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THREE-BODY PROBLEM WITH SEPARABLE POTENTIALS
(I) Bound States

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Abstract: The nuclear 3-body problem in which two particles are identical is investigated, by assuming separable potentials to operate between pairs. The separable interactions between n-p pairs have been taken of the types (1) central s-state and (2) tensor (s and d-states); the corresponding n-n interaction is taken to be in the s-state only. For both these cases the exact wave function of the bound 3-nucleon system is obtained in a particularly elegant form which admits of a simple physical interpretation and is easily amenable to mathematical treatment for applications in nuclear physics at low energies. As an application of the formalism, an approximate calculation of the ground state energy of the triton has been carried out by treating the Schrödinger equation as a direct eigenvalue problem without resort to variational techniques, for the case of pure s-state interactions between all the pairs (with parameters adjusted to fit the low-energy singlet and triplet nucleon-nucleon data). The binding energy so obtained is rather close to the actual value.

1. Introduction

The mathematical insolubility of problems involving more than 2 particles naturally necessitates resort to approximation methods, of which variational ones are by far the most popular. Indeed, variational techniques have so far been the only tools available for the determination of the triton binding energy [1]†. Unfortunately these methods, while fairly reliable for eigenvalue determinations, are of hardly any practical use for estimating the *wave function* of a three-body system, for which a boundary value approach seems to be almost imperative. In recent times some limited attempts have been made to obtain some idea of a 3-body wave function, through a formal Schrödinger type equation, but with extremely idealized types of 2-body interactions. Thus Skoroniakov and Ter-Martirosian [2]) have discussed the 3-body scattering problem with a zero-range 2-body force, which, though capable of exact solution, is unsuitable for the bound state problem as it gives infinite binding energy for a 3-body system. Skoroniakov [3]) later developed an approximation procedure suitable for treating the bound state problem with very short-range potentials. More recently, Eyges [4]) proposed a theory for the bound state 3-body problem with no formal potential but instead a boundary condition that the logarithmic derivative of the wave function be a prescribed constant at a fixed interparticle

† Earlier references are given in ref. [1]).

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distance (hard core). However, in spite of its mathematical elegance, such a model of "interaction" does not appear to be realistic enough to apply to the actual nuclear 3-body problem †.

Considerations of spin and isobaric spin naturally complicate the problem considerably in practice. The most exhaustive treatment so far in this respect has been that of Derrick and Blatt [5]) who made a complete classification of the wave function into states of definite symmetry with respect to rotations in ordinary, spin and charge spaces. They found that the Schrödinger equation for H^3 involving 6 independent coordinates was reducible to 16 coupled partial differential equations involving only 3 coordinates in just the same way as the Schrödinger equation for H^2 is reduced to two total differential equations (involving only the radial distance). It is important to note, however, that while such a simplification for H^3 is no doubt considerable, the fact that the three remaining coordinates are the non-trivial distances between pairs of particles (and not the trivial angular coordinates) makes it of limited practical value, at least from the point of view of determination of the wave function. Such complications can incidentally be traced to the assumption of *local* potentials operative between any two particles, and the work of these authors showed rather clearly the extent beyond which the 3-body problem with local potentials cannot be simplified.

Apart from these aspects of the three-body problem which are essentially of an algebraic nature, there is another, more important, aspect of this problem which bears directly on the very concept of a two-body interaction. Thus if one considers a two-body interaction as a matrix in energy space, it is clear that the elements on the energy shell are essentially related to the scattering phase shifts. To fit the parameters of a potential by scattering data up to fairly high energies does not therefore imply much more than a determination of these diagonal elements (or at most their bordering neighbours). On the other hand, such a procedure throws hardly any light on the validity of the elements far removed from the diagonal though the latter are automatically "expressed" in terms of the parameters of the potential. Now while it is not possible to give specific prescriptions to "determine" these *off-diagonal* elements, it is at least clear that they should have numerous manifestations in problems involving *more* than two particles. One therefore has in principle many indirect tests of the correctness of the off-diagonal elements of a potential matrix through the calculation of various physically measurable quantities (sensitive and insensitive) involving more than two particles. For example, the properties of nuclear matter like binding energy, saturation density, etc., are usually taken to provide such tests for a potential. However, such parameters as the properties of nuclear matter may not be able to distinguish many finer features of an interaction, so that as an alternative one might profitably consider the role of the

† Eyges' latest paper on this subject makes use of a regular potential approach (see ref. [13])).

three-body problem in this context, especially because the latter is rather sensitive to the structure of the two-body potential.

In the context of the three-body problem, it is not difficult to visualize that a two-body potential may fit scattering data quite accurately upto fairly high energies and yet may not provide enough binding for the triton. Indeed, Derrick and Blatt [6]) found that the Gammel-Thaler potential which fits high-energy scattering data is incapable of binding the triton. More recently, Werntz [7]) showed that the analogous Signell-Marshak potential suffers from the same limitation. In the language of the last paragraph, such a discrepancy should be attributed to the failure of the off-diagonal elements of these potentials in the low-energy region. It might well be possible to revise the parameters of these high-energy potentials so as not to affect important low-energy data, retaining at the same time the close agreement with high-energy scattering. To do this in a systematic manner over a wide energy range, would, however, be an extremely ambitious programme. We propose instead the following approach.

Instead of trying to examine the potential matrix over the entire energy range, it is convenient to choose its *low*-energy part for primary investigation. In order that this restricted approach may be meaningful at all, it is of course necessary that the " off-diagonal" elements of the low-energy region are not appreciably affected by the elements in the high-energy region. We *assume* that this is the case. In that event, one needs only a potential which fits scattering data at low and medium energies alone. Such a potential need be operative only in a finite number of angular momentum states, depending on the energies involved. After fixing the parameters of such a low-energy potential through a few experimental partial wave phase shifts, the validity of its off-diagonal elements (and hence of its parameters) can be tested, in principle, through the three-body problem at low energies. In this way one may try to arrive at a consistent "low-energy" picture of a two-body interaction.

To summarize, we see the following possibilities in the three-body problem. Physically it affords a means of testing the off-diagonal elements of a low-energy two-body potential through the calculation of various observable quantities, like magnetic moments, exchange currents, energies of excited states etc., in addition to the binding energy of the triton. In order that this may be done in a reliable way one needs a reasonably accurate three-body wave function which at the same time has a simple enough mathematical structure to lend itself to the above computations without resort to undue approximations. Now from the discussion given earlier we note that conventional local potentials do not hold out much promise in this regard. On the other hand, non-local potentials of the separable type introduced by Yamaguchi [8, 9]) are much more promising from our present point of view, since (1) our low-energy interaction needs only a few partial waves and (2) separable interactions lead to enormous simplifications in the mathematical formalism. As for realistic behaviour of such potentials, our experience with them in recent times [10]) suggests that they can quite satisfactorily account for nuclear phenomena like two-nucleon scattering, deuteron parameters, nuclear saturation, spin-orbit splittings in nuclei and nuclear energy levels.

In view of these considerations we felt that it should be of some interest to examine the extent to which a separable interaction could be employed to simplify the three-body problem and thus facilitate the calculation of observable quantities involving 3 particles in a more or less exact fashion without excessive labour. However, since this idea of employing a separable interaction in a scope of this dimension is comparatively new, the present investigation must necessarily be regarded as a preliminary one intended mainly to focus attention on the mathematical simplification of the three-body problem that is achieved in this new approach. The emphasis in this paper is therefore on the mathematical formulation of the three-body problem with some of the simplest separable potentials known, e.g., the Yamaguchi potentials involving central s-state [8]) and tensor d-state [9]) interactions (which of course fit two-body data at low energies in a limited way). Calculations employing more realistic separable potentials (involving spin-orbit forces and other effects) will be the subject of subsequent publications. In addition, we limit ourselves in this paper to the bound state 3-body problem. The corresponding scattering problem will be considered in a separate paper.

The effects of the neglect of short-range spin-orbit forces of the type found by Gammel and Thaler or Signell and Marshak are discussed qualitatively in sect. 5.

In sect. 2, the 3-body Schrödinger equation is formally solved with separable s-state forces between pairs of nucleons. The 3-body wave function is determined exactly in terms of a one-parameter wave function which satisfies a one-dimensional integral equation. The physical interpretation of the 3-body wave function so obtained is discussed in some detail.

In sect. 3, the same problem is treated by including Yamaguchi type tensor interactions between n-p pairs. For this case, too, the 3-body wave function has a structure very similar to the case of s-state forces and admits of a similar interpretation. The algebra is, however, somewhat more involved, in that one now has two coupled linear one-dimensional equations in two scalar amplitudes.

In sect. 4, the case of s-state interaction (discussed in sect. 2) is applied to the triton problem by adjusting the parameters of the two-body forces to conform to the low energy (triplet and singlet) nucleon-nucleon parameters. The triton ground state energy is then evaluated approximately by treating the Schrödinger equation directly as an eigenvalue problem without resort to variational principles.

2. The Case of S-State Interactions Only

In the overall c.m. system, let the momenta of the proton and 2 neutrons be $\mathbf{P}_3$, $\mathbf{P}_1$ and $\mathbf{P}_2$ respectively, so that

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3 = 0. \tag{2.1}$$

To simplify matters we do not consider the isobaric formalism at this stage, so that we have to antisymmetrize only with respect to the 2 neutrons. This antisymmetrization naturally comes about through the spin part of the wave function which we take as

$$(2)^{-\frac{1}{2}}(\alpha_1\beta_2-\alpha_2\beta_1)\alpha_3, \tag{2.2}$$

with obvious notation. The spatial part of the wave function should therefore be symmetric. As we know from earlier work [5]) the triton ground state is almost entirely space symmetric, so that the above idealization is a rather good approximation.

We find it most convenient to express the kinematics of the problem in terms of the momenta $\mathbf{P}_1$ and $\mathbf{P}_2$ of the 2 neutrons. The kinetic energy is then seen from (2.1) to be

$$(P_1^2+P_2^2+\mathbf{P}_1\cdot\mathbf{P}_2)/M. \tag{2.3}$$

The 3-body wave function is also expressed in terms of $\mathbf{P}_1$ and $\mathbf{P}_2$ via eq. (1). The matrix elements of the respective interactions are then found to be

$$(\mathbf{P}_1\mathbf{P}_3|V_{\mathrm{np}}|\mathbf{P}'_1\mathbf{P}'_3) = \delta(\mathbf{P}_1+\mathbf{P}_3-\mathbf{P}'_1-\mathbf{P}'_3)(\mathbf{P}_1+\tfrac{1}{2}\mathbf{P}_2|V_{\mathrm{np}}|\mathbf{P}'_1+\tfrac{1}{2}\mathbf{P}_2), \tag{2.4}$$

$$(\mathbf{P}_2\mathbf{P}_3|V_{\mathrm{np}}|\mathbf{P}'_2\mathbf{P}'_3) = \delta(\mathbf{P}_2+\mathbf{P}_3-\mathbf{P}'_2-\mathbf{P}'_3)(\mathbf{P}_2+\tfrac{1}{2}\mathbf{P}_1|V_{\mathrm{np}}|\mathbf{P}'_2+\tfrac{1}{2}\mathbf{P}_1), \tag{2.5}$$

$$(\mathbf{P}_1\mathbf{P}_2|V_{\mathrm{nn}}|\mathbf{P}'_1\mathbf{P}'_2) = \delta(\mathbf{P}-\mathbf{P}')(\mathbf{p}|V_{\mathrm{nn}}|\mathbf{p}'), \tag{2.6}$$

where

$$\mathbf{P} = \mathbf{P}_1+\mathbf{P}_2, \qquad \mathbf{P}' = \mathbf{P}'_1+\mathbf{P}'_2, \qquad 2\mathbf{p} = \mathbf{P}_2-\mathbf{P}_1, \qquad 2\mathbf{p}' = \mathbf{P}'_2-\mathbf{P}'_1. \tag{2.7}$$

For s-state interactions alone, we take, like Yamaguchi [8]),

$$(\mathbf{p}|V_{\mathrm{np}}|\mathbf{p}') = -(\lambda_0/M)g(\mathbf{p})g(\mathbf{p}'), \tag{2.8}$$

$$(\mathbf{p}|V_{\mathrm{nn}}|\mathbf{p}') = -(\lambda_1/M)f(\mathbf{p})f(\mathbf{p}'). \tag{2.9}$$

Using these results, the 3-body Schrödinger equation becomes in the above notation

$$\begin{aligned}(P_1^2+P_2^2+\mathbf{P}_1\cdot\mathbf{P}_2+\alpha_0^2)\Psi(\mathbf{P}_1,\mathbf{P}_2) &= \lambda_0\int \mathrm{d}^3\mathbf{P}'_1 g(\mathbf{P}_1+\tfrac{1}{2}\mathbf{P}_2)g(\mathbf{P}'_1+\tfrac{1}{2}\mathbf{P}_2)\Psi(\mathbf{P}'_1,\mathbf{P}_2)+(1\rightleftarrows 2)\\ &+\lambda_1\int \mathrm{d}^3\mathbf{p}' f(\mathbf{p})f(\mathbf{p}')\Psi(\tfrac{1}{2}\mathbf{P}-\mathbf{p}', \tfrac{1}{2}\mathbf{P}+\mathbf{p}'),\end{aligned} \tag{2.10}$$

where $-E = \alpha_0^2/M$ is the binding energy of the triton.

From the right hand side of (2.10) it is clear that $\Psi(\mathbf{P}_1, \mathbf{P}_2)$ has the following structure in the two momenta:

$$\Psi(\mathbf{P}_1, \mathbf{P}_2) = D^{-1}(\mathbf{P}_1, \mathbf{P}_2)[g(\mathbf{P}_1+\tfrac{1}{2}\mathbf{P}_2)\phi(\mathbf{P}_2)+f(\mathbf{p})\chi(\mathbf{P}) \\ +g(\mathbf{P}_2+\tfrac{1}{2}\mathbf{P}_1)\phi(\mathbf{P}_1)], \tag{2.11}$$

$$D(\mathbf{P}_1, \mathbf{P}_2) = P_1^2+P_2^2+P_1\cdot P_2+\alpha_0^2, \tag{2.12}$$

and $\phi(\mathbf{P}_i)$ and $\chi(\mathbf{P})$ are functions of the *single* vectors $\mathbf{P}_i$ and $\mathbf{P}$ respectively. This step essentially decouples the problem to a large extent compared, e.g., to the case of local potentials. Substitution of (2.11) in (2.10) gives in a straightforward way, the following equations for ϕ and χ:

$$(\lambda_0^{-1}-h_0(P_2))\phi(\mathbf{P}_2) \\ = \int d^3\xi D^{-1}(\mathbf{P}_2, \xi)[g(\mathbf{P}_2+\tfrac{1}{2}\xi)g(\xi+\tfrac{1}{2}\mathbf{P}_2)\phi(\xi)+g(\xi+\tfrac{1}{2}\mathbf{P}_2)f(\mathbf{P}_2+\tfrac{1}{2}\xi)\chi(\xi)], \tag{2.13}$$

$$(\lambda_1^{-1}-h_1(P))\chi(\mathbf{P}) = 2\int d^3\xi D^{-1}(\mathbf{P}, \xi)f(\xi+\tfrac{1}{2}\mathbf{P})g(\mathbf{P}+\tfrac{1}{2}\xi)\phi(\xi), \tag{2.14}$$

$$h_0(P_2) = \int d^3\mathbf{p}(p^2+\tfrac{3}{4}P_2^2+\alpha_0^2)^{-1}g^2(p), \tag{2.15}$$

$$h_1(P) = \int d^3\mathbf{p}(p^2+\tfrac{3}{4}P^2+\alpha_0^2)^{-1}f^2(p). \tag{2.16}$$

It is seen from (2.14) that $\chi(\mathbf{P})$ is algebraically expressible in terms of $\phi(\mathbf{P})$, so that by substitution in (2.13) we have essentially *one* integral equation in $\phi(\mathbf{P})$. It is also clear that for the s-state interaction considered here, $\phi(\mathbf{P})$ depends only on the magnitude of $\mathbf{P}$ and not on its direction so that we can now write $\phi(P)$ instead of $\phi(\mathbf{P})$.

From (2.13) and (2.14) explicit elimination of $\chi(P)$ results in the following integral equation for $\phi(P_2)$:

$$\phi(P_2)(\lambda_0^{-1}-h_0(P_2)) = \int d^3\xi D^{-1}(\mathbf{P}_2, \xi)g(\mathbf{P}_2+\tfrac{1}{2}\xi)g(\xi+\tfrac{1}{2}\mathbf{P}_2)\phi(\xi) \\ +2\iint d^3\xi d^3\eta D^{-1}(\mathbf{P}_2, \eta)f(\mathbf{P}_2+\tfrac{1}{2}\eta)g(\eta+\tfrac{1}{2}\mathbf{P}_2) \\ \times(\lambda_1^{-1}-h_1(\eta))^{-1}D^{-1}(\eta, \xi)f(\tfrac{1}{2}\eta+\xi)g(\eta+\tfrac{1}{2}\xi)\phi(\xi). \tag{2.17}$$

This equation is *exact*. We therefore find that the problem is exactly reducible to the so-called equivalent two-body problem of Wigner and Feenberg [11]), as characterized by the quantity $\phi(P_2)$. This fact represents the essential simplification achieved with a separable potential approach to the 3-body problem, and has no analogue in corresponding results obtained with local forces.

The following interpretations can now be given to some of the quantities appearing in eqs. (2.11)—(2.16). A comparison with Yamaguchi's [8]) eq. (6) for the 2-body wave function shows that the term

$$g(\mathbf{P}_1+\tfrac{1}{2}\mathbf{P}_2)[(\mathbf{P}_1+\tfrac{1}{2}\mathbf{P}_2)^2+\tfrac{3}{4}P_2^2+\alpha_0^2]^{-1} \tag{2.18}$$

multiplying $\phi(P_2)$ in (2.11) can be regarded as the two-body wave function of particles 1 and 3, with the quantity $(\frac{3}{4}P_2^2+\alpha_0^2)/M$ (which depends on the momentum of particle 2) now playing the role of the corresponding "binding energy" of 1 and 3. That $(\frac{3}{4}P_2^2+\alpha_0^2)/M$ indeed plays the role of the "binding energy" is further confirmed by comparing eq. (2.15) and the left-hand-side of eq. (2.13) with the eigenvalue equation (7) of ref. [8]). As would of course be expected, the kinematical effect of the 3rd particle is to *increase* the binding energy by the amount $3P_2^2/4M$. Further, $\phi(P_2)$ can be regarded as the wave function of particle 2 in the presence of 1 and 3. An indentical interpretation holds for the function $\chi(P)$ which is the wave function of 3 in the presence of 1 and 2. These "single-particle" wave functions ϕ and χ are in turn affected by the interactions exerted by the other two particles, as manifest from the right-hand-sides of eqs. (2.13) and (2.14). Thus, e.g., the right-hand-side of eq. (2.13) can be regarded as a measure of the extent to which the particle 2 penetrates into the region of 1 and 3, in close analogy with a corresponding interpretation given by Skoroniakov and Ter Martosyan [2]) for the case of zero-range local interaction.

Finally the complete wave function $\Psi(\mathbf{P}_1, \mathbf{P}_2)$ is the result of linear combinations of all possible products of the one and two particle wave functions described above, so as to be *symmetric in* 1 and 2. It is remarkable that the number of such terms is *finite*, viz. three only. On the other hand, with a conventional local interaction, while it is always possible to express a 3-body wave function as a sum of products of two and one-body wave functions, this sum is in general *infinite*. The simplification that arises in the present case is therefore non-trivial. Thus it appears that for any pair of particles, the third particle is *essentially uncorrelated* with the other two, except for the kinematical effects its presence introduces through the conservation of the total momentum. The dynamical effects of the interaction are explicitly absent in the structure (2.11) of the 3-body wave function. Indeed, the former are entirely contained in the right-hand-sides of eqs. (2.13) and (2.14) for the single-particle wave functions ϕ and χ, and these can at most modify their radial behaviour. It is quite conceivable that these features of the 3-body problem have a deeper physical content than the above deduction with separable potentials would warrant but the absence of any simple structure of the 3-body wave function from conventional 2-body interactions is likely to obscure such clear-cut interpretations in the latter case.

3. Inclusion of Tensor Forces

We now introduce tensor n-p forces of the special Yamaguchi type [9]) in the formalism, so that the potential (2.8) is replaced by

$$(\mathbf{p}|V_{13}|\mathbf{p}') = -(\lambda_0/M)g_{13}(\mathbf{p})g_{13}(\mathbf{p}'), \tag{3.1}$$

where

$$g_{13}(\mathbf{p}) = C(p)+8^{-\frac{1}{2}}T(p)S_{13}(\mathbf{p}) \tag{3.2}$$

with

$$S_{13}(\mathbf{p}) = 3(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_3\cdot\mathbf{p})-p^2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3). \tag{3.3}$$

A similar expression holds for V_{23}. For such an interaction which operates only on relative s and d-states, we do not need the elaborate classification schemes given by various authors [5,12]). As we shall see, the symmetry of the entire wave function will be fully determined from the exact solution available for this interaction. Further, since the interaction between the two neutrons is kept the same as (2.9) the overall antisymmetry is still guaranteed by the spin-dependence given by eq. (2.2). For such a spin state, we have

$$S_{23}(\mathbf{p}) \equiv -S_{13}(\mathbf{p}), \tag{3.4}$$

provided this operates *directly* on this spin wave function. For this case as well, the solution (2.11) for the wave function holds formally, except that $g(\mathbf{p})$ is now replaced by expressions like (3.2), and $\phi(\mathbf{p})$ and $\chi(\mathbf{P})$ are now functions of the spin matrices $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$. Thus we have, apart from the spin function (2.2),

$$(P_1^2+P_2^2+\mathbf{P}_1\cdot\mathbf{P}_2+\alpha_0^2)\Psi(\mathbf{P}_1,\mathbf{P}_2)$$
$$= g_{13}(\mathbf{P}_1+\tfrac{1}{2}\mathbf{P}_2)\phi_{13}(\mathbf{P}_2)+f(\mathbf{p})\chi(\mathbf{P})+g_{23}(\mathbf{P}_2+\tfrac{1}{2}\mathbf{P}_1)\phi_{23}(\mathbf{P}_1), \tag{3.5}$$

where we have formally indicated by the subscripts, the σ-dependence of the various terms.

Proceeding as in sect. 2, ϕ and χ are now found to satisfy the equations

$$\phi_{13}(\mathbf{P}_2)(1-\lambda_0 h_0(P_2)) = \lambda_0\int \mathrm{d}^3\xi D^{-1}(\mathbf{P}_2,\xi)g_{13}(\xi+\tfrac{1}{2}\mathbf{P}_2)$$
$$\times[g_{23}(\mathbf{P}_2+\tfrac{1}{2}\xi)\phi_{23}(\xi)+f(\mathbf{P}_2+\tfrac{1}{2}\xi)\chi(\xi)], \tag{3.6}$$

$$\chi(\mathbf{P})(1-\lambda_1 h_1(P)) = \lambda_1\int \mathrm{d}^3\xi D^{-1}(\mathbf{P},\xi)f(\xi+\tfrac{1}{2}\mathbf{P})$$
$$\times[g_{13}(\mathbf{P}+\tfrac{1}{2}\xi)\phi_{13}(\xi)+g_{23}(\mathbf{P}+\tfrac{1}{2}\xi)\phi_{23}(\xi)], \tag{3.7}$$

where

$$D(\mathbf{p},\mathbf{q}) = p^2+q^2+\mathbf{p}\cdot\mathbf{q}+\alpha_0^2. \tag{3.8}$$

Further, $h_1(P)$ is the same as in (2.16), but $h_0(P_2)$ is now given by

$$h_0(P_2) = \int \mathrm{d}^3\mathbf{p}(C^2(p)+p^4T^2(p))/(p^2+\tfrac{3}{4}P_2^2+\alpha_0^2). \tag{3.9}$$

The remaining task is to separate out the $\boldsymbol{\sigma}$-dependence from eq. (3.6) and (3.7), and obtain one-dimensional integral equations as in sect. 2. From inspection, we have

$$\phi_{13}(\mathbf{p}),\ \phi_{23}(\mathbf{p}) = \phi_0(p)\pm 8^{-\frac{1}{2}}S_{13}(\mathbf{p})\phi_1(p), \tag{3.10}$$

where ϕ_0 and ϕ_1 are scalar functions of the momentum involved. As for $\chi(\mathbf{P})$, a little inspection shows that it is a purely scalar function independent of the $\boldsymbol{\sigma}$ matrices †.

Now to exhibit the right-hand-side of (3.6) explicitly as (3.10), the simplest procedure is first to average out over the *azimuthal* dependence of the integration variable ξ, noting that this can come *only* from the tensor operators S_{13} and S_{23}. Thus, e.g., one has

$$\langle S_{13}(\xi)\rangle_{\text{Av}} = \tfrac{1}{2}\xi^2(3\cos^2\theta-1)S_{13}(\hat{P}_2), \qquad \cos\theta = \hat{\xi}\cdot\hat{P}_2, \tag{3.11}$$

which clearly brings out the factor $S_{13}(\mathbf{P}_2)$. Some relevant formulae for a product of two or more tensor operators are listed in appendix 1. Using these relations one arrives at the following coupled one-dimensional integral equations for the scalar amplitudes ϕ_0 and ϕ_1:

$$(\lambda_0^{-1}-h_0(P_2))\phi_0(P_2) = \int d^3\xi D^{-1}(\mathbf{P}_2, \xi)[f_0\phi_0(\xi)+f_1\phi_1(\xi) \\ +C(\xi+\tfrac{1}{2}\mathbf{P}_2)f(\mathbf{P}_2+\tfrac{1}{2}\xi)\chi(\xi)], \tag{3.12}$$

$$P_2^2(\lambda_0^{-1}-h_0(P_2))\phi_1(P_2) = \int d^3\xi D^{-1}(\mathbf{P}_2, \xi)[F_0\phi_0(\xi)+F_1\phi_1(\xi) \\ +f(\mathbf{P}_2+\tfrac{1}{2}\xi)T(\xi+\tfrac{1}{2}\mathbf{P}_2)k_0\chi(\xi)], \tag{3.13}$$

$$\chi(P)(\lambda_1^{-1}-h_1(P)) = 2\int d^3\xi D^{-1}(\mathbf{P}, \xi)f(\xi+\tfrac{1}{2}\mathbf{P}) \\ \times[C(\mathbf{P}+\tfrac{1}{2}\xi)\phi_0(\xi)+\tfrac{1}{8}T(\mathbf{P}+\tfrac{1}{2}\xi)k_1\phi_1(\xi)]. \tag{3.14}$$

The functions f_0, f_1, F_0, F_1, k_0, k_1 are defined in appendix 2.

4. Approximate Treatment of S-State Interaction

For s-state interaction alone, solution of the exact equation (2.17) should give the complete wave function in conjunction with (2.11). Since this integral equation is simply one-dimensional, it can always be solved numerically. As an eigenvalue equation, it is easiest to picture it as one in λ_0, given λ_1 and α_0. It is, however, still desirable to obtain a crude solution by making some approximations in order to obtain some insight into the nature of the solution.

First of all, to write the complete n-p interaction (singlet plus triplet) in the separable form (2.8), we must take, as in ref. [8]), the *same shape* but different strengths for the singlet and triplet interactions, so that the effective n-p interaction becomes, instead of (2.8),

$$-\tfrac{1}{2}M^{-1}(\lambda_0+\lambda_1)f(p)f(p'), \tag{4.1}$$

† On integration, the two terms on the right-hand-side of (3.7) can be expressed as $A+B\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3+CS_{13}(\mathbf{P})$ and $A+B\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3+CS_{23}(\mathbf{P})$, where A, B, C are scalar functions of $\mathbf{P}$. Adding these and remembering that $\boldsymbol{\sigma}_1 \equiv -\boldsymbol{\sigma}_2$, we get simply $2A$.

the function $f(p)$ now being the same as in (2.9). Thus the strength parameter λ_0 in (2.8) and (2.17) should be replaced by $\frac{1}{2}(\lambda_0+\lambda_1)$. For $f(p)$ we take

$$f(p) = (p^2+\beta^2)^{-1}, \tag{4.2}$$

where [8]), in units of the deuteron binding energy parameter α,

$$\beta = 6.255\alpha, \qquad \alpha_0 = 1.952\alpha, \qquad \lambda_1 = 23.43\alpha^3, \qquad \lambda_0 = 33.36\alpha^3. \tag{4.3}$$

Here we shall regard λ_1 and α_0 as given by (4.3), try to determine λ_0 from (2.17), and compare with (4.3).

Now to bring equation (2.17) in a manageable form, we make the following approximations. First we note that the last term of (2.17) represents the effect of function $\chi(P)$ on the behaviour of $\phi(P_2)$. Taking this effect to be small, we replace the portion

$$\int d^3\xi f(\xi+\tfrac{1}{2}\eta)f(\eta+\tfrac{1}{2}\xi)D^{-1}(\eta,\xi)\phi(\xi) \tag{4.4}$$

(now that the f and g functions are the same) by its value which would be obtained with the omission of the last term in (2.17), viz.

$$(\lambda_2^{-1}-h(\eta))\phi(\eta), \tag{4.5}$$

where

$$\lambda_2 = \tfrac{1}{2}(\lambda_0+\lambda_1) \tag{4.6}$$

while $h(\eta)$ now represents the common function (2.15) or (2.16). In this way equation (2.17) reduces to the simple form

$$(\lambda_2^{-1}-h(\eta))\phi(\eta) = \int d^3\xi D^{-1}(\xi,\eta)f(\eta+\tfrac{1}{2}\xi)f(\xi+\tfrac{1}{2}\eta) \times[1+2(\lambda_2^{-1}-h(\xi))(\lambda_1^{-1}-h(\xi))^{-1}]\phi(\xi). \tag{4.7}$$

Our next approximation, aimed at making the kernel of (4.7) separable, consists in neglecting the angular correlation between the vectors ξ and η. Thus we write, according to eqs. (2.12) and (4.2) †,

$$D^{-1}(\xi,\eta) \approx (\xi^2+\eta^2+\alpha_0^2)^{-1}, \tag{4.8}$$

$$f(\mathbf{p}+\mathbf{q}) \approx f(\sqrt{p^2+q^2}) = (\beta^2+p^2+q^2)^{-1}. \tag{4.9}$$

Finally, for (4.8) we make the replacement

$$(\xi^2+\eta^2+\alpha_0^2)^{-1} \approx \alpha_0^2(\xi^2+\alpha_0^2)^{-1}(\eta^2+\alpha_0^2)^{-1} \tag{4.10}$$

and a similar one for (4.9). This approximation is fairly good except when *both* ξ and η are large. However, since the kernel (4.7) falls off very rapidly for large ξ, the error involved is likely to be insignificant ††.

† Such approximations which are fairly good, have also been considered by Eyges [4]) (cf. his equation (39)).

†† Eyges [4]) has also used such an approximation. However, unlike his equation (36), we have here a very rapid convergence for large ξ.

As a result of these manipulations, eq. (4.7) finally reduces to

$$(\lambda_2^{-1}-h(\eta))\phi(\eta) \approx \alpha_0^2\beta^4 f(\eta)f(\tfrac{1}{2}\eta)(\alpha_0^2+\eta^2)^{-1} \times \int d^3\xi \frac{f(\frac{1}{2}\xi)f(\xi)}{\alpha_0^2+\xi^2}\left[1+2\frac{\lambda_2^{-1}-h(\xi)}{\lambda_1^{-1}-h(\xi)}\right]\phi(\xi), \quad (4.11)$$

which explicitly brings out the momentum dependence of the wave function $\phi(\eta)$, viz.,

$$\phi(\eta) = Cf(\eta)f(\tfrac{1}{2}\eta)(\alpha_0^2+\eta^2)^{-1}(\lambda_2^{-1}-h(\eta))^{-1}, \quad (4.12)$$

C being a normalization constant. This structure has certain formal similarities to equations (3.7) and (4.1) of ref. [4]), though the fall off with momentum in the present case is much more rapid.

Regarding (4.11) as an eigenvalue equation for λ_2^{-1}, substitution of (4.12) in (4.11) leads to the equation

$$\alpha_0^{-2}\beta^{-4} = \int d^3\xi f^2(\xi)f^2(\tfrac{1}{2}\xi)(\alpha_0^2+\xi^2)^{-2}((\lambda_2^{-1}-h(\xi))^{-1}+2(\lambda_1^{-1}-h(\xi))^{-1}). \quad (4.13)$$

Now from eqs. (2.15) and (4.2), we have

$$h(\xi) = \pi^2\beta^{-1}(\beta+\sqrt{\alpha_0^2+\tfrac{3}{4}\xi^2})^{-2}. \quad (4.14)$$

This shows that $h(\xi)$ is a monotonically decreasing function of ξ, its largest value being

$$h(0) = \pi^2\beta^{-1}(\beta+\alpha_0)^{-2}. \quad (4.15)$$

If therefore one replaces $h(\xi)$ in (4.13) by $h(0)$, the right-hand-side is somewhat overestimated, but not by any appreciable amount, because of the presence of large convergence factors in ξ in eq. (4.13). Evaluating the rest of the integral in eq. (4.13), one thus obtains the result

$$1 \approx I(\alpha_0, \beta)[(\lambda_2^{-1}-h(0))^{-1}+2(\lambda_1^{-1}-h(0))^{-1}], \quad (4.16)$$

where

$$I(\alpha_0, \beta) = \frac{8\pi^2}{\alpha_0\beta^2}\frac{\alpha_0^3+9\alpha_0^2\beta+29\alpha_0\beta^2+27\beta^3}{(\alpha_0+2\beta)^3(\alpha_0+\beta)^3(3\beta)^3}. \quad (4.17)$$

Substituting from (4.3), (4.15) and (4.17) in (4.16), then gives

$$\lambda_2 = \tfrac{1}{2}(\lambda_0+\lambda_1) \approx 29.85\alpha^3, \quad (4.18)$$

whence

$$\lambda_0 \approx 36.3\alpha^3. \quad (4.19)$$

It is interesting to observe that this estimate is not very far from the value given by (4.3) to fit the 2-body parameters, though this might be somewhat fortuitous in view of the approximations made.

5. Discussion

From the foregoing sections it is quite evident that from the mathematical point of view the simplification of the three-body problem with separable potentials leaves hardly anything to be desired. While the problem in the present case has been reduced to the solution of one, or at most two, linear one-dimensional integral equations, it is clear from the above formalism that for more general types of separable potentials one should eventually be able to reduce the problem to the solution of a *finite* number of one-dimensional linear integral equations (coupled or not). Of greater importance is the fact that our three-body wave function has an extremely simple and elegant structure even in an exact form, which is an essential prerequisite for the calculation of various 3-body physical entities without the need of excessive approximations. This last requirement is especially relevant since the three-body problem happens to be rather sensitive to approximations, unlike, e.g., nuclear matter.

Our second point is that the calculation of the triton binding energy has been carried out in sect. 4 with emphasis on the *eigenvalue* aspect of the problem, rather than on variational techniques, as has been usual with the three-body problem. Of course we are fully aware of the limitations of the above calculations which make use of an extremely simplified model of the interaction. So far these are intended merely as an indication of the practical possibilities of more accurate calculations of the binding energy with such potentials without resort to variational techniques from the very beginning. For the same reason we also recognize that the approximate consistency of the parameters of the two-body potential with the triton binding energy as found in sect. 4, hardly as yet confirms our earlier conjecture concerning the physical validity of the off-energy-shell elements of the potential. To examine this question in a really quantitative manner one needs a more realistic low energy separable potential including tensor and spin-orbit terms, as also the effects of isobaric spin. For example, the special Yamaguchi force including tensor forces treated in sect. 3, approximates more closely to a realistic potential. In recent times we have worked out more realistic separable potentials [10,14]) which take account of more details of the two-body data, and these are now being used to calculate the triton binding energy in the mathematical framework developed in this paper.

Finally, a few remarks on the possible role of a spin-orbit force at low energies may be in order in the context of the 3-body problem. The usual evidence for short-range spin-orbit forces rests essentially on high energy scattering and polarization data. It is *a priori* not clear to what extent such short-range forces should affect low-energy data. For example, the failure of the Gammel-Thaler [6]) and Signell-Marshak [7]) potentials to bind the triton, is traceable directly to the adverse effects of their unusually strong spin-orbit terms which apparently made themselves felt (in spite of short ranges) at the relatively large distances involved in the triton. Similarly the adverse effects of these spin-orbit forces on the deuteron magnetic moment are also generally recognized [15]. It is conceivable that the parameters of these high energy potentials (including spin-orbit forces) can be altered so as not to affect low energy data adversely. In any case it appears so far that these high energy spin-orbit forces have not been playing a particularly positive role in the low energy region. One could perhaps introduce separable spin-orbit forces to fit certain low and medium energy data [10] but in general such forces would have a different physical significance [14] from the short-range ones mentioned above. Indeed the three-body problem should now afford in principle a possibility of testing their existence through the calculation of sensitive entities like exchange currents, magnetic moments and so on. These questions are also under investigation in the present formalism.

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Appendix 1

We define

$$S_{13}(\hat{P}_2) = S, \qquad \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_3 = t, \qquad \cos\theta = \mu = \hat{P}_2 \cdot \hat{\xi},$$
$$k_0(\mathbf{P}_2, \xi) = \tfrac{1}{4}P_2^2 + P_2\xi\mu + \tfrac{1}{2}\xi^2(3\mu^2 - 1),$$
$$k_1(\mathbf{P}_2, \xi) = 9(\tfrac{1}{2}\xi^2 + \xi P_2\mu)^2 - 3\xi^2(\tfrac{1}{2}\xi + \mathbf{P}_2)^2.$$

Then we have

$$\rho_0 \equiv \langle S_{13}(\xi)\rangle_{\text{Av}} = \tfrac{1}{2}\xi^2(3\mu^2-1)S,$$
$$\rho_1 \equiv \langle S_{13}(\xi+\tfrac{1}{2}\mathbf{P}_2)\rangle_{\text{Av}} = k_0(\mathbf{P}_2, \xi)S,$$
$$\rho_2 \equiv \langle S_{13}(\mathbf{P}_2+\tfrac{1}{2}\xi)\rangle_{\text{Av}} = k_0(2\mathbf{P}_2, \tfrac{1}{2}\xi)S,$$
$$\rho_3 \equiv \langle (\boldsymbol{\sigma}_1 \cdot \overline{\xi \times \mathbf{P}_2})(\boldsymbol{\sigma}_3 \cdot \overline{\xi \times \mathbf{P}_2})\rangle_{\text{Av}} = \tfrac{1}{6}(1-\mu^2)\xi^2 P_2^2(2t+S),$$

$$\langle S_{23}(\mathbf{P}_2+\tfrac{1}{2}\xi)S_{23}(\xi)\rangle_{\text{Av}}$$
$$= k_1(\mathbf{P}_2, \xi) + \xi^2\rho_2 + (\tfrac{1}{2}\xi+\mathbf{P}_2)^2\rho_0 + 9\rho_3 - 2t\xi^2(\tfrac{1}{2}\xi+\mathbf{P}_2)^2$$
$$\equiv j_1(\mathbf{P}_2+\tfrac{1}{2}\xi, \xi; t) + Sj_2(\mathbf{P}_2+\tfrac{1}{2}\xi, \xi),$$

$$\langle S_{13}(\mathbf{p})S_{13}(\mathbf{q})\rangle_{\text{Av}} \equiv j_1(\mathbf{p}, \mathbf{q}; -t) - Sj_2(\mathbf{p}, \mathbf{q}),$$

$$\langle S_{13}(\xi+\tfrac{1}{2}\mathbf{P}_2)i\{(\boldsymbol{\sigma}_3-\boldsymbol{\sigma}_1)\cdot \overline{\xi \times \mathbf{P}_2}\}\rangle_{\text{Av}} = 0.$$

Appendix 2

In sect. 3, we have used the following definitions:

$$f_0 = C(\xi+\tfrac{1}{2}\mathbf{P}_2)C(\mathbf{P}_2+\tfrac{1}{2}\xi)-\tfrac{1}{8}T(\xi+\tfrac{1}{2}\mathbf{P}_2)T(\mathbf{P}_2+\tfrac{1}{2}\xi)j_1(\xi+\tfrac{1}{2}\mathbf{P}_2,\ \mathbf{P}_2+\tfrac{1}{2}\xi;\ -t),$$

$$\begin{aligned} f_1 = {} & \tfrac{1}{8}C(\xi+\tfrac{1}{2}\mathbf{P}_2)T(\mathbf{P}_2+\tfrac{1}{2}\xi)j_1(\mathbf{P}_2+\tfrac{1}{2}\xi,\ \xi;\ t) \\ & +\tfrac{1}{8}T(\xi+\tfrac{1}{2}\mathbf{P}_2)\{8^{-\frac{1}{2}}T(\mathbf{P}_2+\tfrac{1}{2}\xi)(\mathbf{P}_2+\tfrac{1}{2}\xi)^2-C(\mathbf{P}_2+\tfrac{1}{2}\xi)\}j_1(\xi+\tfrac{1}{2}\mathbf{P}_2,\ \xi;\ -t) \\ & +8^{-\frac{3}{2}}T(\xi+\tfrac{1}{2}\mathbf{P}_2)T(\mathbf{P}_2+\tfrac{1}{2}\xi)j_3(\xi,\ \mathbf{P}_2;\ t), \end{aligned}$$

$$\begin{aligned} j_3(\xi,\ \mathbf{P}_2;\ t) \equiv {} & 9t(\tfrac{1}{2}P_2^{\,2}+P_2\xi\mu)(\xi^3P_2\mu+\xi^2P_2^{\,2}\mu^2-\tfrac{1}{2}\xi^2P_2^{\,2}) \\ & -9tP_2^{\,2}(\xi^2+\tfrac{1}{2}\xi P_2\mu)^2+\xi^2j_1(\xi+\tfrac{1}{2}\mathbf{P}_2,\ \mathbf{P}_2+\tfrac{1}{2}\xi;\ -t) \\ & +9(\xi+\tfrac{1}{2}\mathbf{P}_2)^2\xi^2P_2^{\,2}(1-\mu^2)(2t-3), \end{aligned}$$

$$\begin{aligned} F_0 = {} & T(\xi+\tfrac{1}{2}\mathbf{P}_2)C(\mathbf{P}_2+\tfrac{1}{2}\xi)k_0(\mathbf{P}_2,\ \xi)-C(\xi+\tfrac{1}{2}\mathbf{P}_2)T(\mathbf{P}_2+\tfrac{1}{2}\xi)k_0(2\mathbf{P}_2,\ \tfrac{1}{2}\xi) \\ & +\tfrac{1}{8}T(\xi+\tfrac{1}{2}\mathbf{P}_2)T(\mathbf{P}_2+\tfrac{1}{2}\xi)j_2(\xi+\tfrac{1}{2}\mathbf{P}_2,\ \mathbf{P}_2+\tfrac{1}{2}\xi), \end{aligned}$$

$$\begin{aligned} F_1 = {} & -C(\xi+\tfrac{1}{2}\mathbf{P}_2)C(\mathbf{P}_2+\tfrac{1}{2}\xi)\tfrac{1}{2}\xi^2(3\mu^2-1) \\ & +8^{-\frac{1}{2}}C(\xi+\tfrac{1}{2}\mathbf{P}_2)T(\mathbf{P}_2+\tfrac{1}{2}\xi)j_2(\mathbf{P}_2+\tfrac{1}{2}\xi,\ \xi) \\ & +\{8^{-\frac{1}{2}}C(\mathbf{P}_2+\tfrac{1}{2}\xi)-\tfrac{1}{8}T(\mathbf{P}_2+\tfrac{1}{2}\xi)(\mathbf{P}_2+\tfrac{1}{2}\xi)^2\}T(\xi+\tfrac{1}{2}\mathbf{P}_2)j_2(\xi+\tfrac{1}{2}\mathbf{P}_2,\ \xi) \\ & +\tfrac{1}{8}T(\xi+\tfrac{1}{2}\mathbf{P}_2)T(\mathbf{P}_2+\tfrac{1}{2}\xi)j_4(\xi,\ \mathbf{P}_2), \end{aligned}$$

$$\begin{aligned} j_4(\xi,\ \mathbf{P}_2) \equiv {} & k_0(\mathbf{P}_2,\ \xi)[9(\tfrac{1}{2}\xi^2+P_2\xi\mu)^2-5\xi^2(\mathbf{P}_2+\tfrac{1}{2}\xi)^2] \\ & -\xi^2j_2(\xi+\tfrac{1}{2}\mathbf{P}_2,\ \mathbf{P}_2+\tfrac{1}{2}\xi)-9(\tfrac{1}{2}P_2^{\,2}+\xi P_2\mu)^2\tfrac{1}{2}\xi^2(3\mu^2-1) \\ & -9\xi^2P_2^{\,2}(1-\mu^2)(\xi+\tfrac{1}{2}\mathbf{P}_2)^2+9(\xi^2+\tfrac{1}{2}P_2\xi\mu)^2(\tfrac{1}{2}P_2^{\,3}\xi\mu+2P_2^{\,2}\mu^2\xi^2-\xi^2P_2^{\,2}). \end{aligned}$$

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