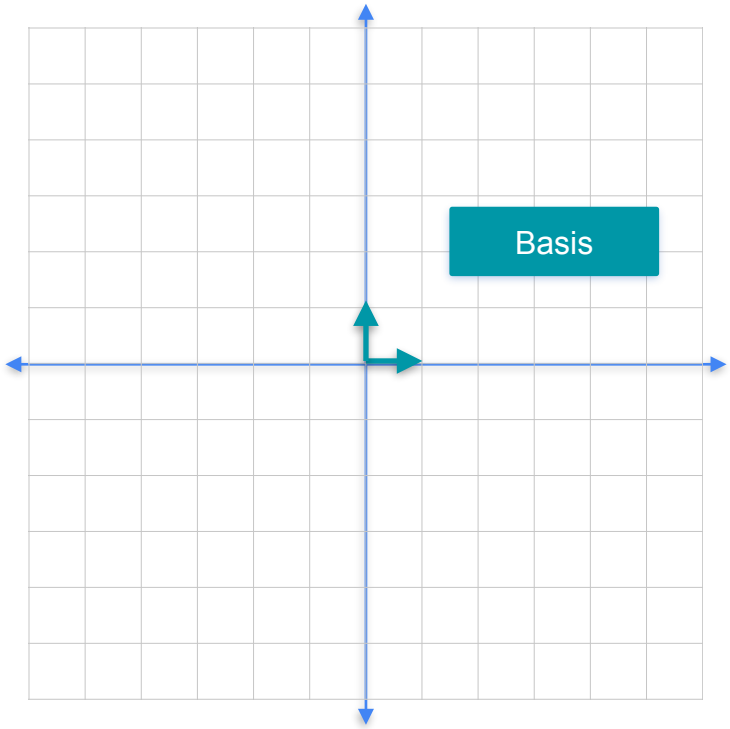
# Bases



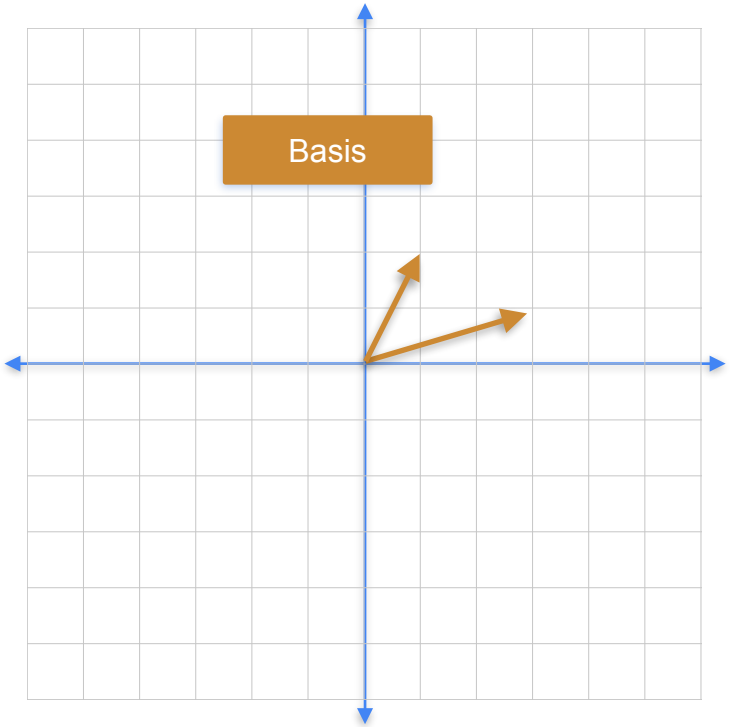

| | |
|---|---|
| 3 | 1 |
| 1 | 2 |



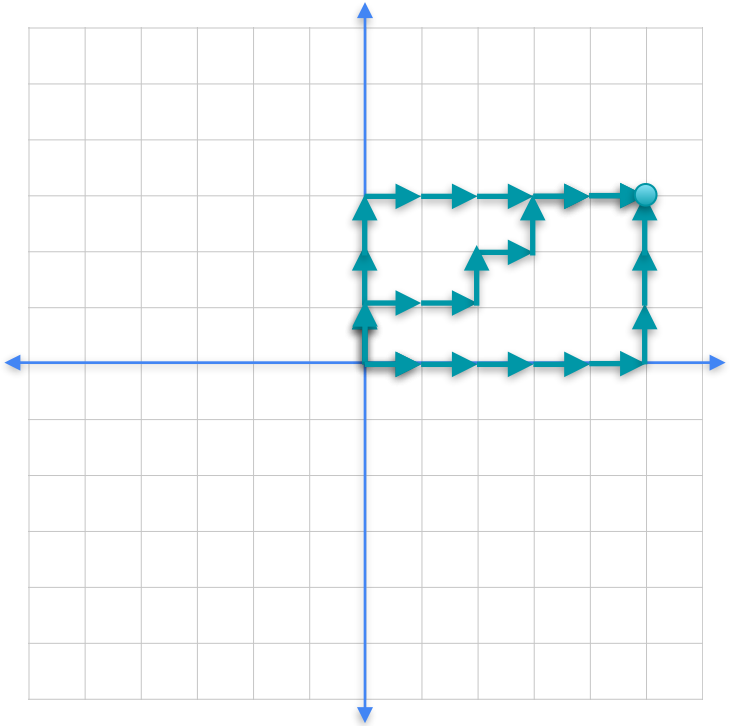
# Bases

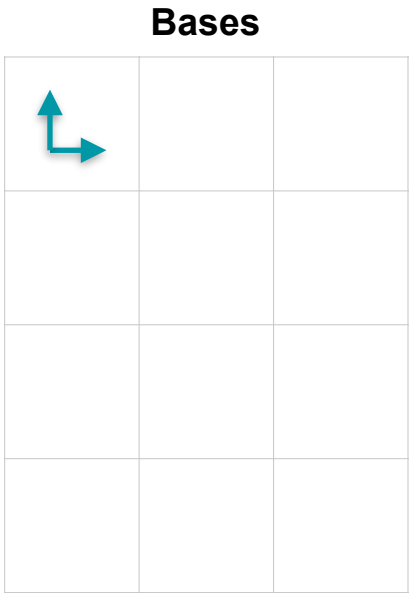



| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

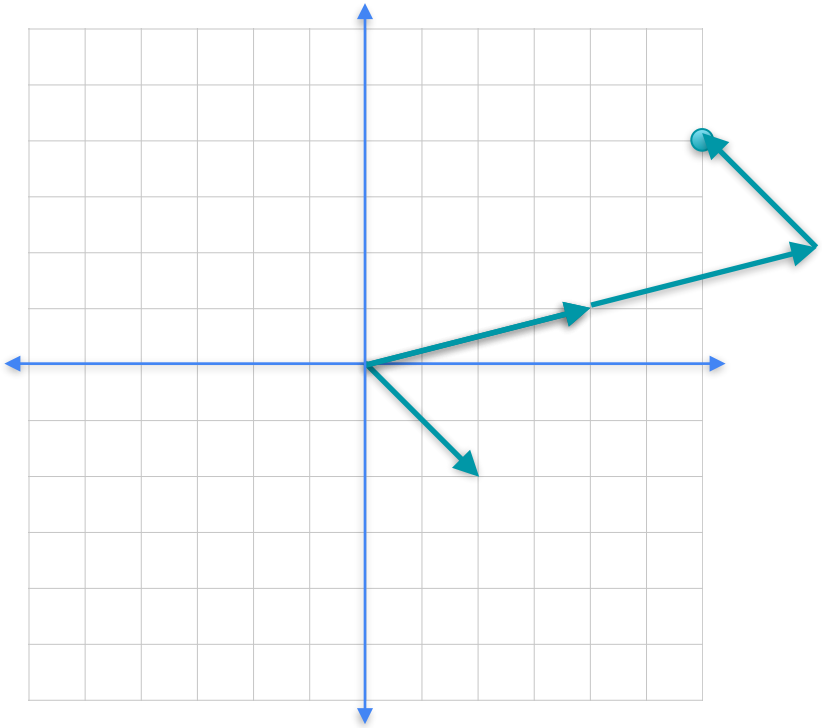


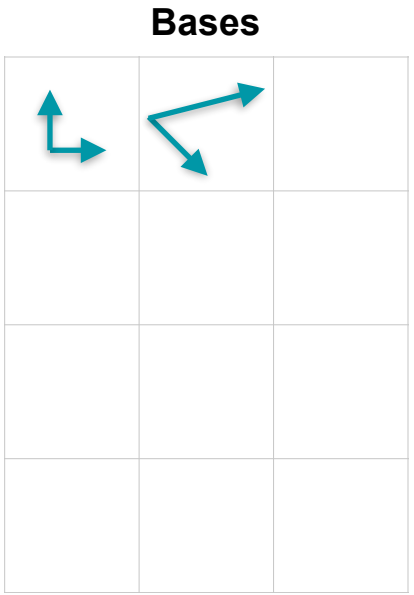
# Bases



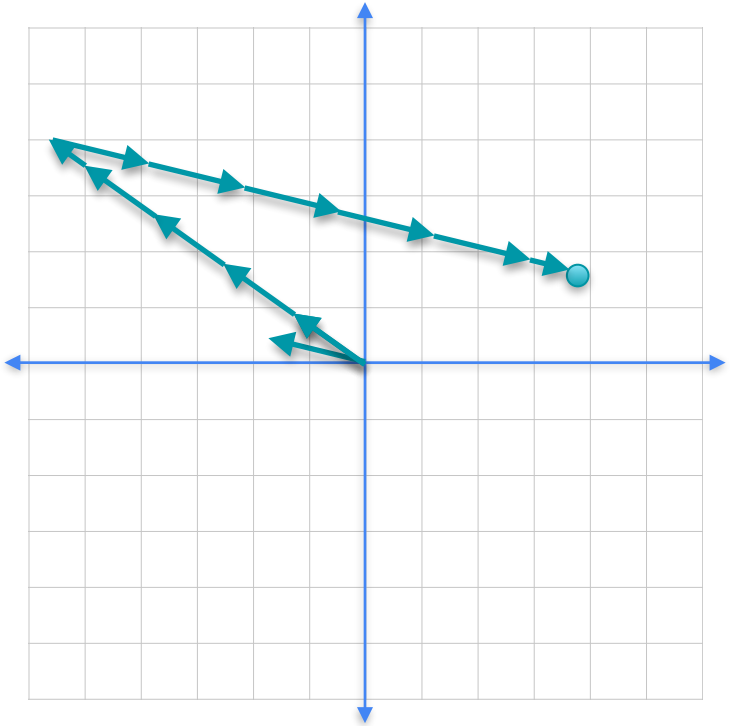


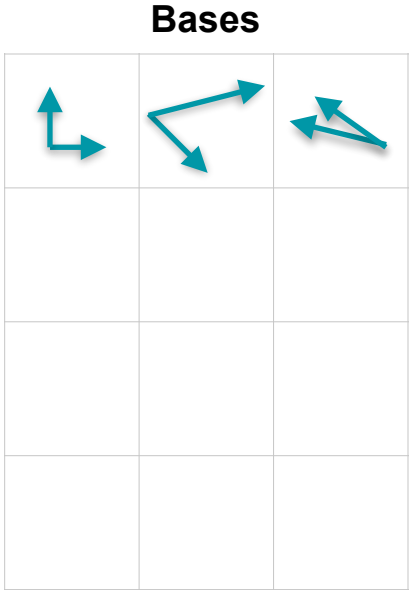
# Bases



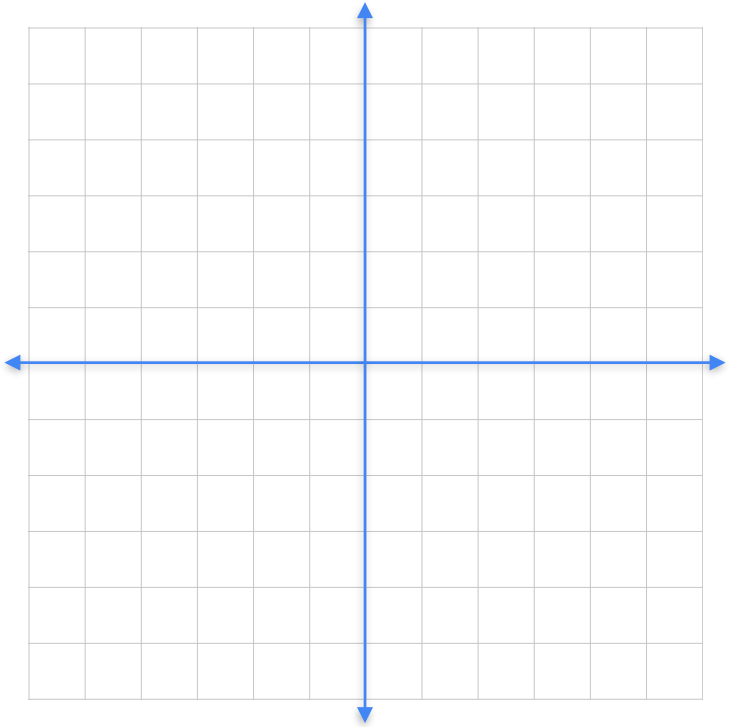


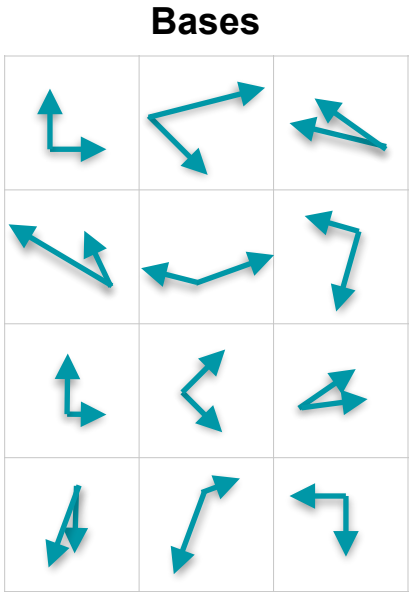
# Bases



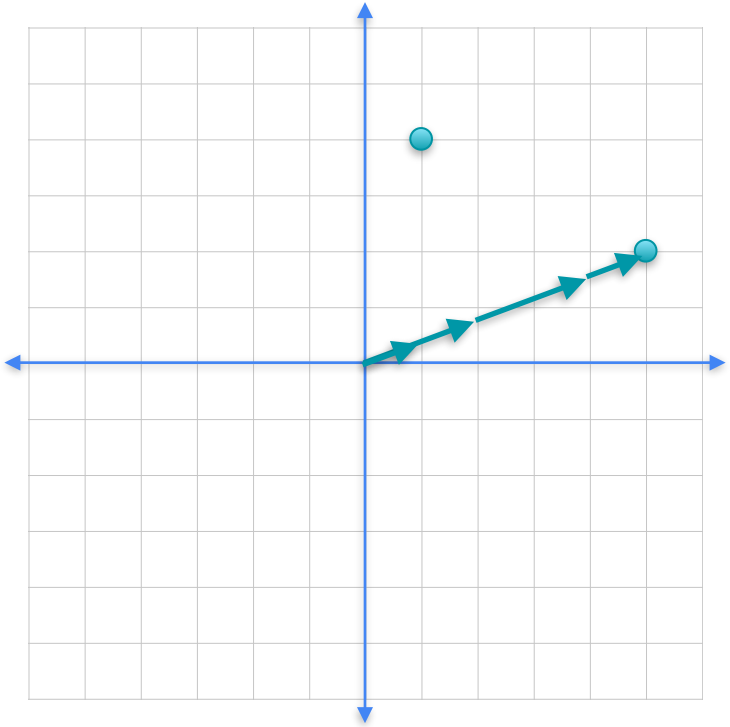


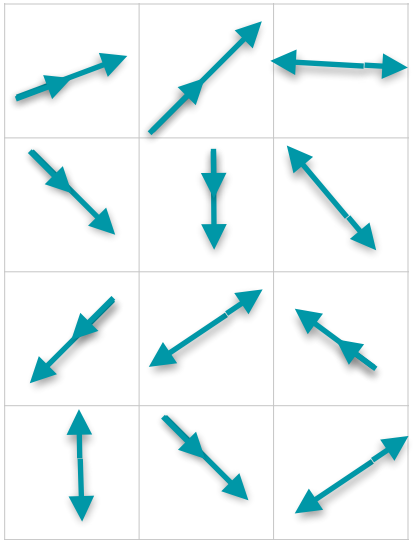
# Bases





# What is not a basis?



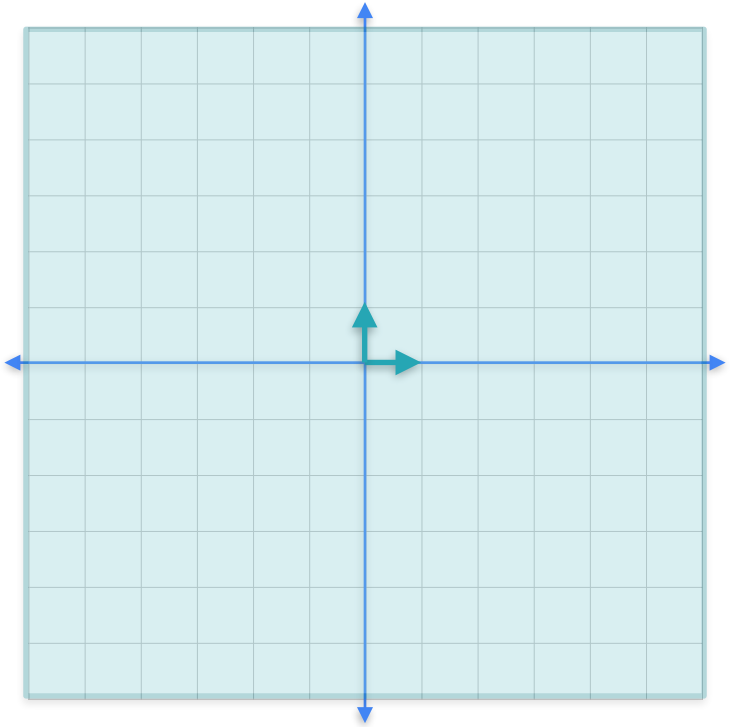


DeepLearning.AI

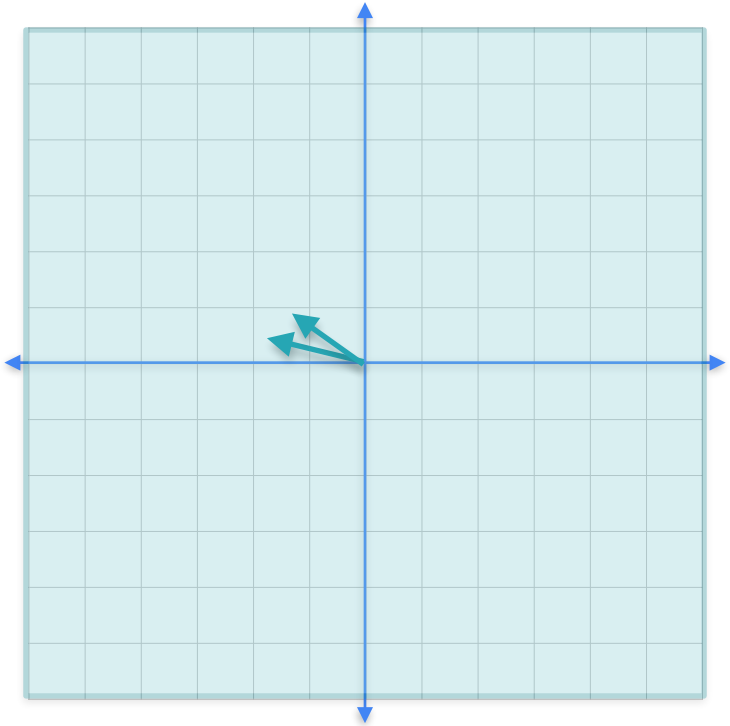
# Determinants and Eigenvectors

## Span

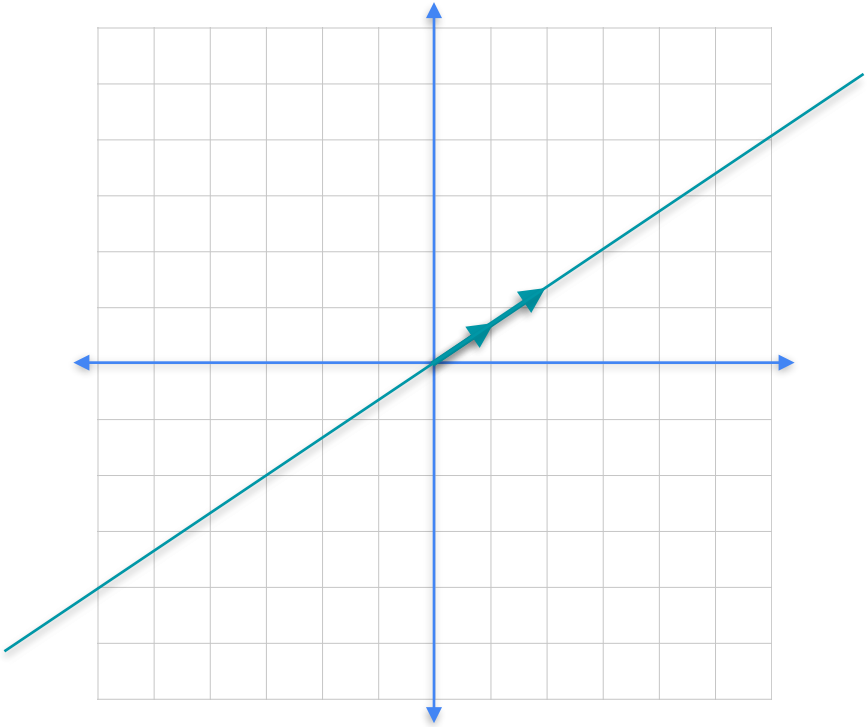
# Span



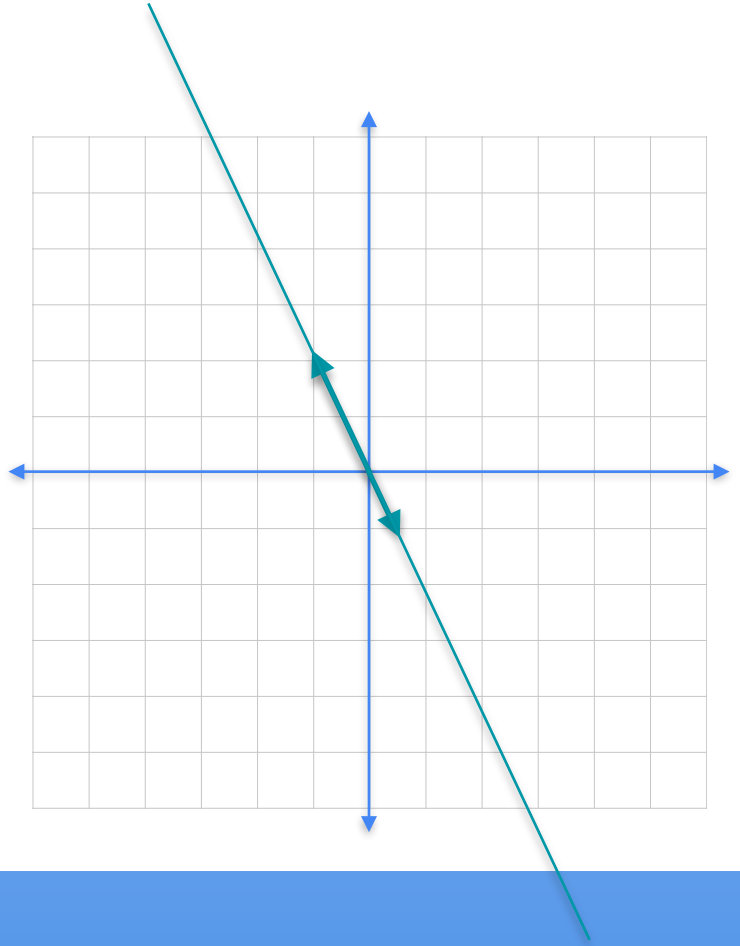
# Span



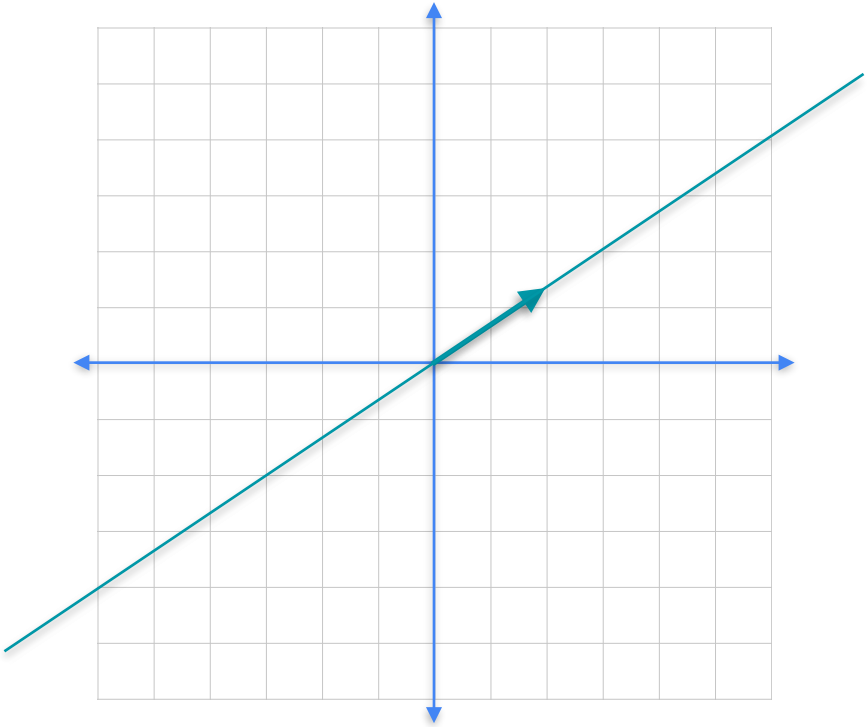
# Span



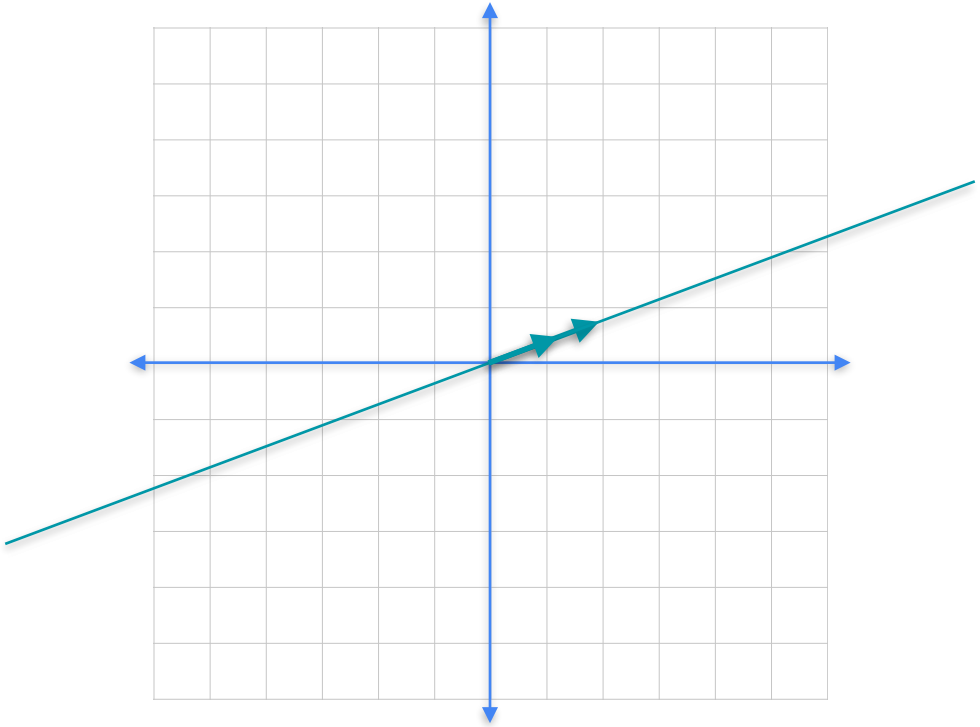
# Span



# Span

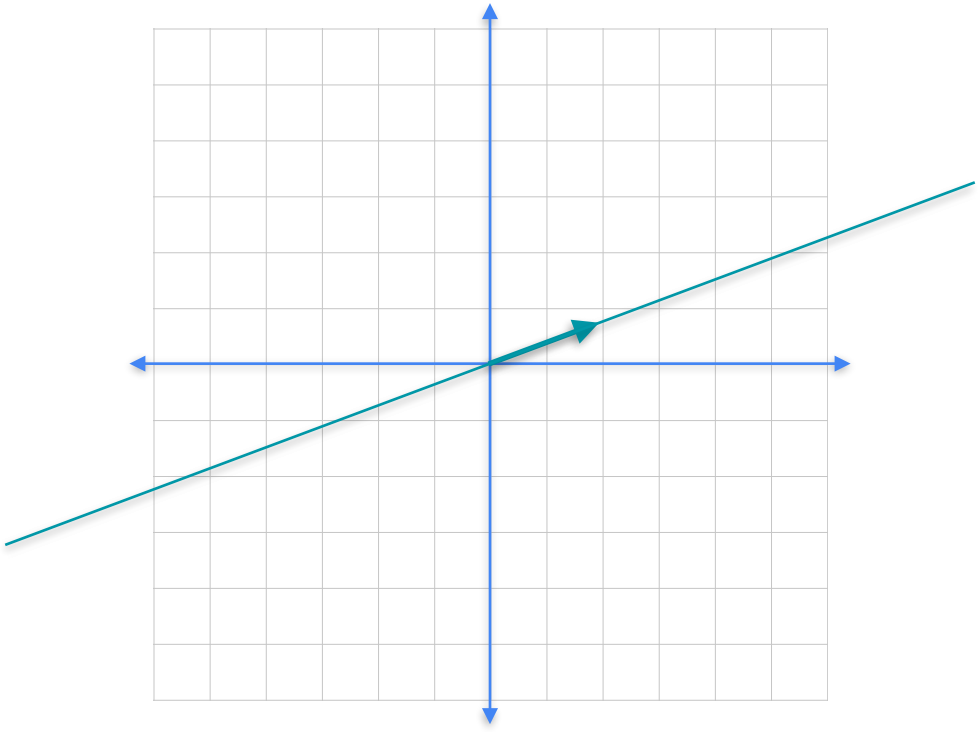


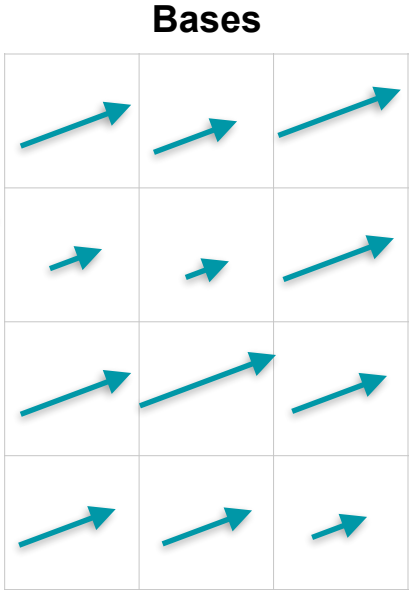
# Is this a basis?



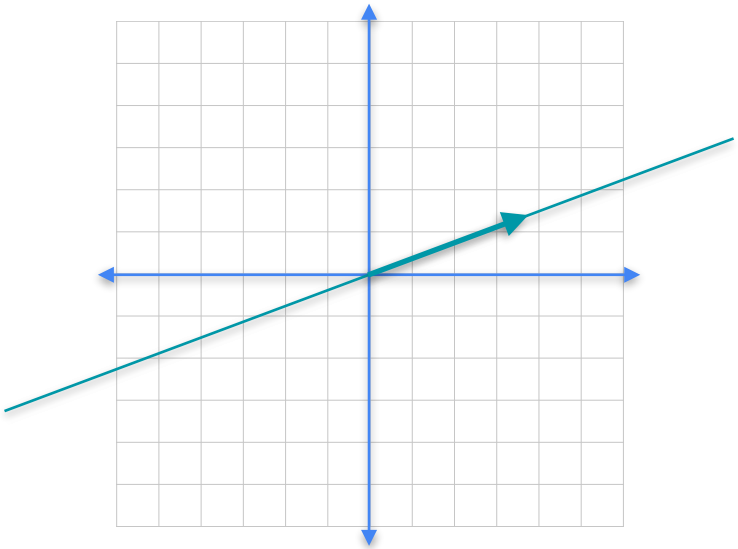
No

# Is this a basis for something?

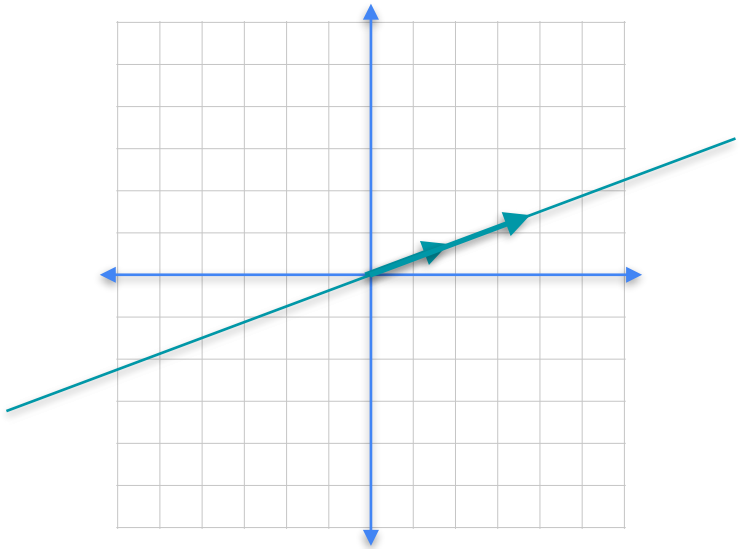




# A basis is a minimal spanning set

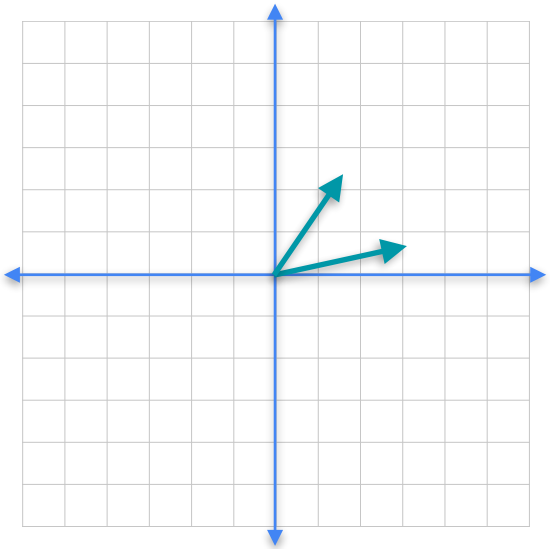


Basis

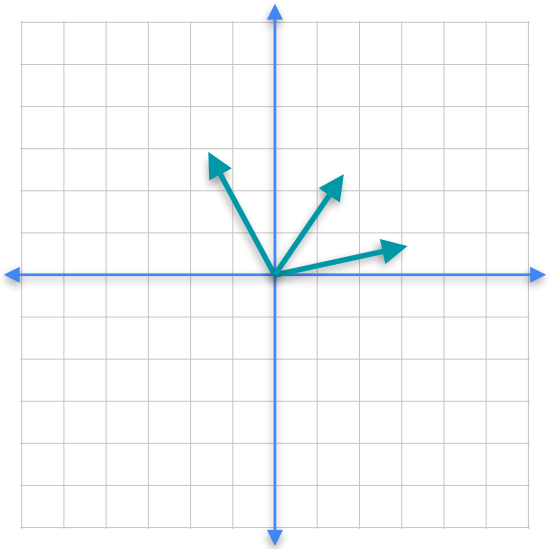


Not a basis

# A basis is a minimal spanning set

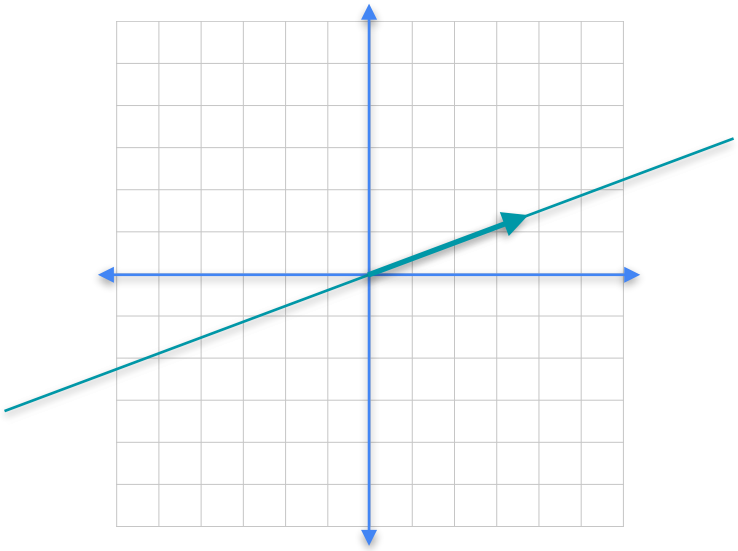


Basis

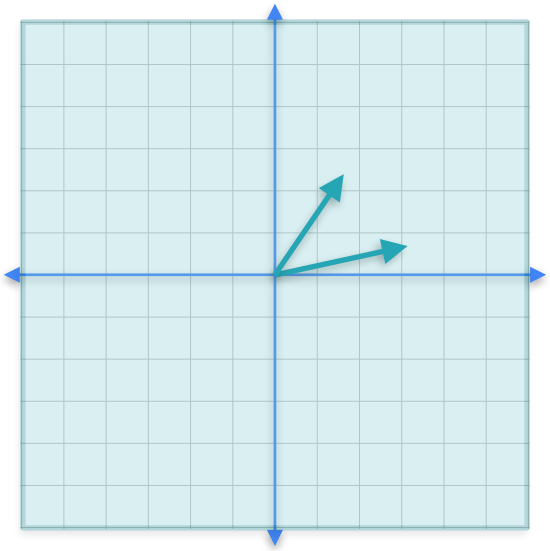


Not a basis

# Number of elements in the basis is the dimension

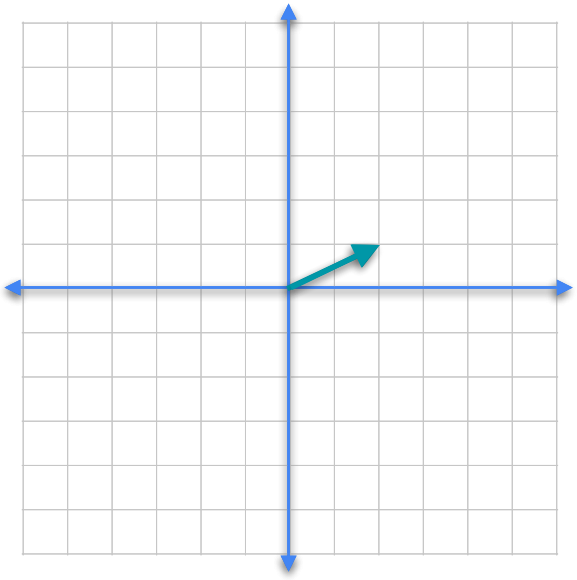


Dimensions: 1
1 element in the basis



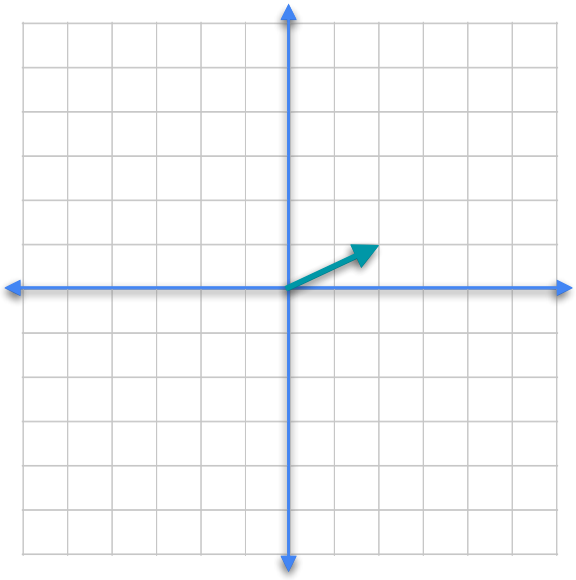
Dimensions: 2
2 elements in the basis

# Linearly independent and linearly dependent vectors

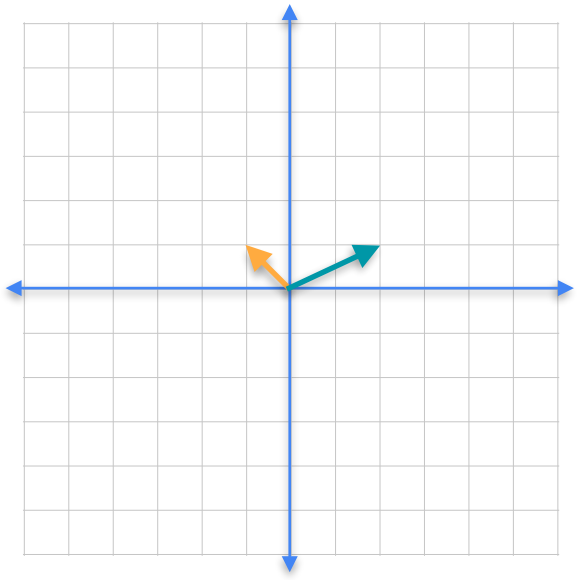


Linearly independent

# Linearly independent and linearly dependent vectors

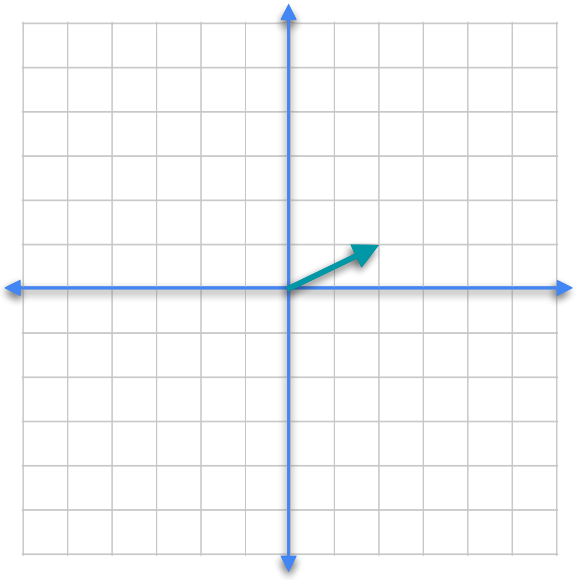


Linearly independent

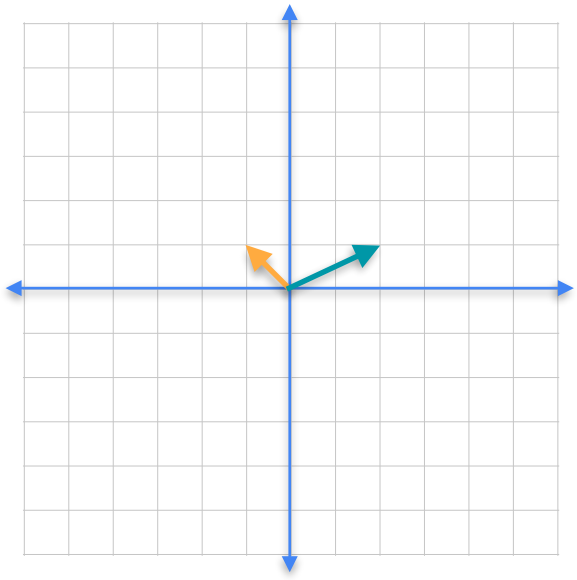


Linearly independent

# Linearly independent and linearly dependent vectors

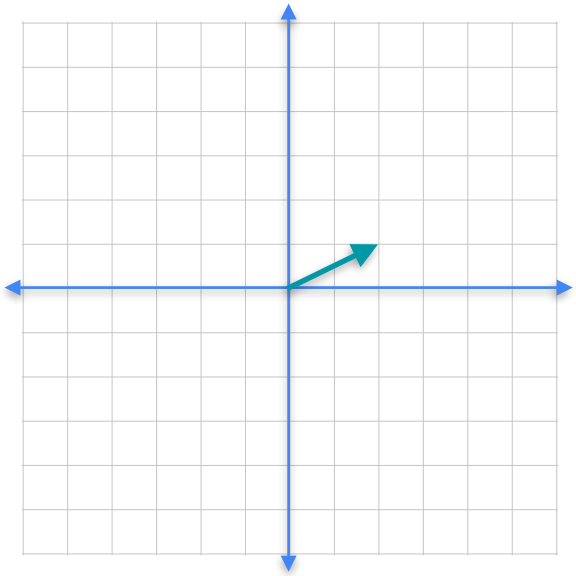


Linearly independent

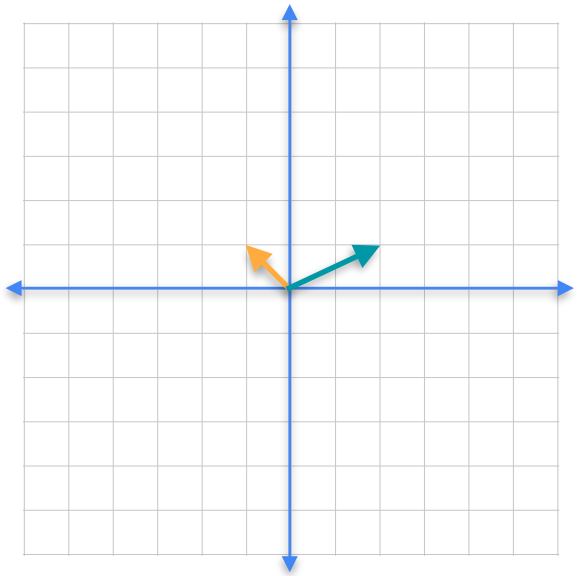


Linearly independent

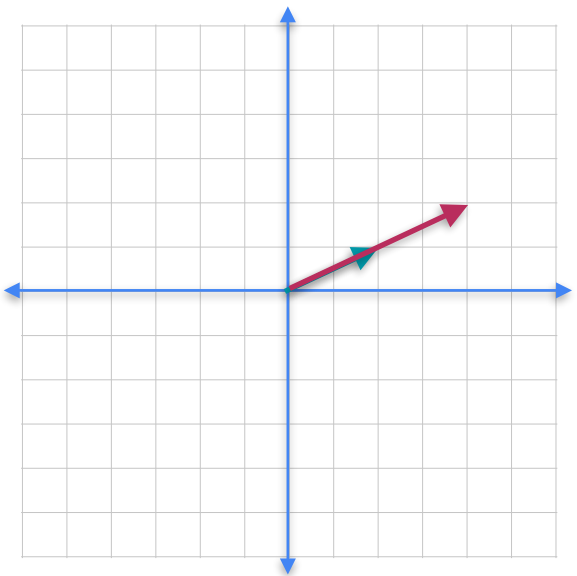
# Linearly independent and linearly dependent vectors



Linearly independent

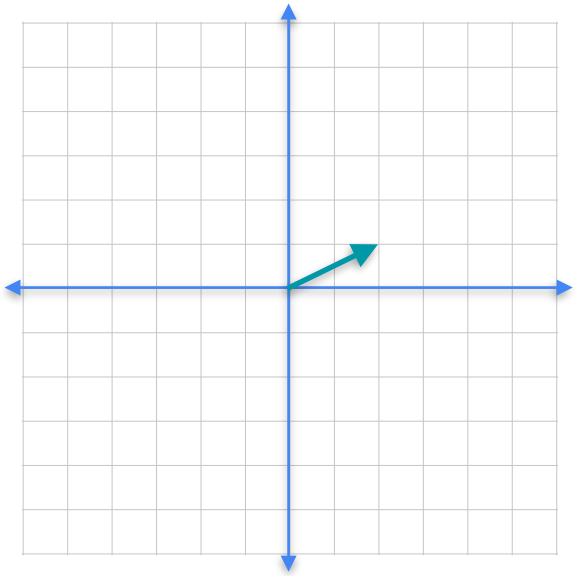


Linearly independent

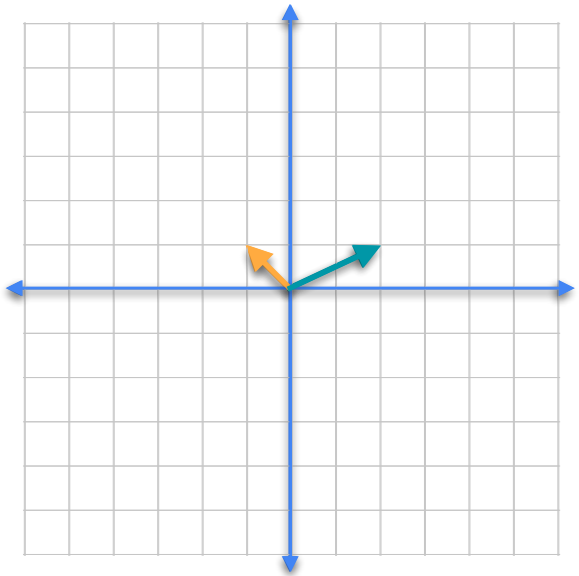


Linearly dependent

# Linearly independent and linearly dependent vectors

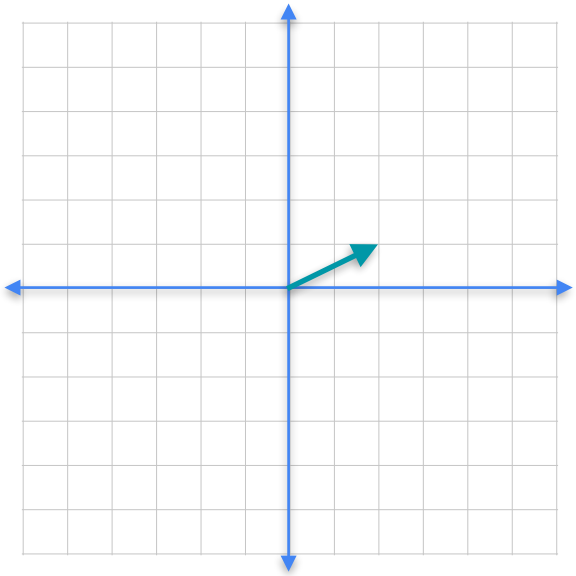


Linearly independent

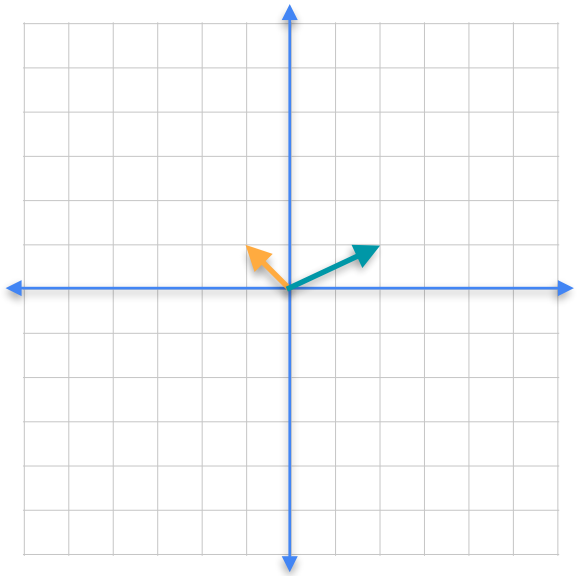


Linearly independent

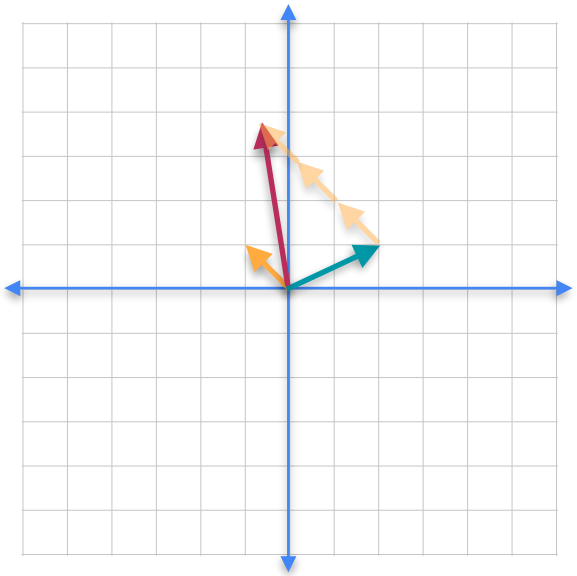
# Linearly independent and linearly dependent vectors



Linearly independent

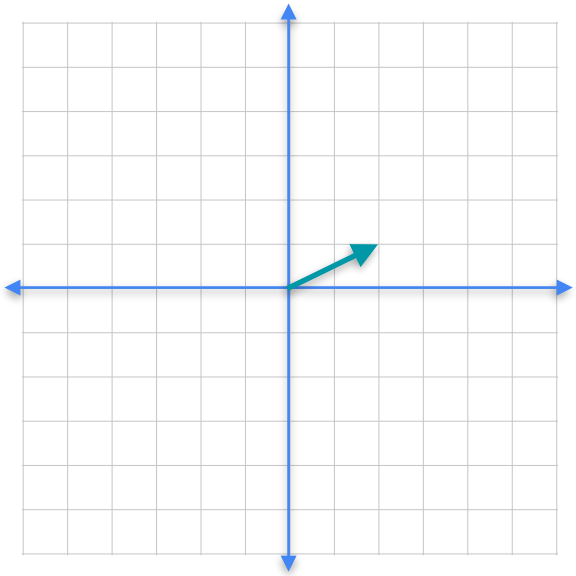


Linearly independent

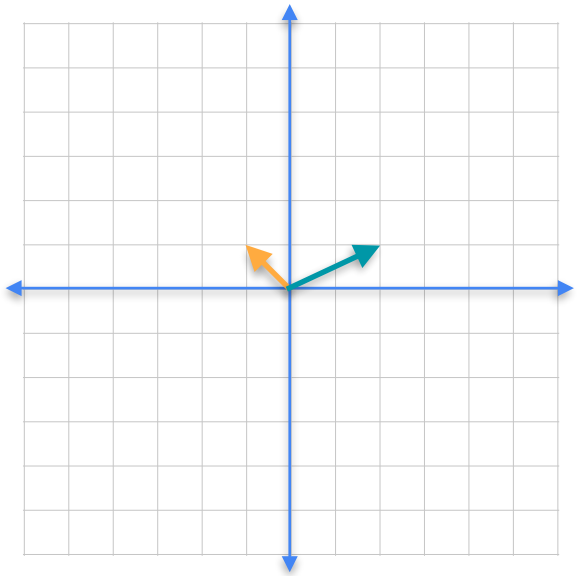


Linearly dependent

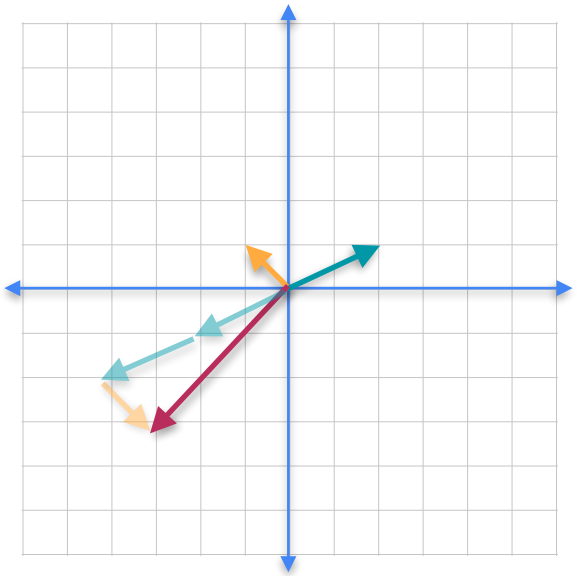
# Linearly independent and linearly dependent vectors



Linearly independent

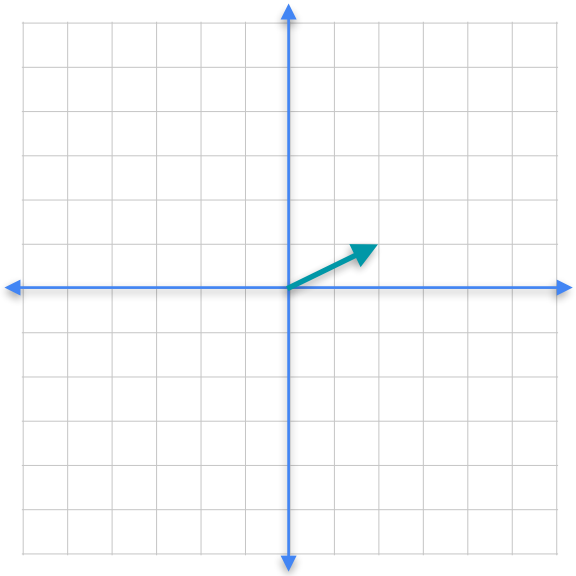


Linearly independent

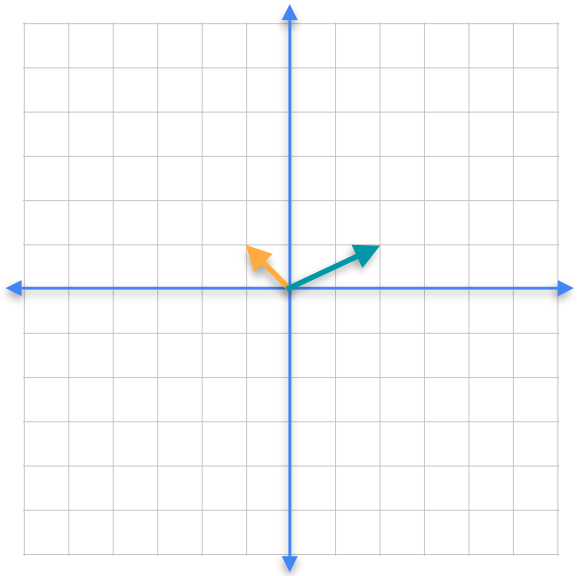


Linearly dependent

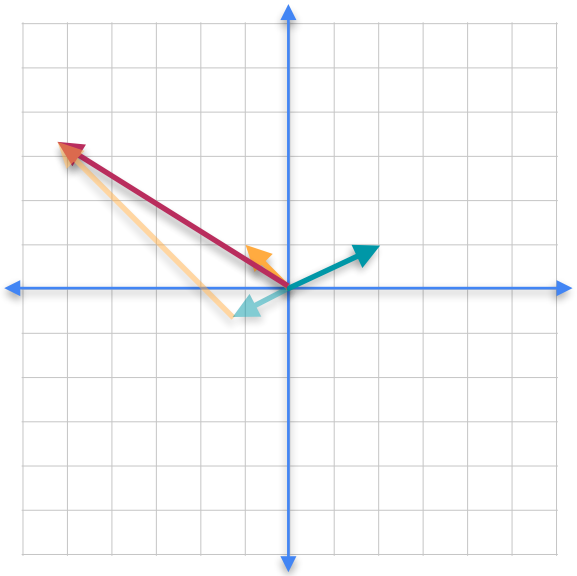
# Linearly independent and linearly dependent vectors



Linearly independent

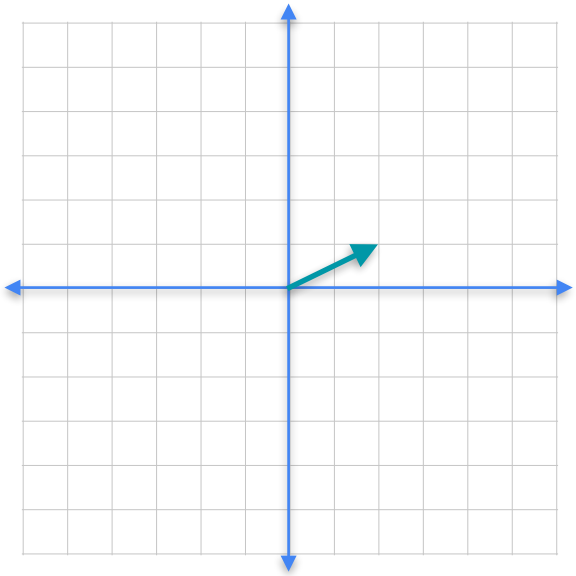


Linearly independent

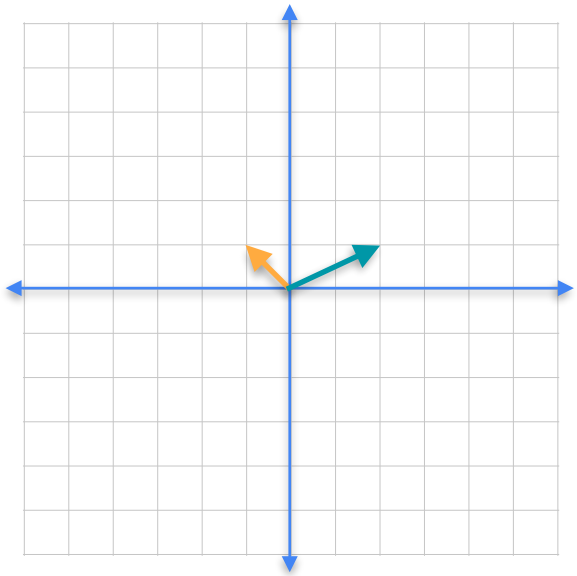


Linearly dependent

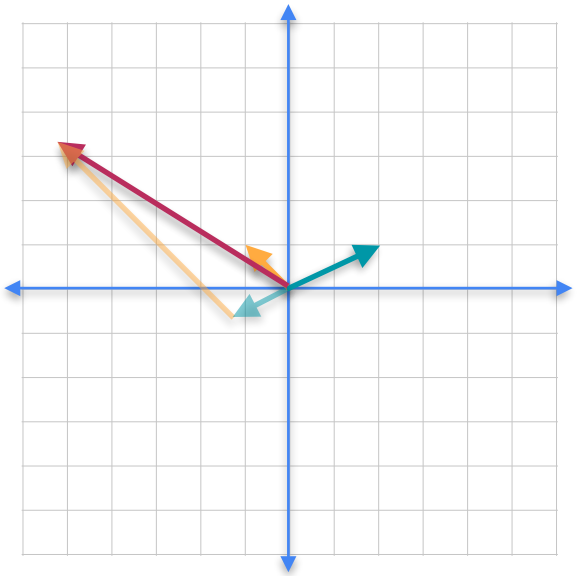
# Linearly independent and linearly dependent vectors



Linearly independent

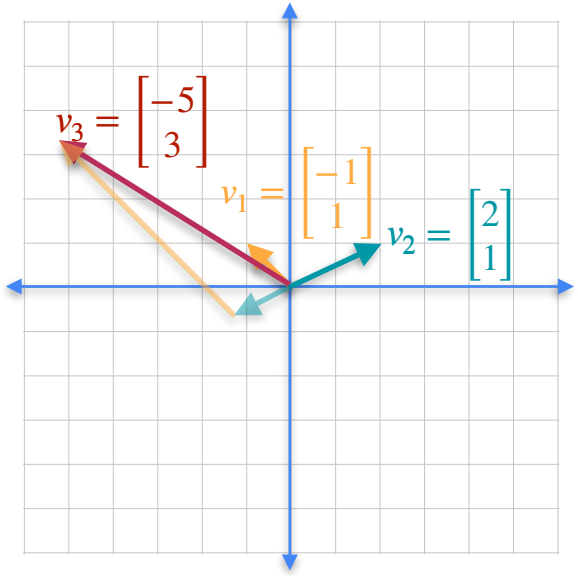


Linearly independent



Linearly dependent

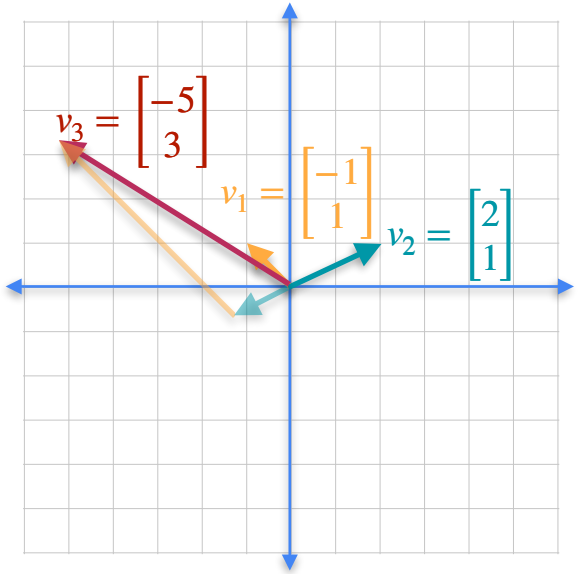
# Let's see how to check for linear dependence




Linearly dependent

$\alpha \quad +\beta \quad =$

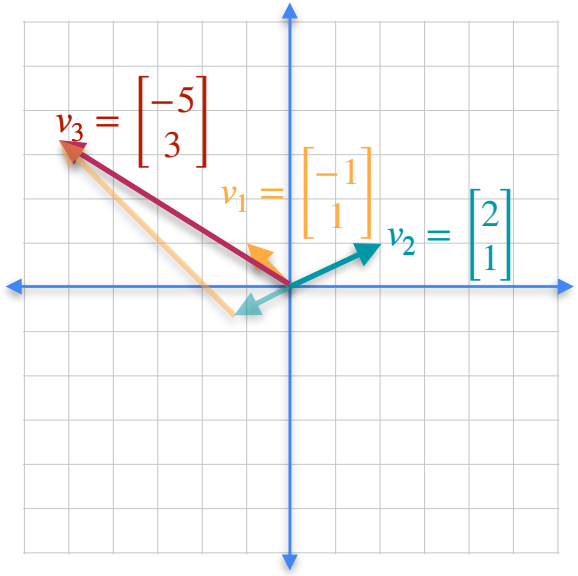
# Let's see how to check for linear dependence




Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

# Let's see how to check for linear dependence



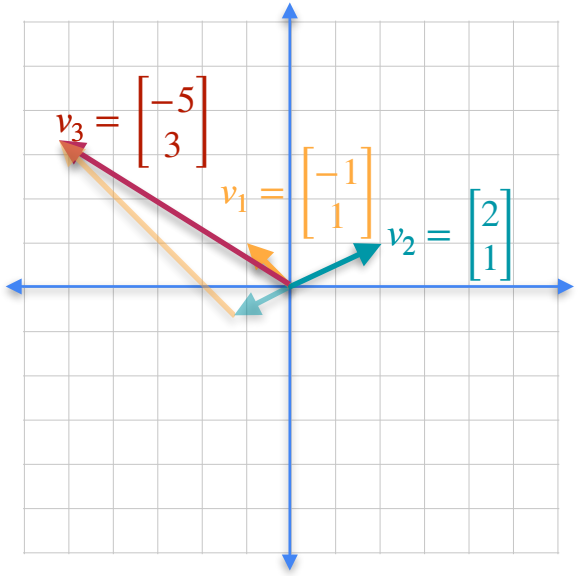

Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-\alpha + 2\beta = -5$
2. $\alpha + \beta = 3$

# Let's see how to check for linear dependence




Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$v_3$ is a linear combination of $v_1$ and $v_2$

1 $-\alpha + 2\beta = -5$

2 $\alpha + \beta = 3$

1 + 2

$3\beta = -2 \rightarrow \beta = -\frac{2}{3}$

2

$\alpha - \frac{2}{3} = 3 \rightarrow \alpha = \frac{11}{3}$

# Quiz

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \qquad -1 \qquad =$$

Linearly dependent

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad -1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly dependent

# Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Not a basis!

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly independent

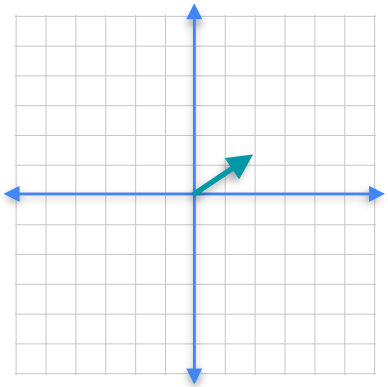
$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

# Basis: a formal definition

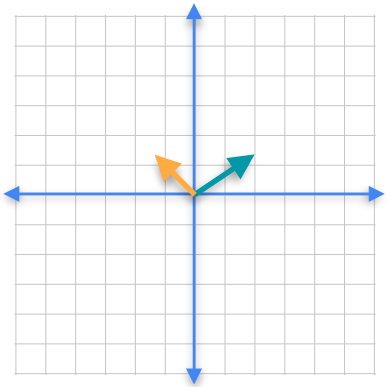
A basis is a set of vectors that:

- Spans a vector space
- Is linearly independent

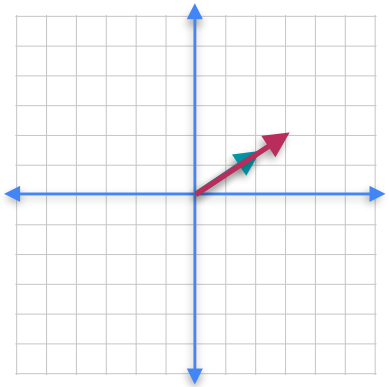
Not all sets of N vectors are a basis
for N-dimensional space



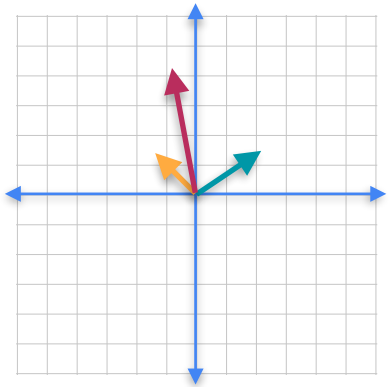
Spans a line
Linearly independent
Is a basis



Spans the plane
Linearly independent
Is a basis



Spans a line
Linearly dependent
Not a basis



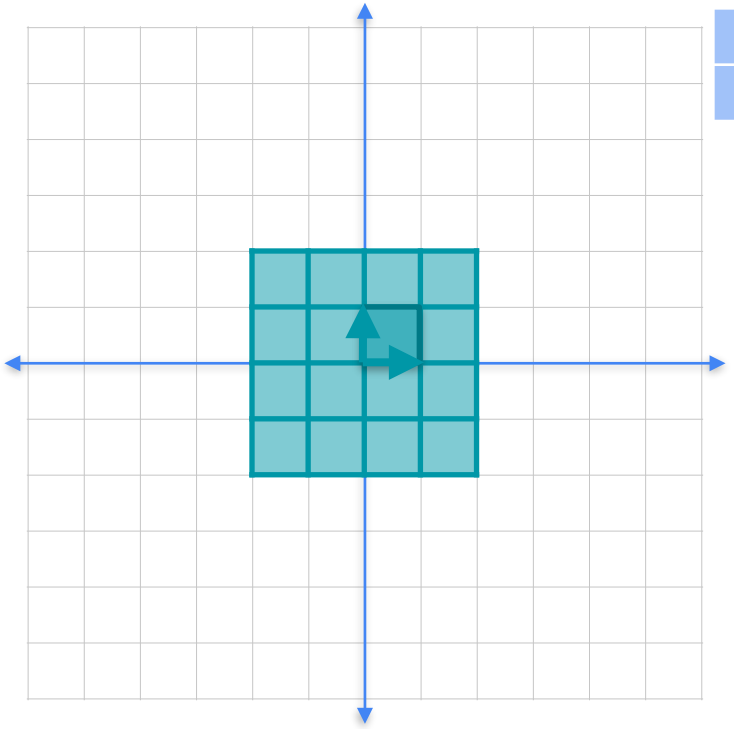
Spans the plane
Linearly dependent
Not a basis

DeepLearning.AI

# Determinants and Eigenvectors

## Eigenbasis

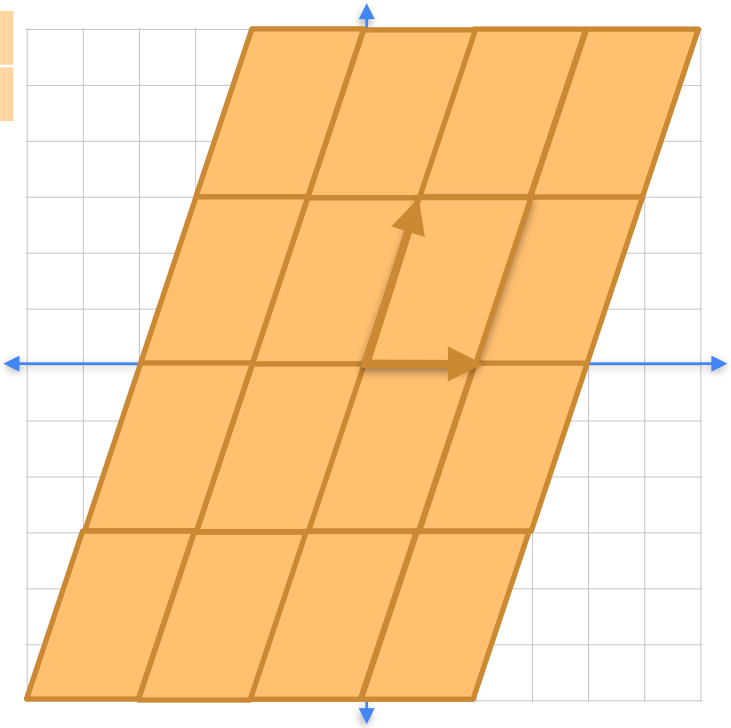
# Basis



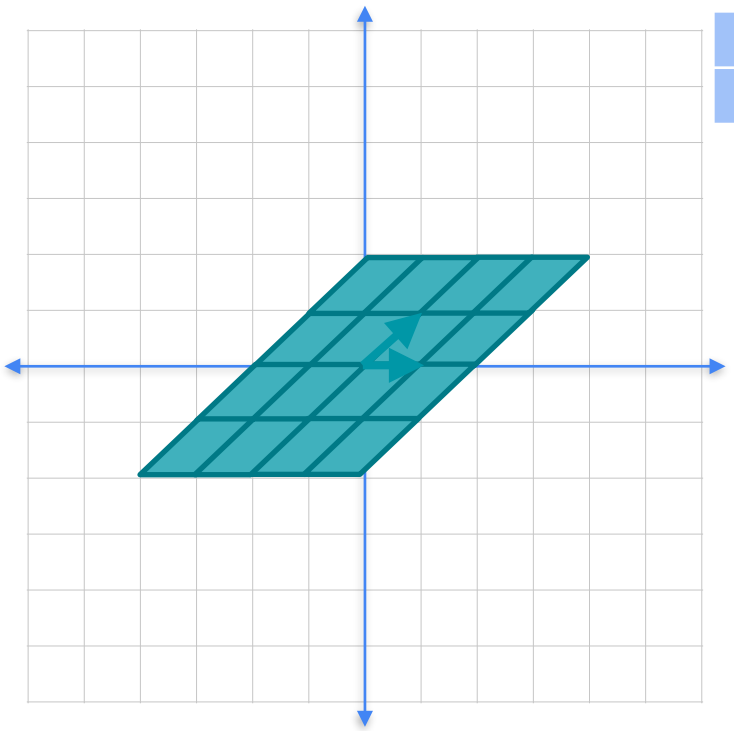
| | | | | |
|---|---|---|---|---|
| 2 | 1 | 0 | = | 1 |
| 0 | 3 | 1 | | 3 |

(1,0) → (2,0)

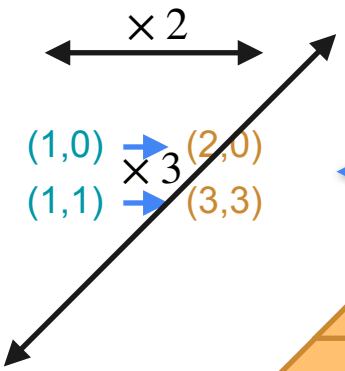
(0,1) → (1,3)

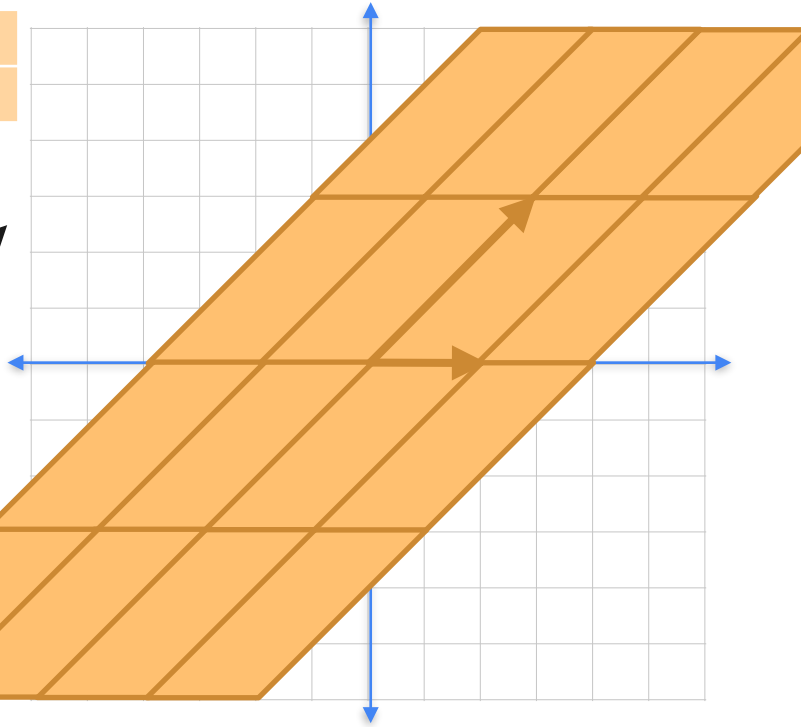


# Eigenbasis

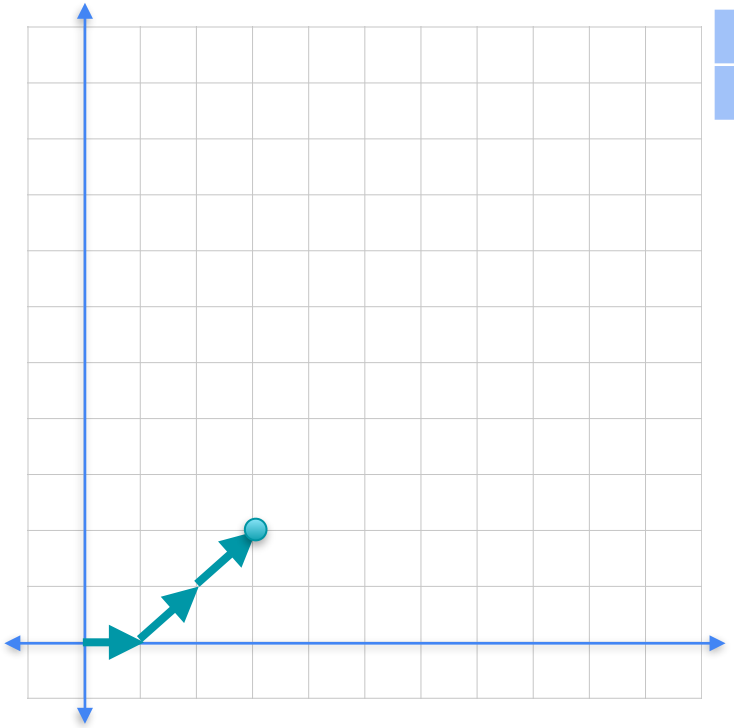


| | | | | |
|---|---|---|---|---|
| 2 | 1 | 1 | = | 3 |
| 0 | 3 | 1 | | 3 |



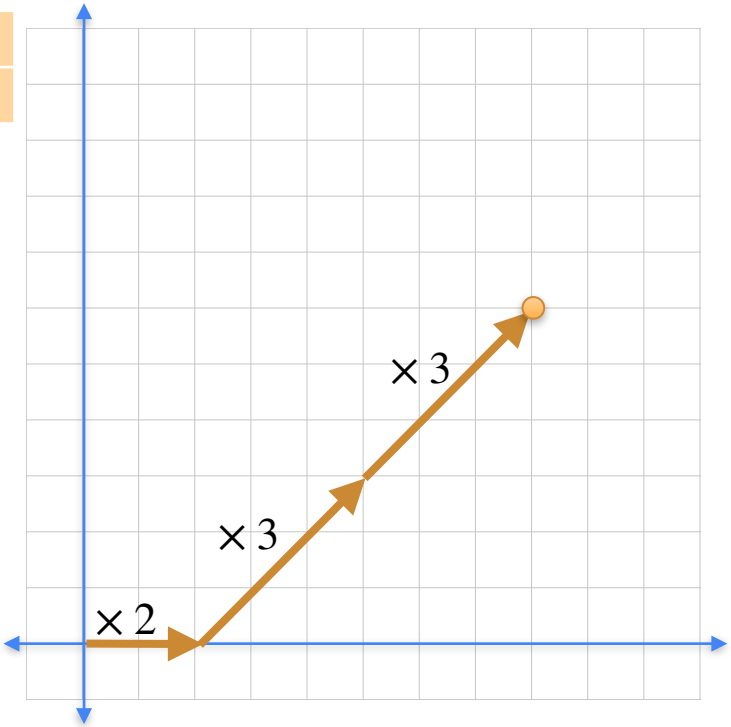



# Eigenbasis



| | | | | |
|---|---|---|---|---|
| 2 | 1 | 3 | = | 8 |
| 0 | 3 | 2 | | 6 |

(3,2) → (8,6)

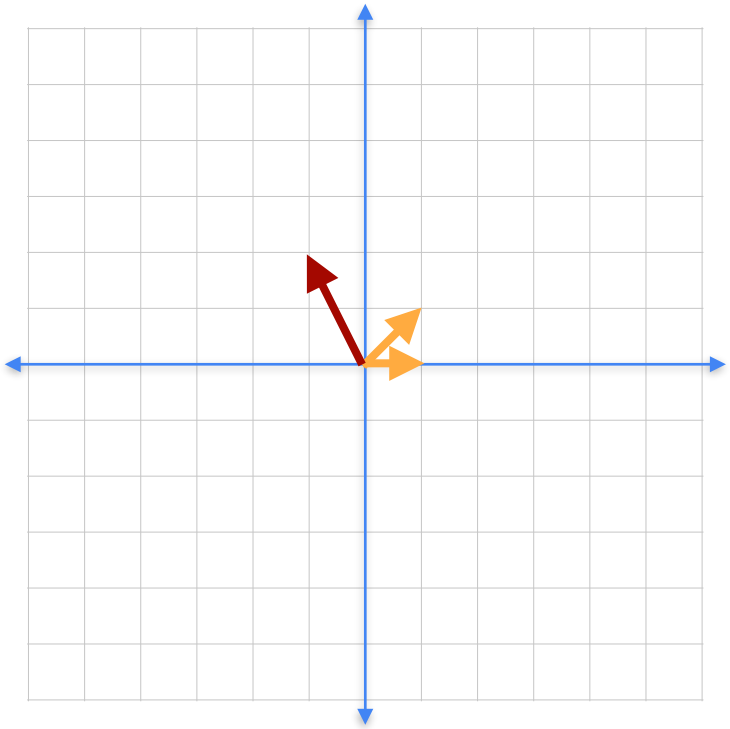


DeepLearning.AI

# Determinants and Eigenvectors

## Eigenvalues and Eigenvectors

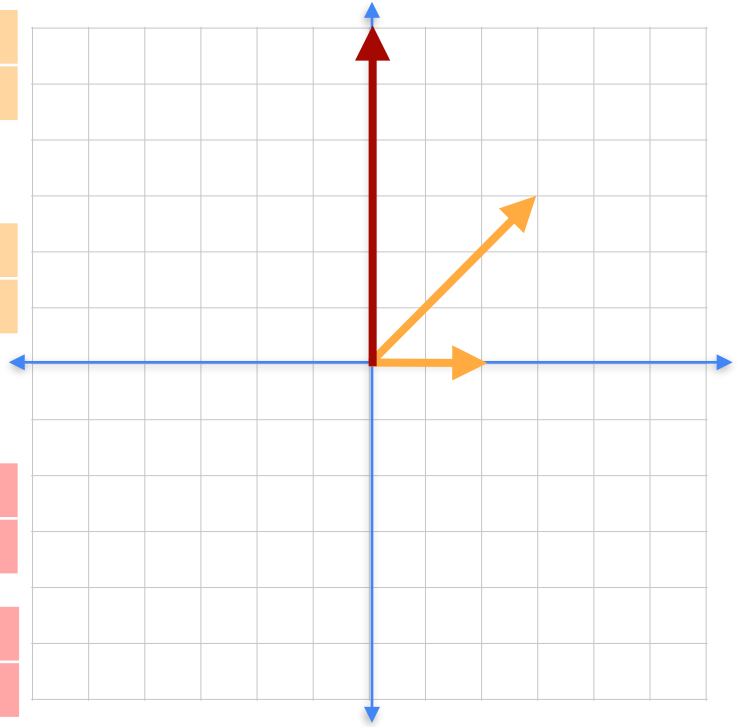
# Eigenvalues and eigenvectors



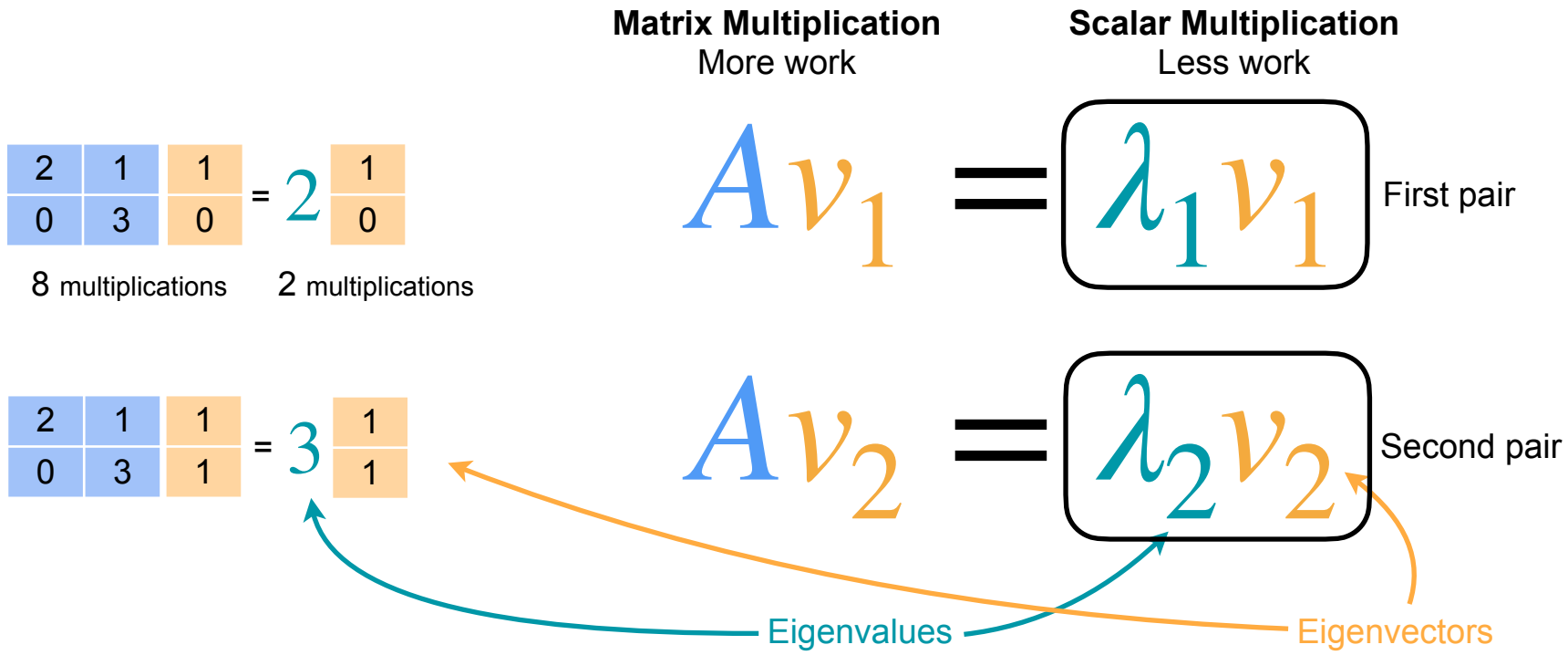
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

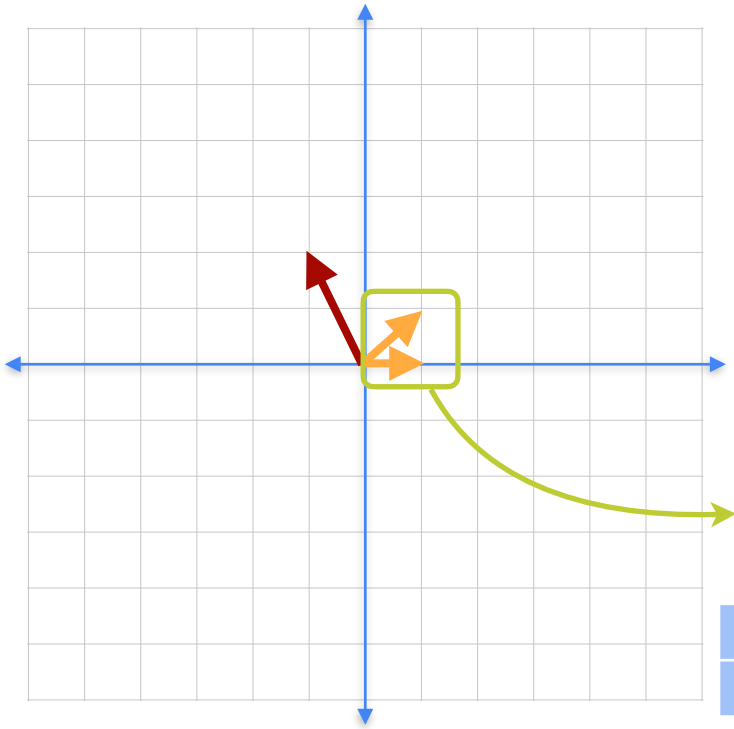
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



# Eigenvalues and eigenvectors



# Eigenvalues and eigenvectors



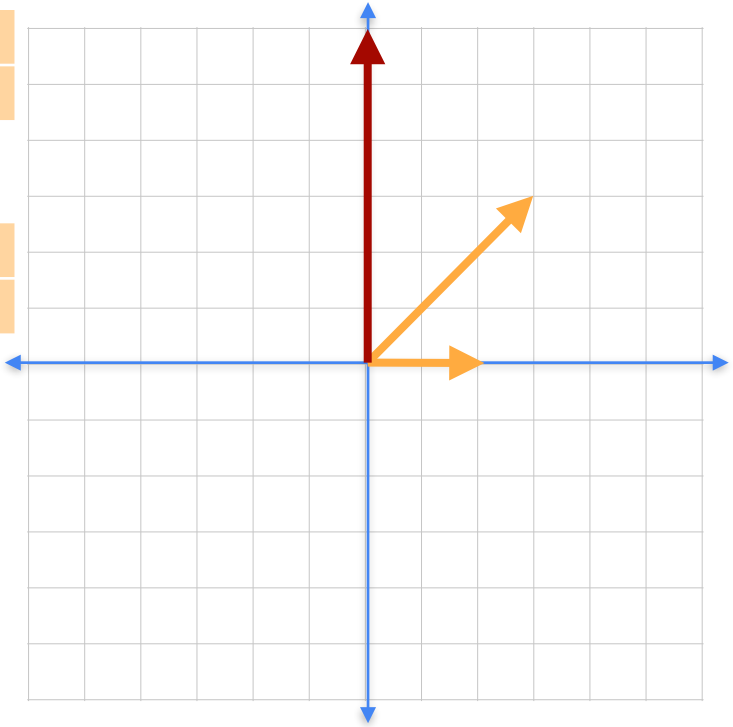
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

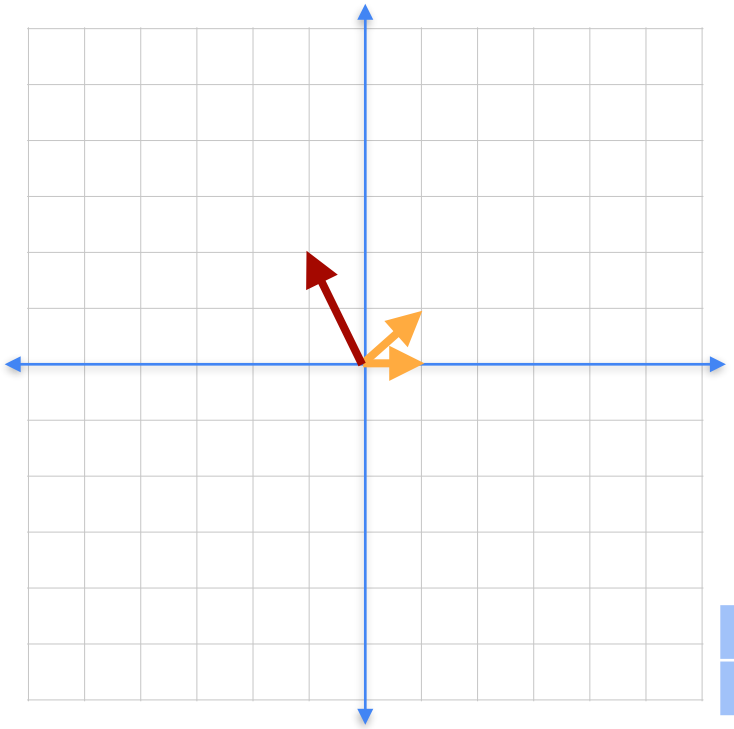
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

form a basis

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



# Eigenvalues and eigenvectors

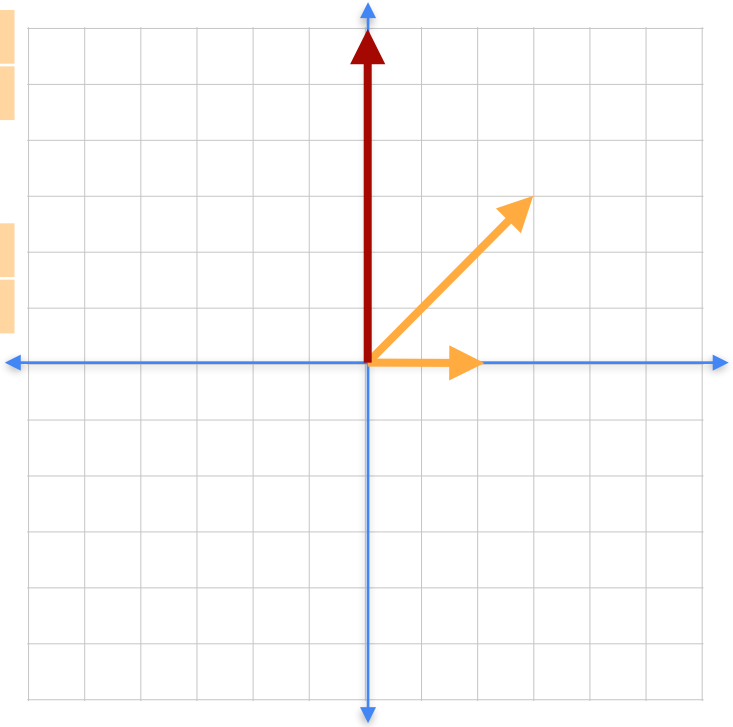


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

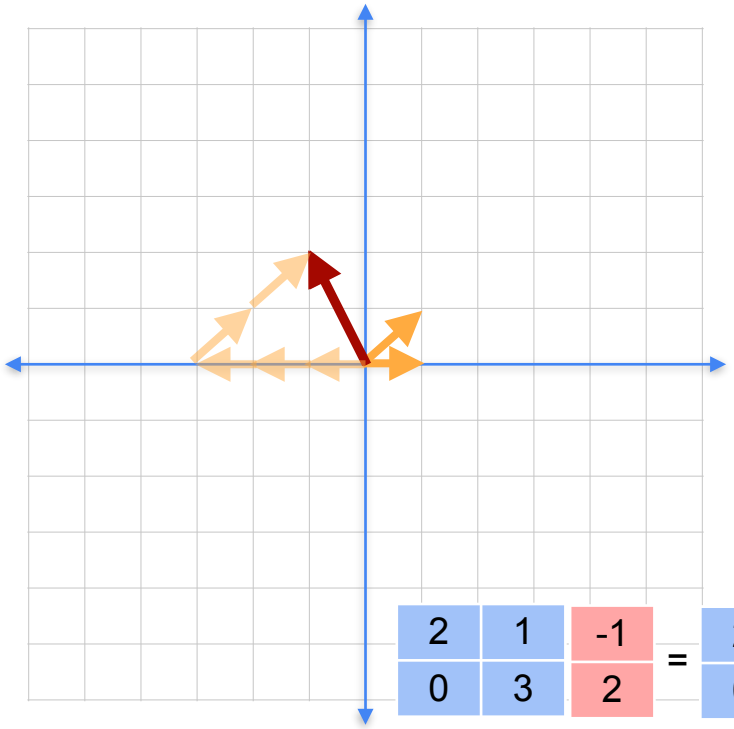
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



# Eigenvalues and eigenvectors

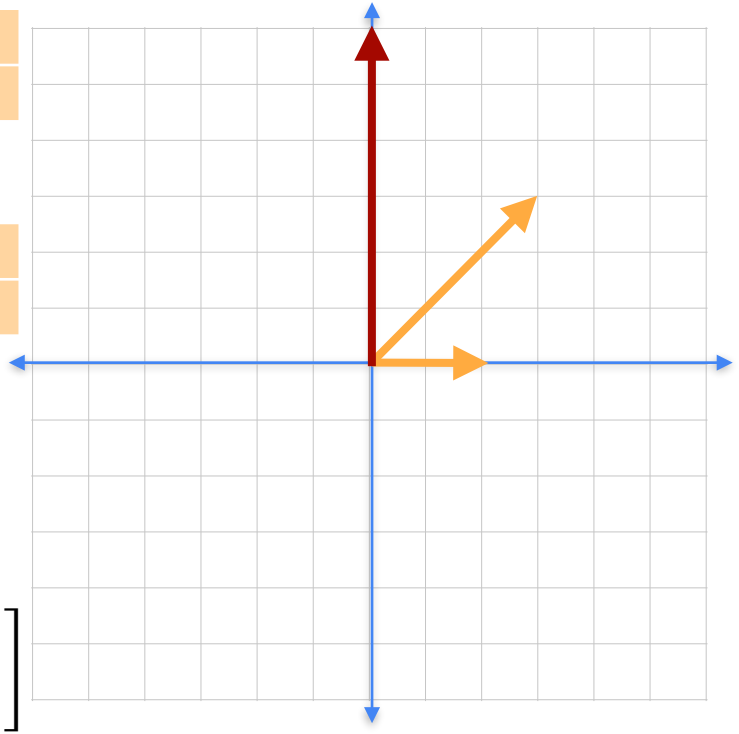


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

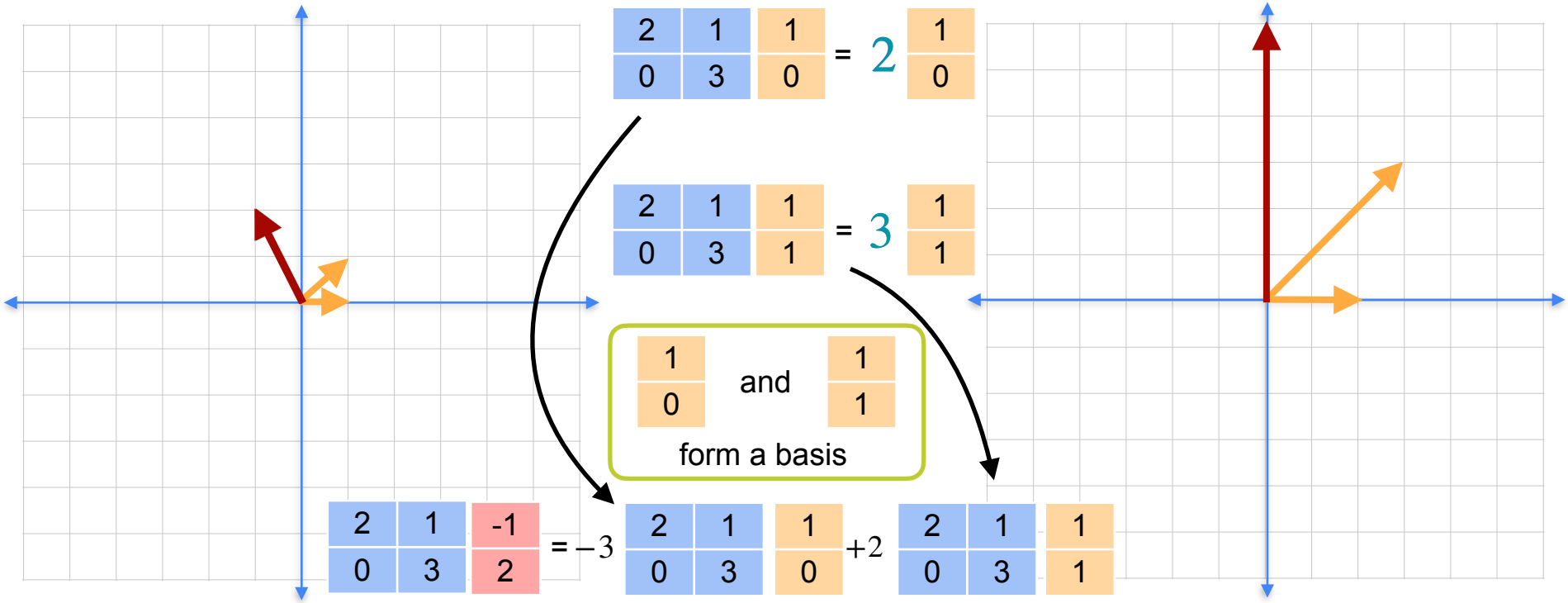
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form a basis

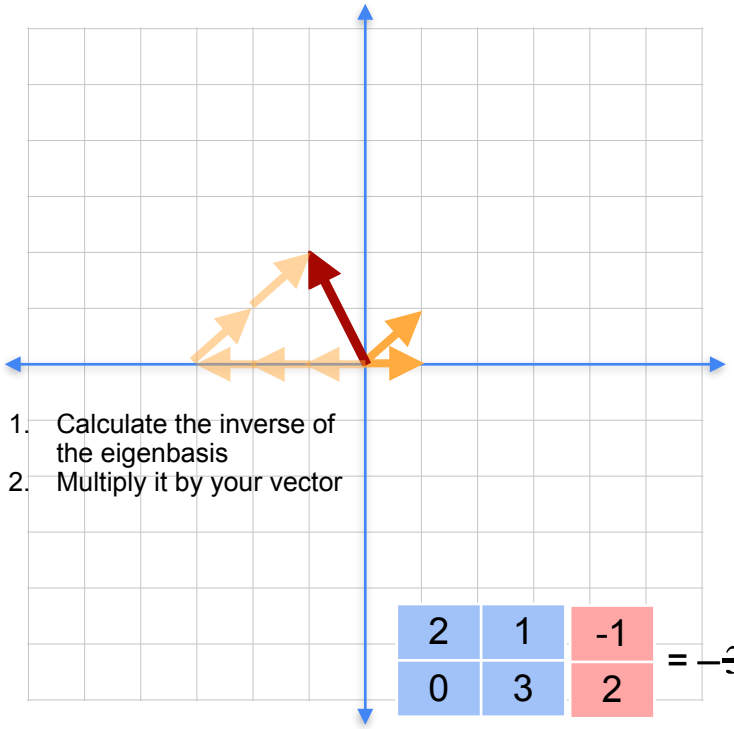
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & +2 \end{bmatrix}$$



# Eigenvalues and eigenvectors



# Eigenvalues and eigenvectors



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

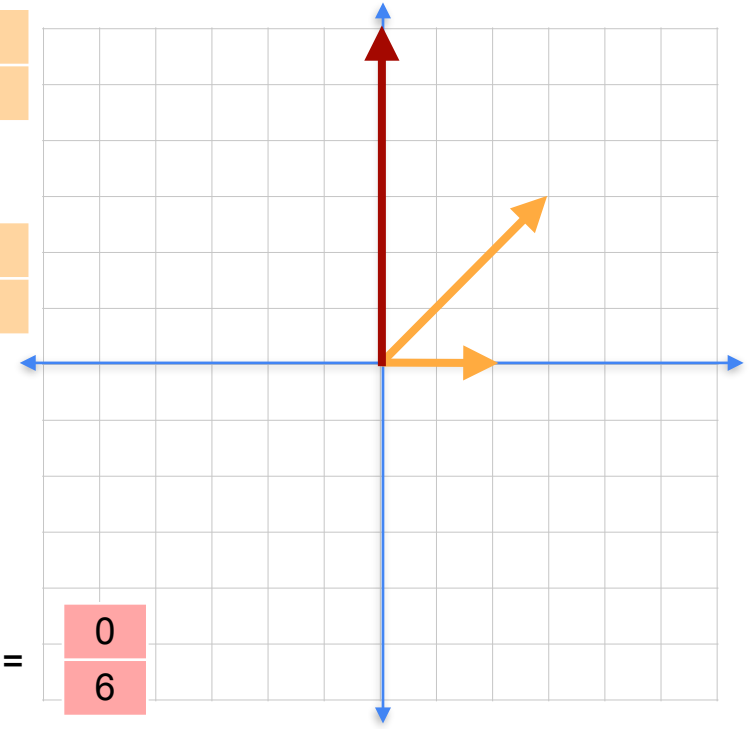
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1. Calculate the inverse of the eigenbasis
2. Multiply it by your vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

form a basis

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



# Eigenvalues and eigenvectors

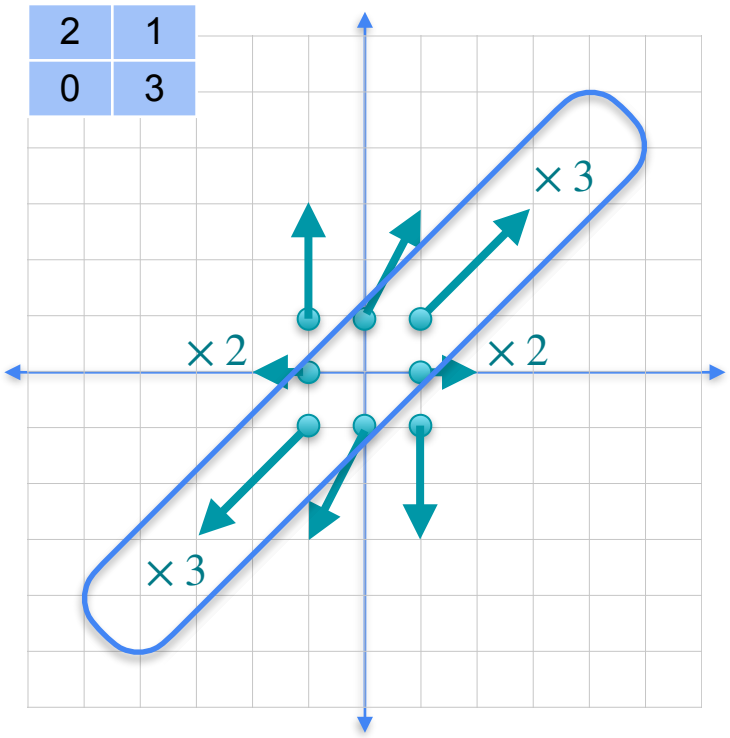
- $Av = \lambda v$ for each eigenvector / eigenvalue
- Eigenvectors: direction of stretch
- Eigenvalues: how much stretch
- Eigenbasis: the set of a matrix's eigenvectors, can be arranged as a matrix with one eigenvector in each column
- Save work and characterize a transformation

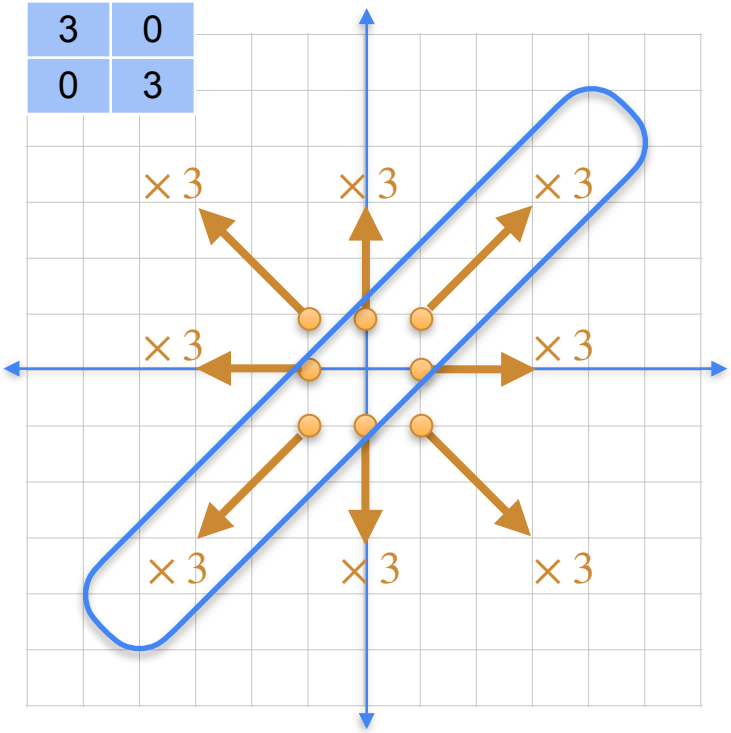
DeepLearning.AI

# Determinants and Eigenvectors

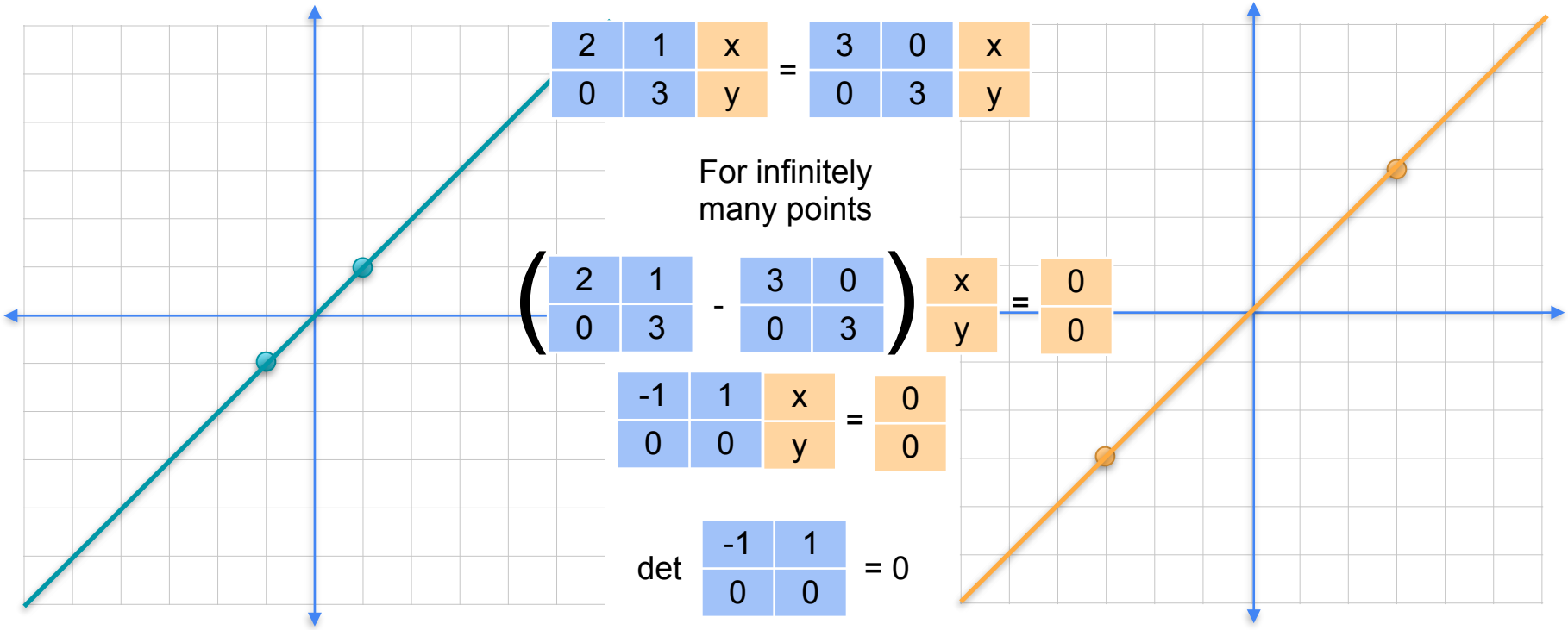
## Calculating eigenvalues and eigenvectors

# Finding eigenvalues

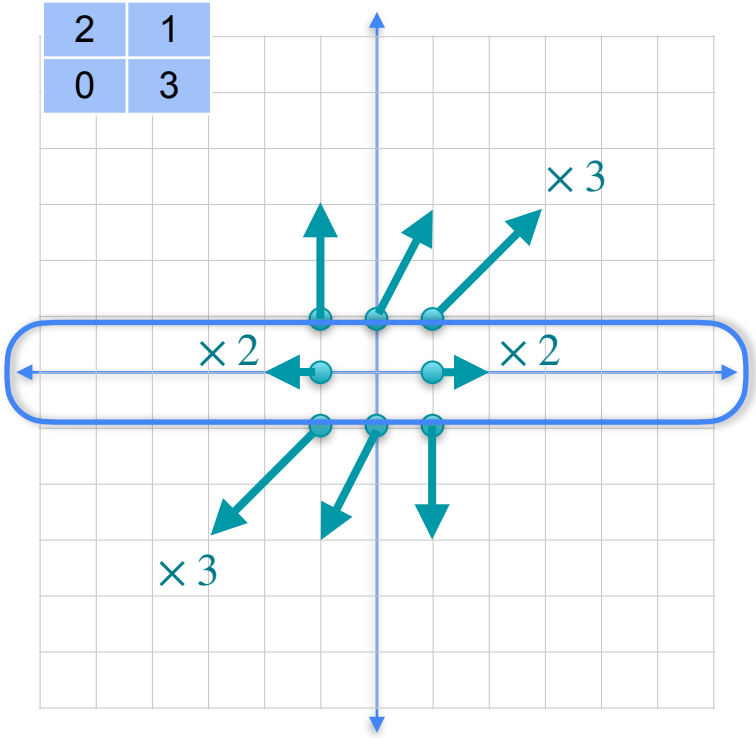


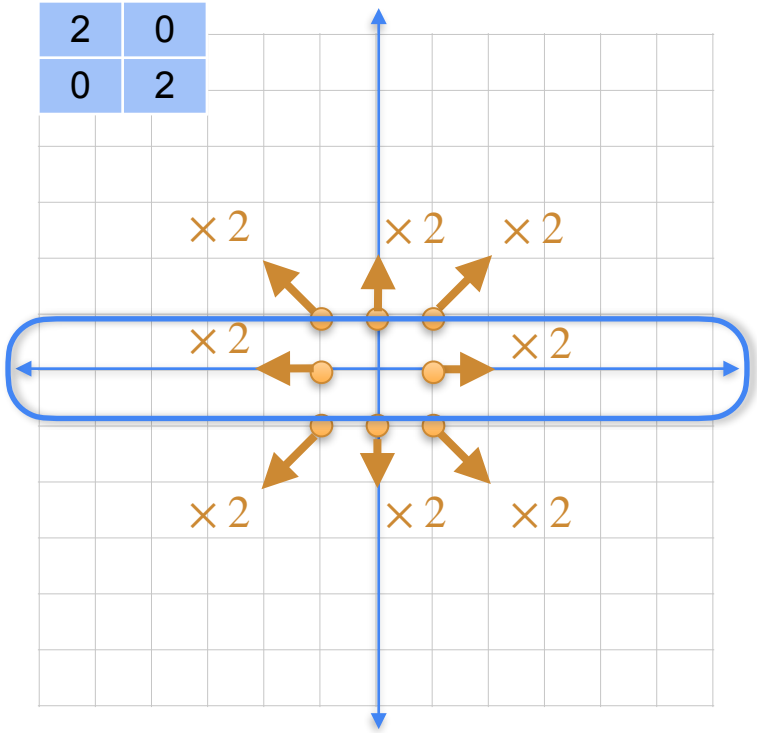



# Finding eigenvalues

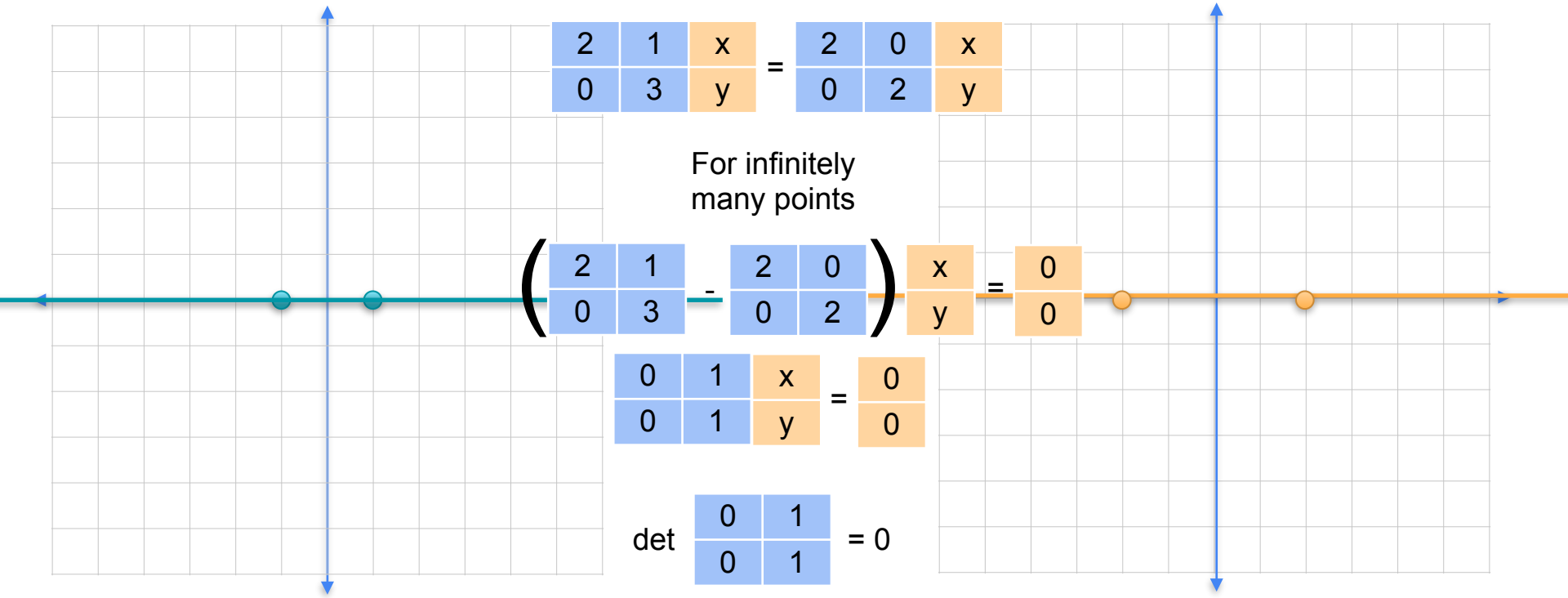


# Finding eigenvalues



# Finding eigenvalues



# Finding eigenvalues

If $\lambda$ is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2-\lambda)(3-\lambda) - 1 \cdot 0 = 0$$

$$\lambda = 2$$
$$\lambda = 3$$

# Finding eigenvectors

Eigenvalues: $\lambda = 2$, $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x \qquad x = 1$$

$$0x + 3y = 2y \qquad y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x \qquad x = 1$$

$$0x + 3y = 3y \qquad y = 1$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

# Quiz

- Find the eigenvalues and eigenvectors of this matrix:

| | |
|---|---|
| 9 | 4 |
| 4 | 3 |

# Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

| | |
|---|---|
| 9 | 4 |
| 4 | 3 |

- The characteristic polynomial is $\det \begin{vmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} = (9-\lambda)(3-\lambda) - 4 \cdot 4 = 0$
- Which factors as $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$
- The solutions are $\lambda = 11$, $\lambda = 1$

# Finding eigenvalues

$A =$

| | | |
|---|---|---|
| 2 | 1 | -1 |
| 1 | 0 | -3 |
| -1 | -3 | 0 |

$\lambda I =$

| | | |
|---|---|---|
| $\lambda$ | 0 | 0 |
| 0 | $\lambda$ | 0 |
| 0 | 0 | $\lambda$ |

**Characteristic polynomial:** $\det(A - \lambda I) = 0$

$\det$

| | | |
|---|---|---|
| 2 - $\lambda$ | 1 | -1 |
| 1 | -$\lambda$ | -3 |
| -1 | -3 | -$\lambda$ |

$= 0$

$$(2-\lambda)\lambda^2 \; +3 \; +3 \; -9(2-\lambda) \; +\lambda \; +\lambda \quad = -\lambda^3 + 2\lambda^2 + 11\lambda - 12 = 0$$

$$-(\lambda + 3)(\lambda - 1)(\lambda - 4) = 0$$

**Eigenvalues:** $-3, 1, 4$

# Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

**Eigenvalues:** $-3, 1, 4$

$$Av = \lambda v$$

$$\underbrace{\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

# Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

**Eigenvalues:** $-3, 1, 4$

$$Av = \lambda v$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4x_1 \\ x_1 - 3x_3 &= 4x_2 \\ -x_1 - 3x_2 &= 4x_3 \end{aligned}$$

$$\begin{aligned} R_1 \quad & -2x_1 + x_2 - x_3 = 0 \\ R_2 \quad & x_1 - 4x_2 - 3x_3 = 0 \\ R_3 \quad & -x_1 - 3x_2 - 4x_3 = 0 \end{aligned}$$

$$\begin{aligned} R_2 + R_3 \\ -7x_2 - 7x_3 = 0 \\ x_2 = -x_3 \end{aligned} \qquad \begin{aligned} 3R_1 + R_3 \\ -7x_1 - 7x_3 = 0 \\ x_1 = -x_3 \end{aligned}$$

$$\begin{aligned} x_1 &= k \\ x_2 &= k \\ x_3 &= -k \end{aligned}$$

infinite solutions of this form

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

this works!

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 2 \\ x_3 &= -2 \end{aligned}$$

so does this!

**Eigenvector:** $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

# Finding eigenvalues

$A =$

| | | |
|---|---|---|
| 2 | 1 | -1 |
| 1 | 0 | -3 |
| -1 | -3 | 0 |

**Eigenvalues** $\lambda_1 = 4$ $\lambda_2 = 1$ $\lambda_3 = -3$

**Eigenvectors**

| | | |
|---|---|---|
| 1 | 0 | 2 |
| 1 | 1 | $-1$ |
| $-1$ | 1 | 1 |

# Note on dimensions

Eigenvalues → Determinant → Square Matrix

| | |
|---|---|
| 9 | 4 |
| 4 | 3 |

| | | |
|---|---|---|
| 9 | 4 | 5 |
| 4 | 3 | -2 |

DeepLearning.AI

# Determinants and Eigenvectors

## On the number of eigenvectors

# Number of eigenvectors

$A =$

| | | |
|---|---|---|
| 2 | 1 | -1 |
| 1 | 0 | -3 |
| -1 | -3 | 0 |

3 by 3 matrix → ? → 3 distinct eigenvalues → ? → 3 distinct eigenvectors

Eigenvalues

$\lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_3 = -3$

Eigenvectors

| | | |
|---|---|---|
| 1 | 0 | 2 |
| 1 | 1 | −1 |
| −1 | 1 | 1 |

# Repeated eigenvalues - Example 1

$A =$

| | | |
|---|---|---|
| 2 | 0 | 0 |
| -1 | 4 | -0.5 |
| 0 | 0 | 2 |

Characteristic polynomial = $\det(A - \lambda I)$ = det

| | | |
|---|---|---|
| 2 - $\lambda$ | 0 | 0 |
| 1 | 4 - $\lambda$ | 0.5 |
| 0 | 0 | 2 - $\lambda$ |

$$(2-\lambda)^2(4-\lambda)+0+0-0-0-0=0$$

Eigenvalues: $4, \boxed{2,\ 2}$ Repeated eigenvalue

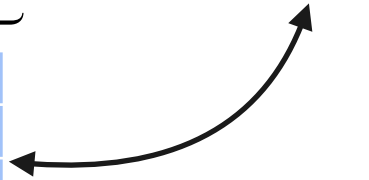
# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: $4$

$$Av = 4v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$



# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ -2x_3 &= 0 \end{aligned}$$

$x_1 = 0$

$x_2 =$ any number

$x_3 = 0$

Eigenvector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: $2$

$$Av = 2v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: $2$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$x_1 = 2x_2 - 0.5x_3$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Point in different directions

Different eigenvectors

# Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 4$ $\lambda_2 = 2$ $\lambda_3 = 2$

Eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

# Repeated eigenvalues - Example 2

$A =$

| | | |
|---|---|---|
| 2 | 0 | 0 |
| -1 | 4 | -0.5 |
| 4 | 0 | 2 |

Characteristic polynomial = $\det(A - \lambda I)$ = det

| | | |
|---|---|---|
| 2-$\lambda$ | 0 | 0 |
| 1 | 4-$\lambda$ | 0.5 |
| -4 | 0 | 2-$\lambda$ |

$$(2-\lambda)^2(4-\lambda) + 0 \;+0\; -0\; -0 -0$$

Eigenvalues: $4, \boxed{2,\ 2}$ Repeated eigenvalue

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: $4$

$$Av = 4v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 4x_1 + 2x_3 &= 4x_3 \end{aligned}$$

$$Av = 4v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ 4x_1 - 2x_3 &= 0 \end{aligned}$$

$x_1 = 0$ $\quad x_3 = 0$ $\quad x_2 =$ any number

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

Same as before!

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: $2$

$$Av = 2v$$

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: $2$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

$x_1 = 0 \qquad x_3 = 4x_2$

$$\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \quad x_1 = 0,\ x_2 = 1,\ x_3 = 4 \qquad \begin{bmatrix} 0 \\ 0.5 \\ 2 \end{bmatrix} \quad x_1 = 0,\ x_2 = 0.5,\ x_3 = 2$$

On the same line

Same eigenvector

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: $2$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$Av = 2v$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$x_1 = 0$ $\qquad$ $x_3 = 4x_2$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

# Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalues $\lambda_1 = 4$ $\lambda_2 = 2$ $\lambda_3 = 2$

Eigenvectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

Can’t create an eigenbasis from this matrix

# Summary

| a | b |
|---|---|
| c | d |

Eigenvalues $\lambda_1, \lambda_2$

If $\lambda_1 \neq \lambda_2$ → 2 eigenvectors (2 different directions)

If $\lambda_1 = \lambda_2$ → 1 eigenvector (1 direction)

If $\lambda_1 = \lambda_2$ → 2 eigenvectors (2 different directions)

| a | b | c |
|---|---|---|
| d | e | f |
| g | h | i |

$\lambda_1, \lambda_2, \lambda_3$

If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ → 3 eigenvectors (3 different directions)

If $\lambda_1 = \lambda_2 \neq \lambda_3$ → 2 eigenvectors (2 different directions)

If $\lambda_1 = \lambda_2 \neq \lambda_3$ → 3 eigenvectors (3 different directions)

If $\lambda_1 = \lambda_2 = \lambda_3$ → 1 eigenvector (1 direction)

If $\lambda_1 = \lambda_2 = \lambda_3$ → 2 eigenvectors (2 different directions)

If $\lambda_1 = \lambda_2 = \lambda_3$ → 3 eigenvectors (3 different directions)

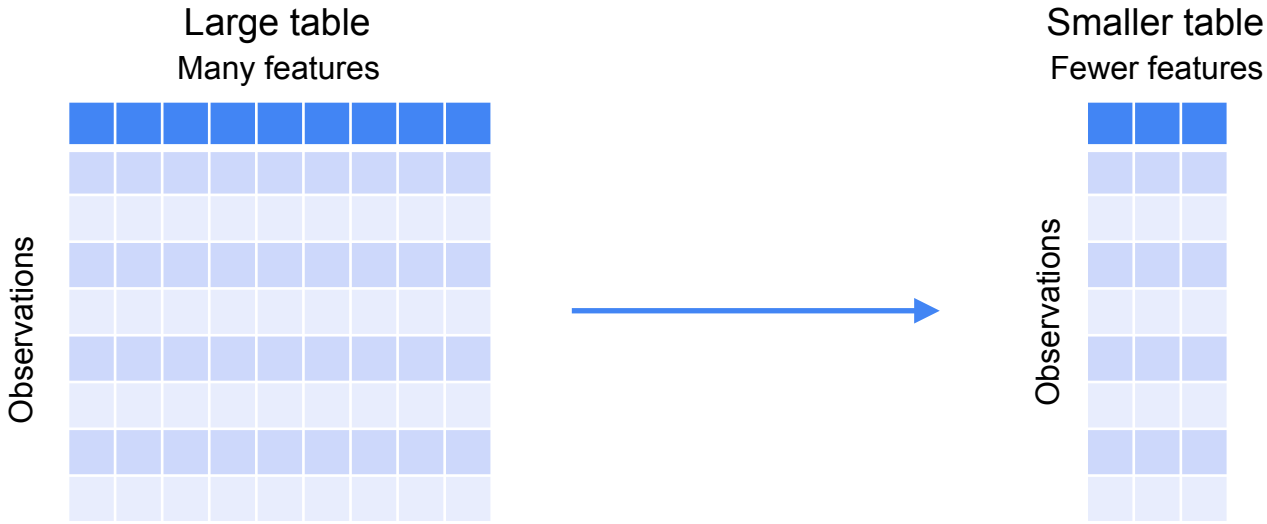
DeepLearning.AI

# Determinants and Eigenvectors

## Dimensionality reduction annd projection

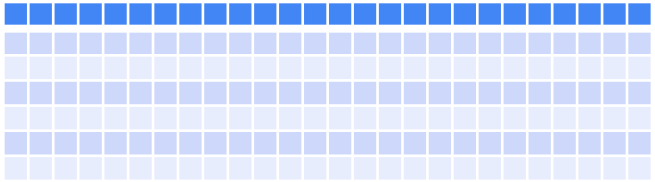
# Dimensionality Reduction

- Reduce dimensions (# of columns) of dataset
- Preserve as much information as possible



# Dimensionality Reduction

- Leads to smaller datasets
- Easier to visualize



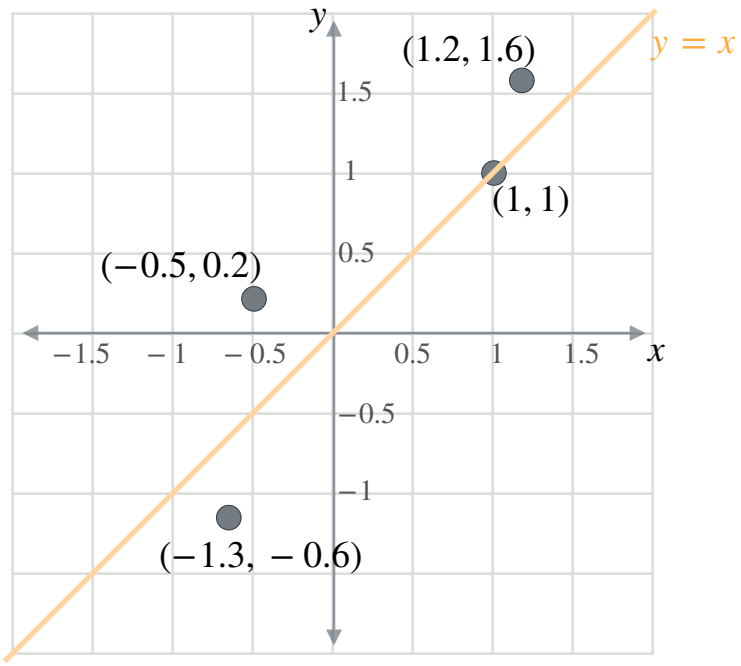
| Customer Age | Account Age | Days Since Login | Total Purchases | Total $ Spent |
|---|---|---|---|---|
| 23 | 1 month | 10 days | 1 | $100 |
| 71 | 45 months | 2 days | 5 | $150 |
| 54 | 30 months | 15 days | 2 | $70 |
| 36 | 22 months | 12 days | 4 | $210 |

Easy approach - just delete columns

Loses valuable information

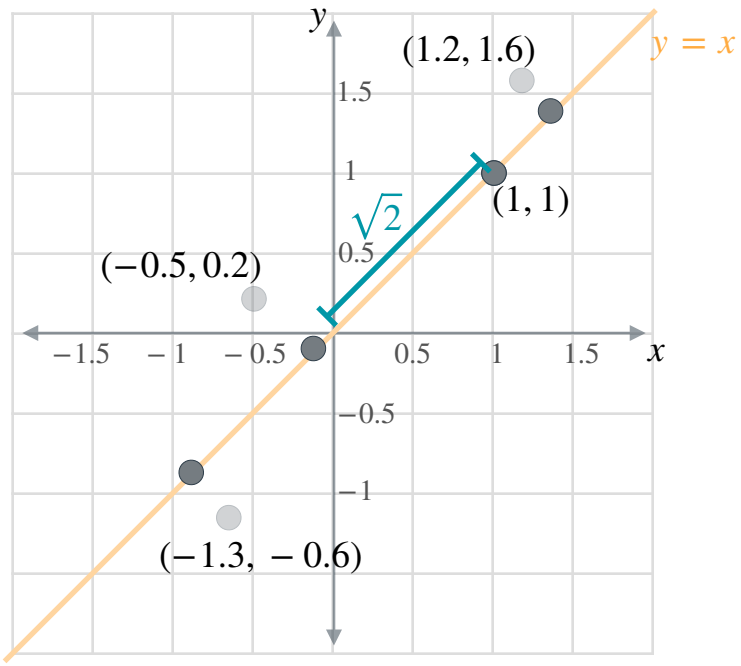
# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |



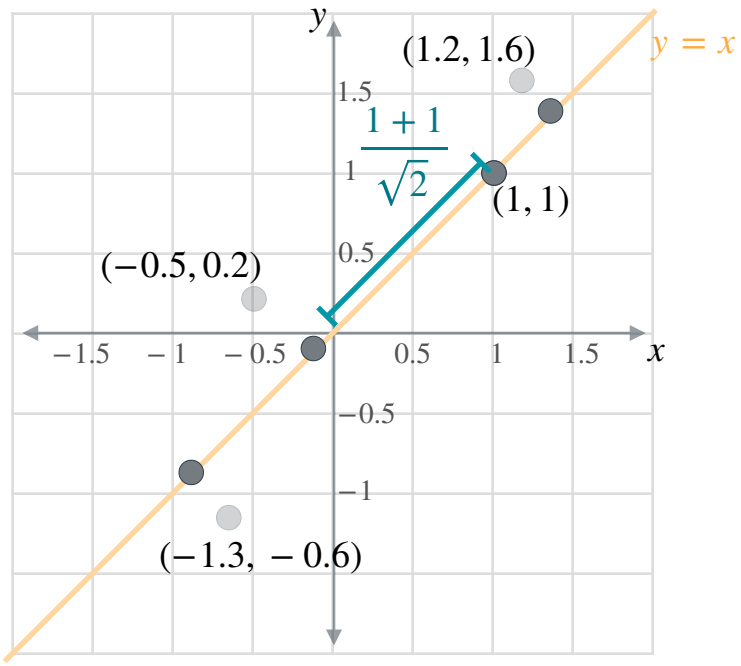
# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |



# Projections

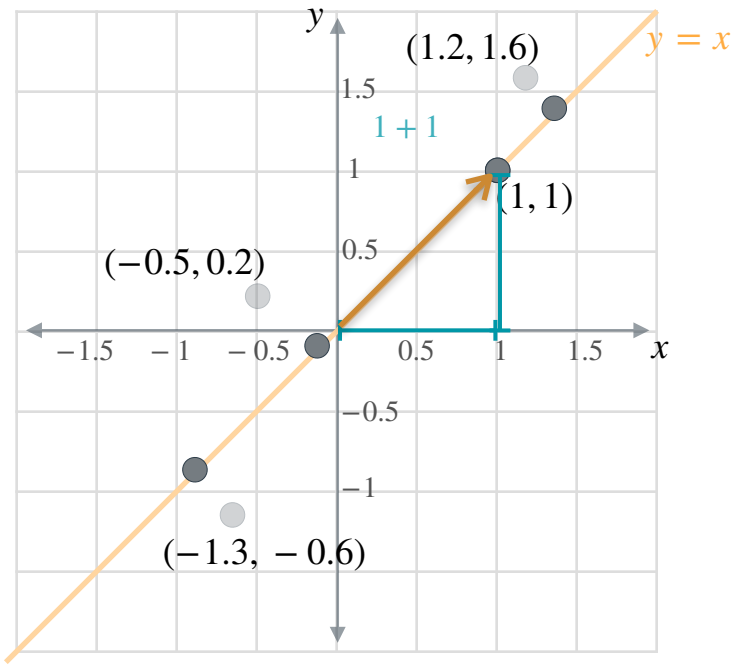
| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

1

1

=

$(1 + 1)$



# Projections

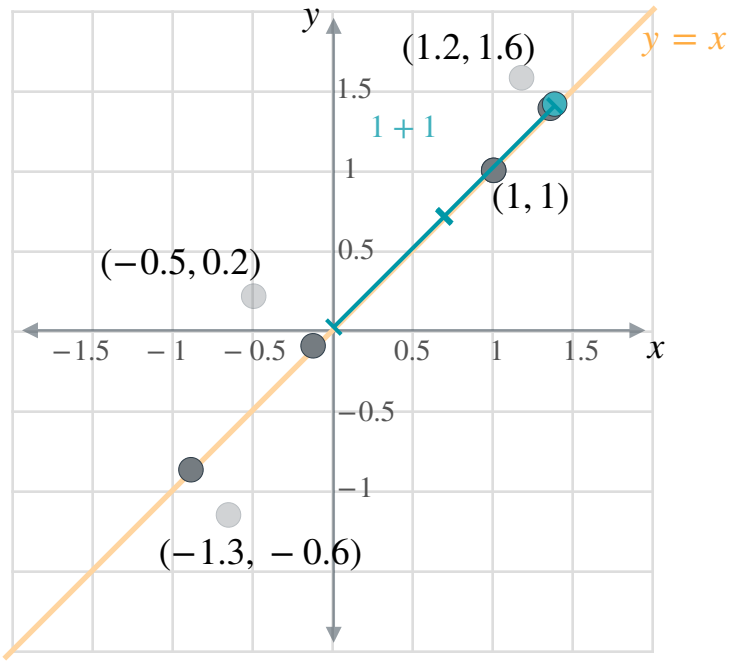
| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

1

1

=

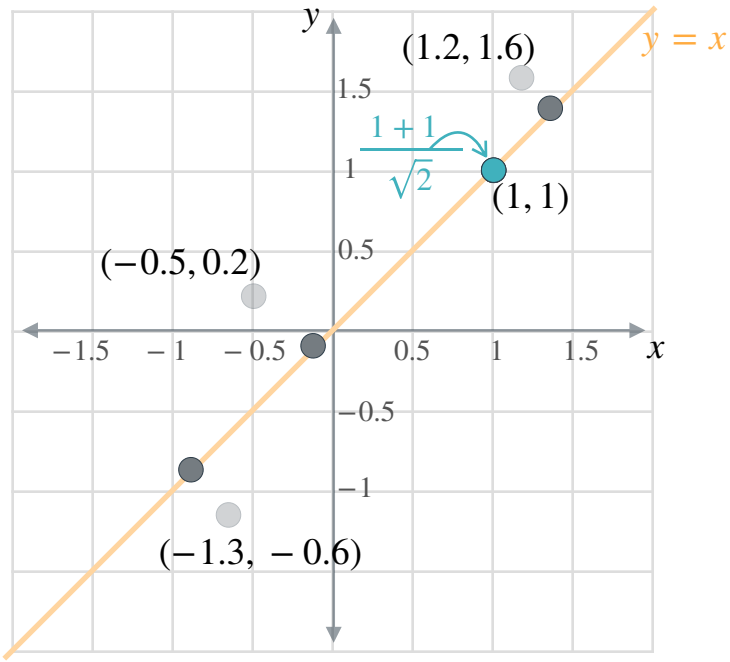
$(1 + 1)$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{matrix}1\\1\end{matrix} \frac{1}{\sqrt{2}} = (1+1)/\sqrt{2}$$



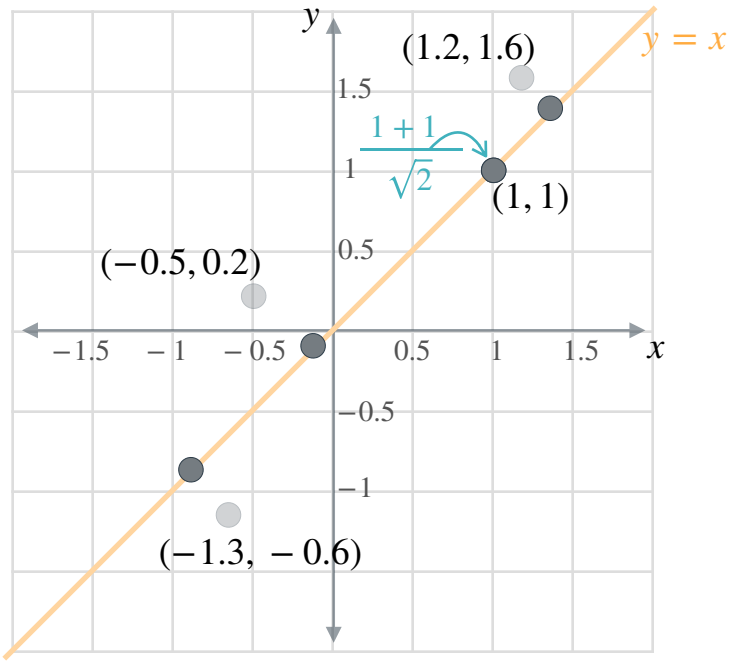
# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

Norm of 1

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2}$$

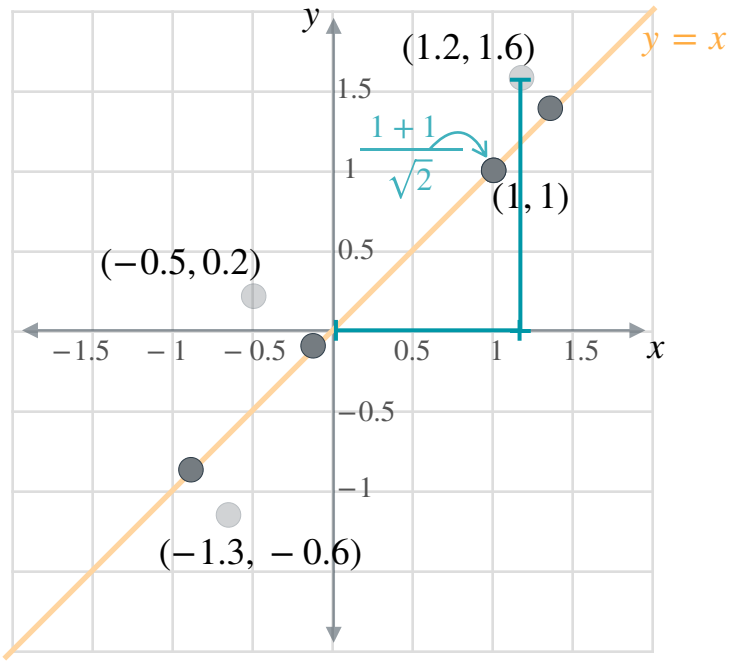
$$(1+1)/\sqrt{2}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

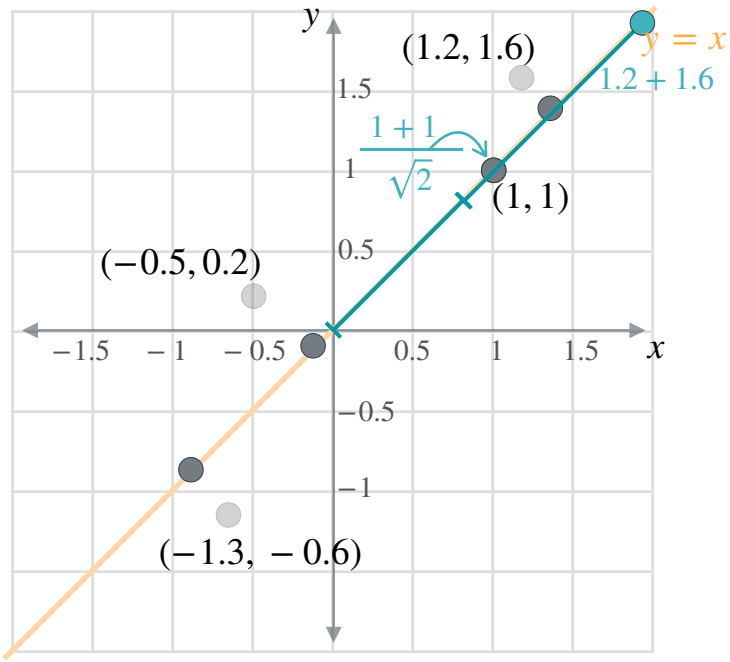
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ \end{bmatrix}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

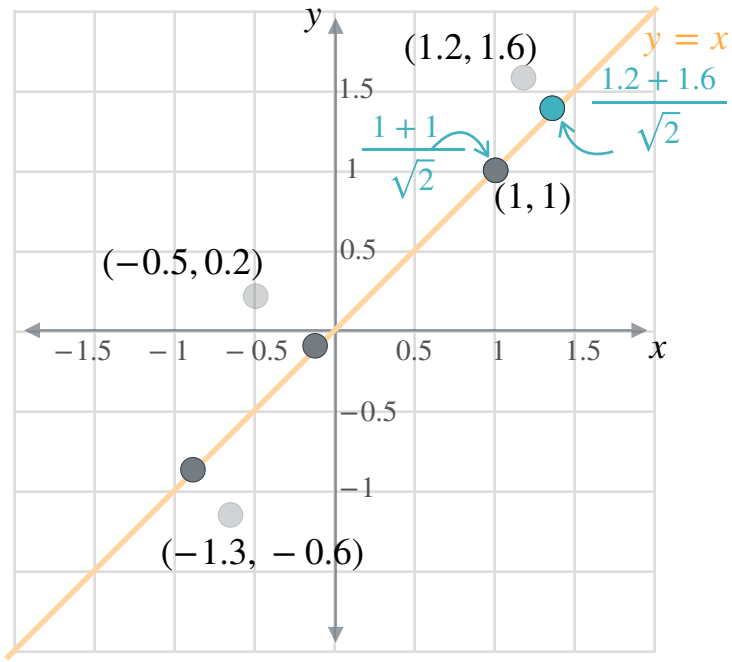
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6) \end{bmatrix}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

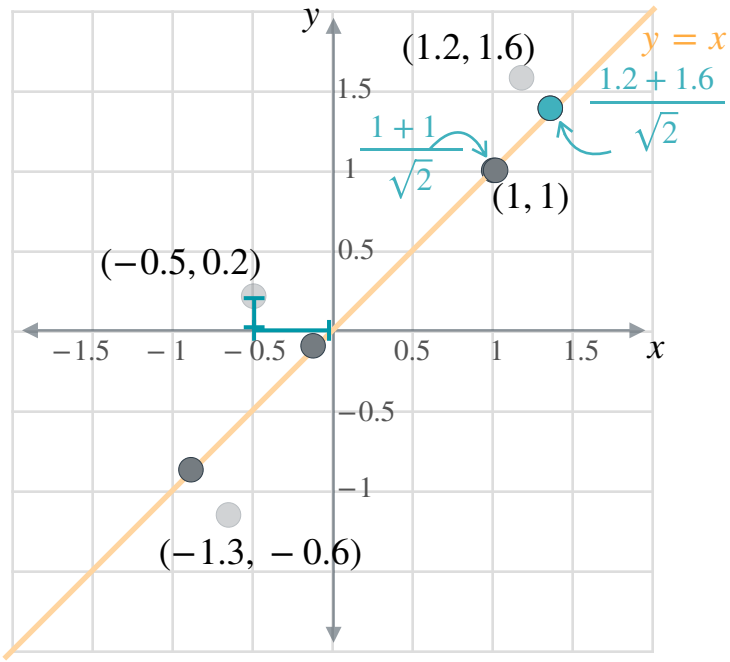
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \end{bmatrix}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

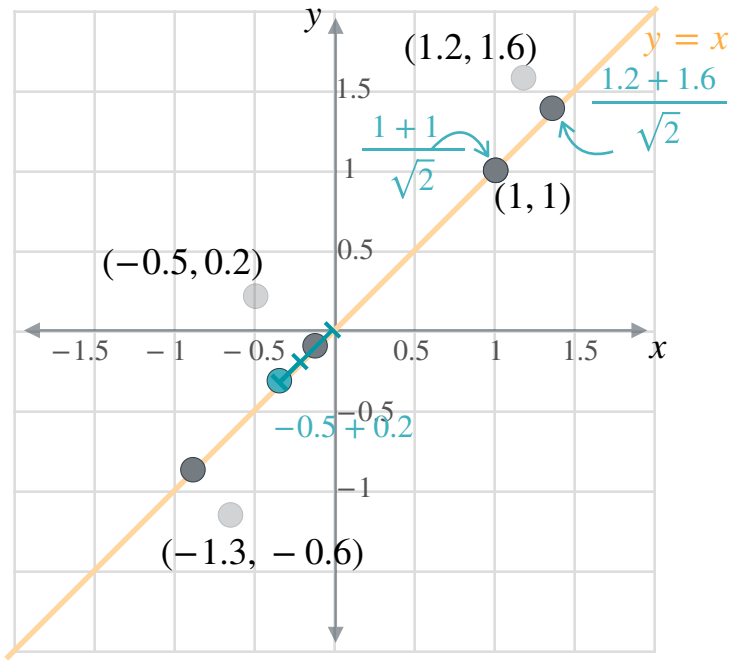
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ \end{bmatrix}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

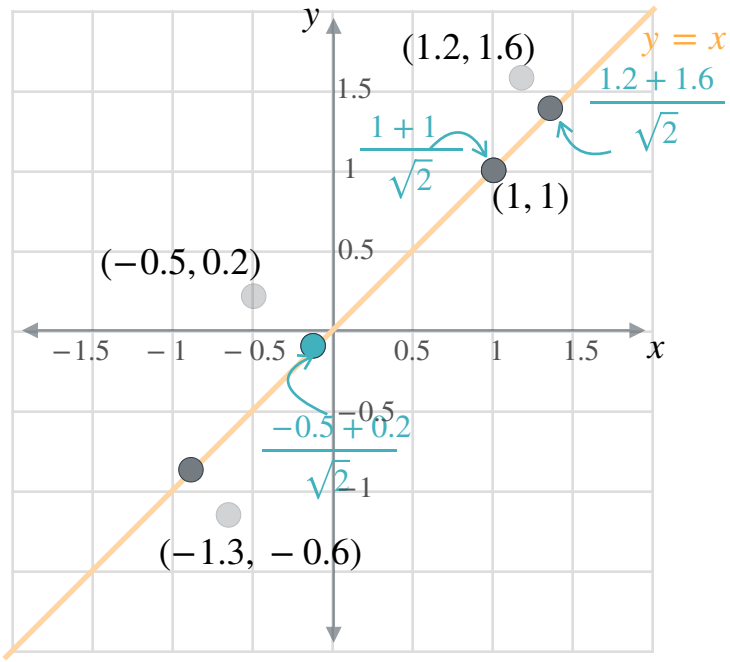
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2) \end{bmatrix}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

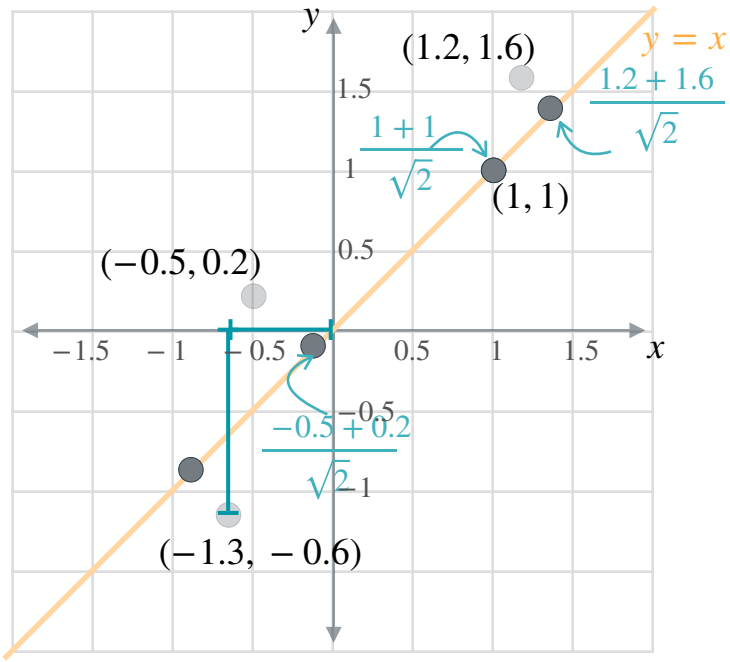
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \end{bmatrix}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

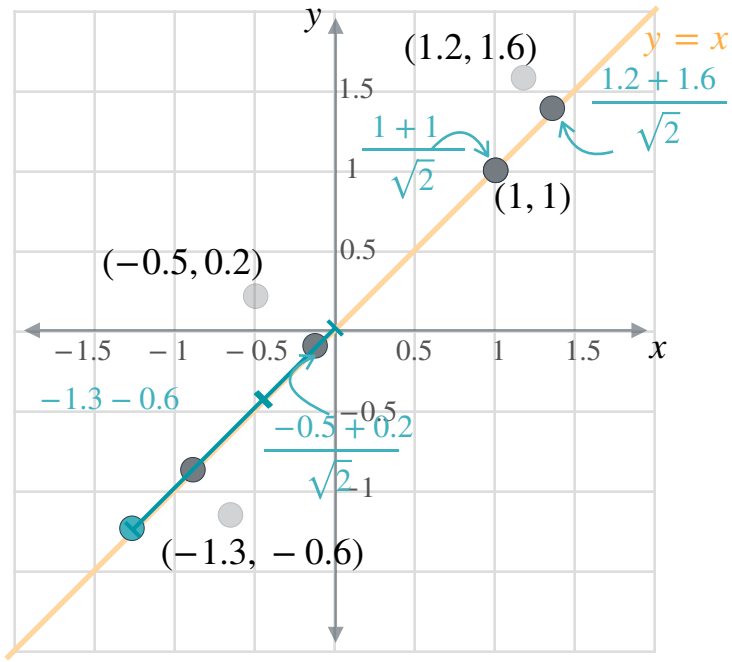
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \\ \end{bmatrix}$$



# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} (1+1)/\sqrt{2} \\ (1.2+1.6)/\sqrt{2} \\ (-0.5+0.2)/\sqrt{2} \\ (-1.3,-0.6) \end{bmatrix}$$

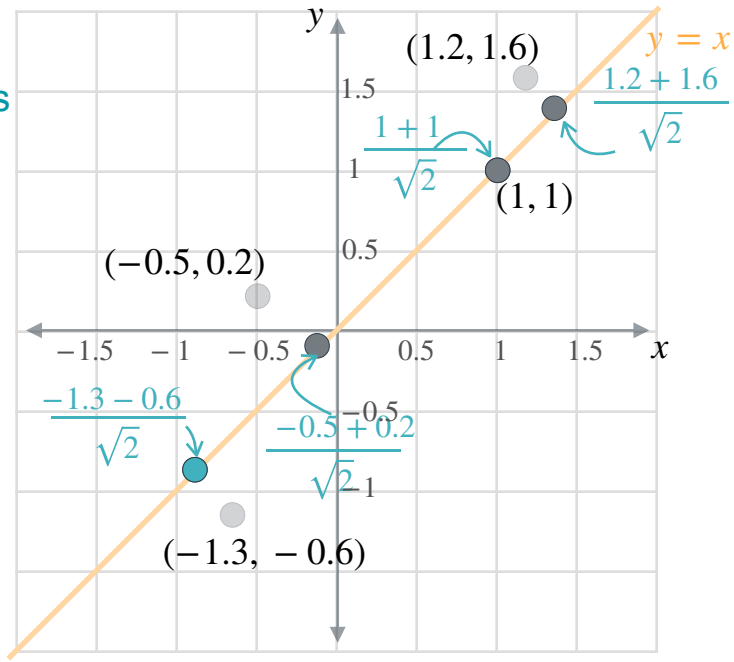


# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{matrix}1\\1\end{matrix}\ \frac{1}{\sqrt{2}} = \begin{matrix}(1+1)/\sqrt{2}\\(1.2+1.6)/\sqrt{2}\\(-0.5+0.2)/\sqrt{2}\\(-1.3,-0.6)/\sqrt{2}\end{matrix}$$

Final coordinates



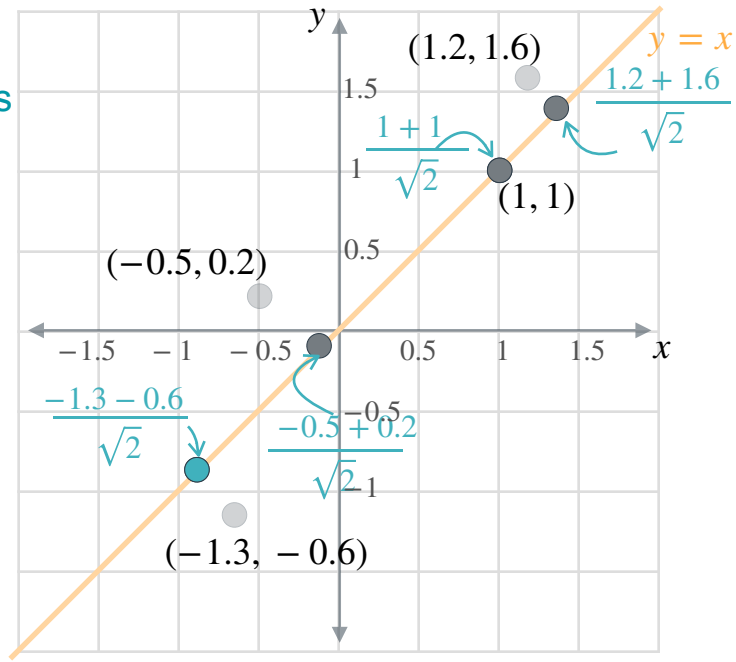
# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

| |
|---|
| 1.4142 |
| 1.9799 |
| -0.2121 |
| -1.344 |



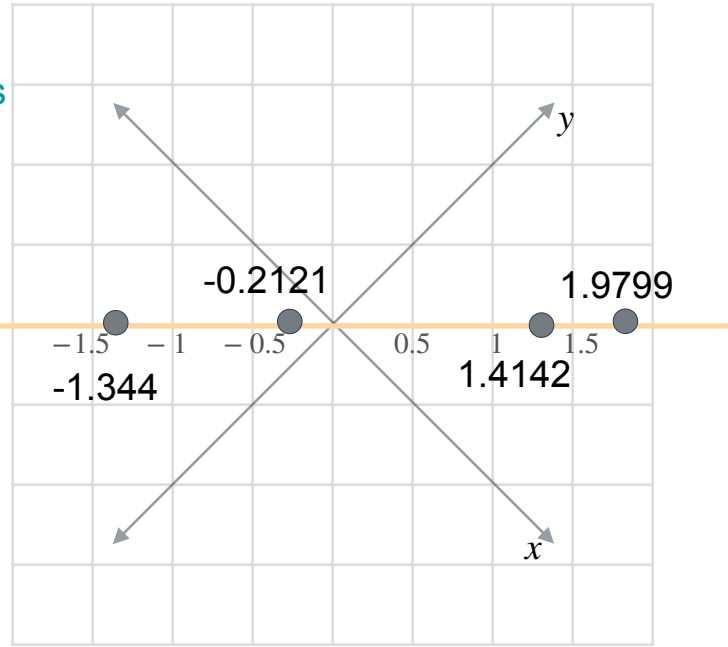
# Projections

| x | y |
|---|---|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

| |
|---|
| 1.4142 |
| 1.9799 |
| -0.2121 |
| -1.344 |



# Projections

To project a matrix $A$ onto a vector $v$

$$A_P = A \frac{v}{\|v\|_2}$$

$r \times 1 \quad r \times c \quad c \times 1$

# Projections

To project a matrix $A$ onto vectors $v_1$ and $v_2$

$$A_P = A \overbrace{\left[ \frac{v_1}{\|v_1\|_2} \quad \frac{v_2}{\|v_2\|_2} \right]}^{V}$$

$r \times 2 \qquad r \times c \qquad c \times 2$

# Projections

To project a matrix $A$ onto vectors $v_1$ and $v_2$

$$A_P = A \overbrace{\left[ \frac{v_1}{\|v_1\|_2} \quad \frac{v_2}{\|v_2\|_2} \right]}^{V}$$

$r \times 2 \qquad r \times c \qquad c \times 2$

# Projections

To project a matrix $A$ onto vectors $v_1$ and $v_2$

$$A_P = AV$$

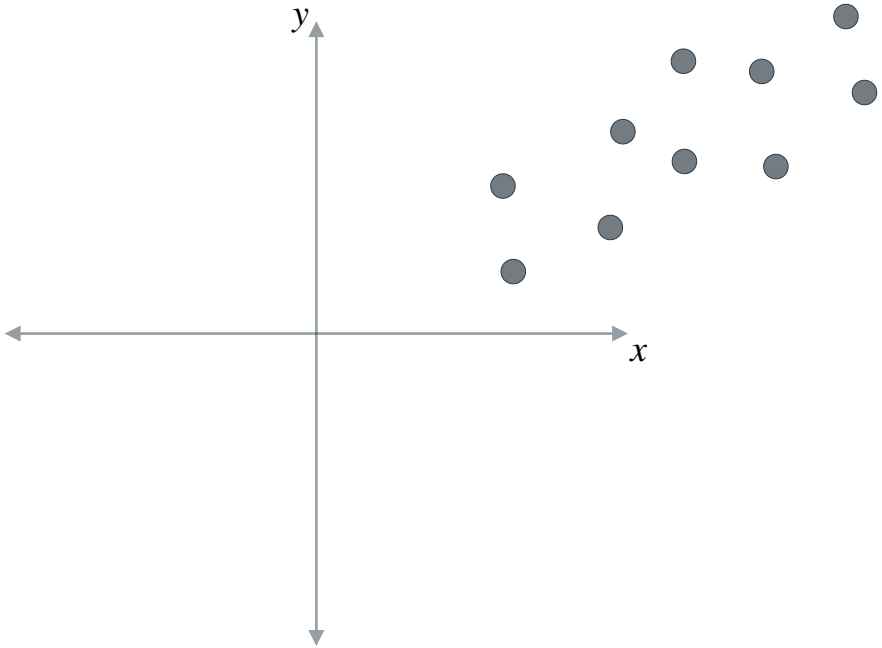
$r \times 2 \qquad r \times c \;\; c \times 2$

DeepLearning.AI

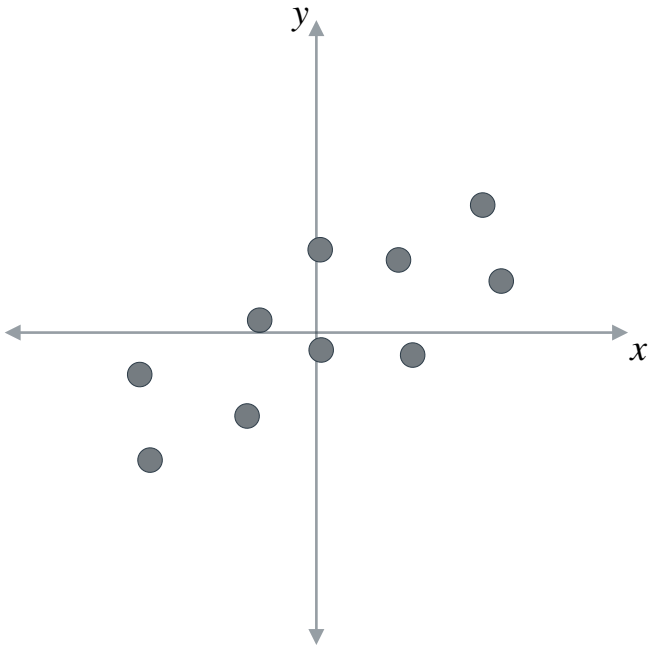
# Determinants and Eigenvectors

## Motivating PCA

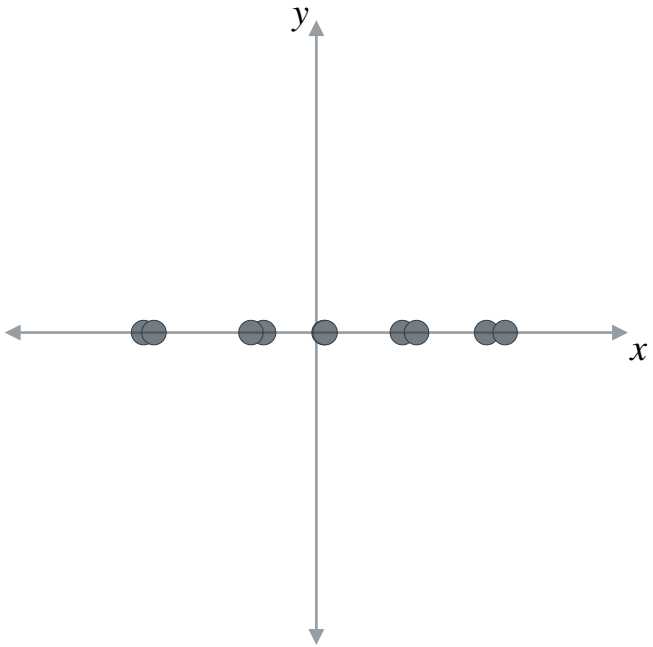
# Dimensionality Reduction



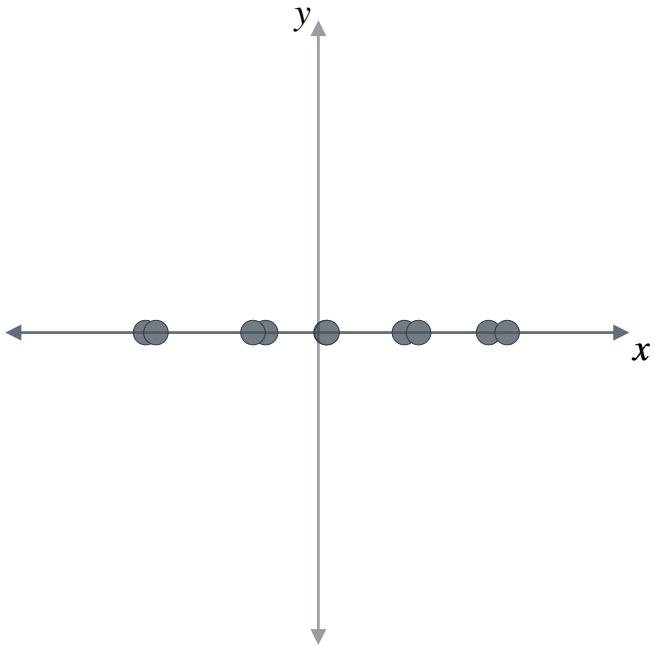
# Principal Component Analysis (PCA)



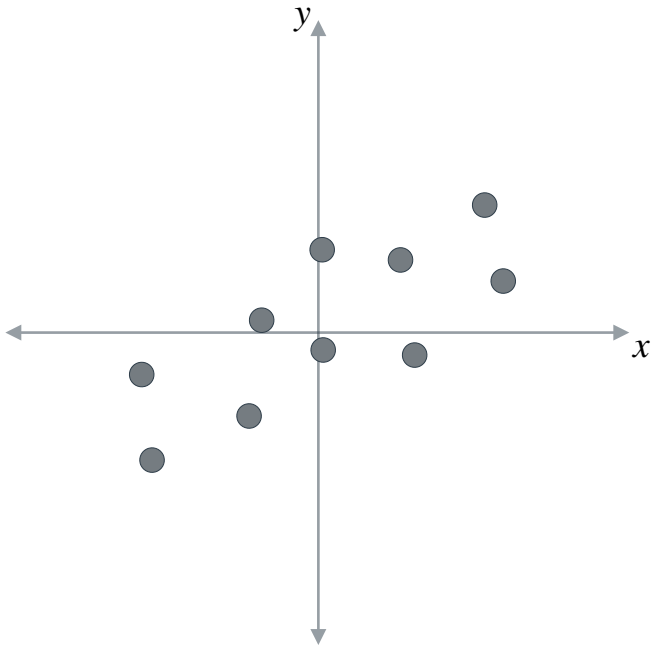
# Principal Component Analysis (PCA)



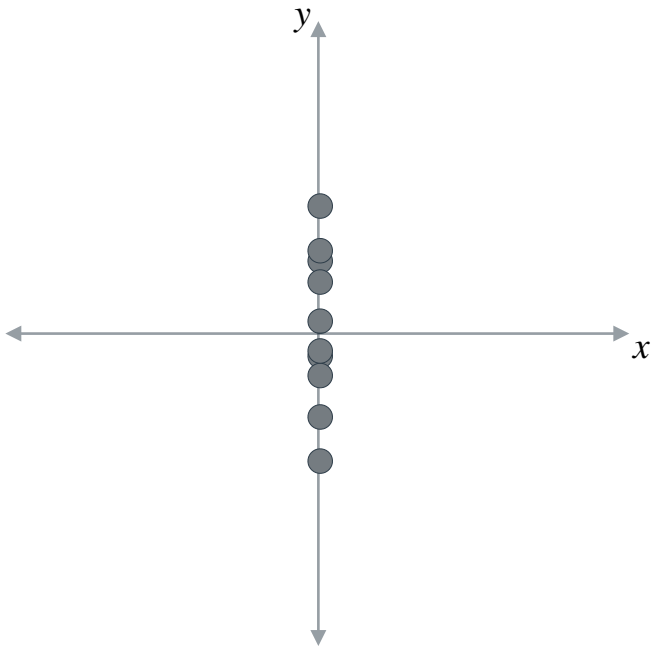
# Principal Component Analysis (PCA)



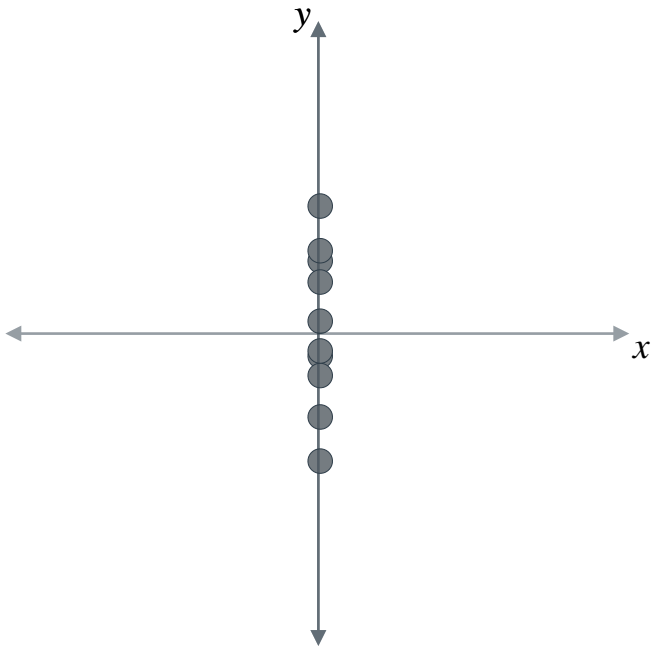
# Principal Component Analysis (PCA)



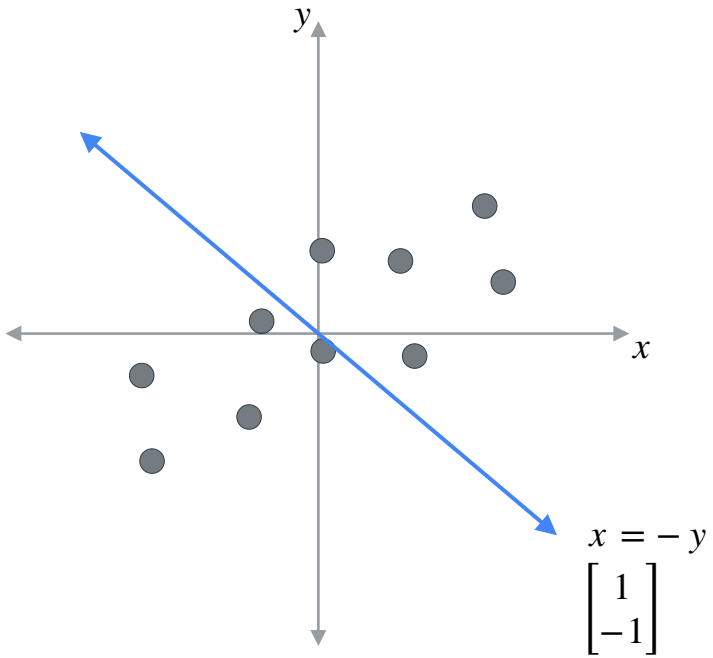
# Principal Component Analysis (PCA)

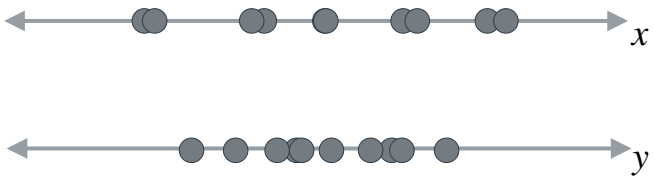


# Principal Component Analysis (PCA)

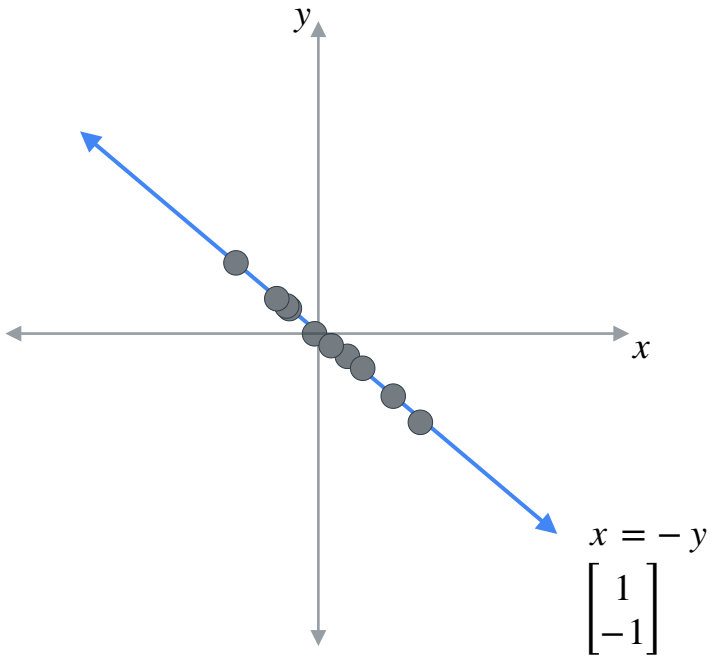


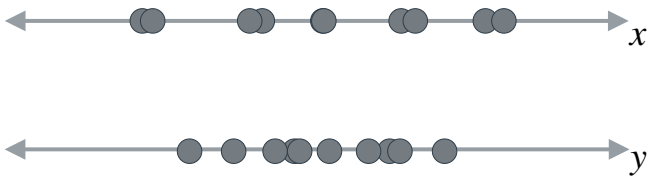
# Principal Component Analysis (PCA)



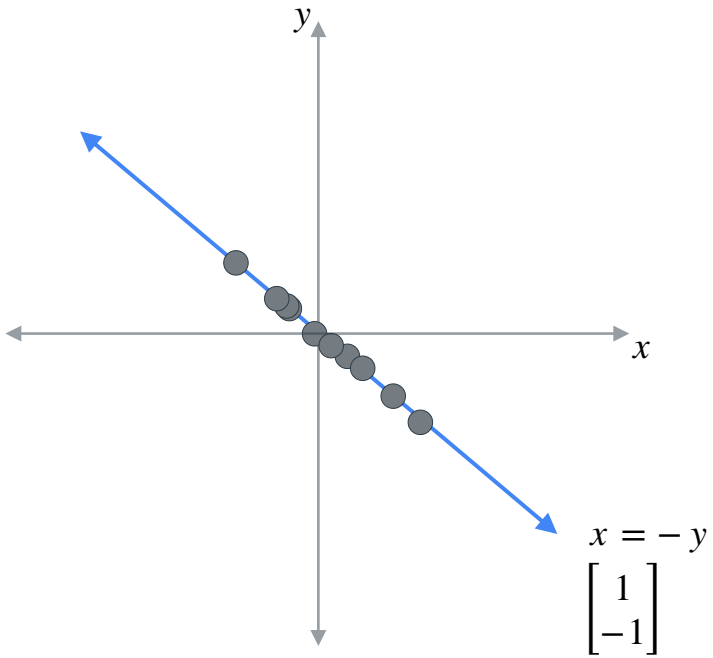



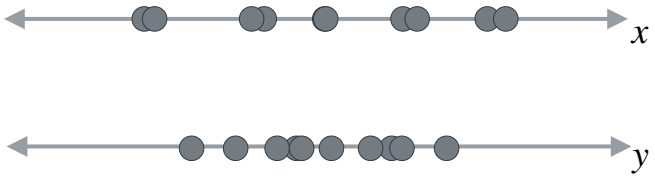
# Principal Component Analysis (PCA)



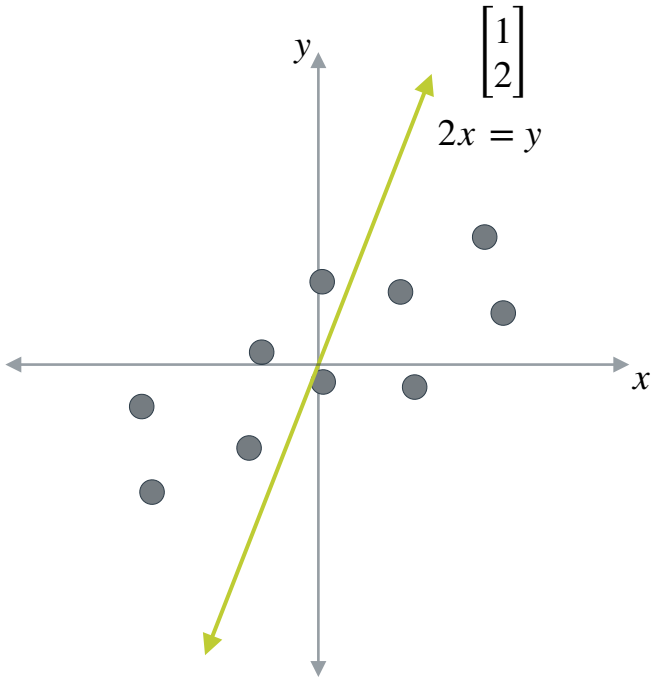



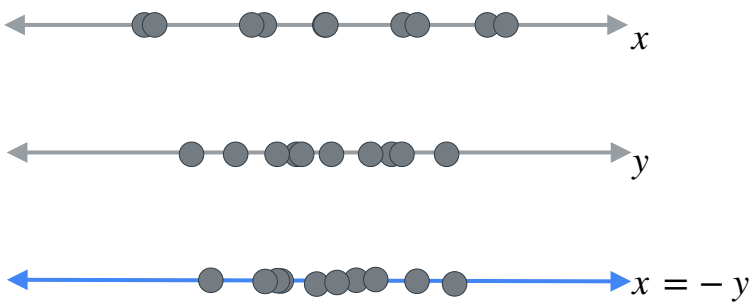
# Principal Component Analysis (PCA)



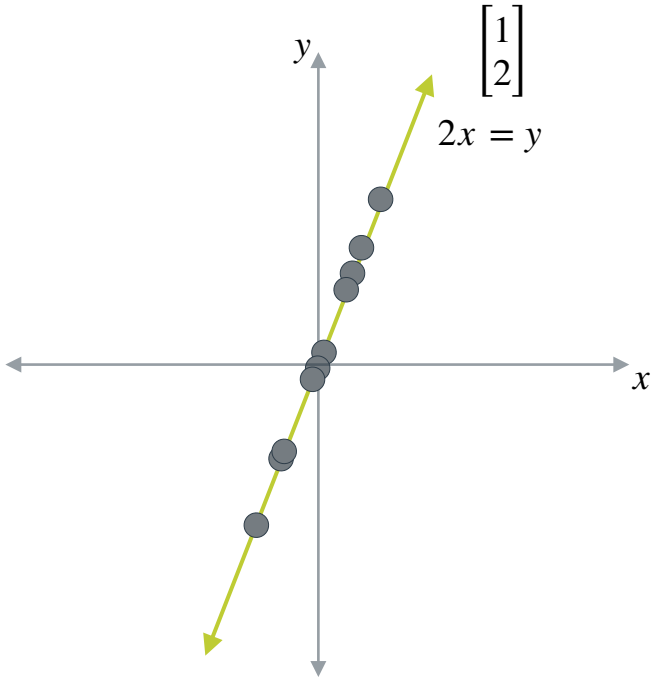



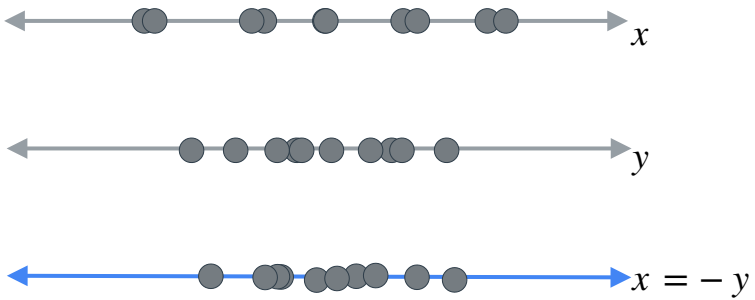
# Principal Component Analysis (PCA)



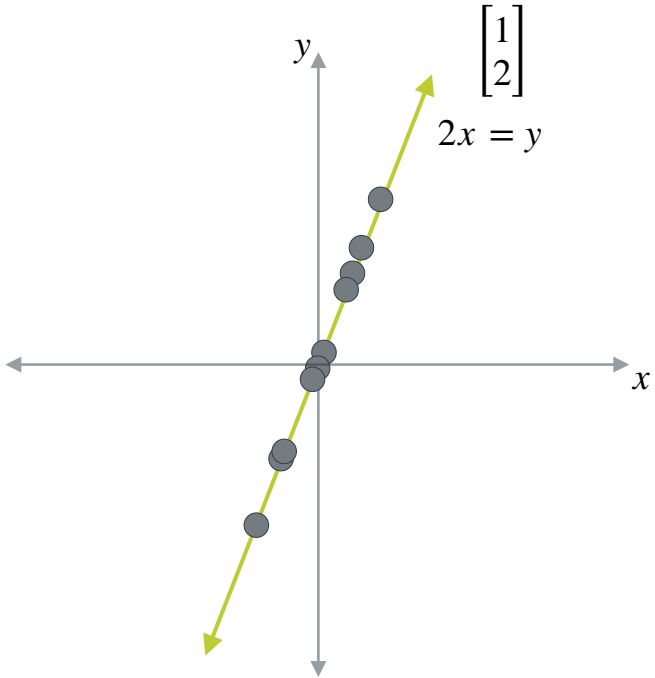



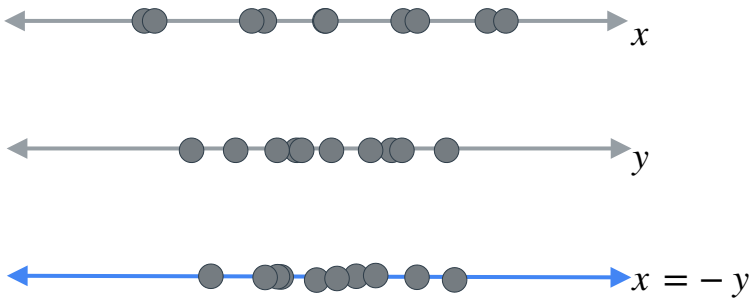
# Principal Component Analysis (PCA)



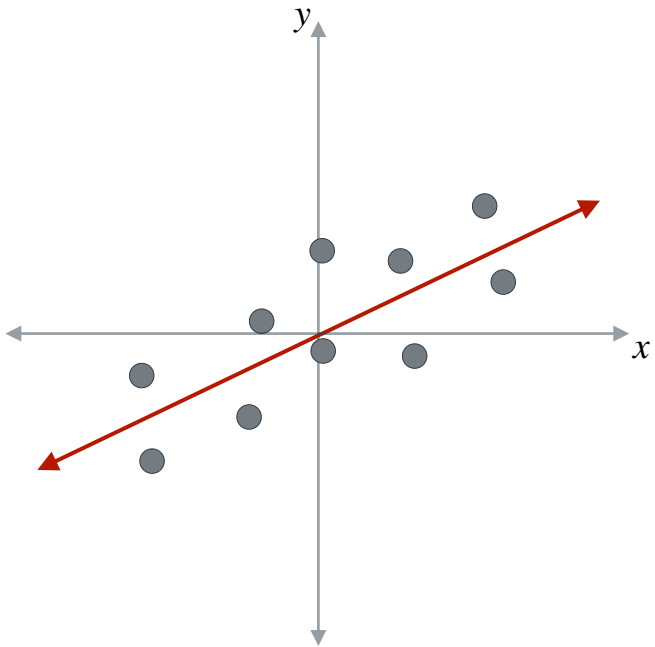



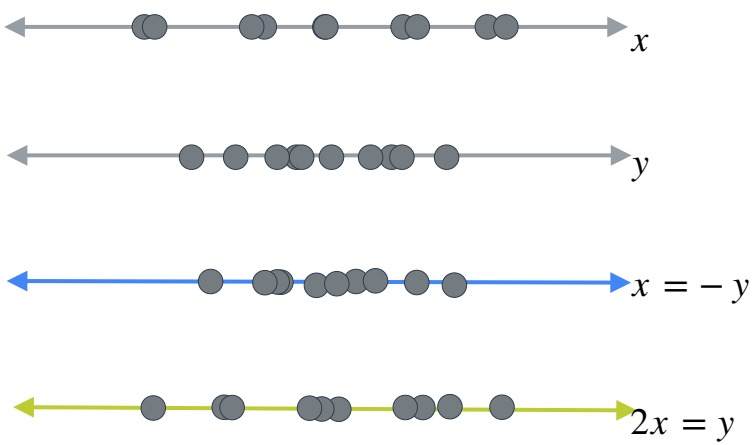
# Principal Component Analysis (PCA)



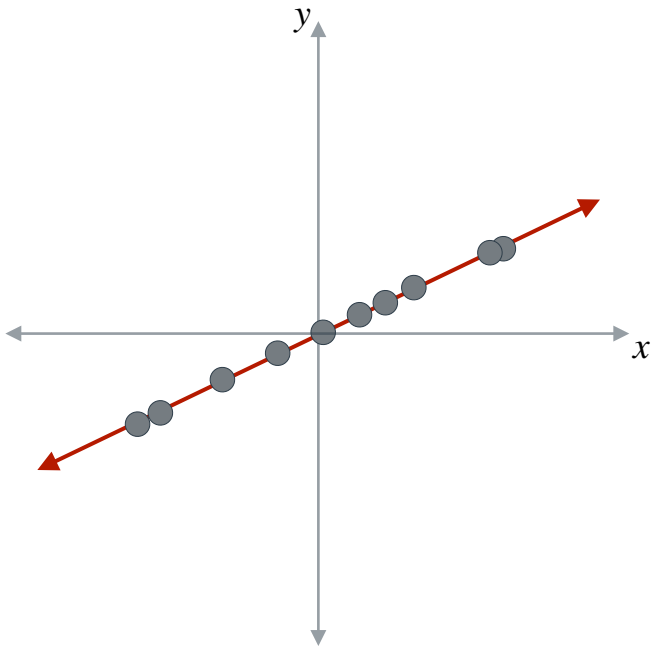



# Principal Component Analysis (PCA)

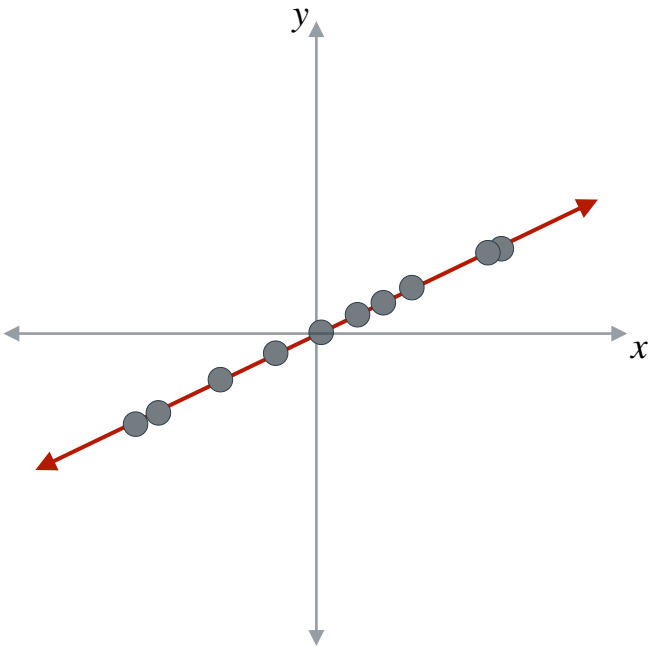


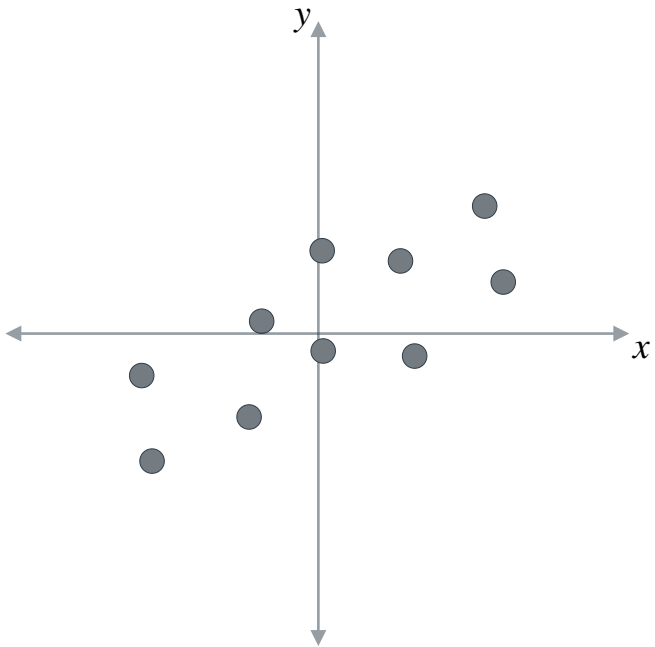
# Principal Component Analysis (PCA)

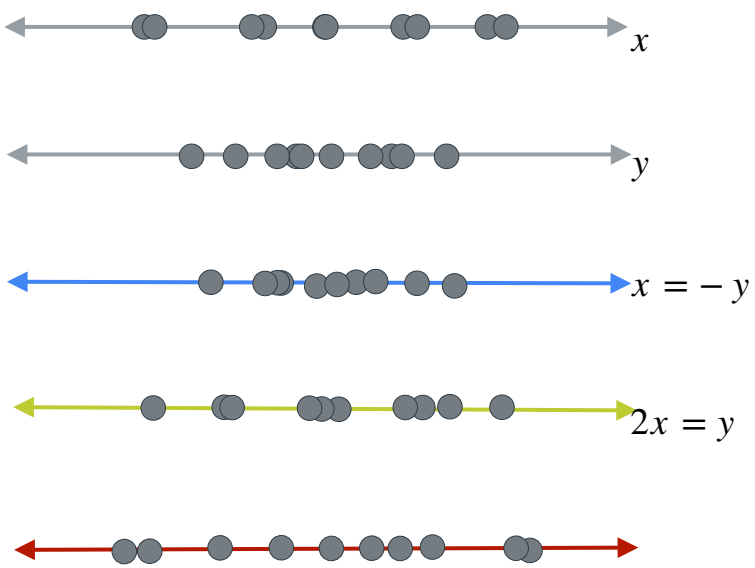


# Principal Component Analysis (PCA)

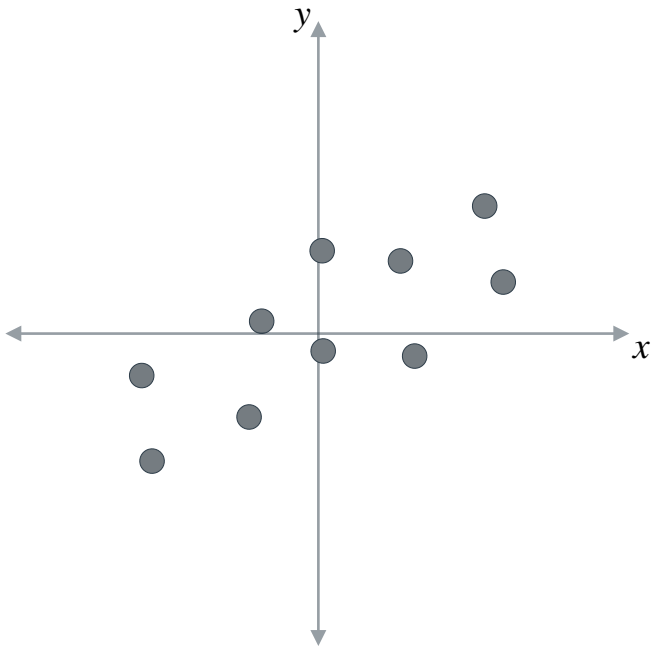


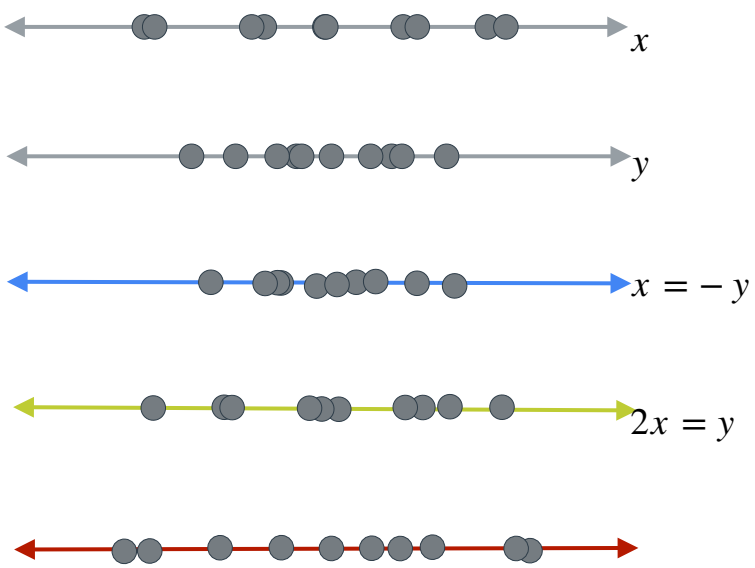
# Principal Component Analysis (PCA)



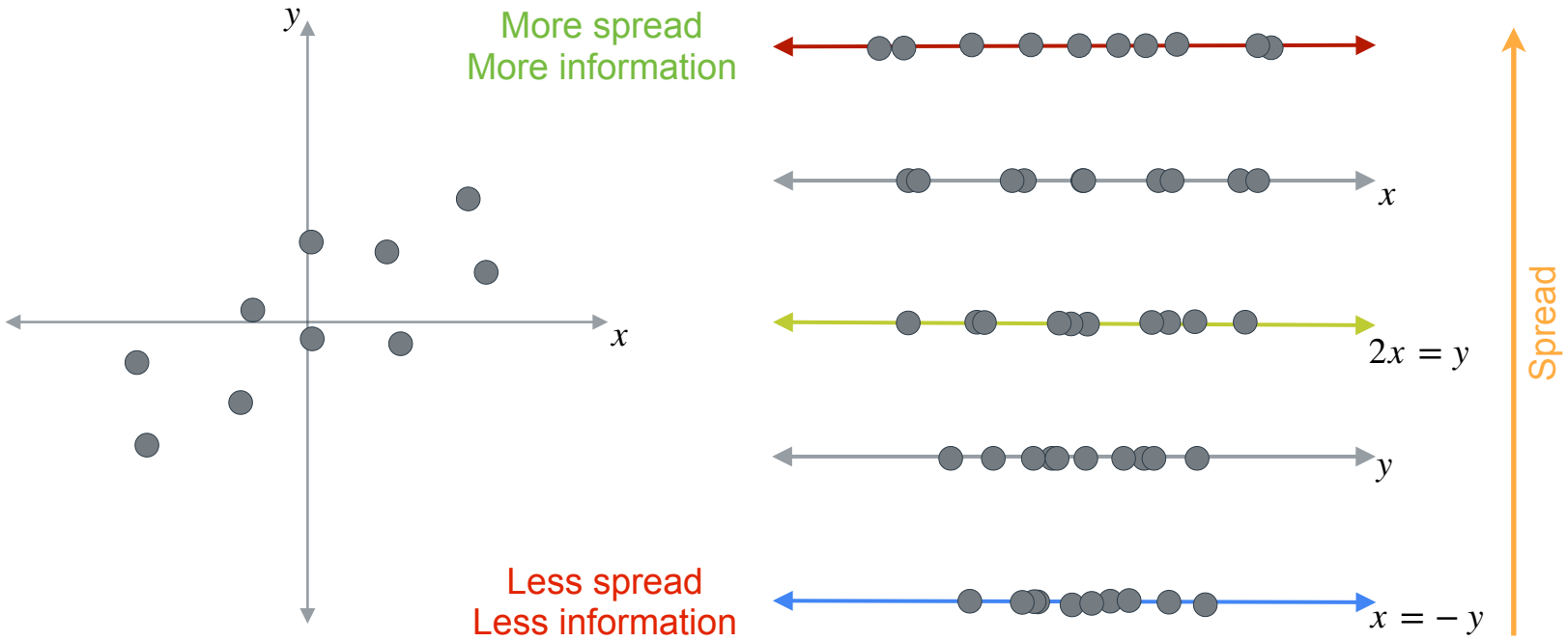



# Principal Component Analysis (PCA)



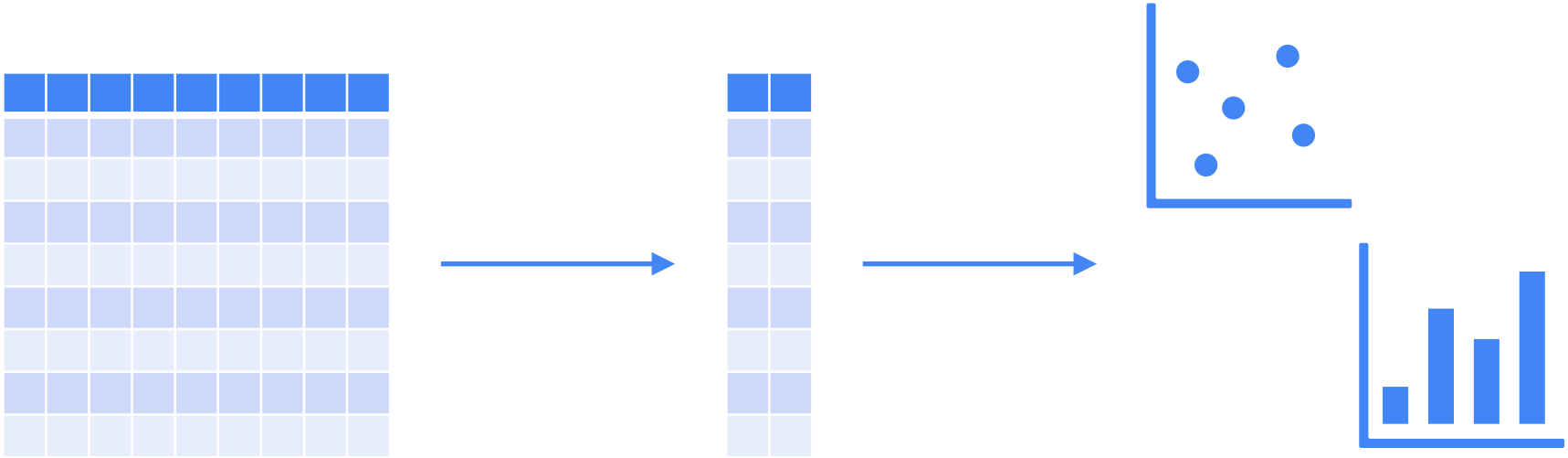



# Principal Component Analysis (PCA)



# Benefits of Dimensionality Reduction

- Easier dataset to manage
- PCA reduces dimensions while minimizing information loss
- Simpler visualization



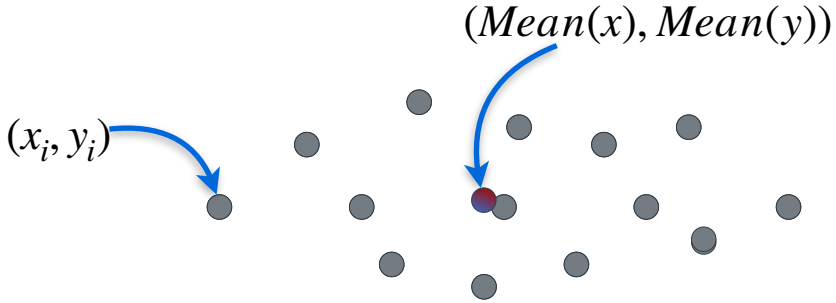
DeepLearning.AI

# Determinants and Eigenvectors

## Variance and covariance

# Mean

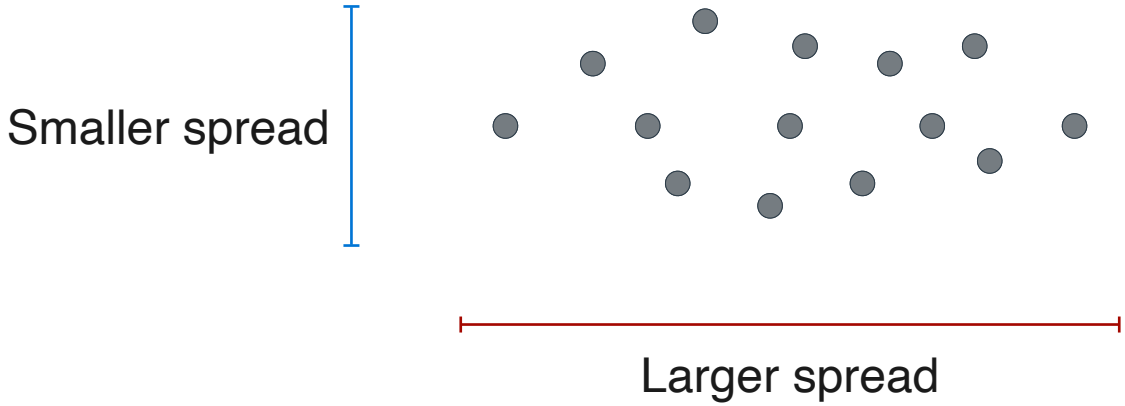
"The average of the data"



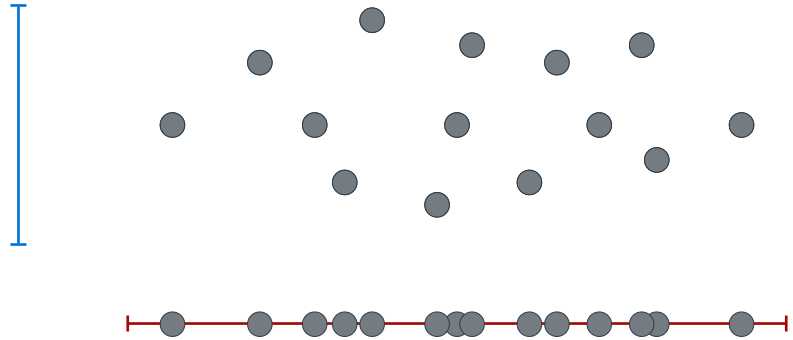

$$Mean(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$Mean(y) = \frac{1}{n}\sum_{i=1}^{n} y_i$$

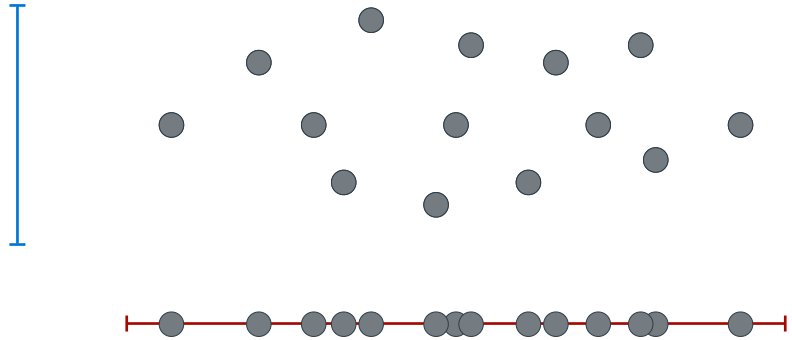
# Variance



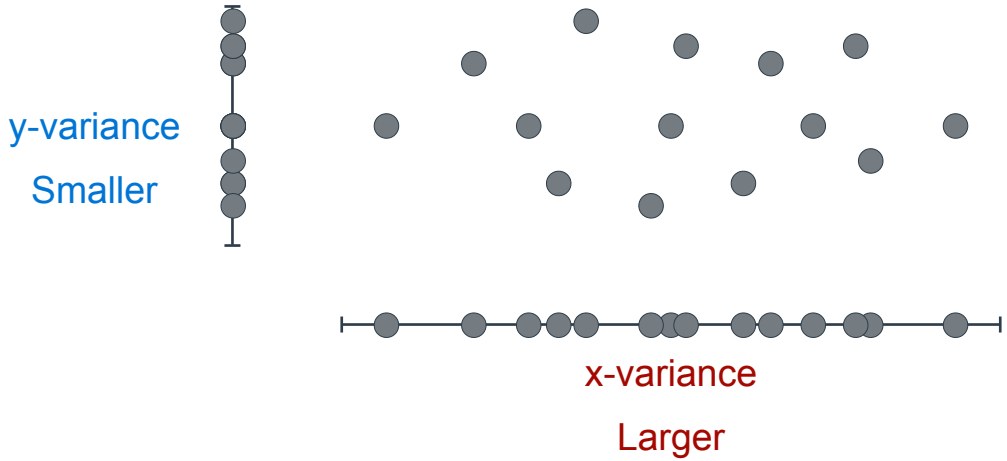
# Variance



# Variance



# Variance



# Variance

$$Variance(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - Mean(x))^2 \quad = 16$$

| | $x_i$ | $x_i$ - $Mean(x)$ | $(x_i - Mean(x))^2$ |
|---|---|---|---|
| 1 | 10 | 1 | 1 |
| 2 | 4 | -5 | 25 |
| 3 | 11 | 2 | 4 |
| 4 | 14 | 5 | 25 |
| 5 | 6 | -3 | 9 |

→ 64

$$Mean(x) = 9$$

# Variance

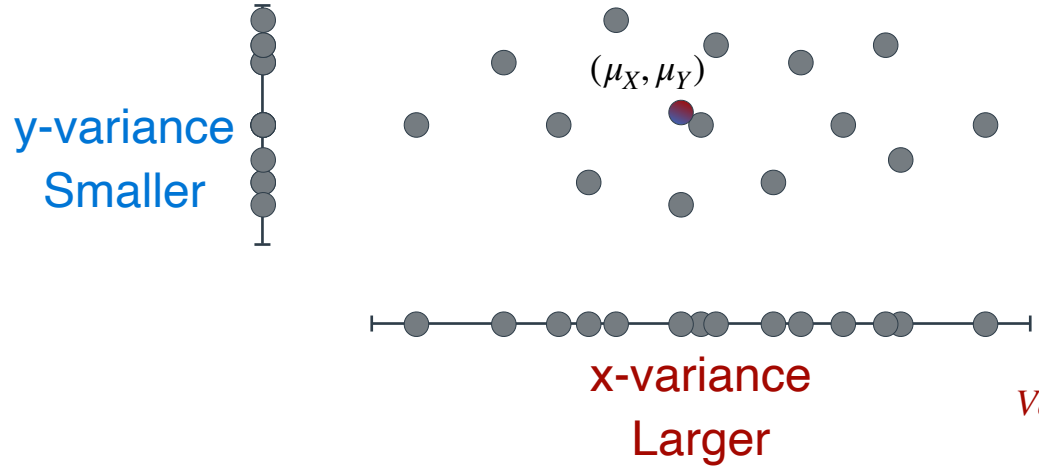
$$Variance(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - Mean(x))^2$$

$Var(x)$ $\mu_x$

“The average squared distance from the mean”

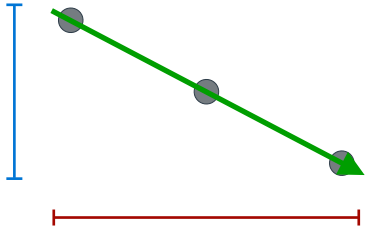
# Variance

$$Var(y) = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \mu_Y)^2$$

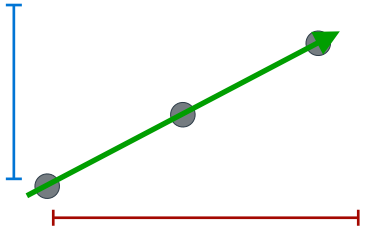



$$Var(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2$$

# Problem



Solution: Covariance



Negative covariance

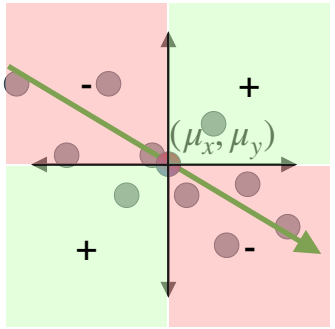
Positive covariance

# Covariance

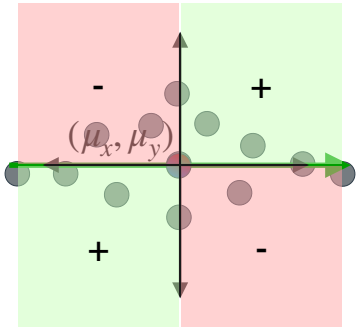
"Take the average"

$$Cov(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)$$

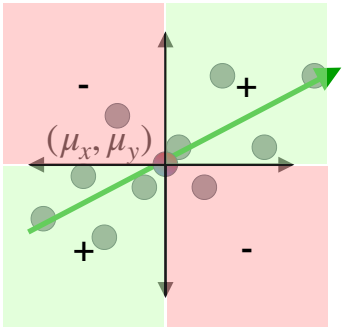
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)^2 (x_i - \mu_x)$$




negative covariance




covariance zero (or very small)

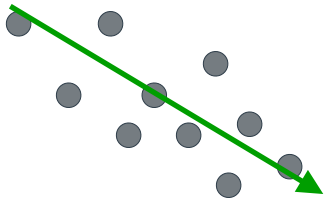



positive covariance

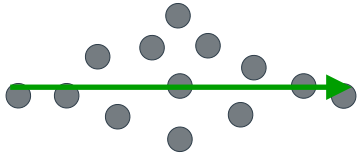
# Covariance

$$Cov(x, y) = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \mu_x)(y_i - \mu_y)$$

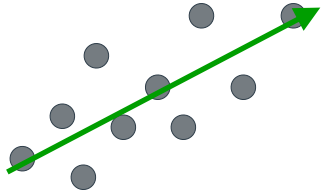
“The direction of the relationship between two variables”



negative
covariance



covariance zero
(or very small)



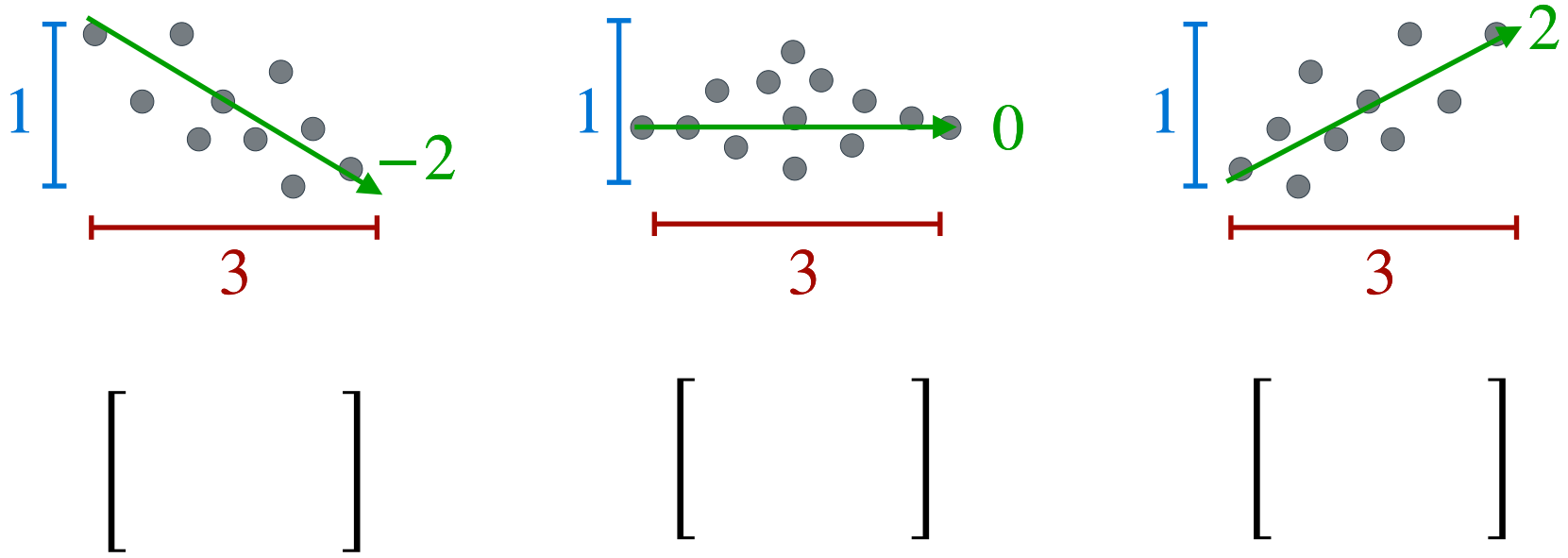
positive
covariance

DeepLearning.AI

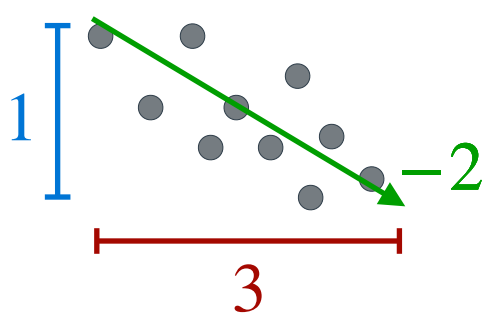
# Determinants and Eigenvectors

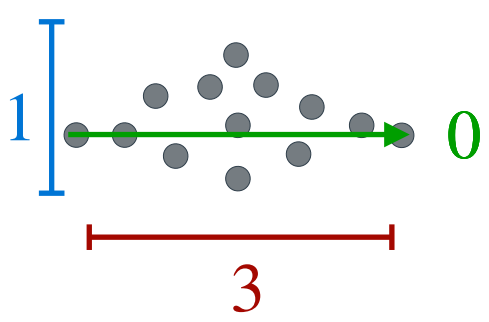
## The covariance matrix

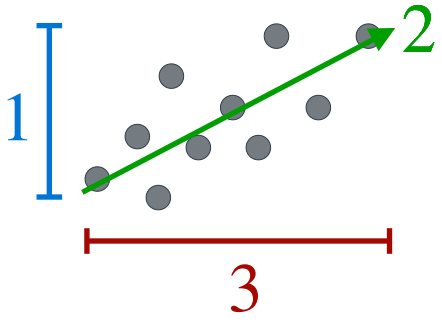
# Covariance matrix



# Covariance matrix



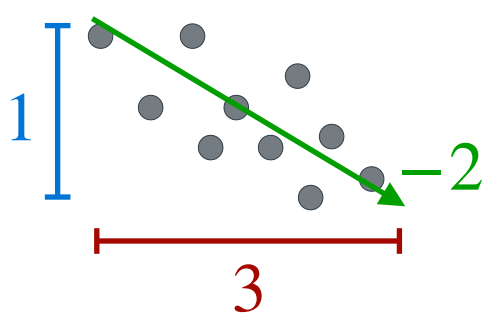




$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

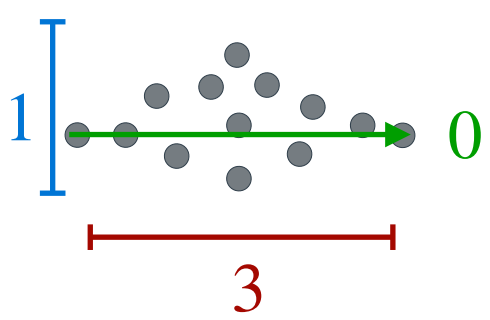
$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

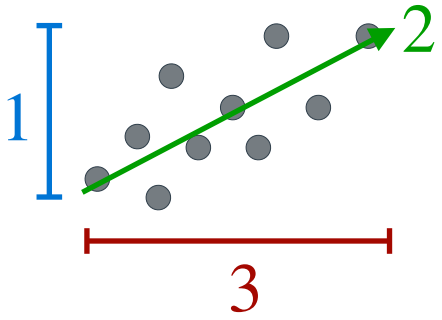
# Covariance matrix




$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

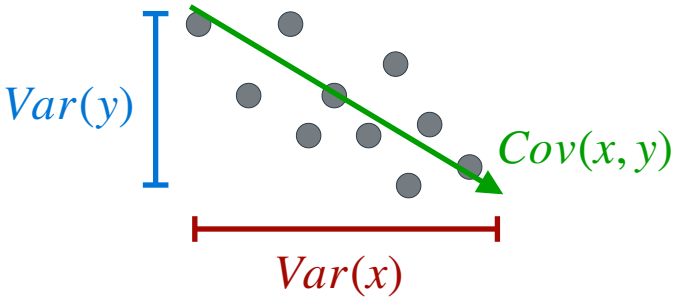



$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$




$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

# Covariance matrix




$$C = \begin{matrix} & \begin{matrix} x & \quad y \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(y,x) & Var(y) \end{bmatrix} \end{matrix}$$

$$Cov(x, x) = Var(x)$$

# Covariance matrix

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(\quad - \quad)^T(\quad - \quad)$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu)$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left( \quad - \quad \right)^T \left( \quad - \quad \right)$$

$$= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}$$

# Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\underbrace{\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix}}_{2 \times n} \underbrace{\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}}_{n \times 2}$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$(x_1-\mu_x)(x_1-\mu_x) + (x_2-\mu_x)(x_2-\mu_x) + \dots + (x_n-\mu_x)(x_n-\mu_x)$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$\sum_{i=1}^{n}(x_i-\mu_x)^2 = Var(x)$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \\ & \end{bmatrix}$$

$$\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)^2 = Var(x)$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \\ & \end{bmatrix}$$

$$(x_1-\mu_x)(y_1-\mu_y) + (x_2-\mu_x)(y_2-\mu_y) + \dots + (x_n-\mu_x)(y_n-\mu_y)$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & \\ & \end{bmatrix}$$

$$\sum_{i=1}^{n}(x_i-\mu_x)(y_i-\mu_y) = Cov(x,y)$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & Cov(x,y) \\ & \end{bmatrix}$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\mu_x)(y_i-\mu_y) = Cov(x,y)$$

# Covariance matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(y,x) & \end{bmatrix}$$

# Covariance matrix

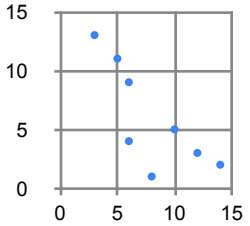
$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu) = \frac{1}{n-1}\left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}\right)$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}^T \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix}$$

$$= \frac{1}{n-1}\begin{bmatrix} x_1-\mu_x & x_2-\mu_x & \dots & x_n-\mu_n \\ y_1-\mu_y & y_2-\mu_y & \dots & y_n-\mu_y \end{bmatrix} \begin{bmatrix} x_1-\mu_x & y_1-\mu_y \\ x_2-\mu_x & y_2-\mu_y \\ \vdots & \vdots \\ x_n-\mu_x & y_n-\mu_y \end{bmatrix} = \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(y,x) & Var(y) \end{bmatrix}$$

# Matrix formula

$$A - \mu = \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \qquad C = \frac{1}{n-1}(A - \mu)^T(A - \mu)$$




Similar X and Y variances
Negative covariance

$A$

| $x$ | $y$ |
|---|---|
| 10 | 5 |
| 12 | 3 |
| 6 | 9 |
| 6 | 4 |
| 5 | 11 |
| 14 | 2 |
| 8 | 1 |
| 3 | 13 |

$\mu_x = 8 \quad \mu_y = 6$

$A - \mu$

| $x - \mu_x$ | $y - \mu_y$ |
|---|---|
| 2 | -1 |
| 4 | -3 |
| -2 | 3 |
| -2 | -2 |
| -3 | 5 |
| 6 | -4 |
| 0 | -5 |
| -5 | 8 |

$\frac{1}{7}$

$(A - \mu)^T$

| $x - \mu_x$ | 2 | 4 | -2 | -2 | -3 | 6 | 0 | -5 |
|---|---|---|---|---|---|---|---|---|
| $y - \mu_y$ | -1 | -3 | 3 | -2 | 5 | -4 | -5 | 8 |

$A - \mu$

| $x - \mu_x$ | $y - \mu_y$ |
|---|---|
| 2 | -1 |
| 4 | -3 |
| -2 | 3 |
| -2 | -2 |
| -3 | 5 |
| 6 | -4 |
| 0 | -5 |
| -5 | 8 |

$$C = \begin{bmatrix} 14 & -11.86 \\ -11.86 & 19.71 \end{bmatrix}$$

# Matrix formula

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \qquad C = \frac{1}{n-1}(A-\mu)^T(A-\mu)$$

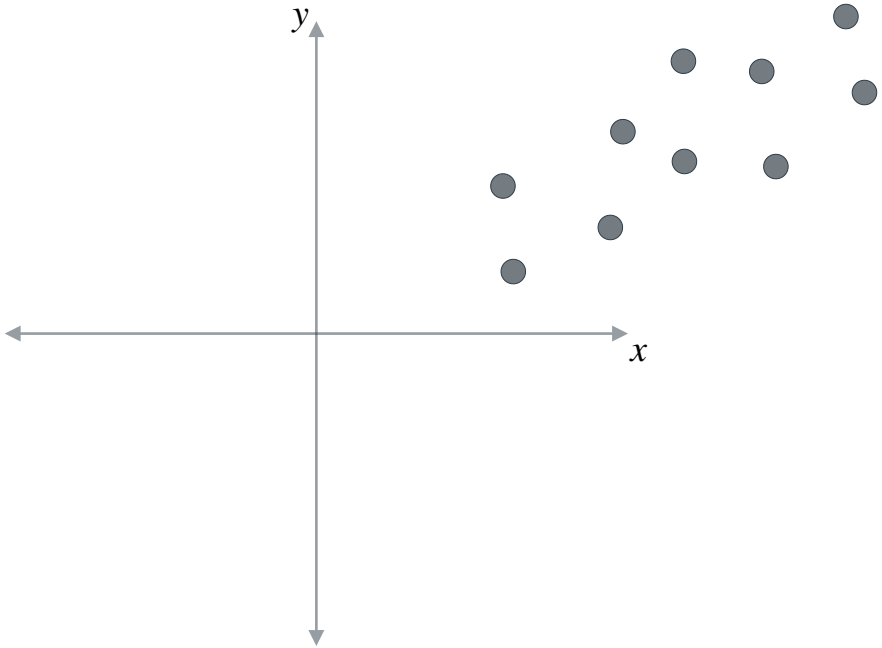
1. Arrange data with a different feature in each column
2. Calculate column averages
3. Subtract each average from their respective column to generate $A - \mu$
4. $\frac{1}{n-1}(A-\mu)^T(A-\mu)$ gives the covariance matrix $C$

DeepLearning.AI

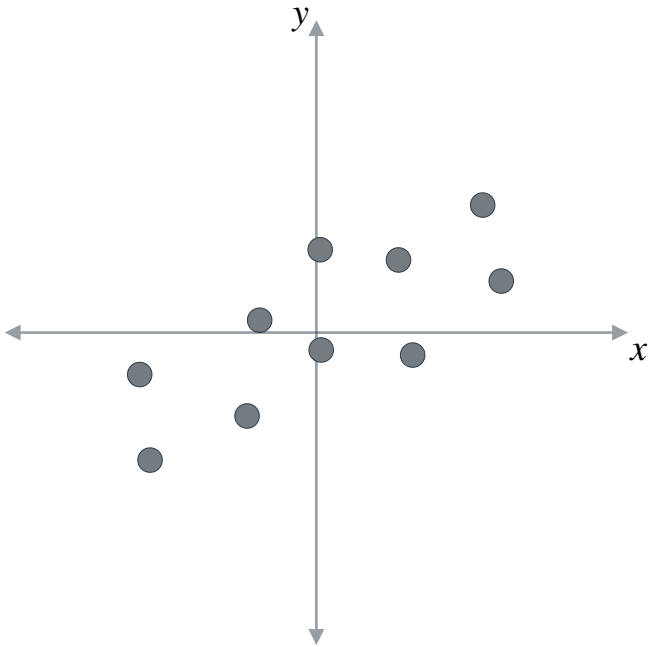
# Determinants and Eigenvectors

## PCA - Overview

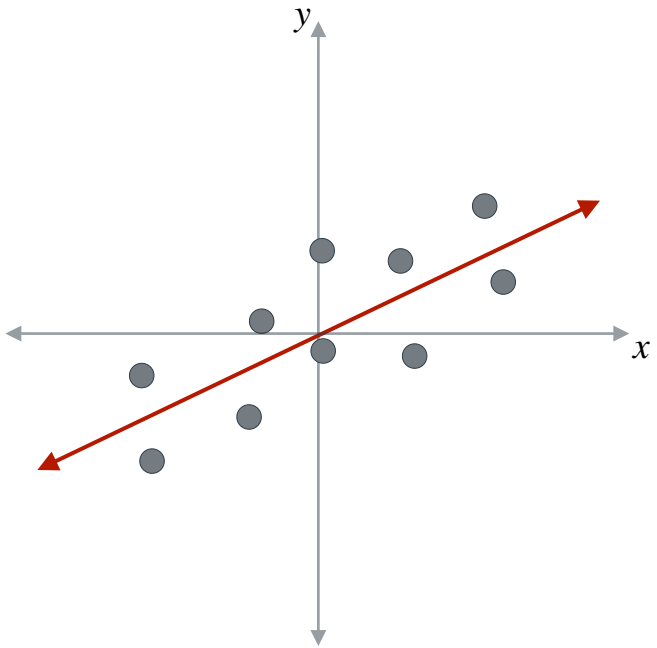
# Principal Component Analysis (PCA)



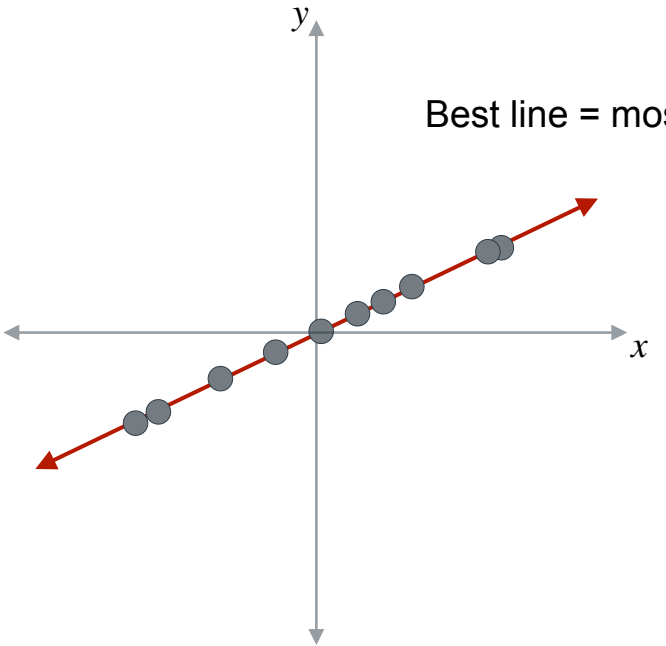
# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)

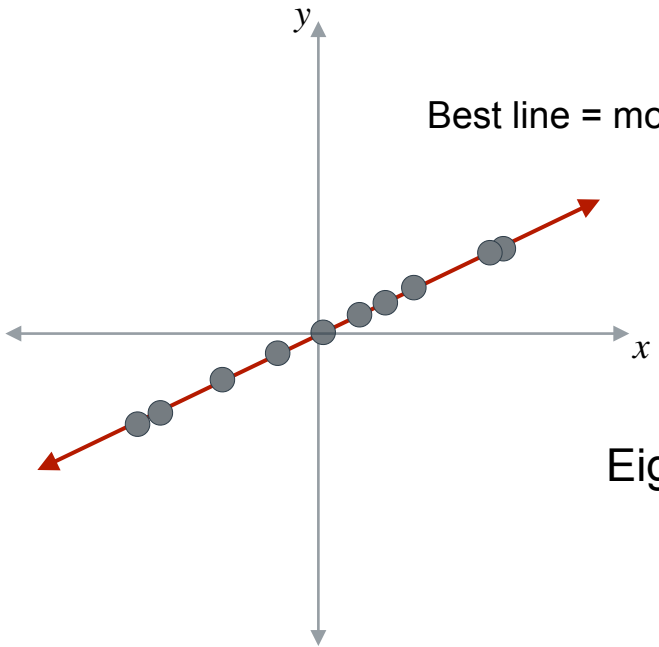


# Principal Component Analysis (PCA)




Best line = most variance

# Principal Component Analysis (PCA)




Best line = most variance

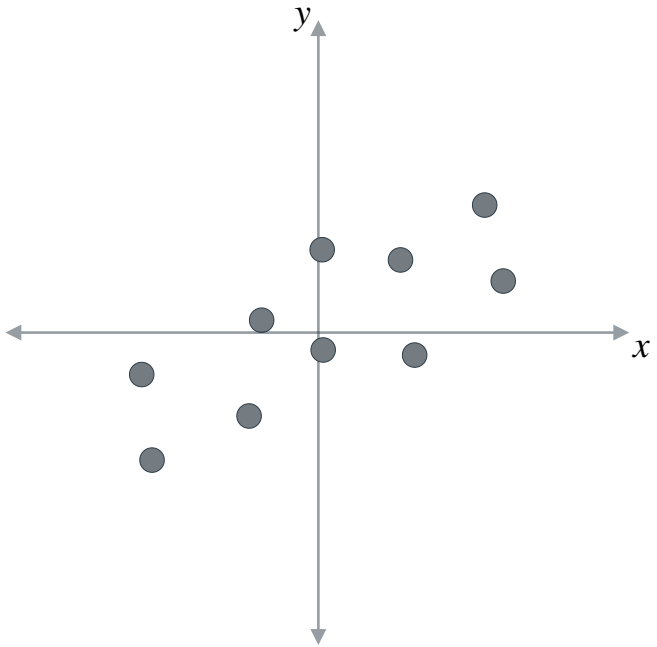
Projections

Eigenvalues / Eigenvectors

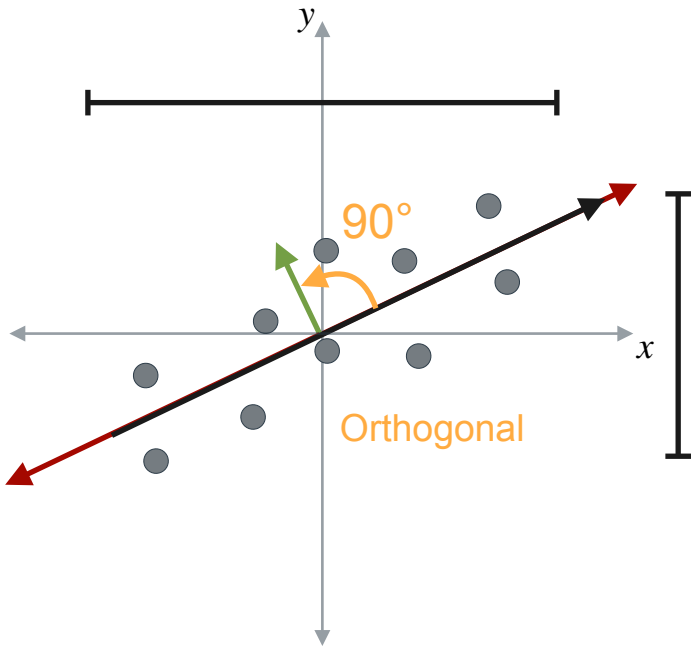
Covariance Matrix

PCA

# Principal Component Analysis (PCA)



# Principal Component Analysis (PCA)




$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

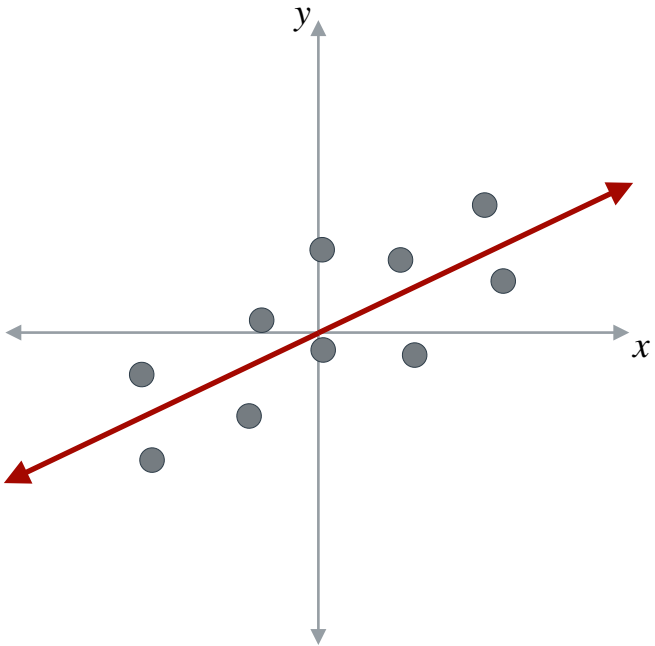
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Eigenvectors (direction)

11 1

Eigenvalues (magnitude)

# Principal Component Analysis (PCA)




$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

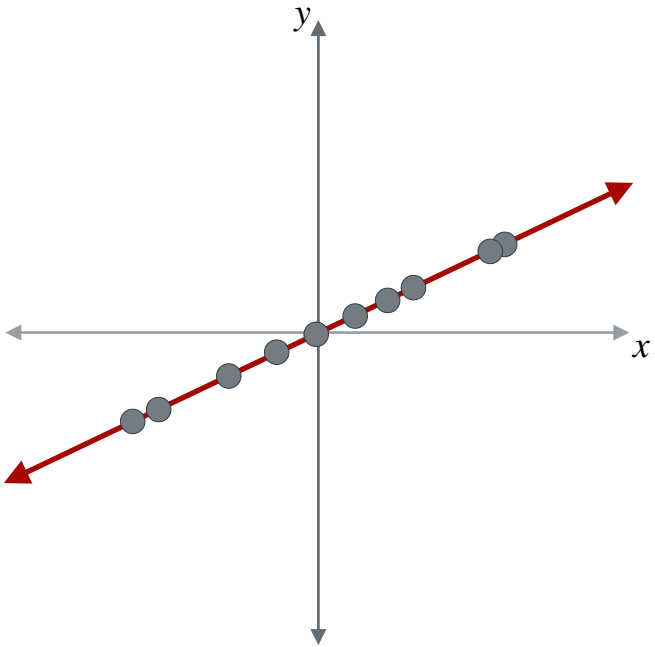
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors
(direction)

11

Eigenvalues
(magnitude)

# Principal Component Analysis (PCA)




$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

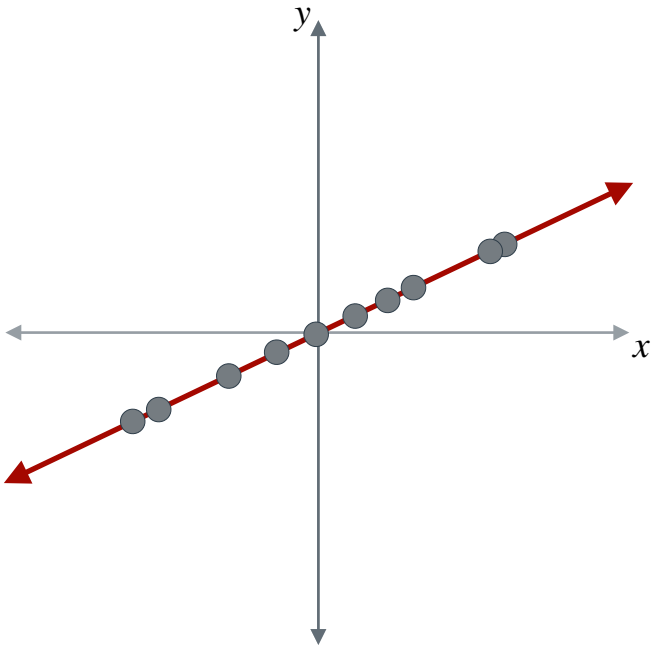
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors
(direction)

11

Eigenvalues
(magnitude)

# Principal Component Analysis (PCA)




$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

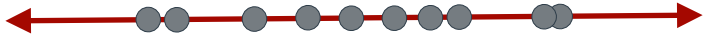
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors (direction)

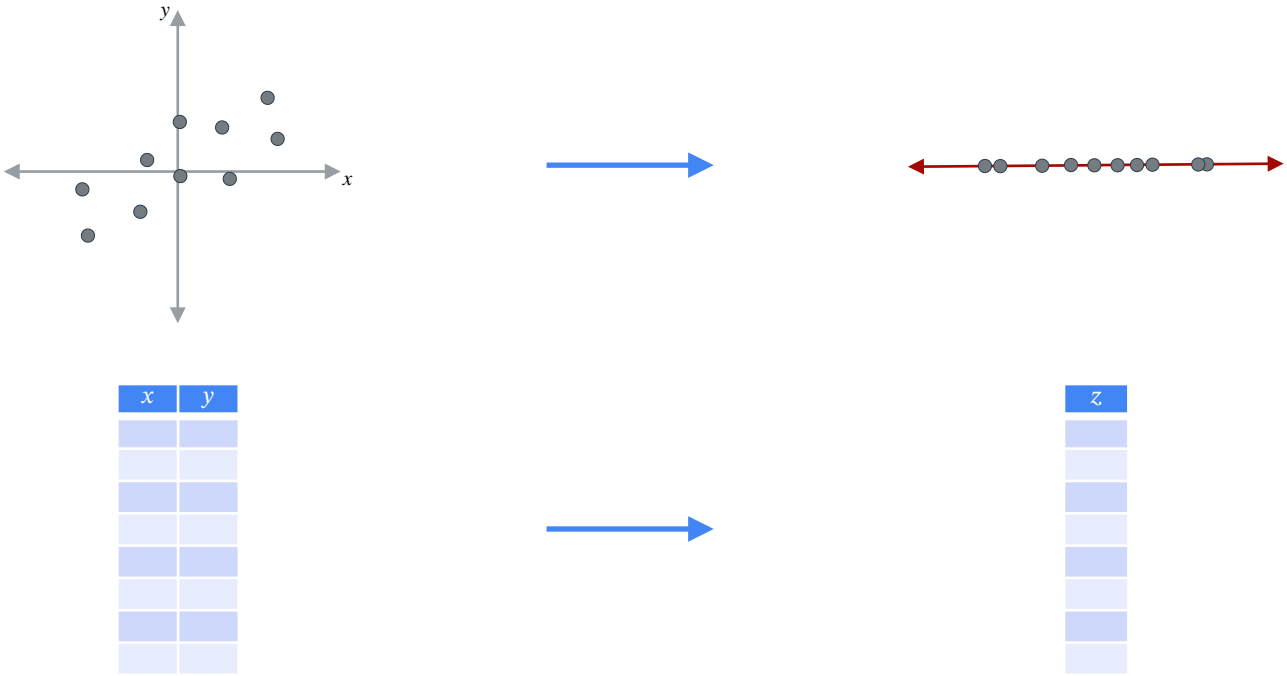
11

Eigenvalues (magnitude)

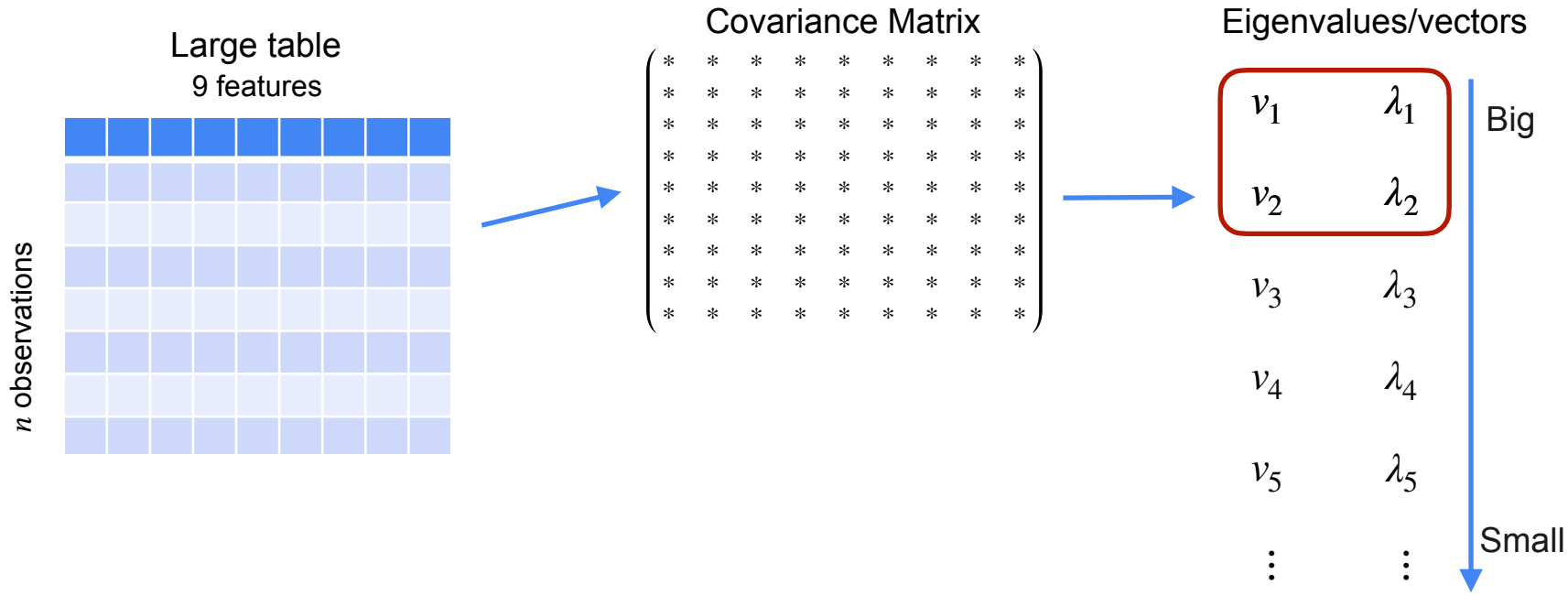
# Principal Component Analysis (PCA)



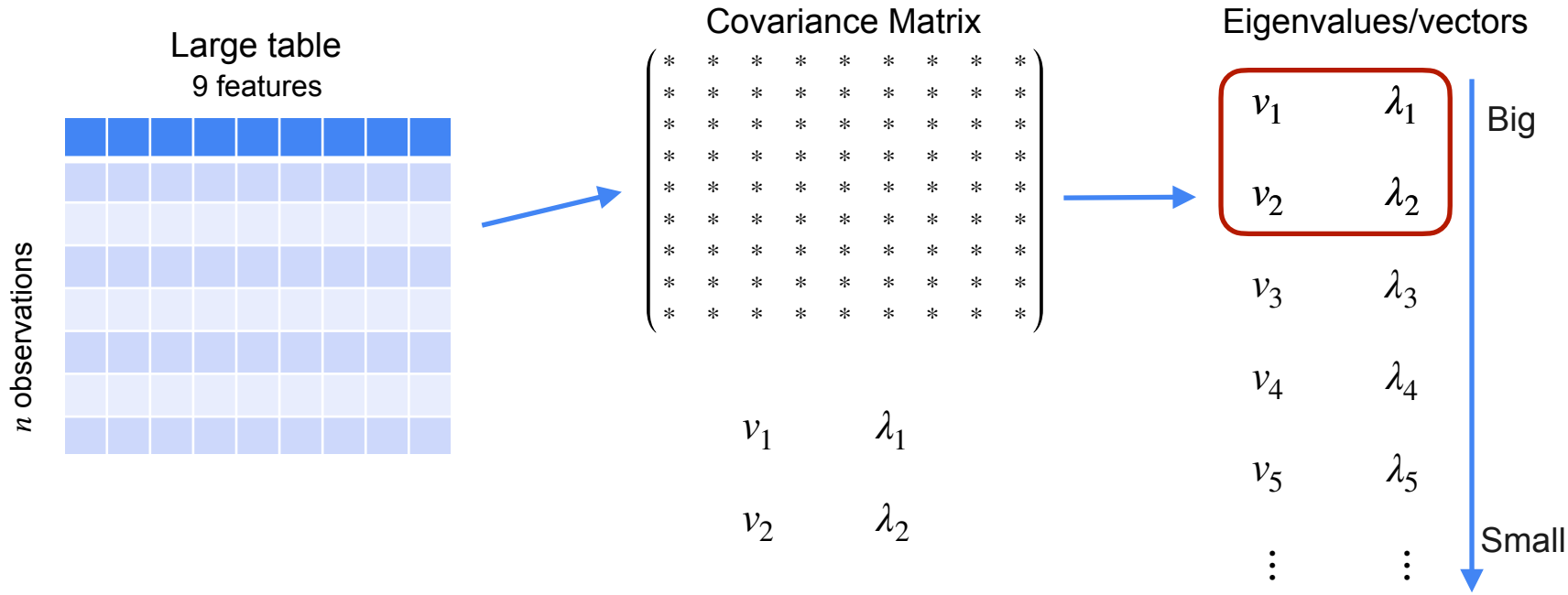
# Principal Component Analysis (PCA)



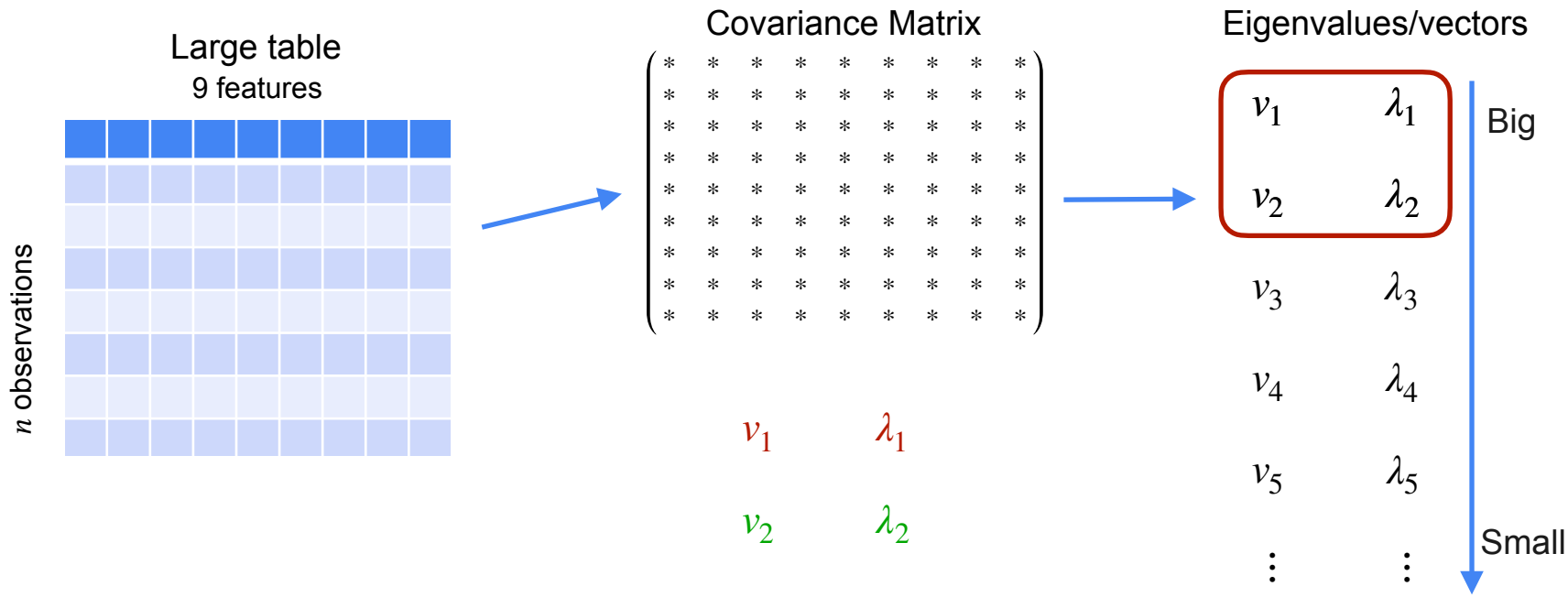
# PCA: Principal Component Analysis



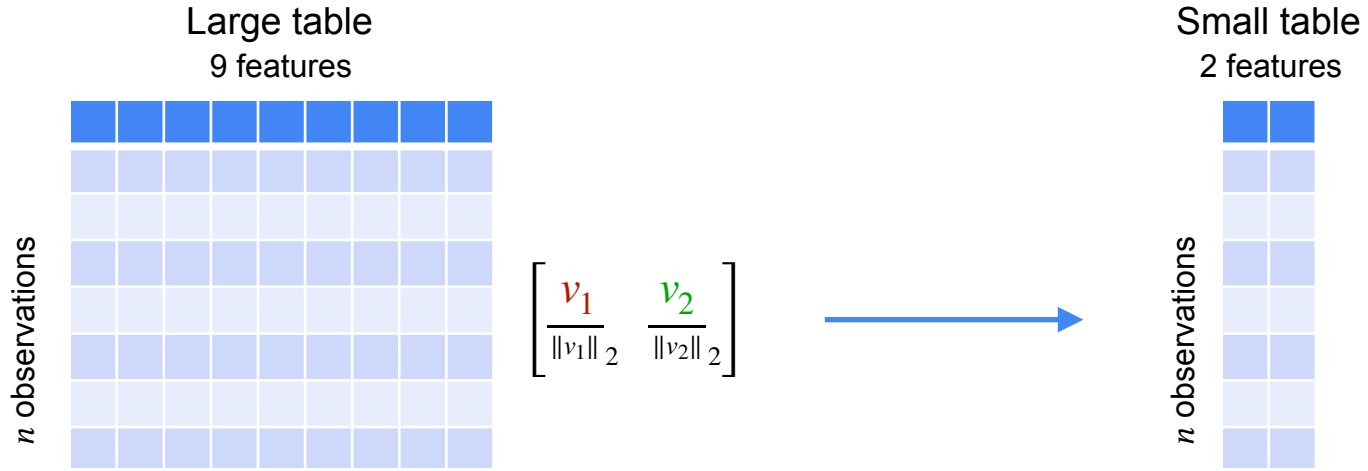
# PCA: Principal Component Analysis



# PCA: Principal Component Analysis



# PCA: Principal Component Analysis

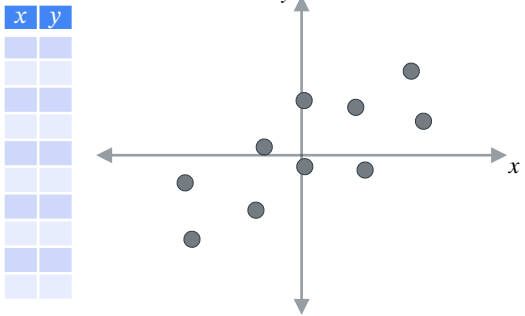


DeepLearning.AI

# Determinants and Eigenvectors

## PCA - Why it works

# PCA: Why It Works

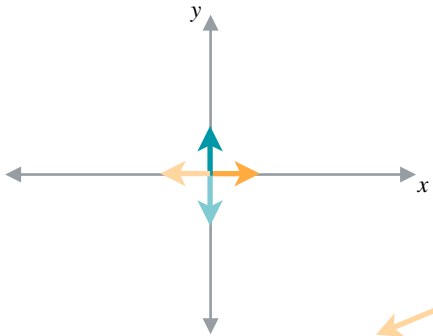


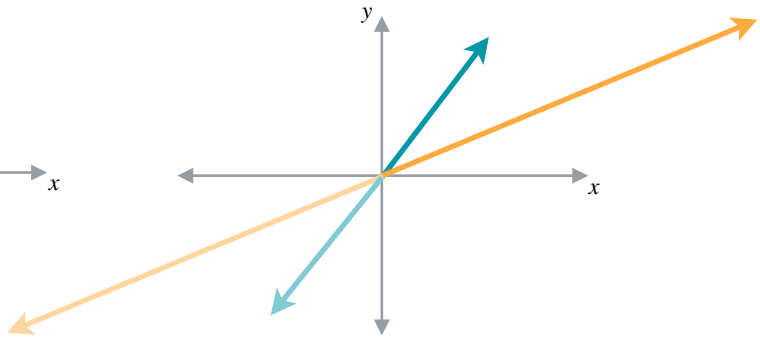

Covariance matrix

$$C$$

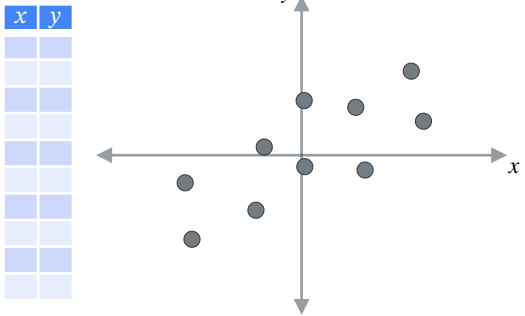
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

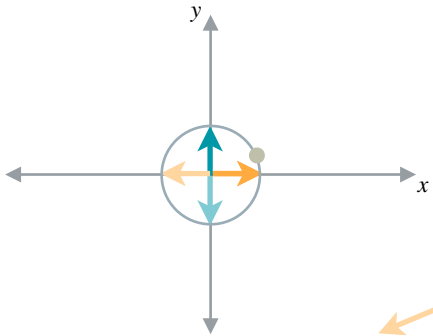


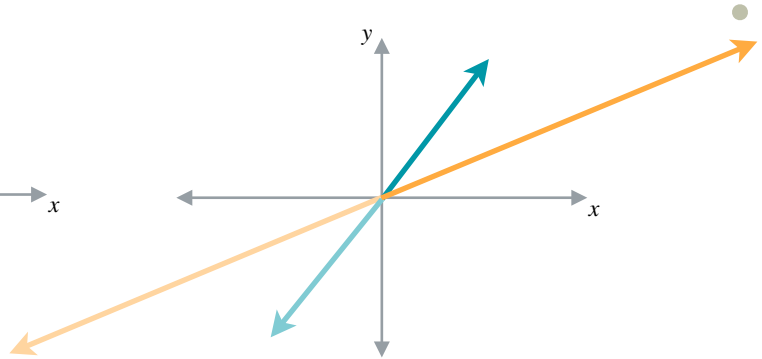

Covariance matrix

$$C$$

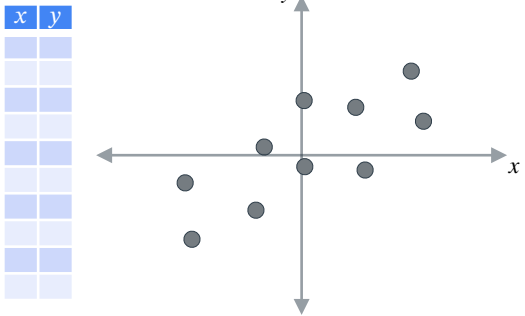
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

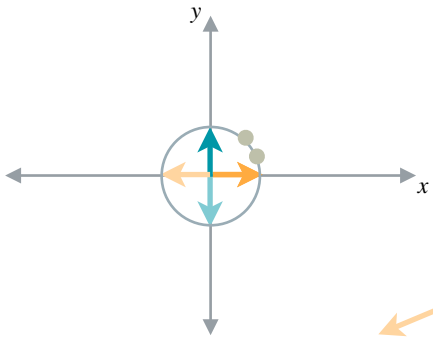


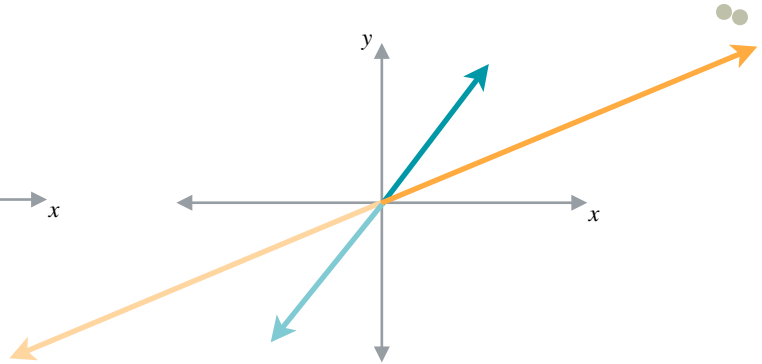

Covariance matrix

$$C$$

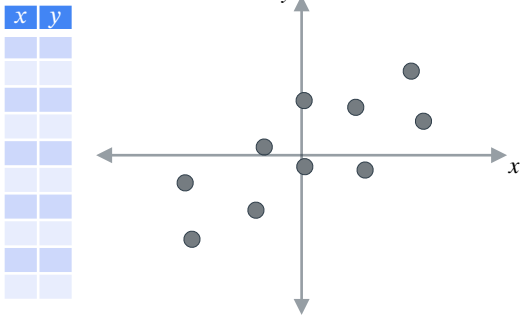
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

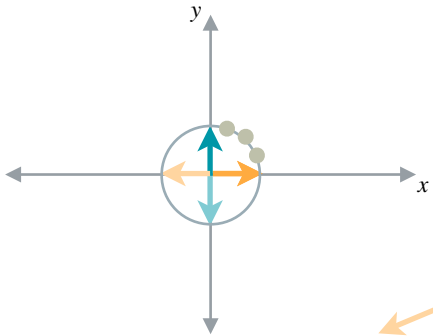


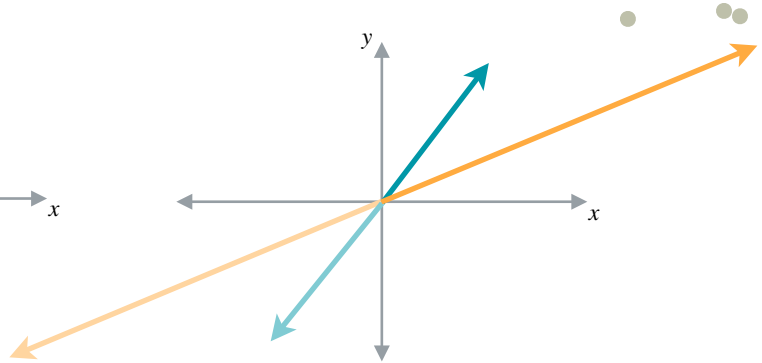

Covariance matrix

$$C$$

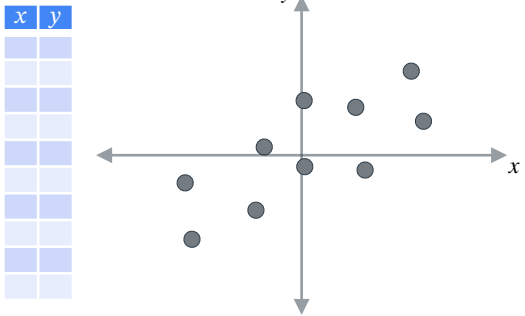
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

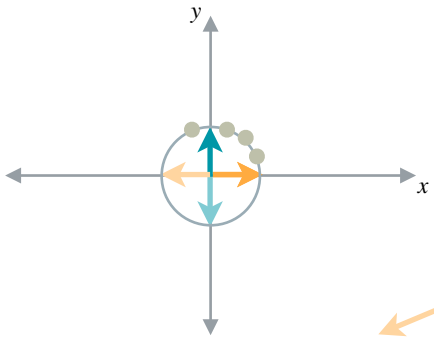


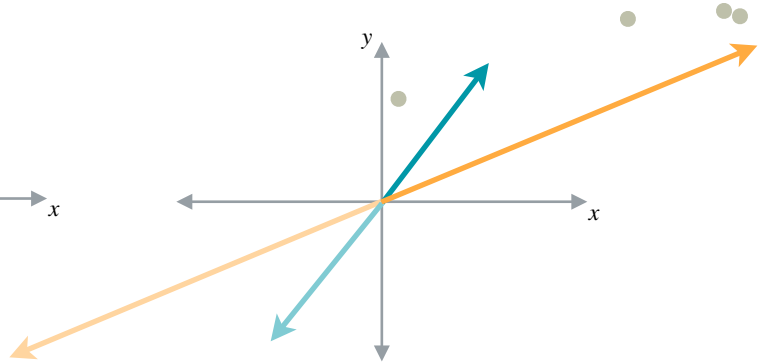

Covariance matrix

$$C$$

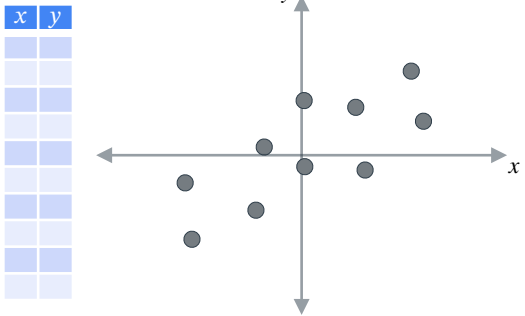
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

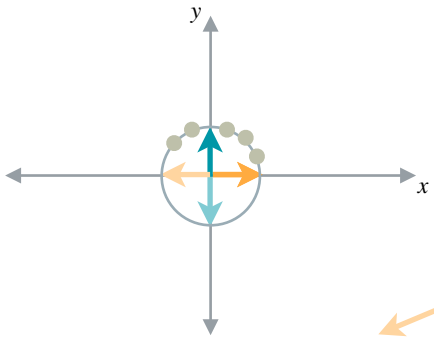


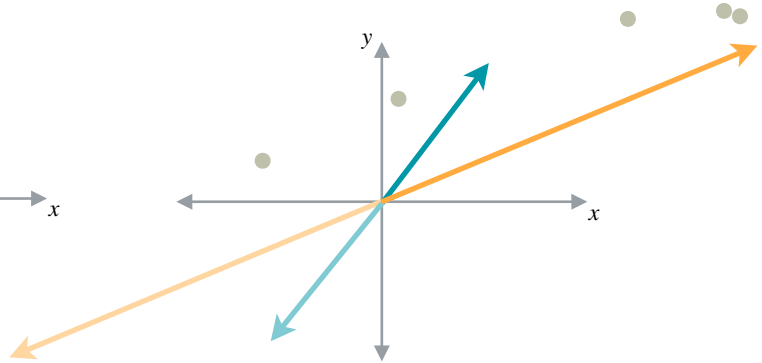

Covariance matrix

$$C$$

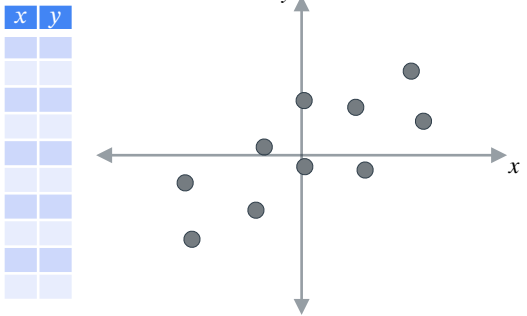
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

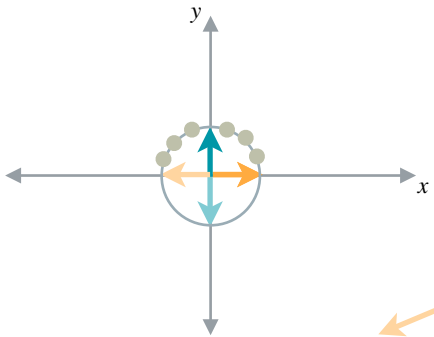


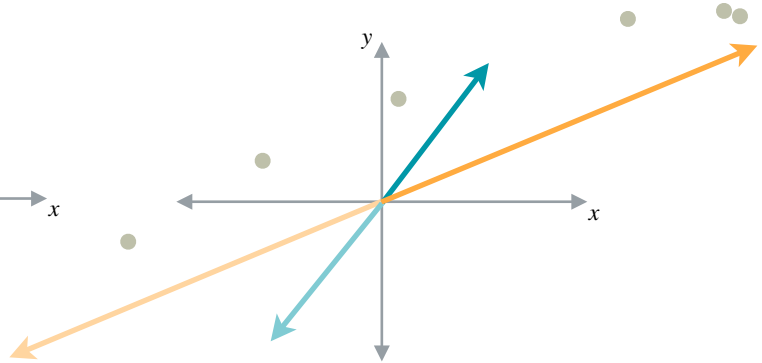

Covariance matrix

$$C$$

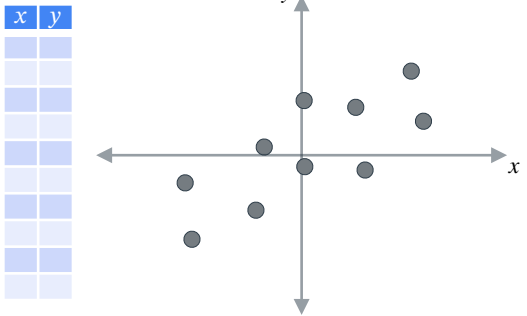
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

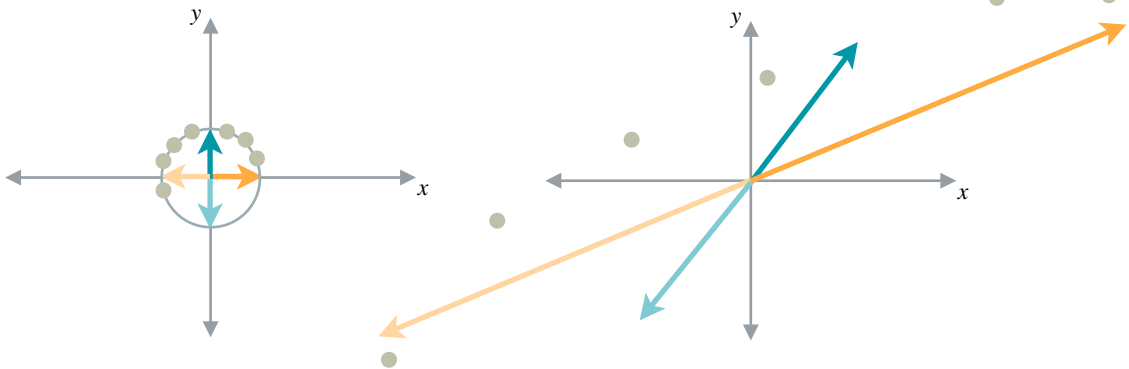



Covariance matrix

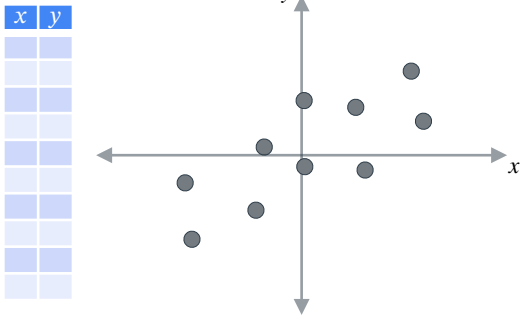
$$C$$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

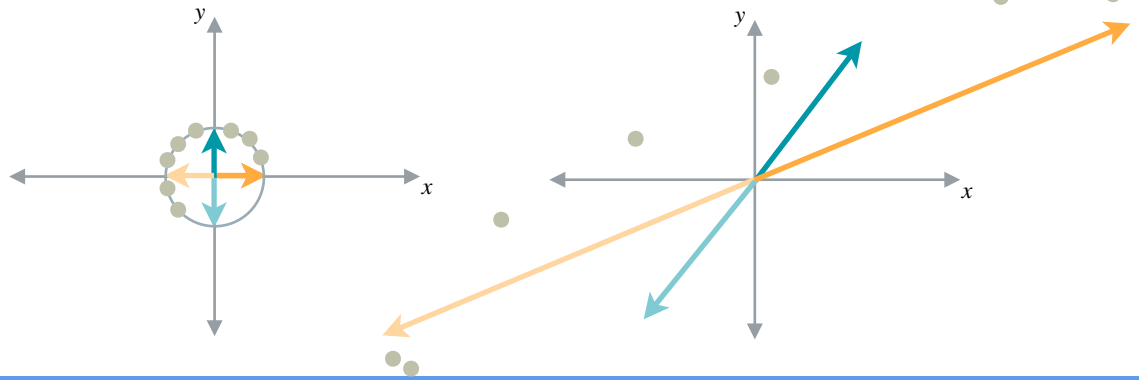



Covariance matrix

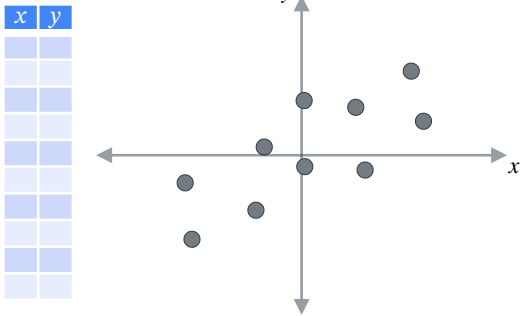
$$C$$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

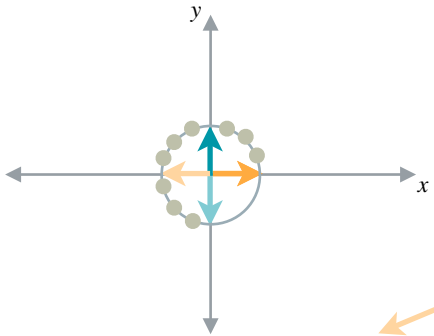


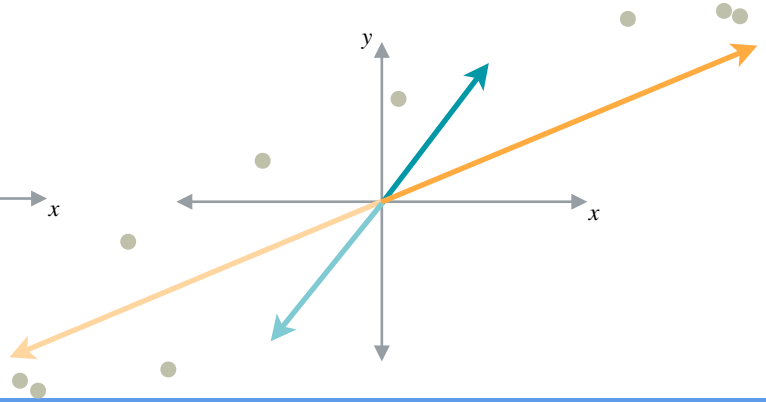

Covariance matrix

$$C$$

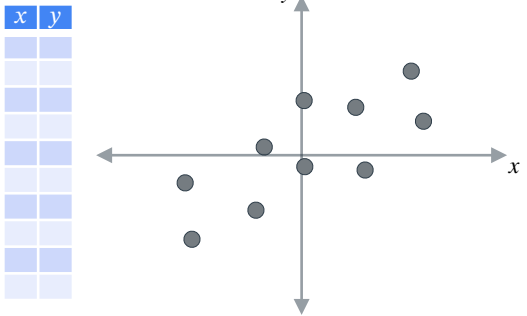
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

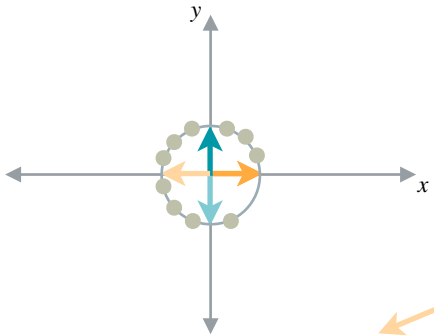


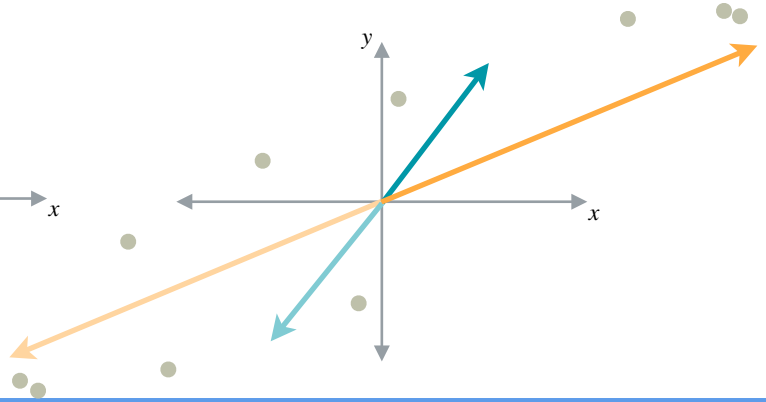

Covariance matrix

$$C$$

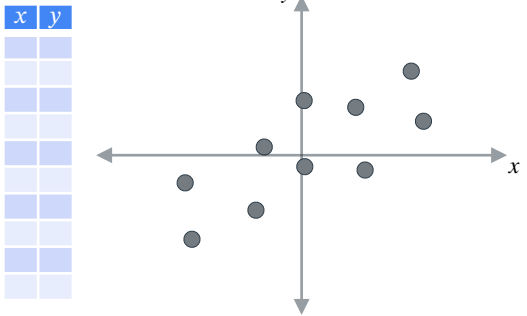
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

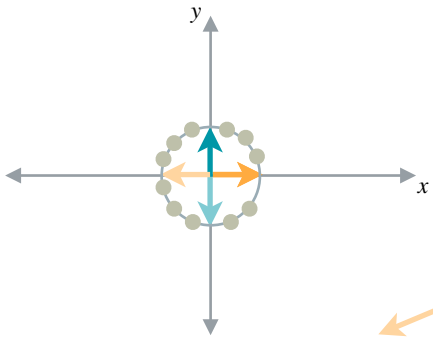


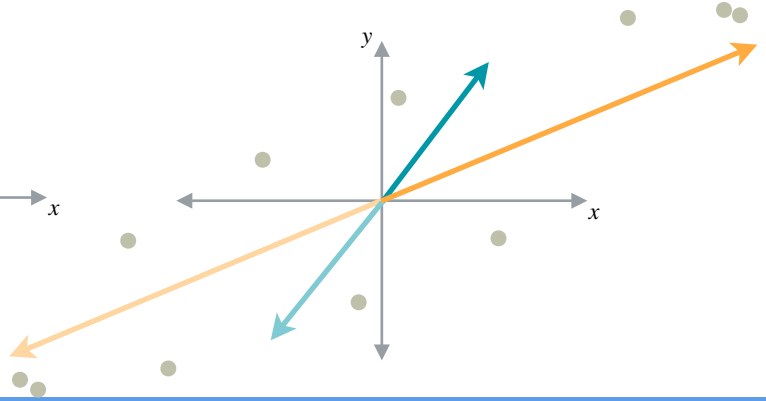

Covariance matrix

$$C$$

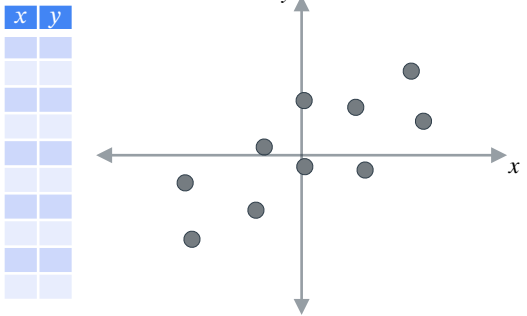
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

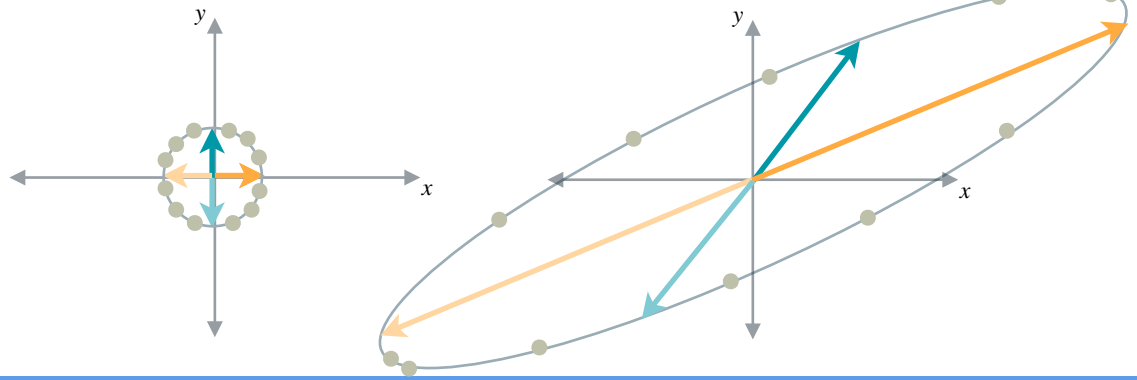



Covariance matrix

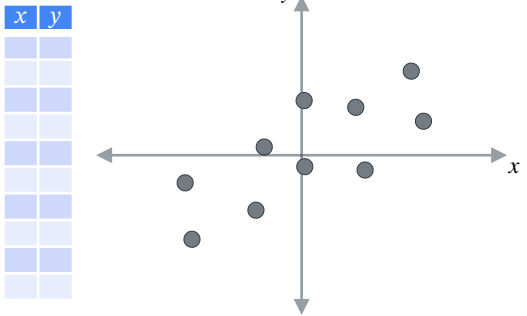
$$C$$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

“How spread out is the data?”



# PCA: Why It Works

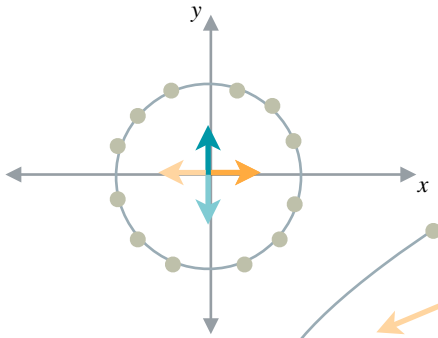


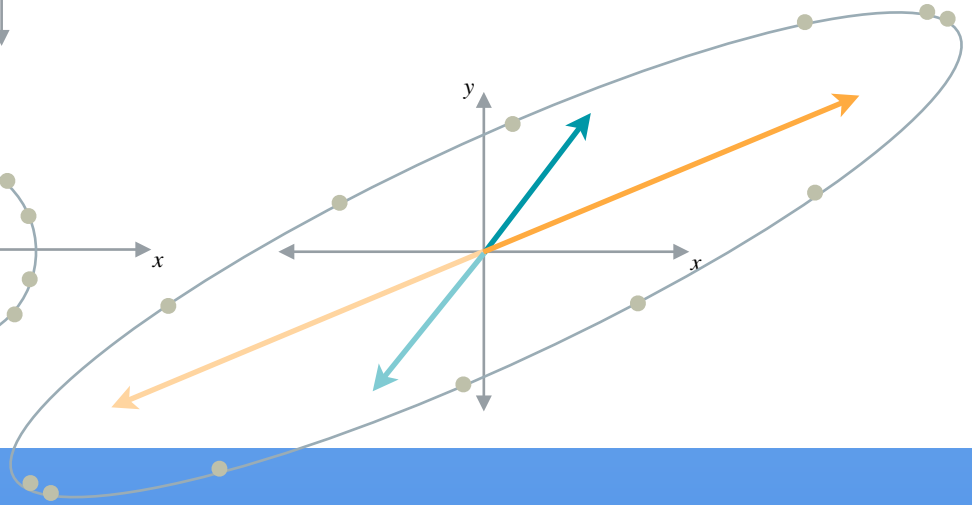

Covariance matrix

$$C$$

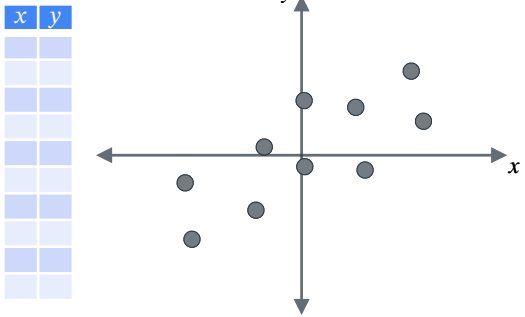
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works

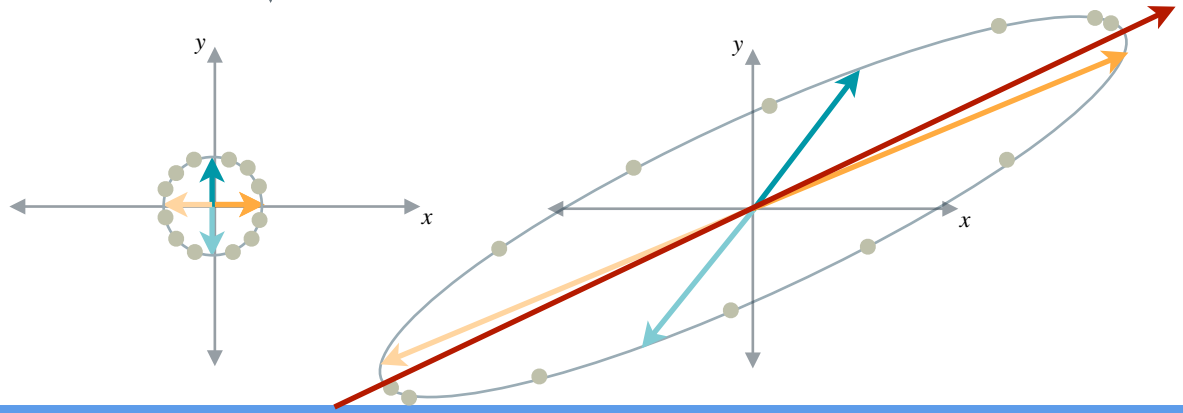



Covariance matrix

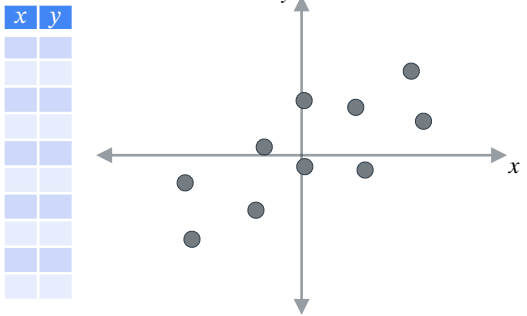
$$C$$

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"



# PCA: Why It Works




Covariance matrix

$C$

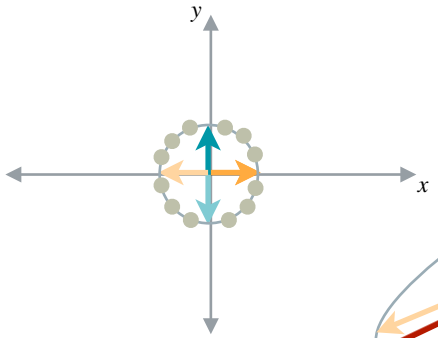
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

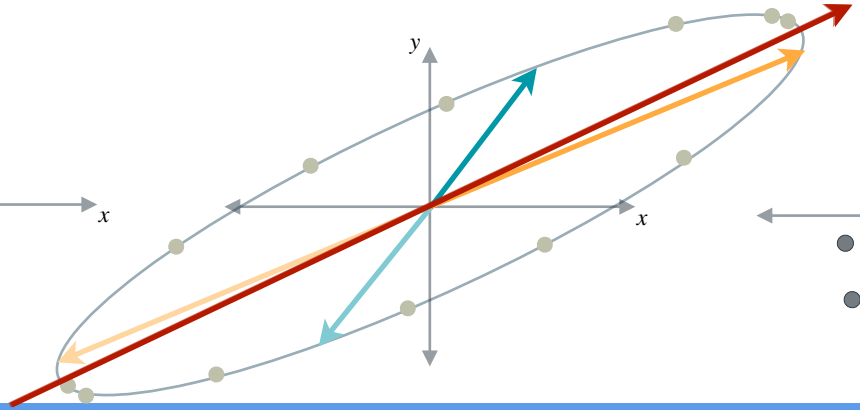
"How spread out is the data?"

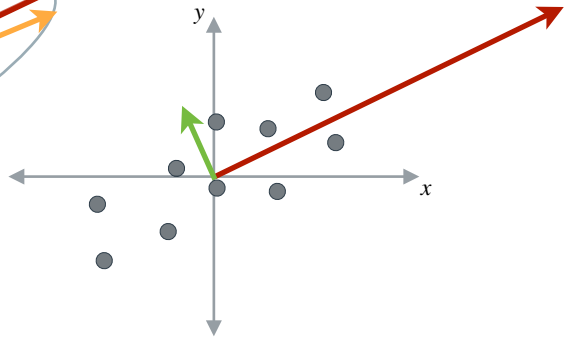
Eigenvalues and Eigenvectors

$$\begin{matrix} 11 \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{matrix} \quad \begin{matrix} 1 \\ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{matrix}$$

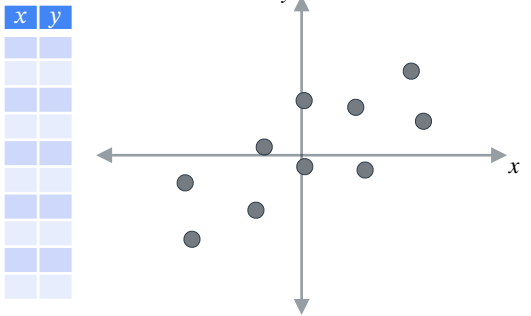
"Everything is just stretches"



# PCA: Why It Works




Covariance matrix

$C$

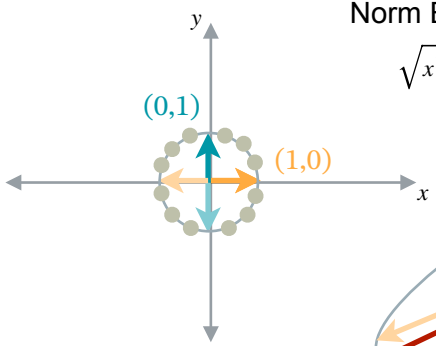
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

"How spread out is the data?"

Eigenvalues and Eigenvectors

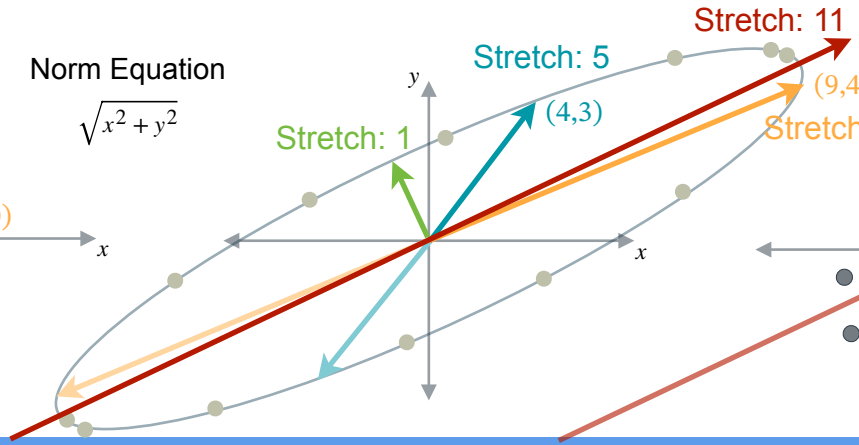
$$\begin{matrix} 11 \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{matrix} \quad \begin{matrix} 1 \\ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{matrix}$$

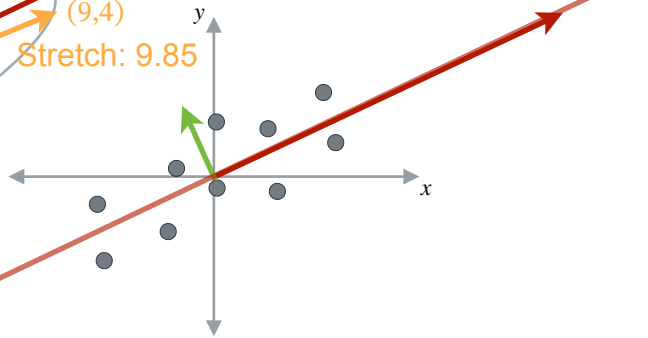
"Everything is just stretches"




Norm Equation

$$\sqrt{x^2 + y^2}$$



DeepLearning.AI

Determinants and Eigenvectors

# PCA - Mathematical formulation

# PCA Mathematical formulation

You have $n$ observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

1 Create matrix

5 variables

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{15} \\ x_{21} & x_{22} & \cdots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n5} \end{bmatrix}$$

$n$ Observations

2 Center the data

$$X - \mu = \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \cdots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \cdots & x_{25} - \mu_5 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & \cdots & x_{n5} - \mu_5 \end{bmatrix}$$

# PCA Mathematical formulation

You have $n$ observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

3 Calculate Covariance Matrix

$$C = \frac{1}{n-1}(X-\mu)^T(X-\mu) = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & Cov(X_1,X_3) & Cov(X_1,X_4) & Cov(X_1,X_5) \\ Cov(X_1,X_2) & Var(X_2) & Cov(X_2,X_3) & Cov(X_2,X_4) & Cov(X_2,X_5) \\ Cov(X_1,X_3) & Cov(X_2,X_3) & Var(X_3) & Cov(X_3,X_4) & Cov(X_3,X_5) \\ Cov(X_1,X_4) & Cov(X_2,X_4) & Cov(X_3,X_4) & Var(X_4) & Cov(X_4,X_5) \\ Cov(X_1,X_5) & Cov(X_2,X_5) & Cov(X_3,X_5) & Cov(X_4,X_5) & Var(X_5) \end{bmatrix}$$

# PCA Mathematical formulation

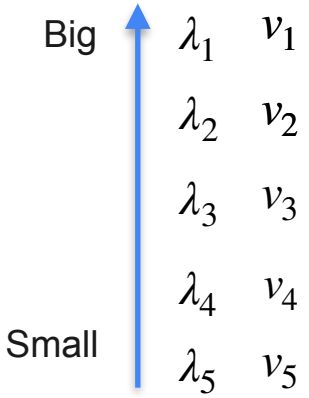
You have $n$ observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal: Reduce to 2 variables

4 Calculate Eigenvectors and Eigenvalues

5 Create Projection Matrix

6 Project Centered Data




$$V = \left[ \frac{}{\|v_1\|_2} \quad \frac{}{\|v_2\|_2} \right]$$

$$X_{PCA} = (X - \mu)V$$

DeepLearning.AI

# Determinants and Eigenvectors

## Conclusion