4.5 The Virasoro Algebra and the Central Charge

The stress-tensor $T_{\alpha\beta}$ is conserved. In exercise 4.12 on page 119 you are invited to show that in such a case the stress-tensor has a scaling dimension that is exactly 2. In particular, $T(z)$ has conformal weight (2,0) and $\bar{T}(\bar{z})$ (0,2). They are obviously quasi-primary fields since they cannot be derivatives of other fields.[4] From these properties we can write the most general OPE between two stress-tensors compatible with conservation (holomorphicity) and conformal invariance:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + 2\frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots. \tag{4.5.1}$$

The fourth-order pole can only be proportional to a constant from dimension counting. This constant has to be positive in a unitary theory since $\langle T(z)T(w)\rangle = c/2(z-w)^4$. There can be no third-order pole since the OPE has to be symmetric under $z \leftrightarrow w$. Finally the rest of the singular terms are fixed by the fact that T has conformal weight (2,0). We have a similar OPE for $\bar{T}$ with $z \to \bar{z}$ and $c \to \bar{c}$ and

$$T(z)\bar{T}(\bar{w}) = \text{regular}. \tag{4.5.2}$$

Comparing (4.5.1) with (4.3.15) on page 56 we can conclude that $T(z)$ itself is not a primary field due to the presence of the most singular term. The constant c is called the (left) central charge and $\bar{c}$ the right central charge. Invariance under two-dimensional world-sheet parity requires $c = \bar{c}$.

The operator product expansions of $T(z)T(w)$ and $\bar{T}(\bar{z})\bar{T}(\bar{w})$ can now be written in terms of the modes. We have

$$\begin{aligned}
[L_n, L_m] &= \left(\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i}\oint \frac{dz}{2\pi i}\right) z^{n+1}T(z)w^{m+1}T(w) \\
&= \oint \frac{dw}{2\pi i}\oint_{C_w} \frac{dz}{2\pi i} z^{n+1}w^{m+1}\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots\right) \\
&= \oint \frac{dw}{2\pi i}\left(\frac{c}{12}(n+1)n(n-1)w^{n-2}w^{m+1}\right. \\
&\qquad \left. +2\,(n+1)w^n w^{m+1}T(w) + w^{n+1}w^{m+1}\partial T(w)\right).
\end{aligned} \tag{4.5.3}$$

The residue of the first term comes from $\frac{1}{3!}\partial_z^3 z^{n+1}\big|_{z=w} = \frac{1}{6}(n+1)n(n-1)w^{n-2}$. We integrate the last term by parts and combine it with the second term. This gives $(n-m)w^{n+m+1}T(w)$. Performing the w integration leads to the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{4.5.4}$$

[4] The case where they are derivatives of currents is a degenerate case that will not interest us further. Its consequences are explored in exercise 4.17 on page 120.

The analogous calculation for $\bar{T}(\bar{z})$ yields

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}(n^3 - n)\delta_{n+m,0}. \tag{4.5.5}$$

Since $T\bar{T}$ has no singularities in its OPE,

$$[L_n, \bar{L}_m] = 0. \tag{4.5.6}$$

Equations (4.5.4)–(4.5.6) are generalizations of the centerless algebras in (4.1.17).

Therefore, every conformally invariant theory realizes the conformal algebra, and its spectrum decomposes into its representations. For $c = \bar{c} = 0$, it reduces to the classical algebra. In a diffeomorphism invariant theory $c = \bar{c}$.

4.6 The Hilbert Space

To describe the Hilbert space of a CFT, we will use the standard formalism of in- and out-states of quantum field theory adapted to our coordinate system. For quasi-primary fields $A(z, \bar{z})$, the in-states are defined as

$$|A_{\rm in}\rangle = \lim_{\tau \to -\infty} A(\tau, \sigma)|0\rangle = \lim_{z \to 0} A(z, \bar{z})|0\rangle. \tag{4.6.1}$$

For the out-states, we need a description in the neighborhood of $z \to \infty$. If we define $z = \frac{1}{w}$, then $z = \infty$ corresponds to the point $w = 0$. The map $f : w \to z = \frac{1}{w}$ is a conformal transformation, under which $A(z, \bar{z})$ transforms as

$$\tilde{A}(w, \bar{w}) = A(f(w), \bar{f}(\bar{w}))(\partial f(w))^{\Delta}(\bar{\partial}\bar{f}(\bar{w}))^{\bar{\Delta}}. \tag{4.6.2}$$

Substituting $f(w) = \frac{1}{w}$, we find

$$\tilde{A}(w, \bar{w}) = A\left(\frac{1}{w}, \frac{1}{\bar{w}}\right)(-w^{-2})^{\Delta}(-\bar{w}^{-2})^{\bar{\Delta}}. \tag{4.6.3}$$

It is natural to define

$$\langle A_{\rm out}| = \lim_{w, \bar{w} \to 0} \langle 0|\tilde{A}(w, \bar{w}). \tag{4.6.4}$$

We would like $\langle A_{\rm out}|$ to be the Hermitian conjugate of $|A_{\rm in}\rangle$. Hermitian conjugation of operators of weight $(\Delta, \bar{\Delta})$ is defined by

$$[A(z, \bar{z})]^{\dagger} = A\left(\frac{1}{\bar{z}}, \frac{1}{z}\right)\bar{z}^{-2\Delta}z^{-2\bar{\Delta}}. \tag{4.6.5}$$

This definition finds its justification in the continuation from Euclidean space back to Minkowski space. The missing factor of i in Euclidean time evolution $A(\sigma, \tau) = e^{\tau H}$ $A(\sigma, 0)\ e^{-\tau H}$ must be compensated for in the definition of the adjoint by a Euclidean