Linear Independence, Basis & Dimensions

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1 Linear Combination

Let’s start with this first

Linear Combination

Let the vectors,

$\mathbf{v_1, v_2, v_3, \ldots\ldots . v_n}$ be vectors in $\mathbf{R^n}$

$\mathbf{c_1, c_2, c_3, \ldots\ldots . c_n}$ be scalars

Then the vector b, where

$\mathbf{b = c_1 v_1 + c_2 v_2 + c_3 v_3 + , \ldots\ldots . + c_n v_n}$ is called a *linear combination*

In general, a linear combination is a particular way of combining vectors using scalar multiplication and addition.

2 Linear Independence

Linear Independence

- An indexed set of vectors $\{v_1, \dots, v_p\}$ in V is said to be linearly independent if the vector equation $$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0 \qquad \text{----(1)}$$ has *only* the trivial solution.

- Consider the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_m v_m = 0$$

where the x's are unknown scalars.

Suppose this is the only solution i.e.,

$$x_1 v_1 + x_2 v_2 + \cdots + x_m v_m = 0 \qquad \Longrightarrow \quad x_1 = 0 \,,\, x_2 = 0 \,,\, \dots \,,\, x_m = 0$$

Then the vectors $V_1, V_2, \dots, V_m$ are linearly independent.

Example

Q. Check whether the given vector sets are linearly independent or not.
V_1 =(2,3,-1), V_2 =(-1,4,-2), V_3 =(1,18,-4).

Sol. Let $a_1, a_2, a_3 \in F$

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$ ---(1) [Linear Combination]

Now putting the values of V_1, V_2, V_3 in the equation 1 we get,

$$a_1(2,3,-1) + a_2(-1,4,-2) + a_3(1,18,-4) = 0$$

Now after solving the above equation we get,

$\Longrightarrow \{(2a_1,3a_1,-1a_1) + (-1a_2,4a_2,-2a_2) + (1a_3,18a_3,-4a_3)\} = 0$

$\Longrightarrow \{2a_1-a_2+a_3,\ 3a_1+4a_2+18a_3,\ -a_1-2a_2-4a_3\}= 0$

Equations obtained from above data are:

$$2a_1 - a_2 + a_3 = 0$$
$$3a_1 - 4a_2 + 18a_3 = 0$$
$$-a_1 - 2a_2 - 4a_3 = 0$$

Now using the equations we will form the matrix A

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 18 \\ -1 & -2 & -4 \end{bmatrix}$$

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Now we will find the rank of the above matrix using Minor Method:

$$|A| = 2(-16+36) - \{-1(-12+18)+1(-6+4)\}$$

$$= 40 + 6 - 2$$

$$= 44$$

Rank of Matrix A = 3

So, now as

V_3 (F)

Rank of A = 3

Therefore, the given Vector sets are linearly independent.

3 Basis & Dimensions

Basis of Vector Space

Any subset S of a vector space V(F) is called basis of V(F) if,

- S is linearly independent
- S generates V i.e; L(S) = V

Standard basis of $V_2(F) = \{(1,0),(0,1)\}$

& $V_n(F) = \{(1,0,0\ldots,0),(0,1,0\ldots,0),(0,0,\ldots,1)\}$

Example

Show that the vectors (1, 0, 0), (1, 1, 0), (1, 1, 1) form a basis for R^3

Solution : Let S = {(1, 0, 0), (1, 1, 0), (1, 1, 1)}

Again let $a_1, a_2, a_3 \in R$ be such that

$a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) = 0$

$\Rightarrow (a_1 + a_2 + a_3, a_2 + a_3, a_3) = (0, 0, 0)$

$\Rightarrow a_1 + a_2 + a_3 = 0, a_2 + a_3 = 0, a_3 = 0$

Coefficient matrix of these equation is $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |A| = 1 \neq 0$

Hence vector are LI and only solution of these equation is

$\Rightarrow a_1 = 0, a_2 = 0, a_3 = 0$. Clearly $L(S) = V_3(R)$

Hence S = {(1, 0, 0), (1, 1,), (1, 1, 1)} is a basis of R^3

Another example

- Finding a Basis for Row Space

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

1^{st} Step $\longrightarrow$ Reduce A to echelon form

$R_5 \rightarrow R_5 - 2R_1$
$R_4 \rightarrow R_4 - R_1$

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$$

$R_4 \rightarrow R_4 + 2R_2$
$R_5 \rightarrow R_5 + R_2$

$R_3 \rightarrow R_3 + R_5 \quad R_4 \rightarrow R_4 + R_5$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank} = 3}$

Then $\omega_1 = [1, 1, 4, 1, 2]$
$\omega_2 = [0, 1, 2, 1, 1]$
$\omega_3 = [0, 0, 0, 1, 2]$ form a basis for row space of A.

Finding a basis for the column space of A

Step 1- Finding the row-echelon form of A

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

Now,after converting in echelon form after series of steps

$$A= \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- ❖ The columns from the original matrix which have leading ones when reduced, form a basis for the column space of A.

- ❖ In the above example, columns 1, 2, and 4 have leading ones.

-Therefore, columns 1, 2, and 4 of the original matrix form a basis for the column space of A.

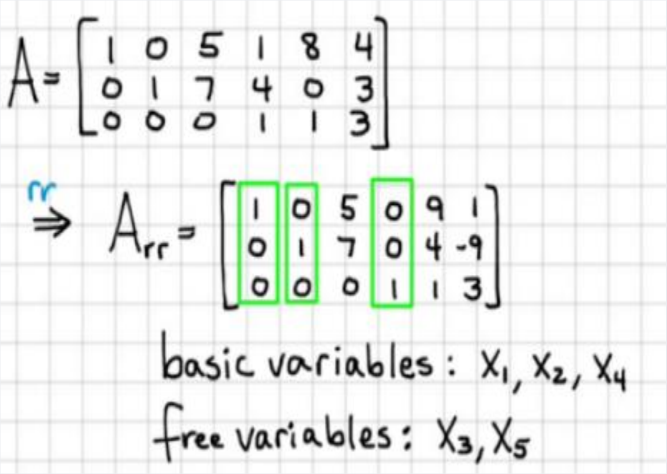
So,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

form a basis for the column space of A. The dimension of the column space of A is 3.

Basis for null space

- ❑ Let A∈Fm×n be a matrix in reduced row-echelon form.
- ❑ We can get all the solutions to Ax=0 by setting the **free variables** to distinct parameters.
- ❑ Then the set of solutions can be written as a linear combination of n-tuples where the parameters are the scalars.
- ❑ These n-tuples give a basis for the nullspace of A. Hence, the dimension of the nullspace of A, called the nullity of A, is given by the number of non-pivot columns.



Finding a basis for null space N(A)

Let $A \in \mathbb{R}^{2\times 4}$ be given by $\begin{bmatrix} 1 & -1 & -1 & 3 \\ 2 & -2 & 0 & 4 \end{bmatrix}$

performing elementary row operations-

$$\begin{bmatrix} 1 & -1 & -1 & 3 \\ 2 & -2 & 0 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\downarrow R_2 \leftarrow \frac{1}{2} R_2$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} +1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \xleftarrow{R_1 \to R_1 + R_2} \begin{bmatrix} 1 & -1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Now, $Ax = 0$ for obtaining solutions-

Now, x_2 & $x_4 \longrightarrow$ free variables, so set $\boxed{x_2 = s}$ $\boxed{x_4 = t}$

So, Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s - 2t \\ s \\ t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence basis for N(A) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Dimensions

Definition

The number of elements in any basis is the dimension of the vector space. or it can be explained as no.of linearly independent vectors . We denote it dim V .

Examples

1. dim R n = n
2. dim Mm×n(R) = mn
3. dim{0} = 0

Types of dimensions :

A vector space is called finite dimensional if it has a basis with a finite number of elements, or infinite dimensional otherwise

Theorem

If $\dim V = n$, then any set of n linearly independent vectors in V is a basis.

Theorem

If $\dim V = n$, then any set of n vectors that spans V is a basis.

Corollary

If S is a subspace of a vector space V then $\dim S \leq \dim V$ and $S = V$ only if $\dim S = \dim V$.

Example

Let W be a subspace of the real space $\mathbb{R}3$, then dim $\mathbb{R}3$ = 3.
Theorem 9 tells us that the dimension of W can only be 0, 1, 2, or 3.
The following cases apply:

a) If dim W = 0, then W = {0}, a point

b) If dim W = 1, then W is a line through the origin 0.

c) If dim W = 2, then W is a plane through the origin 0.

d) If dim W = 3, then W is the entire space $\mathbb{R}3$.

If a vector space V has a basis of n vectors, then every basis of V must consist of n vectors.

Proof: Suppose β1 is a basis for V consisting of exactly n vectors.
Now suppose β2 is any other basis for V.
By the definition of a basis, we know that β1 and β2 are both linearly independent sets.
By Theorem 9, if β1 has more vectors than β2, then is a linearly dependent set (which cannot be the case).
Again by Theorem 9, if β2 has more vectors than β1, then is a linearly dependent set (which cannot be the case).
Therefore β2 has exactly n vectors

Thank You!