Some formal aspects of two dimensional gravity and its field theory dual

PHN-600B Presentation

Shivam Singh Aswal

Supervisor: Prof. Dibakar Roychowdhary

`shivam_sa@ph.iitr.ac.in`

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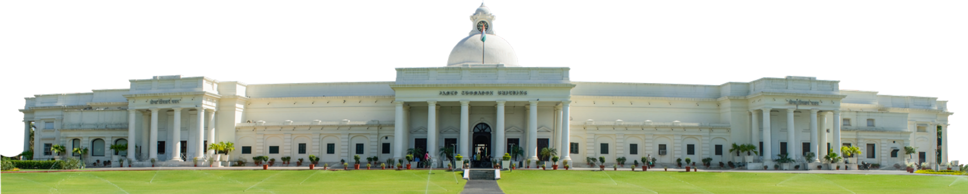

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Introduction to ModMax Electrodynamics ([1],[8

- ModMax is a **nonlinear generalization of Maxwell theory** in $D = 4$ that preserves:
  - Conformal invariance
  - Electric-magnetic duality invariance
- ModMax Lagrangian:

$$\mathcal{L} = \cosh\gamma\,\mathcal{S} + \sinh\gamma\sqrt{\mathcal{S}^2 + \mathcal{P}^2}$$

where:

$$\mathcal{S} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2), \quad \mathcal{P} = -\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} = \vec{E}\cdot\vec{B}$$

- γ is a dimensionless parameter, $\gamma \to 0$ recovers linear Maxwell theory.
- Despite being nonlinear, exact solutions like plane waves and Liénard-Wiechert fields persist.

Conformal and Duality Invariance I

Conformal Invariance:

- The action is invariant under scaling:

$$x^\mu \to b x^\mu, \quad A_\mu \to b^{-1} A_\mu, \quad F_{\mu\nu} \to b^{-2} F_{\mu\nu}$$

Duality Invariance:

- The field equations are invariant under $SO(2)$ electric-magnetic duality rotations:

$$\begin{pmatrix} G'_{\mu\nu} \\ \widetilde{F}'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} G_{\mu\nu} \\ \widetilde{F}_{\mu\nu} \end{pmatrix}$$

- Duality condition:

$$\mathcal{L}_S^2 - 2S\mathcal{L}_S\mathcal{L}_P - \mathcal{L}_P^2 = 1$$

where $\mathcal{L}_S = \partial\mathcal{L}/\partial\mathcal{S}$ and similarly for $\mathcal{L}_P$.

ModMax Coupled to Gravity I

- ModMax theory admits coupling to gravitational backgrounds via standard minimal coupling:

$$\mathcal{L} = \sqrt{-g}\left(\cosh\gamma\,\mathcal{S} + \sinh\gamma\sqrt{\mathcal{S}^2 + \mathcal{P}^2}\right)$$

- Coupling to point charges (electric, magnetic, dyons) preserves exact Liénard-Wiechert field solutions.
- A reformulation using auxiliary axion-dilaton-like scalar fields suggests a possible **effective field theory origin** and link to string theory.
- Thermodynamic and geometric effects of ModMax (e.g., black holes, Taub-NUT metrics) differ from Maxwell, signaling new physics in strong-field regimes.

Mathematical Definition of Central Charge I

Central Charge ([2],[5], [7]) in 2D CFT:

- The energy-momentum tensor generates the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

- The constant c is the **central charge**.

Gravity Perspective (Brown–Henneaux [2]):

- In 3D gravity with AdS_3 boundary conditions:

$$c = \frac{3\ell}{2G_3}$$

where ℓ is the AdS radius and G_3 is Newton's constant in 3D.

2D Gravity on AdS_2 [3] (e.g., Jackiw-Teitelboim gravity):

Mathematical Definition of Central Charge II

- With Maxwell-dilaton gravity and a twisted energy-momentum tensor:

$$\tilde{T}_{\pm\pm} = T_{\pm\pm} \pm A\partial_{\pm} j_{\pm}$$

the central charge becomes:

$$c = 12kG^2Q^2\ell^4$$

where Q is electric charge, ℓ the AdS_2 radius, and k the level of the U(1) current.

Physical Significance of Central Charge I

1. Holography and AdS/CFT:

- The central charge governs the density of states in a 2D CFT.
- In AdS/CFT, it encodes the number of degrees of freedom on the boundary.

2. Entropy via Cardy Formula:

$$S = 2\pi\sqrt{\frac{c}{6}\left(\Delta - \frac{c}{24}\right)}$$

- This formula matches black hole entropy when c comes from asymptotic symmetries.
- Example: AdS_2/CFT_1 — entropy from twisted CFT matches gravity result.

3. Indicator of Anomalies:

- Central charge appears in quantum anomalies (e.g., conformal anomaly).
- Shows failure of classical symmetry at the quantum level.

4. Near-Horizon Symmetries:

Physical Significance of Central Charge II

- Emergence of a CFT near black hole horizons with well-defined central charge.
- Supports holographic descriptions even in 2D (AdS_2 cases).

Central Charge for AdS_2 Gravity [Hartman et al.][5] I

Setup: 2D Maxwell-dilaton gravity on AdS_2 with constant electric field E, AdS radius ℓ, and gauge field A_μ.

1. Classical Action in Conformal Gauge:

$$S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left(\eta \left(R + \frac{8}{\ell^2} \right) - \frac{\ell^2}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

2. Twisted Stress Tensor: Conformal diffeomorphisms do not preserve boundary conditions $\rightarrow$ must be accompanied by a $U(1)$ gauge transformation. This leads to a *twisted* energy-momentum tensor:

$$\widetilde{T}_{\pm\pm} = T_{\pm\pm} \pm \frac{E\ell^2}{4} \partial_\pm j_\pm$$

3. Commutator of Twisted Stress Tensor:

$$\widetilde{T}_{--}(x), \widetilde{T}_{--}(y)] = \cdots + \frac{\pi k E^2 \ell^4}{8} \partial_y^3 \delta(x - y)$$

Central Charge for AdS_2 Gravity [Hartman et al.][5] II

4. Central Charge: From the above commutator, extract the central term:

$$c = \frac{3kE^2\ell^4}{4}$$

Conclusion: Twisting by a conserved $U(1)$ current yields a Virasoro algebra with nonzero central charge, despite pure AdS_2 gravity having $c = 0$.

Central Charge for JT Gravity ([6],[9]) coupled to ModMax [HR,DRC][8] I

The authors in this paper started with the *ModMax* lagrangian, which is an example of non-linear electrodynamics with one free parameter, β, so formed to possess the usual $\mathbb{SO}(2)$ symmetry of Maxwell's theory along with the conformal symmetry. Further in the weak field limit ($\beta \to \infty$), it must yield the Maxwell theory. The *ModMax* lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left(S\cosh\beta - \sqrt{S^2 + P^2}\sinh\beta\right) \tag{1}$$

where $S = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $P = \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}$. $\tilde{F}^{\mu\nu}$ is the hodge dual of the electromagnetic field tensor defined as $\tilde{F}^{\mu\nu} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.
The 4D action for the gravity coupled to the *ModMax* lagrangian is given as

$$I = \frac{1}{16\pi G_4}\int d^4x\sqrt{-g}\,(R - 2\Lambda - 4\alpha\mathcal{L}_{MM}) \tag{2}$$

Central Charge for JT Gravity ([6],[9]) coupled to ModMax [HR,DRC][8] II

where α is the coupling constant and Λ is the cosmological constant. The authors performed suitable dimensional reduction to obtain action for a $(1+1)$D *JT* gravity theory. The ansatz for the metric was taken to be

$$ds^2_{(3+1)} = g_{\mu\nu}(x^\rho)dx^\mu dx^\nu + \Phi(x^\mu)dx_i^2 \tag{3}$$

$$A_\mu \equiv A_\mu(x^\nu), \quad A_z \equiv A_z(x^\mu), \tag{4}$$

Here μ, ν are the indices for coordinates in the reduced dimensions and i denotes the compact dimensions. The $2D$ projected *ModMax* action is written as

$$I_2 = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g_{(2)}}\left(\Phi R^{(2)} - 2\Lambda\Phi - 4\alpha\Phi\mathcal{L}^{(2)}_{(M)}\right) \tag{5}$$

$$\mathcal{L}^{(2)}_{\text{MM}} = \frac{1}{2}\left(s\cosh\beta - \sqrt{s^2+p^2}\sinh\beta\right) \tag{6}$$

$$s = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \Phi^{-1}\left((\partial\chi)^2 + (\partial\zeta)^2\right), \quad p = -2\Phi^{-1}\epsilon^{\mu\nu}\partial_\mu\chi\partial_\nu\zeta \tag{7}$$

Central Charge for JT Gravity ([6],[9]) coupled to ModMax [HR,DRC][8] III

An approach similar to [3] was employed here, wherein the gauge transformation were obtained by imposing boundary condition preservation upon diffeomorphisms. And thus the coefficient of the modification term in the stress-energy tensor was identified as the central charge. The central charge thus obtained is given as

$$c_{3D} = \frac{1}{144\sqrt{3}\pi G_2}\left(\alpha - 12\beta\alpha + 2\alpha^2\right), \tag{8}$$

The authors thus rightly concluded that in the limiting case of weak field ($\beta \to 0$) the central charge asymptotes to $\frac{1}{G_2}$ which matches with the central charge obtained by [3] for the case of Maxwell theory.

4D to 3D reduction [4] I

We start with the 4D action for *EH* gravity coupled to *ModMax* electrodynamics. We then perform suitable dimensional reduction to obtain the action in 3D and solve for the system thereafter.
The ansatz for the metric is taken to be

$$ds^2_{(4)} = e^{2\alpha\phi} ds^2_{(3)} + e^{2\beta\phi}(dx^3)^2 \tag{9}$$

$$dx^2_{(3)} = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu \tag{10}$$

$$R_{(4)} = e^{-2\alpha\phi}\left(R_{(3)} - \frac{1}{2}(\partial\phi)^2 - d\alpha\Box\phi\right) \tag{11}$$

μ, ν are indices representing reduced dimensions and x^3 is the compact dimension. Note that we can drop the dalembertian term in the Ricci scalar as it is a total derivative.
Further for this form of choice for dissection of the metric to one lower dimension (from $d+1$ to d), we observe that $\mathcal{L}$ becomes $e^{(\beta+(d-2)\alpha)\phi}\sqrt{-g}\mathcal{R} + \dots$ where $\mathcal{R}$ is the Ricci scalar in d dimensions. So it is

4D to 3D reduction [4] II

required to set $\beta + (d-2)\alpha = 0$, which gives us $\beta = -\alpha$ in $4D$ to $3D$ reduction. Further to ensure that we obtain a term of the form $\frac{1}{2}\sqrt{-g}\,(\partial\phi)^2$ in the action we require $\alpha^2 = \dfrac{1}{2(d-1)(d-2)}$ which gives $\alpha = \frac{1}{2}$ for our case. The reduced metric determinant can be determined using

$$\sqrt{-g_{(4)}} = \sqrt{-g_{(3)}}e^{(\beta+d\alpha)\phi} = e^{2\alpha\phi}\sqrt{-g_{(3)}}$$

The 3D projected *ModMax* lagrangian can now be calculated as

$$\begin{aligned} s &= \frac{1}{2}F_{AB}F^{AB} = \frac{1}{2}F_{AB}F_{CD}g^{AC}g^{BD} \\ &= \frac{1}{2}F_{\mu\nu}F^{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} + e^{-2\beta\phi}\left(F_{M3}F_{P3}g^{PM}\right) \\ &= \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + e^{-2\beta\phi}(\partial_\mu\chi)^2 \end{aligned}$$

4D to 3D reduction [4] III

$$
\begin{aligned}
p &= \frac{1}{2} F_{AB}\tilde{F}^{AB} = \frac{1}{2} F_{AB}F_{CD}\epsilon^{ABCD} \\
&= 2e^{-2\beta\phi}\sqrt{-g_{(3)}}\,(F_{01}F_{23} + F_{02}F_{31} + F_{21}F_{03}) \\
&= e^{-2\beta\phi}\epsilon^{abc}F_{ab}F_{c3} \\
&= 2e^{-2\beta\phi}\epsilon^{abc}F_{ab}\partial_c\chi
\end{aligned}
$$

$$
\mathcal{L}^{(3)}_{\text{MM}} = \frac{1}{2}\left(s\cosh\zeta - \sqrt{s^2 + p^2}\sinh\zeta\right) \tag{12}
$$

here ζ is the *ModMax* parameter.
The 4D lagrangian is integrated along the compact dimension and we get the 3D reduced action as

$$
I = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g_{(3)}}\left(R - 2\Lambda - 4\kappa\mathcal{L}^{(3)}_{\text{MM}}\right) \tag{13}
$$

4D to 3D reduction [4] IV

The variation of the action yields (and setting them all to zero to find the stationary action)

$$\delta I = \frac{1}{16\pi G_3} \int d^3x \sqrt{-g} \left(\Psi_{\mu\nu} \delta g^{\mu\nu} + \Psi_\mu \delta A^\mu + \Psi_\phi \delta\phi + \Psi_\chi \delta\chi \right) \tag{14}$$

$$\Psi_\phi = -2\kappa \left[-2\beta e^{-2\beta\phi} (\partial\chi)^2 \cosh\zeta + \frac{\sinh\zeta}{\sqrt{s^2+p^2}} \left(s e^{-2\beta\phi} (\partial\chi)^2 + p^2 \right) \right] = 0 \tag{15}$$

$$\Psi_\chi = -4\kappa \nabla_\mu \left[-e^{-2\beta\phi} \partial^\mu \chi \cosh\zeta + e^{-2\beta\phi} \frac{\sinh\zeta}{\sqrt{s^2+p^2}} \left(s(\partial\chi)^2 + \frac{p}{2} \epsilon^{ab\mu} F_{ab} \right) \right] = 0 \tag{16}$$

$$\Psi_\nu = 4\kappa \nabla_\mu \left[F^\mu_\nu \cosh\zeta - \frac{\sinh\zeta}{\sqrt{s^2+p^2}} \left(s F^\mu_\nu + p e^{-2\beta\phi} \epsilon^{\mu bc} g_{b\nu} \partial_c \chi \right) \right] = 0 \tag{17}$$

4D to 3D reduction [4] V

$$\Psi_{\mu\nu} = \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \Lambda g_{\mu\nu} + 2\kappa\mathcal{L}_{MM}g_{\mu\nu} -$$
$$\kappa\left[2\left(\cosh\zeta - \frac{s\sinh\zeta}{\sqrt{s^2+p^2}}\right)\left(F_\mu^\beta F_{\nu\beta} + e^{-2\beta\phi}\partial_\mu\chi\partial_\nu\chi\right) - \frac{p^2\sinh\zeta}{\sqrt{p^2+s^2}}g_{\mu\nu}\right] = 0 \tag{18}$$

We can extract some information from the Eq 18 upon contracting it with the contravariant metric tensor. We thus obtain

$$-\frac{R}{2} + 3\Lambda - 2\kappa\left[-3\mathcal{L}_{MM} + 2\left(\cosh\zeta - \frac{s\sinh\zeta}{\sqrt{s^2+p^2}}\right)\left(F^2 + e^{-2\beta\phi}(\partial\chi)^2\right) - \frac{3p^2}{\sqrt{p^2+s^2}}\sinh\zeta\right] = 0 \tag{19}$$

In the FG gauge, upon expanding the fields

$$\Xi = \Xi^{(0)} + \kappa\Xi^{(1)} + \gamma\kappa\Xi^{(2)} + \kappa^2\Xi^{(3)} + \ldots \tag{20}$$

$$\Sigma = \Sigma^{(1)} + \gamma\Sigma^{(2)} + \kappa\Sigma^{(3)} + ... \tag{21}$$

here Ξ represents the metric field and the dilaton field whereas Σ denotes the reduced gauge field and the auxiliary scalar field coming from the A_μ term. Now if we expand the fields according to the above expansion and collect like order terms then we obtain the following equations

$$R^{(0)} = 6\Lambda \tag{22}$$

$$G^{(0)}_{\mu\nu} + \Lambda g^{(0)}_{\mu\nu} = 0 \tag{23}$$

4D to 3D reduction [4] VII

All the dynamical variables are from the reduced 3D space. We note that the zeroth order equations are in agreement with the equations we obtain from the EH action for 3D AdS space.

$$\nabla_\mu\left(-e^{-2\beta\phi^{(0)}}\partial^\mu\chi^{(1)}\cosh\zeta+\frac{e^{-2\beta\phi^{(0)}}\sinh\zeta}{\sqrt{(s^{(1)})^2+(p^{(1)})^2}}\times\right.$$
$$\left.\left(s^{(0)}\partial^\mu\chi^{(1)}+p^{(0)}/2\epsilon^{ab\mu}F^{(1)}{}_{ab}\right)\right)=0 \tag{24}$$

4D to 3D reduction [4] VIII

$$\nabla_\mu\left(F^{(1)\mu}_{\ \ \nu}\cosh\zeta - \frac{\sinh\zeta}{\sqrt{(s^{(1)})^2+(p^{(1)})^2}}\times\right.$$

$$\left.\left(s^{(0)}(F^{(1)})^{\mu}_{\nu} + p^{(1)}e^{-2\beta\phi}\epsilon^{\mu bc}(g^{(0)})_{b\nu}\partial_c\chi^{(1)}\right)\right) = 0 \tag{25}$$

$$2\mathcal{L}^{(1)}_{MM} - 2\left(\cosh\zeta - \frac{s^{(1)}\sinh\zeta}{\sqrt{s^2+p^2}}\right)\left(F^{(1)\beta}_{\ \ \mu}F^{(1)}_{\nu\beta} + e^{-2\beta\phi^{(0)}}\partial_\mu\chi^{(1)}\partial_\nu\chi^{(1)}\right) -$$

$$\frac{(p^{(1)})^2}{\sqrt{(s^{(1)})^2+(p^{(1)})^2}\sinh\zeta g^{(0)}_{\mu\nu}} = 0 \tag{26}$$

Contracting the last of the above equations by $g^{\mu\nu}$ gives

$$2\mathcal{L}^{(1)}_{MM} - \left(\cosh\zeta - \frac{s^{(1)}\sinh\zeta}{\sqrt{s^2+p^2}}\right)\left(F^{(1)^2} + e^{-2\beta\phi^{(0)}}(\partial\chi)^2\right) - \frac{3(p^{(1)})^2}{\sqrt{(s^{(1)})^2+(p^{(1)})^2}}\sinh\zeta =$$

4D to 3D reduction [4] IX

These are highly coupled partial differential equations and obtaining analytical solution to them is not possible. We will have to solve them in the IR fixed point limit as done in [8].

3D to 2D reduction I

x We further perform dimensional reduction to get the action for a 2D spacetime and compute the central charge in the 2D to be able to make a contrast with the charge obtained from the dimensional reduction of the 4D *ModMax* action as mentioned in [8].
The choice of ansatz for the metric reduction as highlighted in **??** will not word for the reduction from 3D to 2D. We thus need to choose a different form of the metric. The ansatz for the metric, in reference to [4] and [8], is taken to be

$$ds^2_{(3)} = \mathcal{G}_{\mu\nu} dx^\mu dx^\nu + \Phi dz^2 \tag{27}$$

μ, ν are the indices of the reduced spacetime and z is the compact dimension with volume L_z. Further the ricci scalar is given by

$$R_{(3)} = \left(R_{(2)} - \frac{1}{4}(\partial\Phi)^2 - \frac{1}{2}\Phi^{-1}\Box\Phi \right) \tag{28}$$

and

$$\sqrt{-g_{(3)}} = \sqrt{-\mathcal{G}}\sqrt{\Phi} \tag{29}$$

3D to 2D reduction II

and the action we obtain is

$$I = \frac{1}{16\pi G_2} \int d^3x \sqrt{-g_{(2)}} \sqrt{\Phi} \left(R - 2\Lambda - 4\kappa \mathcal{L}^{(2)}_{\text{MM}} \right) \tag{30}$$

where the definition of the projected *ModMax* lagrangian is similar to the one written in [8]

Thesis work I

What I could do?

- I have studied the ModMax theory and its coupling to gravity from the papers cited.
- I could verify the central charge for the cases of constant electric field and modmax EM

What I could not do?

- I could not find the exact solutions for the ModMax theory in projected 3D and further projected 2D cases.
- I could not find the physical implications of the central charge for the ModMax theory.
- Further I also couldn't establish any relation between the 2D and 3D central charge.

Future work

- I will try to solve the equations and obtain the central charge.
- I will study the physical implications of the central charge for the ModMax theory.
- I can try to analyse the effect of the choice of metric on the central charge

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