Chapter 1

Introduction

What am I going to do in the thesis?

1.1 Background

The background for a theory of *Quantum Gravity* is set by a very famous principle, known as the *Holographic Principle*. $AdS_n - CFT_{n-1}$ duality is the prime example of this principle.

The complexity of the mathematical models that we have to deal with increases exponentially with the dimensionality of space-time. So it is fruitful to study the consequences of these principles in lower dimensions. Thus it is customary to explore the $(1+1)D$ (known as the *Jachiw-Teitelboim* gravity) or $(2+1)D$ theories of gravity.

1.1.1 Holographic Principle

The Holographic Principle: A Scientific Explanation The Holographic Principle is a revolutionary conjecture in theoretical physics that suggests the information content of a volume of space can be fully described by a theory defined on its boundary, much like a hologram encodes a three-dimensional image on a two-dimensional surface. Proposed in the 1990s by Gerard 't Hooft and Leonard Susskind, it emerged from studies of black hole thermodynamics and quantum gravity, challenging our intuitive understanding of space, information, and the fundamental nature of the universe. The principle posits that the degrees of freedom within a region of space are not proportional to its volume (as one might expect in a three-dimensional world) but to the area of its boundary, implying that our seemingly three-dimensional reality might be a "projection" of a lower-dimensional system. This idea has profound implications for reconciling quantum mechanics with general relativity, particularly in the context of black holes and string theory, and it underpins modern approaches to quantum gravity like the AdS/CFT correspondence. The Holographic Principle originated from the study of black holes, specifically the paradox of information loss. In the 1970s, Stephen Hawking showed that black holes emit radiation (now called Hawking radiation) due to quantum effects near the event horizon, leading to their eventual evaporation. This process suggested that information inside the black hole might be lost, violating quantum mechanics' principle of unitarity, which demands that information is preserved. Jacob Bekenstein's work on black hole entropy provided a crucial clue: the entropy (S) of a black hole is proportional to the area (A) of its event horizon, not its volume. This is encapsulated in the Bekenstein-Hawking entropy formula:

$$S = \frac{kc^3 A}{4\hbar G},$$

where k is Boltzmann's constant, c is the speed of light, $\hbar$ is the reduced Planck constant, G is the gravitational constant, and $A = 4\pi r^2$ for a Schwarzschild black hole with radius $r = \frac{2GM}{c^2}$. Since entropy measures the number of microstates (or information content) of a system, this formula implies that the information inside a black hole scales with the two-dimensional area of its horizon (in Planck units, $A/4l_p^2$, where $l_p = \sqrt{\frac{\hbar G}{c^3}}$ is the Planck's length), not its three-dimensional volume. This was surprising because, in most physical systems, entropy scales with volume, reflecting the number of particles or states within. 't Hooft and Susskind generalized this observation, proposing that the information content of any region of space, not just black holes, is limited by the area of its boundary. Specifically, the maximum entropy in a region enclosed by a surface of area (A) is given by the Bekenstein bound:

$$S \leq \frac{kc^3 A}{4\hbar G}.$$

This bound suggests that the number of quantum states (or bits of information) needed to describe everything inside a volume is encoded on its boundary, with roughly one bit per Planck area (l_p^2). For example, a spherical region of radius r has a boundary area $A = 4\pi r^2$, so its maximum entropy is proportional to r^2, not r^3. This is the essence of the Holographic Principle: the physics of a d-dimensional volume can be described by a theory in $d-1$ dimensions on its boundary, much like a hologram projects a 3D image from a 2D film. The principle gained traction with the development of the AdS/CFT correspondence, proposed by Juan Maldacena in 1997, which provides a concrete realization of holography.

AdS/CFT conjectures a duality between a gravitational theory in (d)-dimensional Anti-de Sitter (AdS) space (a universe with a negative cosmological constant) and a conformal field theory (CFT) on its ((d-1))-dimensional boundary. For instance, in the most studied case, type IIB string theory in $AdS_5 \times S^5$ (a five-dimensional AdS space times a five-dimensional sphere) is dual to a four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on the boundary. The CFT, a quantum field theory without gravity, fully encodes the dynamics of the AdS bulk, including gravity, black holes, and quantum effects. The duality implies that bulk phenomena, like the formation of a black hole, correspond to specific states or operators in the boundary CFT. Mathematically, the partition functions of the two theories are equal:

$$Z_{\text{AdS}} = Z_{\text{CFT}},$$

where $Z = \text{Tr}(e^{-\beta H})$ is the partition function, H is the Hamiltonian, and $\beta = 1/(kT)$. This equivalence allows physicists to study complex gravitational phenomena, like quantum gravity, using well-understood quantum field theories. The Holographic Principle has far-reaching implications. First, it suggests that gravity, traditionally described by general relativity in the bulk, may be an emergent phenomenon arising from quantum interactions on the boundary. In AdS/CFT, the metric of the AdS space is encoded in the CFT's correlation functions, and the radial dimension of AdS corresponds to the energy scale in the CFT via the renormalization group flow. This is often expressed through the Ryu-Takayanagi formula, which relates the entanglement entropy S_{EE} of a region in the CFT to the area of a minimal surface in the AdS bulk:

$$S_{\text{EE}} = \frac{\text{Area of minimal surface}}{4G\hbar}.$$

This formula generalizes the Bekenstein-Hawking entropy to arbitrary regions and highlights the deep connection between quantum entanglement and geometry. Second, the principle challenges our understanding of spacetime. If a 3D universe can be described by a 2D boundary, the extra dimension may be an illusion, much like a hologram creates the appearance of depth. This raises questions about the fundamental nature of reality: is our universe holographic, with physical laws emerging from a lower-dimensional theory? While AdS/CFT applies to AdS spaces, efforts are underway to extend holography to flat spacetimes (like our universe) or de Sitter spaces, though these are less understood. Third, the Holographic Principle constrains quantum gravity theories. Any consistent theory must respect the area scaling of entropy, ruling out models where information scales with volume. This has influenced string theory, loop quantum gravity, and other approaches, pushing physicists to rethink locality and causality. Despite its elegance, the Holographic

Principle faces challenges. Outside AdS/CFT, explicit holographic dualities for realistic spacetimes are lacking. The principle also raises philosophical questions: if reality is a hologram, what is the "true" dimensionality of the universe? Moreover, encoding bulk dynamics on a boundary requires non-local interactions, which are hard to reconcile with local field theories. In conclusion, the Holographic Principle is a cornerstone of modern theoretical physics, bridging black hole thermodynamics, quantum mechanics, and gravity. By suggesting that the universe's information is encoded on a lower-dimensional boundary, it offers a path to unify quantum mechanics and general relativity. Equations like the Bekenstein-Hawking entropy, Bekenstein bound, and Ryu-Takayanagi formula quantify this idea, while AdS/CFT provides a concrete framework. As research progresses, the principle may unlock deeper truths about the universe, perhaps revealing that reality, like a hologram, is a projection of a more fundamental truth.

1.1.2 AdS$_n$-CFT$_{n-1}$ Duality

1.1.2.1 AdS$_n$ spacetime

Anti-de Sitter (AdS) spacetimes are maximally symmetric solutions to Einstein's field equations with a negative cosmological constant, pivotal in theoretical physics, notably in the AdS/CFT correspondence and Kaluza-Klein reductions. In D dimensions, AdS$_D$ exhibits constant negative curvature, described in Poincaré coordinates by the metric:

$$ds^2 = e^{-2kz}\eta_{\mu\nu}dx^\mu dx^\nu + dz^2,$$

where z is the radial coordinate, k relates to the cosmological constant, and $\eta_{\mu\nu}$ is the $(D-1)$-dimensional Minkowski metric. The curvature tensors satisfy:

$$R_{ABCD} = k^2(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}),$$

yielding a Ricci tensor:

$$R_{AB} = -(D-1)k^2\eta_{AB}.$$

The Einstein equation, with cosmological constant $\Lambda = -\frac{(D-1)(D-2)}{2}k^2$, is:

$$R_{AB} - \frac{1}{2}Rg_{AB} + \Lambda g_{AB} = 0.$$

AdS spacetimes feature a timelike conformal boundary at $z \to \infty$, distinguishing them from Minkowski or de Sitter spacetimes.

In gravity and Kaluza-Klein reductions, AdS spacetimes are central to brane-world scenarios, like the Randall-Sundrum model, where gravity localizes on a lower-dimensional brane within a higher-dimensional AdS bulk. A brane-world reduction from D-dimensional AdS gravity to $(D-1)$-dimensional gravity without a cosmological constant uses the ansatz:

$$d\hat{s}_D^2 = e^{-2k|z|} ds_{D-1}^2 + dz^2.$$

This yields the D-dimensional Ricci tensor:

$$\hat{R}_{ab} = e^{2k|z|} R_{ab} - (D-1)k^2 \eta_{ab} + 2k\delta(z)\eta_{ab},$$

$$\hat{R}_{zz} = -(D-1)k^2 + 2k(D-1)\delta(z).$$

If the bulk satisfies $\hat{R}_{AB} = -(D-1)k^2\eta_{AB}$, the $(D-1)$-dimensional metric satisfies $R_{ab} = 0$, indicating pure Einstein gravity. Delta-function terms are attributed to brane sources, ensuring consistency.

AdS spacetimes also appear in sphere reductions, such as $D = 11$ supergravity on S^4, yielding $SO(5)$-gauged supergravity in $D = 7$. The $\mathrm{AdS}_7 \times S^4$ background arises from the negative cosmological constant induced by the 4-form field strength. The bosonic Lagrangian includes:

$$\mathcal{L}_7 = \hat{R} * 1 - \frac{1}{4} T_{ij}^{-1} * DT_{jk} \wedge T_{k\ell}^{-1} DT_{\ell i} - V * 1,$$

with potential:

$$V = \frac{1}{2} g^2 \left(2T_{ij}T_{ij} - (T_{ii})^2\right).$$

This AdS geometry facilitates consistent truncation to massless modes, vital for holography and unified theories.

1.1.2.2 CFT$_{n-1}$

Conformal Field Theory (CFT) is a cornerstone of theoretical physics, particularly in two dimensions, where it describes systems invariant under conformal transformations—coordinate changes that preserve angles but rescale the metric by a factor $\Omega^2(\sigma)$. Formally, for a metric $g_{\alpha\beta}$, a conformal transformation satisfies:

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma). \tag{1.1}$$

This scale invariance ensures that the physics is identical across all length scales, making CFTs essential for modeling critical phenomena in statistical mechanics and worldsheet dynamics in string theory. In two dimensions, the conformal group is infinite-dimensional, comprising holomorphic transformations $z \to f(z)$, unlike the finite-dimensional $SO(p+1, q+1)$ in higher dimensions ($p+q > 2$). This leads to the Virasoro algebra, with generators L_n that encode symmetries like translations ($z \to z + a$) and dilatations ($z \to \zeta z$).

A defining feature of CFTs is the absence of massive excitations, as any mass scale would violate scale invariance. Instead, the focus is on correlation functions and operator transformations. The stress-energy tensor $T_{\alpha\beta}$, which captures conserved currents from translational invariance, is traceless in conformal theories:

$$T^{\alpha}_{\alpha} = 0. \tag{1.2}$$

In complex coordinates $z = \sigma^1 + i\sigma^2$, the tensor splits into holomorphic $T_{zz}(z) = T(z)$ and antiholomorphic $T_{\bar{z}\bar{z}}(\bar{z}) = \bar{T}(\bar{z})$ components, with conservation laws $\bar{\partial}T(z) = 0$ and $\partial\bar{T}(\bar{z}) = 0$. The operator product expansion (OPE) is a key tool, expressing the product of operators at nearby points:

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \sum_k C^k_{ij}(z-w, \bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}). \tag{1.3}$$

Primary operators, with weights $(h, \bar{h})$, transform simply under conformal symmetries:

$$\mathcal{O}(z,\bar{z}) \to \left(\frac{\partial\bar{z}}{\partial z}\right)^{-h} \left(\frac{\partial\bar{\bar{z}}}{\partial\bar{z}}\right)^{-\bar{h}} \mathcal{O}(\bar{z},\bar{\bar{z}}). \tag{1.4}$$

These weights define the scaling dimension $\Delta = h + \bar{h}$ and spin $s = h - \bar{h}$.

In the quantum regime, two-dimensional CFTs are exactly solvable, a rare trait for interacting field theories. The central charge c in the Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{1.5}$$

quantifies degrees of freedom, appearing in phenomena like the Weyl anomaly, where the trace of the stress-energy tensor in curved space is:

$$\langle T^{\alpha}_{\alpha}\rangle = -\frac{c}{12}R. \tag{1.6}$$

The c-theorem states that c decreases along renormalization group flows, reflecting information loss. Radial quantization, where states live on circles and evolve via the dilatation operator $D = L_0 + \bar{L}_0$, facilitates state analysis. The state-operator map uniquely links states to local operators, with primary states corresponding to primary operators.

In string theory, CFTs describe the worldsheet, ensuring gauge invariance in the Polyakov formalism. Boundary CFTs, relevant for open strings, impose conditions like $T_{zz} = T_{\bar{z}\bar{z}}$ at the boundary, reducing the number of states. The free scalar field, with action:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X \partial^\alpha X, \tag{1.7}$$

illustrates these concepts, its propagator revealing conformal properties via the OPE. CFTs thus bridge symmetry principles with physical predictions, offering profound insights into quantum field theory and string theory.

1.1.2.3 AdS/CFT correspondence

AdS/CFT Duality Explanation The AdS/CFT correspondence, a cornerstone of modern theoretical physics, posits a duality between a conformal field theory (CFT) in (d)-dimensional Minkowski space and a gravitational theory in ((d+1))-dimensional Anti-de Sitter (AdS) space. Introduced by Maldacena, this holographic principle suggests that a strongly coupled CFT, such as $\mathcal{N} = 4$ super Yang-Mills (SYM) in four dimensions, is equivalent to a weakly coupled supergravity theory in $AdS_5 \times S^5$. The correspondence leverages the isometry group $SO(2,4)$ of AdS_5, which matches the conformal group of the CFT, enabling a dictionary between CFT operators and AdS fields. For instance, a CFT operator $O(x)$ with conformal dimension Δ couples to a bulk field $\phi(x,u)$ via boundary interactions, described by the action term $\int d^4x, \phi_0(x)O(x)$, where $\phi_0(x)$ is the boundary value of ϕ. Correlation functions in the CFT are computed using the bulk partition function, approximated classically as

$$Z_{\mathrm{AdS}}[\phi_0] = e^{-S_{\mathrm{AdS}}[\phi]} \approx \langle e^{\int \phi_0 O} \rangle_{\mathrm{CFT}}$$

The large (N) limit of the gauge theory, with 't Hooft coupling $\lambda = g^2_{\mathrm{YM}} N$, corresponds to the classical supergravity limit when $\lambda \gg 1$, while the string coupling $g_s \sim 1/N$. This duality allows quantum effects in the CFT, like correlation functions

$$\langle O(x_1) \cdots O(x_n) \rangle$$

to be computed via classical gravitational dynamics, providing insights into strongly coupled systems, confinement, and even condensed matter physics, by mapping complex quantum phenomena to tractable geometric problems in AdS space.

1.1.3 AdS_2-CFT$_1$ correspondence

1.1.3.1 2D gravity models - JT gravity

Jackiw-Teitelboim (JT) gravity is a model of two-dimensional (2D) gravity that provides a valuable framework for understanding quantum gravity and black hole thermodynamics in a simplified setting. Unlike in four dimensions where Einstein's equations produce dynamic degrees of freedom for the metric, in 2D spacetime the Einstein tensor vanishes identically due to topological constraints. Specifically, for any 2D metric $g_{\mu\nu}$, the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is identically zero because the Ricci tensor $R_{\mu\nu}$ is completely determined by the scalar curvature R, and the variation of the Einstein-Hilbert action yields no dynamics. Therefore, to have a nontrivial theory of gravity in 2D, auxiliary fields such as the *dilaton* ϕ are introduced.

In JT gravity, the action is constructed as

$$S_{\rm JT} = \int d^2x\sqrt{-g}\,\phi(R + 2\Lambda),$$

where ϕ is the dilaton field, R is the Ricci scalar of the 2D spacetime metric $g_{\mu\nu}$, and Λ is a cosmological constant (often negative, corresponding to an AdS$_2$ background). This action yields second-order field equations for the metric and first-order equations for the dilaton, making the theory solvable. Varying the action with respect to ϕ gives

$$R + 2\Lambda = 0,$$

which fixes the geometry to a constant curvature spacetime—typically anti-de Sitter (AdS$_2$) for $\Lambda < 0$. Varying the action with respect to the metric gives another equation involving derivatives of ϕ, which governs how the dilaton profiles across the spacetime.

A key feature of JT gravity is that, despite having no propagating degrees of freedom in the metric, it supports black hole solutions, and its boundary dynamics are nontrivial.

These dynamics are governed by the Schwarzian action, which appears when considering the low-energy limit of the boundary mode:

$$S_{\mathrm{Sch}}[f] = -C \int dt \, \{f(t), t\},$$

where $\{f(t), t\}$ is the Schwarzian derivative of the boundary reparametrization $f(t)$, and $C \propto \phi_r$ is related to the value of the dilaton at the boundary. This boundary action captures the low-energy dynamics of near-extremal black holes in higher-dimensional theories and connects JT gravity to the SYK model—a disordered quantum mechanical system with similar infrared behavior.

The JT model also arises from dimensional reduction of higher-dimensional gravity theories. For instance, spherically reducing four-dimensional Einstein gravity under the ansatz

$$ds^2_{(4)} = g_{\mu\nu}(x)dx^\mu dx^\nu + \Phi^2(x)d\Omega_2^2$$

produces an effective 2D theory for $g_{\mu\nu}$ and the scalar Φ, leading to actions of the form

$$S = \int d^2x\sqrt{-g}\left[\Phi^2 R - \frac{1}{2}(\nabla\Phi)^2 - V(\Phi)\right],$$

which includes the JT gravity model in specific limits where $\Phi \sim \phi$, and kinetic terms may be neglected.

The beauty of JT gravity lies in its exact solvability and the ability to study non-perturbative aspects of quantum gravity, black hole entropy, and holography in a tractable setting. The model encapsulates the essence of diffeomorphism invariance, black hole thermodynamics, and quantum effects such as Hawking radiation, despite its apparent simplicity.

1.1.3.2 1D CFT model - SYK model

The Sachdev-Ye-Kitaev (SYK) model is a quantum mechanical system of N Majorana fermions $\chi_i(\tau)$ with all-to-all random q-body interactions, most commonly $q = 4$. Its action is given by:

$$S = \int d\tau \left[\frac{1}{2} \sum_{i=1}^{N} \chi_i \partial_\tau \chi_i - \frac{1}{4!} \sum_{i,j,k,l} J_{ijkl} \chi_i \chi_j \chi_k \chi_l \right],$$

where J_{ijkl} are real, antisymmetric, Gaussian-random couplings with zero mean and variance

$$\langle J_{ijkl}^2 \rangle = \frac{3! J^2}{N^3}.$$

At large N, the dominant Feynman diagrams contributing to the two-point function are "melon" diagrams, which can be resummed via a Schwinger-Dyson equation involving the full two-point function $G(\tau) = \frac{1}{N} \sum_i \langle \chi_i(\tau) \chi_i(0) \rangle$ and its self-energy $\Sigma(\tau)$:

$$\Sigma(\tau) = J^2 G(\tau)^3, \qquad G(i\omega)^{-1} = -i\omega - \Sigma(i\omega).$$

In the infrared (IR) limit, where $J|\tau| \gg 1$, the kinetic term ∂_τ becomes negligible and the equations become conformally invariant. The solution to the Schwinger-Dyson equations in this regime is:

$$G(\tau) = b \frac{\mathrm{sgn}(\tau)}{|J\tau|^{2\Delta}}, \quad \Delta = \frac{1}{4}, \quad b^4 = \frac{1}{4\pi}.$$

This emergent conformal symmetry is spontaneously and explicitly broken, leading to the appearance of a soft mode governed by the Schwarzian action. By considering reparametrizations $\tau \to f(\tau)$, one finds that the leading IR effective action for these modes is:

$$S_{\mathrm{Sch}} = -\alpha_S N \int d\tau \, \mathrm{Sch}(f(\tau), \tau), \quad \text{where} \quad \mathrm{Sch}(f(\tau), \tau) = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2} \left(\frac{f''(\tau)}{f'(\tau)} \right)^2.$$

This Schwarzian action governs the breaking of reparametrization symmetry from Diff(R) to $SL(2, \mathbb{R})$, and it plays a central role in the model's connection to two-dimensional dilaton gravity and holography.

The four-point function is given by a sum over ladder diagrams built from full propagators and a kernel:

$$K(\tau_1, \tau_2; \tau_3, \tau_4) = -J^2(q-1)G(\tau_{13})G(\tau_{24})G(\tau_{34})^{q-2}.$$

This kernel acts on bilocal functions, and its eigenfunctions are the conformal partial waves $\mathcal{F}_h(\tau_i)$ of the $SL(2, \mathbb{R})$ group. The resulting four-point function in the conformal limit takes the form:

$$\langle \chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle = G(\tau_{12})G(\tau_{34}) + \frac{1}{N}\int \frac{dh}{2\pi i}\rho(h)\mathcal{F}_h(\tau_i),$$

with

$$\rho(h) = \mu(h)\frac{k(h)}{1-k(h)},$$

where $k(h)$ are the eigenvalues of the kernel and $\mu(h)$ is a measure factor involving gamma functions. The poles of $\rho(h)$, given by $k(h) = 1$, determine the dimensions h of bilinear operators exchanged in the four-point function, such as $\mathcal{O}_h = \sum_i \chi_i \partial_\tau^{2n+1}\chi_i$.

Of particular importance is the $h = 2$ mode, which corresponds to the Schwarzian sector. It leads to a divergence in the conformal four-point function and signals the need to include the full effective action, incorporating non-conformal corrections. The dominance of the $h = 2$ exchange in out-of-time-ordered correlators implies maximal quantum chaos, with a Lyapunov exponent:

$$\lambda_L = \frac{2\pi}{\beta},$$

which saturates the bound on chaos in quantum systems and matches the behavior of black holes in Einstein gravity. This profound connection makes the SYK model a valuable tool in exploring the AdS/CFT correspondence and quantum aspects of gravity.

1.1.4 Virosoro Algebra and the Central Charge

The Witt algebra and the Virasoro algebra are fundamental structures in theoretical physics, particularly in conformal field theory (CFT), which describes systems with conformal symmetry, such as those in string theory or critical phenomena in statistical mechanics.

The Witt algebra is an infinite-dimensional Lie algebra that captures the infinitesimal conformal transformations of the complex plane or the circle (S^1). Physically, it represents the symmetries of a system that remain invariant under angle-preserving transformations, like stretching or rotating parts of a plane while preserving its local structure. Its basis elements, labeled L_n, correspond to vector fields that generate these transformations, with a Lie bracket $[L_n, L_m] = (n-m)L_{n+m}$, reflecting how these symmetries compose.

The Virasoro algebra, however, is the central player in CFT, extending the Witt algebra to account for quantum effects. In classical systems, symmetries like those in the Witt algebra are straightforward, but quantization introduces anomalies—quantum corrections that modify the symmetry algebra. The Virasoro algebra is the Witt algebra's *central extension*, meaning it adds a new element, a central charge Z, to the algebra, which doesn't transform under the symmetries but affects their composition. The modified Lie bracket becomes $[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n(n^2-1)}{12}Z$, where the extra term, proportional to Z, appears only when $n+m=0$. This central charge Z is a scalar that commutes with all L_n ($[L_n, Z] = 0$), making it "central" to the algebra.

The transition from the Witt to the Virasoro algebra is like upgrading a classical machine to a quantum one: the Witt algebra handles the smooth, classical symmetries, but the Virasoro algebra, with its central charge, captures the richer, quantum-deformed structure. This extension is unique (up to equivalence), making the Virasoro algebra the universal framework for describing conformal symmetries in quantum physics, with the central charge acting as a fingerprint of the system's quantum nature.

We have adopted the following approach to compute the central-charge. From the twisted stress-energy tensor, we can compute the central charge using the following formula:

$$\delta_\epsilon T_{zz}(z) = \frac{c}{12}\partial_z^3\eta(z) + 2\partial_z\eta(z)T_{zz}(z) + \eta(z)\partial_z T_{zz}(z) \tag{1.8}$$

1.2 Motivation

Central charge calculation ...

Chapter 2

Literature Review

It is natural to assume that if a problem has been solved in the $(3+1)\,D$ then the solution in reduced number of dimensions should follow similarly. However, this is not the case for the *Einstein-Hilbert* field equations which were originally formulated in 4D spacetime background. Especially the reduction to the $(1+1)\,D$ requires modification in the action as the 4D action gives no information on the dynamics and the equations instead express mathematical identities in the 2D spacetime.

Instead of writing the action from scratch in lower dimensions we have another alternative which is to use the *Kaluza-Klein* reduction scheme. [A]

In the following reviewed papers, we will see how the authors have formulated the 2D gravity theories coupled with the electromagentic theories (Maxwell and ModMax). Further due to Hartman and Strominger [**?**] we know that the conformal diffeomorphisms have to be provided with appropriate gauge transformations in order to preserve the boundary conditions. Further they noted that these transformations satisfy the *Virosoro algebra* (1.1.4) and have a corresponding central charge.

We use the relations

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta I}{\delta g_{ab}}, \quad J^{a} = \frac{1}{\sqrt{-g}}\frac{\delta I}{\delta A_{a}}. \tag{2.1}$$

where T^{ab} is the energy-momentum tensor, J^a is the current density on the boundary. Further h_{ab} is the induced metric on the boundary.

We solve for the form of the required $U(1)$ gauge transformation so as to preserve the boundary conditions upon diffeomorphisms. Further the transformed stress-energy tensor under the combined effect of both the transformations reveals the central-charge.

2.1 Brown-Heanneux Central charge [?]

2.2 $(1+1)D$ gravity coupled to constant EM field [?]

In a paper authored by *Castro et al.* [?], the authors explore the dynamics of a $(1+1)D$ JT gravity theory coupled to a constant electromagnetic field strength. The paper delves into the implications of this coupling on the gravitational dynamics and the resulting spacetime structure.

2.2.1 The bulk action and the equations of motion

$$I_{bulk} = \frac{\alpha}{2\pi}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[e^{-2\phi}\left(R+\frac{8}{L^2}\right)-\frac{L^2}{4}F^2\right]$$

To obtain the equations of motion, we vary the action and get the following expression

$$\delta I_{\text{bulk}} = \frac{\alpha}{2\pi}\int_{\mathcal{M}} d^2x\sqrt{-g}\left[E^{\mu\nu}\delta g_{\mu\nu}+E^{\phi}\delta\phi+E^{\mu}\delta A_{\mu}\right]+\text{boundary terms.} \tag{2.2}$$

$$E_{\mu\nu} = \nabla_\mu\nabla_\nu e^{-2\phi} - g_{\mu\nu}\nabla^2 e^{-2\phi} + \frac{4}{L^2}e^{-2\phi}g_{\mu\nu} + \frac{1}{2}F_\mu^\lambda F_{\nu\lambda} - \frac{1}{8}g_{\mu\nu}F^2 = 0. \tag{2.3}$$

$$E_\phi = -2e^{-2\phi}\left(R+\frac{8}{L^2}\right) = 0 \text{ and } E_\mu = L^2\nabla^\nu F_{\nu\mu} = 0 \tag{2.4}$$

Further considering a constant electromagnetic field strength $F_{\mu\nu} = 2\mathcal{E}\epsilon_{\mu\nu}$, we obtain for the dilaton field $e^{-2\phi} = \frac{L^4}{4}\mathcal{E}^2$

Further analyzing and the formulation of complete solution requires framing the problem in the *Fefferman-Graham gauge* [?] and considering the asymptotic expansion of the fields in this gauge.

2.2.2 Calculation of the central charge

Under the diffeomorphic transformation $x^\mu \to x^\mu + \theta^\mu$, requiring the boundary condition preservation, the authors obtained the gauge transformation as

$$\theta^\eta = -\frac{L}{2}\partial_t\zeta(t), \quad \theta^t = \zeta(t) + \frac{L^2}{2}\left(e^{4\eta/L}+g_1(t)\right)^{-1}\partial_t^2\zeta(t) \tag{2.5}$$

here ζ is an arbitrary function of time. To preserve $A_\eta = 0$ the transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ gets the solution

$$\Lambda = -L\, e^{-\phi} \left(e^{2\eta/L} + g_1(t)\, e^{-2\eta/L} \right)^{-1} \partial_t^2 \zeta(t). \tag{2.6}$$

The current and stress tensor's transformation can be expressed as

$$(\delta_\epsilon + \delta_\Lambda) A_t = -e^{-2\eta/L} e^{-\phi} \left(\frac{1}{L} \xi(t) \partial_t h_1(t) + \frac{2}{L} h_1(t) \partial_t \xi(t) + \frac{L}{2} \partial_t^3 \xi(t) \right). \tag{2.7}$$

$$(\delta_\theta + \delta_\Lambda) T_{tt} = 2T_{tt} \partial_t \zeta + \zeta \partial_t T_{tt} - \frac{c}{24\pi} L \partial_t^3 \zeta(t). \tag{2.8}$$

From these equations, the relation between A_μ and $T^{\mu\nu}$ and noting from the Eq 1.8 we can identify $c = -24\alpha e^{-2\phi} = \dfrac{6}{G_2} = \dfrac{3}{2} k \mathcal{E}^2 L^4$.

Proceeding further, the authors performed a dimensional reduction of the $(2+1)D$ theory using $ds^2 = e^{-2\Phi} \ell^2 (dz + \mathcal{A}_\mu dx^\mu)^2 + \mathcal{G}_{\mu\nu} dx^\mu dx^\nu$ and obtained a relation between the central charge in 3D to one that in 2D as

$$c_{2D} = 2\pi e^{-\Phi} c_{3D} \tag{2.9}$$

2.3 $(1+1)D$ gravity coupled to the ModMax EM field [?]

The authors in this paper started with the *ModMax* lagrangian, which is an example of non-linear electrodynamics with one free parameter, β, so formed to possess the usual $\mathbb{SO}(2)$ symmetry of Maxwell's theory along with the conformal symmetry. Further in the weak field limit ($\beta \to \infty$), it must yield the Maxwell theory. The *ModMax* lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \left(S \cosh \beta - \sqrt{S^2 + P^2} \sinh \beta \right) \tag{2.10}$$

where $S = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}$ and $P = \frac{1}{2} F_{\mu\nu} \tilde{F}^{\mu\nu}$. $\tilde{F}^{\mu\nu}$ is the hodge dual of the electromagnetic field tensor defined as $\tilde{F}^{\mu\nu} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$.

2.3.1 The bulk action and the equations of motion

The 4D action for the gravity coupled to the *ModMax* lagrangian is given as

$$I = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g} \left(R - 2\Lambda - 4\alpha \mathcal{L}_{MM}\right) \tag{2.11}$$

where α is the coupling constant and Λ is the cosmological constant. The authors performed suitable dimensional reduction to obtain action for a $(1+1)$D JT gravity theory. The ansatz for the metric was taken to be

$$ds^2_{(3+1)} = g_{\mu\nu}(x^\rho)dx^\mu dx^\nu + \Phi(x^\mu)dx_i^2 \tag{2.12}$$

$$A_\mu \equiv A_\mu(x^\nu), \quad A_z \equiv A_z(x^\mu), \tag{2.13}$$

Here μ, ν are the indices for coordinates in the reduced dimensions and i denotes the compact dimensions. The $2D$ projected *ModMax* action is written as

$$I_2 = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g_{(2)}} \left(\Phi R^{(2)} - 2\Lambda\Phi - 4\alpha\Phi\mathcal{L}^{(2)}_{(M)}\right) \tag{2.14}$$

$$\mathcal{L}^{(2)}_{\text{MM}} = \frac{1}{2}\left(s\cosh\beta - \sqrt{s^2+p^2}\sinh\beta\right) \tag{2.15}$$

$$s = \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \Phi^{-1}\left((\partial\chi)^2 + (\partial\zeta)^2\right), \quad p = -2\Phi^{-1}\epsilon^{\mu\nu}\partial_\mu\chi\partial_\nu\zeta \tag{2.16}$$

2.3.2 Calculation of the central charge

An approach similar to 2.2.2 was employed here, wherein the gauge transformation were obtained by imposing boundary condition preservation upon diffeomorphisms. And thus the coefficient of the modification term in the stress-energy tensor was identified as the central charge. The central charge thus obtained is given as

$$c_{3D} = \frac{1}{144\sqrt{3}\pi G_2}\left(\alpha - 12\beta\alpha + 2\alpha^2\right), \tag{2.17}$$

The authors thus rightly concluded that in the limiting case of weak field ($\beta \to 0$) the central charge asymptotes to $\frac{1}{G_2}$ which matches with the central charge obtained by 2.2 for the case of Maxwell theory.

Chapter 3

ModMax lagrangian and comparision between central charge in 2D and 3D

3.1 4D to 3D reduction and central charge

We start with the 4D action for *EH* gravity coupled to *ModMax* electrodynamics [**?**] given by 2.10. We then perform suitable dimensional reduction to obtain the action in 3D and solve for the system thereafter.

3.1.1 Reduction of the ModMax Lagrangian and bulk action

The ansatz for the metric is taken to be

$$ds^2_{(4)} = e^{2\alpha\phi} ds^2_{(3)} + e^{2\beta\phi}(dx^3)^2 \tag{3.1}$$

$$dx^2_{(3)} = g_{\mu\nu}(x^\rho) dx^\mu dx^\nu \tag{3.2}$$

$$R_{(4)} = e^{-2\alpha\phi}\left(R_{(3)} - \frac{1}{2}(\partial\phi)^2 - d\alpha\Box\phi\right) \tag{3.3}$$

μ, ν are indices representing reduced dimensions and x^3 is the compact dimension. Note that we can drop the dalembertian term in the Ricci scalar as it is a total derivative.

Further for this form of choice for dissection of the metric to one lower dimension (from $d+1$ to d), we observe that $\mathcal{L}$ becomes $e^{(\beta+(d-2)\alpha)\phi}\sqrt{-g}\mathcal{R}+\ldots$ where $\mathcal{R}$ is the Ricci scalar in d dimensions. So it is required to set $\beta+(d-2)\alpha=0$, which gives us $\beta=-\alpha$ in $4D$ to $3D$ reduction. Further to ensure that we obtain a term of the form $\frac{1}{2}\sqrt{-g}\,(\partial\phi)^2$ in the action we require $\alpha^2=\dfrac{1}{2(d-1)(d-2)}$ which gives $\alpha=\frac{1}{2}$ for our case. The reduced metric determinant can be determined using

$$\sqrt{-g_{(4)}}=\sqrt{-g_{(3)}}e^{(\beta+d\alpha)\phi}=e^{2\alpha\phi}\sqrt{-g_{(3)}}$$

The 3D projected *ModMax* lagrangian can now be calculated as

$$\begin{aligned}
s&=\frac{1}{2}F_{AB}F^{AB}=\frac{1}{2}F_{AB}F_{CD}g^{AC}g^{BD}\\
&=\frac{1}{2}F_{\mu\nu}F^{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}+e^{-2\beta\phi}\left(F_{M3}F_{P3}g^{PM}\right)\\
&=\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+e^{-2\beta\phi}(\partial_\mu\chi)^2
\end{aligned}$$

$$\begin{aligned}
p&=\frac{1}{2}F_{AB}\tilde{F}^{AB}=\frac{1}{2}F_{AB}F_{CD}\epsilon^{ABCD}\\
&=2e^{-2\beta\phi}\sqrt{-g_{(3)}}\,(F_{01}F_{23}+F_{02}F_{31}+F_{21}F_{03})\\
&=e^{-2\beta\phi}\epsilon^{abc}F_{ab}F_{c3}\\
&=2e^{-2\beta\phi}\epsilon^{abc}F_{ab}\partial_c\chi
\end{aligned}$$

$$\mathcal{L}^{(3)}_{\mathrm{MM}}=\frac{1}{2}\left(s\cosh\zeta-\sqrt{s^2+p^2}\sinh\zeta\right) \tag{3.4}$$

here ζ is the *ModMax* parameter.

3.1.2 Equations of motion

The 4D lagrangian is integrated along the compact dimension and we get the 3D reduced action as

$$I=\frac{1}{16\pi G_3}\int d^3x\sqrt{-g_{(3)}}\left(R-2\Lambda-4\kappa\mathcal{L}^{(3)}_{\mathrm{MM}}\right) \tag{3.5}$$

The variation of the action yields (and setting them all to zero to find the stationary action)

$$\delta I=\frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\,(\Psi_{\mu\nu}\delta g^{\mu\nu}+\Psi_\mu\delta A^\mu+\Psi_\phi\delta\phi+\Psi_\chi\delta\chi) \tag{3.6}$$

where

$$\Psi_\phi = -2\kappa \left[-2\beta e^{-2\beta\phi}(\partial\chi)^2 \cosh\zeta + \frac{\sinh\zeta}{\sqrt{s^2+p^2}} \left(s e^{-2\beta\phi}(\partial\chi)^2 + p^2 \right) \right] = 0 \tag{3.7}$$

$$\Psi_\chi = -4\kappa \nabla_\mu \left[-e^{-2\beta\phi}\partial^\mu\chi \cosh\zeta + e^{-2\beta\phi} \frac{\sinh\zeta}{\sqrt{s^2+p^2}} \left(s(\partial\chi)^2 + \frac{p}{2}\epsilon^{ab\mu}F_{ab} \right) \right] = 0 \tag{3.8}$$

$$\Psi_\nu = 4\kappa \nabla_\mu \left[F^\mu_\nu \cosh\zeta - \frac{\sinh\zeta}{\sqrt{s^2+p^2}} \left(sF^\mu_\nu + pe^{-2\beta\phi}\epsilon^{\mu bc} g_{b\nu}\partial_c\chi \right) \right] = 0 \tag{3.9}$$

$$\Psi_{\mu\nu} = \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \Lambda g_{\mu\nu} + 2\kappa \mathcal{L}_{MM} g_{\mu\nu} - \tag{3.10}$$

$$\kappa \left[2 \left(\cosh\zeta - \frac{s\sinh\zeta}{\sqrt{s^2+p^2}} \right) \left(F^\beta_\mu F_{\nu\beta} + e^{-2\beta\phi}\partial_\mu\chi\partial_\nu\chi \right) - \frac{p^2 \sinh\zeta}{\sqrt{p^2+s^2}} g_{\mu\nu} \right] = 0 \tag{3.11}$$

We can extract some information from the Eq 3.10 upon contracting it with the contravariant metric tensor. We thus obtain

$$-\frac{R}{2} + 3\Lambda - 2\kappa \left[-3\mathcal{L}_{MM} + 2 \left(\cosh\zeta - \frac{s\sinh\zeta}{\sqrt{s^2+p^2}} \right) \left(F^2 + e^{-2\beta\phi}(\partial\chi)^2 \right) - \frac{3p^2}{\sqrt{p^2+s^2}} \sinh\zeta \right] = 0 \tag{3.12}$$

3.1.3 Perturbative solutions in FG gauge

3.1.4 Boundary terms and counterterms

3.1.5 Central charge

3.2 3D to 2D reduction and central charge

3.2.1 Reduction of the ModMax Lagrangian and bulk action

3.2.2 Equations of motion

3.2.3 Perturbative solutions in FG gauge

3.2.4 Boundary terms and counterterms

3.2.5 Central charge