DAY - 27

Problem data (given):

Number of layers: 6.
Innermost layer thickness: $t_1 = 1$ cm.
Each new layer is 20% thicker $\Rightarrow$ common ratio $r = 1 + 0.20 = 1.2$.
Therefore the thickness of the k-th layer is $t_k = t_1 \cdot r^{k-1} = 1 \cdot r^{k-1}$.

(a) Total radius after all 6 layers

The total radius R is the sum of the thicknesses of the 6 layers:

$$R = \sum_{k=1}^{6} t_k = \sum_{k=0}^{5} 1 \cdot r^k.$$

This is a finite geometric series with first term 1, ratio $r = 1.2$, and $n = 6$ terms. Use the geometric series formula:

$$\sum_{k=0}^{n-1} ar^k = a\,\frac{r^n - 1}{r - 1}.$$

With $a = 1$, $r = 1.2$, $n = 6$:

$$R = \frac{r^6 - 1}{r - 1}.$$

Compute powers of r step by step:

$$\begin{aligned}
r^1 &= 1.2,\\
r^2 &= 1.2^2 = 1.44,\\
r^3 &= 1.44 \times 1.2 = 1.728,\\
r^4 &= 1.728 \times 1.2 = 2.0736,\\
r^5 &= 2.0736 \times 1.2 = 2.48832,\\
r^6 &= 2.48832 \times 1.2 = 2.985984.
\end{aligned}$$

Now substitute into the formula:

$$R = \frac{r^6 - 1}{r - 1} = \frac{2.985984 - 1}{1.2 - 1} = \frac{1.985984}{0.2} = 1.985984 \times 5 = 9.92992 \text{ cm}.$$

$$\boxed{R = 9.92992 \text{ cm}}$$

(b) Fraction of volume left after peeling off the outer two layers

Peeling off the outer two layers removes layers 6 and 5. The remaining onion consists of the first 4 layers. The remaining radius is

$$R_{\text{inner}} = \sum_{k=1}^{4} t_k = \sum_{k=0}^{3} r^k = \frac{r^4 - 1}{r - 1}.$$

We already computed $r^4 = 2.0736$. Thus

$$R_{\text{inner}} = \frac{2.0736 - 1}{0.2} = \frac{1.0736}{0.2} = 1.0736 \times 5 = 5.368 \text{ cm}.$$

The volume of a sphere scales as the cube of its radius. If V is the original total volume and V_{inner} is the volume remaining, then

$$\frac{V_{\text{inner}}}{V} = \left(\frac{R_{\text{inner}}}{R}\right)^3.$$

Substitute the exact geometric-sum expressions first:

$$\frac{V_{\text{inner}}}{V} = \left(\frac{\frac{r^4 - 1}{r - 1}}{\frac{r^6 - 1}{r - 1}}\right)^3 = \left(\frac{r^4 - 1}{r^6 - 1}\right)^3.$$

Now numerical values with $r = 1.2$:

$$\begin{aligned}
r^4 - 1 &= 2.0736 - 1 = 1.0736, \\
r^6 - 1 &= 2.985984 - 1 = 1.985984, \\
\frac{r^4 - 1}{r^6 - 1} &= \frac{1.0736}{1.985984} \approx 0.5405884438142503, \\
\left(\frac{r^4 - 1}{r^6 - 1}\right)^3 &\approx (0.5405884438142503)^3 \approx 0.1579793318035828.
\end{aligned}$$

$$\boxed{\frac{V_{\text{inner}}}{V} \approx 0.1579793318 \approx 15.7979\%}$$

(Exact algebraic expression: $\left(\frac{r^4-1}{r^6-1}\right)^3$.)

(c) Estimating number of layers for an onion with total radius ≈ 25 cm

For a general number of layers n (with the same ratio $r = 1.2$), the total radius after n layers is

$$R_n \;=\; \frac{r^n - 1}{r - 1}.$$

If $R_n \approx 25$ cm, then

$$r^n - 1 \approx (r-1) \cdot 25 = 0.2 \times 25 = 5, \qquad \text{so} \qquad r^n \approx 6.$$

Taking natural logarithms gives

$$n \approx \frac{\ln(6)}{\ln(1.2)}.$$

Such that:

$$\ln(6) \approx 1.791759469, \qquad \ln(1.2) \approx 0.182321557,$$

$$n \approx \frac{1.791759469}{0.182321557} \approx 9.83.$$

Since the number of layers must be an integer, Lila can estimate the onion has about

$$\boxed{\text{approximately 10 layers.}}$$

Final answers:

(a) Total radius: $R = 9.92992$ cm.

(b) Fraction of volume remaining after removing outer two layers: $\dfrac{V_{\text{inner}}}{V} \approx 0.1579793318.$

(c) For an onion with total radius ≈ 25 cm and the same 20% growth pattern, estimated number of layers: $\boxed{\text{about 10 layers.}}$