Portfolio optimization and mean-variance framework

Course: Portfolio Management

Instructor: Abhinava Tripathi

Expected returns on a portfolio

- Actual returns on the portfolio can be represented by the following
- $\mathbf{R_{Pt}} = \sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{X_i R_{it}}$ **(1)**
- Where 'i' depicts one of the 'N' securities and 'Xi' is the weight invested in the security 'i'
- Now, the expected returns of the portfolio can also be written as:
- $\bar{R}_P = E(R_{\mathrm{P}t}) = E(\sum_{i=1}^{N} X_i R_{it})$
- This can be also written as follows: $\sum_{i=1}^{N} E(X_i R_{it})$ or $\sum_{i=1}^{N} X_i E(R_{it})$
- $\boldsymbol{\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i}$ **(2)**

Risk of a two security-portfolio

- $\sigma_p^2 = E\left(R_{pt} - \bar{R}_p\right)^2 = E[X_1 R_{1t} + X_2 R_{2t} - (X_1 \bar{R}_1 + X_2 \bar{R}_2)]^2$
- $= E[\, X_1(R_{1t} - \bar{R}_1) + X_2(R_{2t} - \bar{R}_2)]^2$
- $= E[\, X_1^2(R_{1t} - \bar{R}_1)^2 + X_2^2(R_{2t} - \bar{R}_2)^2 + 2X_1X_2(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$
- $= X_1^2 E[\, (R_{1t} - \bar{R}_1)]^2 + X_2^2 E[(R_{2t} - \bar{R}_2)^2] + 2X_1X_2 E[(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$
- The third term "$E[(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$" is called covariance and can be depicted as σ_{12} (here $\sigma_{12} = \sigma_{21}$)
- $\boldsymbol{\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}}$ **(3)**

Use the same steps to derive the 3-security formula

- $\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + X_3^2\sigma_3^2 + 2X_1X_2\sigma_{12} + 2X_1X_3\sigma_{13} + 2X_2X_3\sigma_{23}$ **(4)**

Few words on covariance

- Please note that this covariance is the product of two deviations $E[(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$
- If both the securities move together, i.e., positive deviations and negative deviations are observed for both of these securities together, then covariance is expected to be positive
- Conversely if positive deviations of one security occur together with negative deviations of the other security, then the covariance is expected to be negative

Few words on covariance

- If the securities do not move together then the covariance is expected to be low
- This covariance is standardized in the following manner to obtain the correlation coefficient, as follows.
- $$\boldsymbol{\rho_{ik}} = \frac{\boldsymbol{\sigma_{ik}}}{\boldsymbol{\sigma_i \sigma_k}} \qquad (5)$$
- The standardized measure known as correlation coefficient
- It varies between +1 and -1

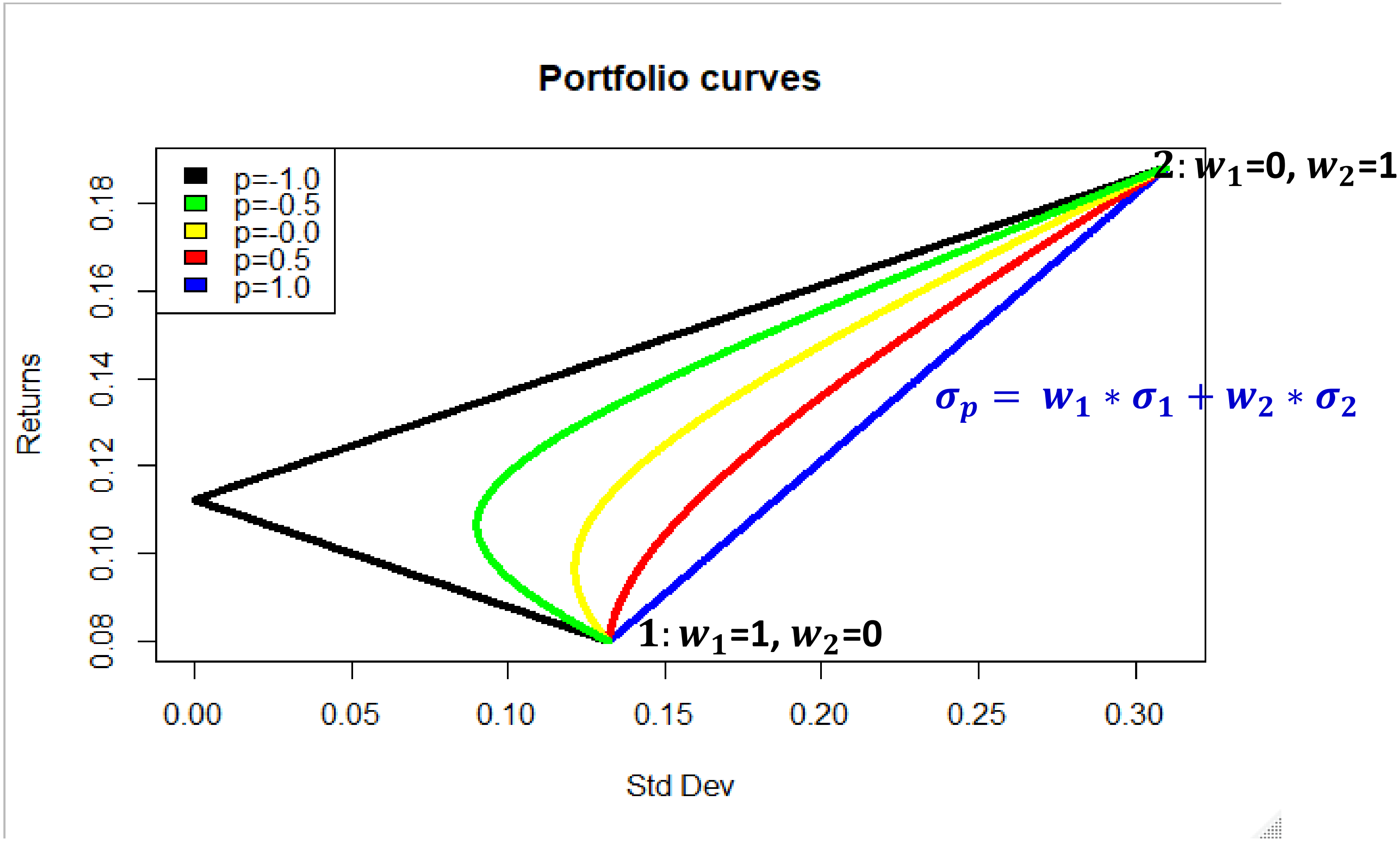
Portfolio risk and return profile

- $\overline{R_p} = w_1 * \overline{R_1} + w2 * \overline{R_2}$ (1)
- $\sigma_p^2 = w_1^2 * \sigma_1^2 + w_2^2 * \sigma_2^2 + 2 * w_1 * w_2 * \rho_{12} * \sigma_1 * \sigma_2$ (2)
- Consider two securities A and B. Security A offers 8% expected return and B offers 18.8% return. SD of A is 13.2% and that of B is 31%
- We will examine how the risk-return profile looks for ρ_{12}= 1.0 (blue), ρ_{12}=0.5 (red), ρ_{12}=0 (yellow), ρ_{12}=-0.5 (green), and ρ_{12}=-1.0 (black).
- We will vary the proportionate amounts, that is w_1 and w_2 between 0 and 1. Where w_1+w_2=1

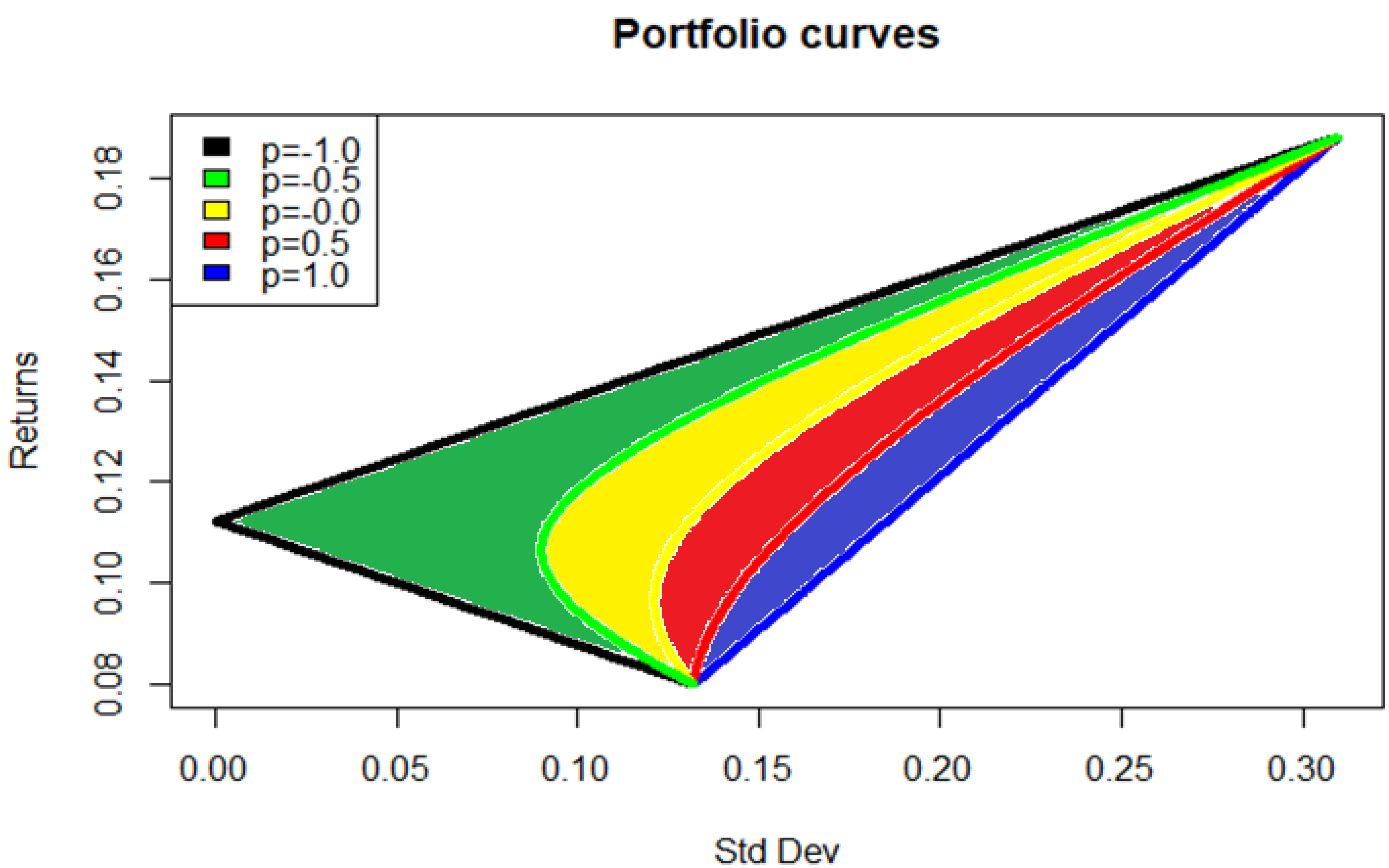
Portfolio risk and return profile

- $\overline{R_p} = w_1 * \overline{R_1} + w2 * \overline{R_2}$ (1)
- $\sigma_p^2 = w_1^2 * \sigma_1^2 + w_2^2 * \sigma_2^2 + 2 * w_1 * w_2 * \rho_{12} * \sigma_1 * \sigma_2$ (2)
- Consider two securities A and B. Security A offers 8% expected return and B offers 18.8% return. SD of A is 13.2% and that of B is 31%
- We will examine how the risk-return profile looks for ρ_{12}= 1.0 (blue), ρ_{12}=0.5 (red), ρ_{12}=0 (yellow), ρ_{12}=-0.5 (green), and ρ_{12}=-1.0 (black).
- We will vary the proportionate amounts, that is w_1 and w_2 between 0 and 1. Where w_1+w_2=1

Portfolio risk and return profile



Portfolio risk and return profile



Portfolio risk and return profile

- Consider the blue line with ρ_{12}=1 correlation
- In this special case, the equation becomes a straight line with
- $\sigma_p = w_1 * \sigma_1 + w_2 * \sigma_2$
- Across all the graphs, the lowest amount of diversification (highest portfolio risk, σ_p^2) for a given level of return are associated with this line
- Next, we examine the other extreme case corresponding to ρ_{12}=-1 correlation shown in black
- This case offers the highest diversification, as it carries the lowest levels of risk for a given level of returns
- In this case, the equation for risk effectively becomes
- $\sigma_p^2 = (w_1 * \sigma_1 - w_2 * \sigma_2)^2$

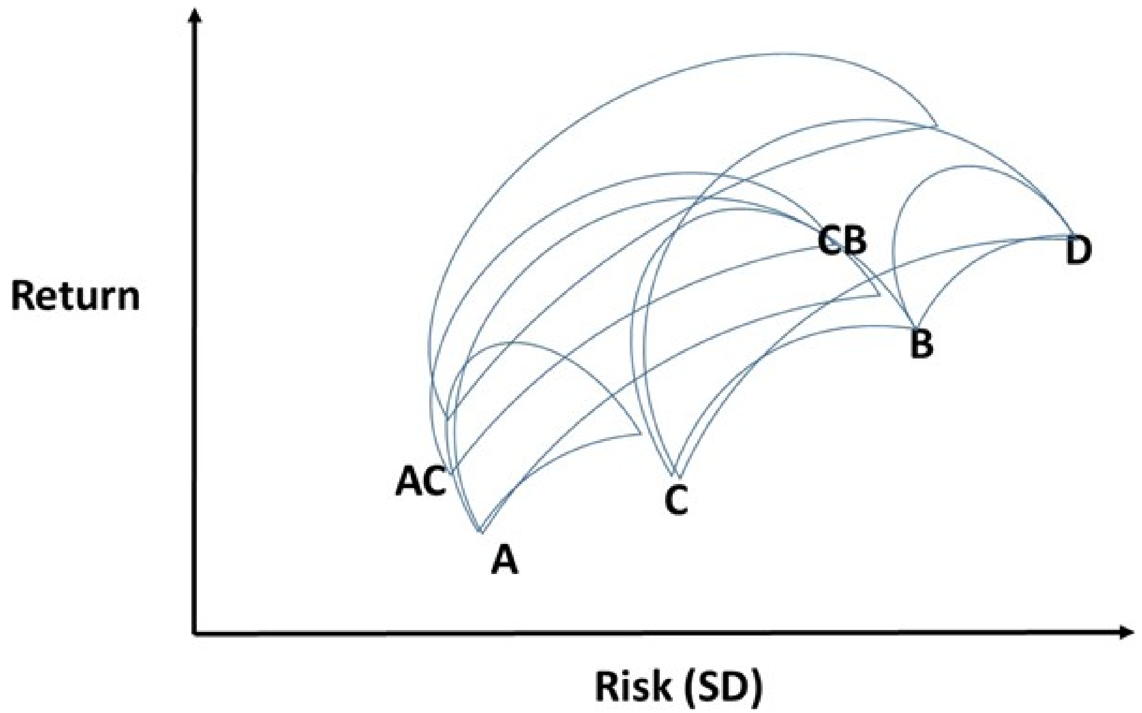
Portfolio risk and return profile

- In this case, the equation for risk effectively becomes
- $\sigma_p^2 = (w_1 * \sigma_1 - w_2 * \sigma_2)^2$
- This equation has two solutions, each representing a straight line
- (a) $\sigma_p = (w_1 * \sigma_1 - w_2 * \sigma_2)$ when $(w_1 * \sigma_1 - w_2 * \sigma_2)$ >=0; and $\sigma_p = -(w_1 * \sigma_1 - w_2 * \sigma_2)$ when $(w_1 * \sigma_1 - w_2 * \sigma_2)$<0
- These two lines intersect at $\sigma_p = 0, where\ (w_1 * \sigma_1 = w_2 * \sigma_2)$
- This is a special though impractical case where we attained complete diversification with zero risks
- The cases, where ρ_{12} lies between -1 and +1 are concave kind of curves in-between the two extreme cases

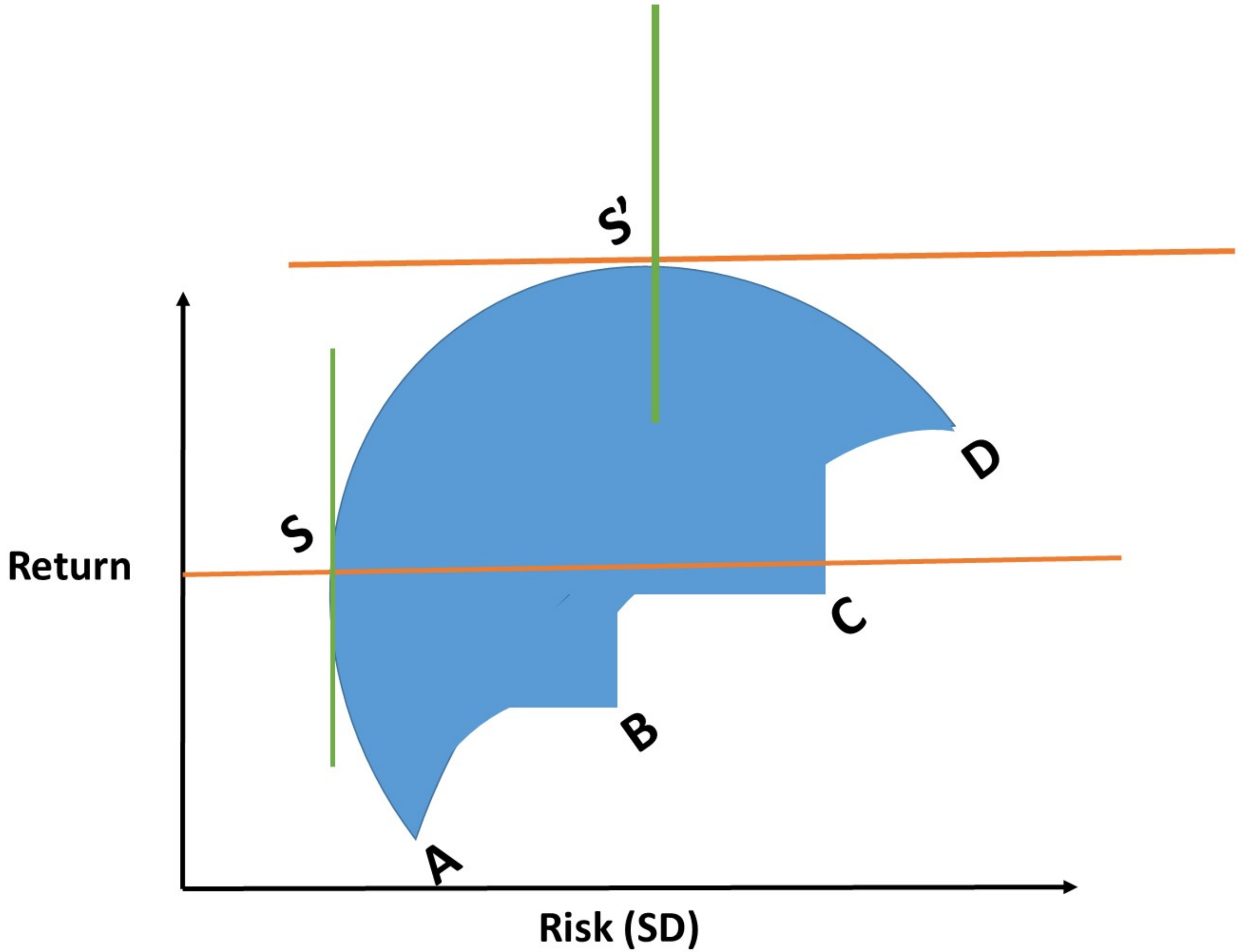
Portfolio risk and return profile

- An important observation here is that the risk of the portfolio, for a given level of returns, is sometimes even less than the least risky security in the portfolio
- And this is particularly the case when the correlation between the securities in the portfolio is low
- Adding more securities to the portfolio surely lowers the specific risk of the portfolio. Even say 15-20 stocks can offer a considerable amount of diversification
- What happens when we add more and more securities, how does the feasible region of area of possibilities changes

Portfolio risk and return profile



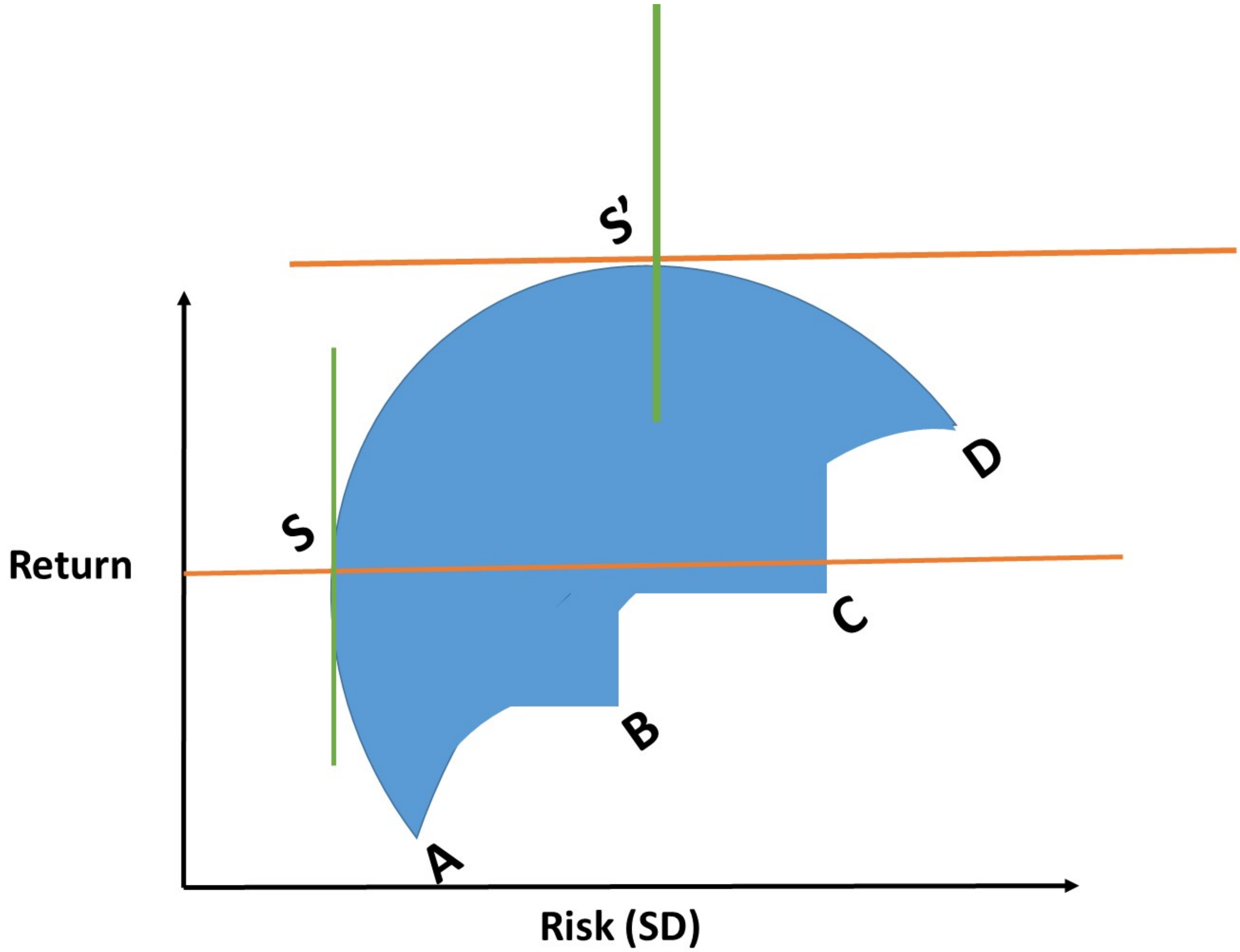
Portfolio risk and return profile



Portfolio risk and return profile

- As we keep on forming these combinations infinitely, we will get the following convex egg-cut shape
- The region of possibilities is shown in blue
- The blue area is effectively the region of expected return and risk possibilities that an investor can attain
- Each point represents the combination of risk and returns that is available to investors in the form of investment into portfolios
- Together, all these points (portfolios) comprise the region of possibilities (or the feasible region)

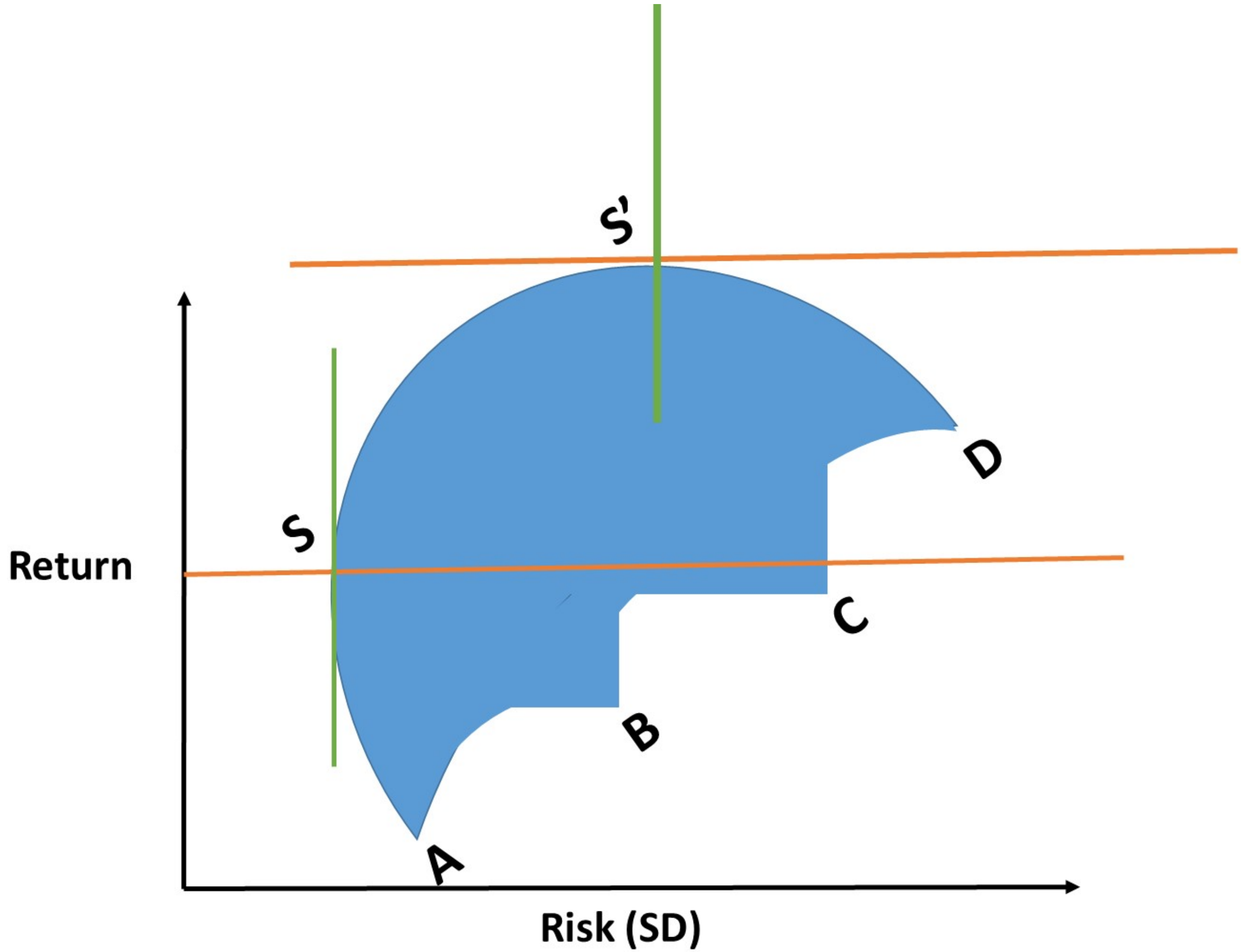
How to improve our position in this region?



How to improve our position in this region?

- We want to move up (increase returns) and move to the left (reduce risk)
- As we do that, we reach the top surface of the region of possibilities, that is, the surface SS'
- There are no more points where we can move further left or up on this curve (SS')
- This region would be called the efficient frontier. And all the points on this region offer the highest return for the given level of risk (or lowest risk for a given level of returns)
- Also, each investor depending upon his risk preference, may choose a specific risk level
- Once he decides a specific risk level, he will have a given certain expected return level on the surface SS'

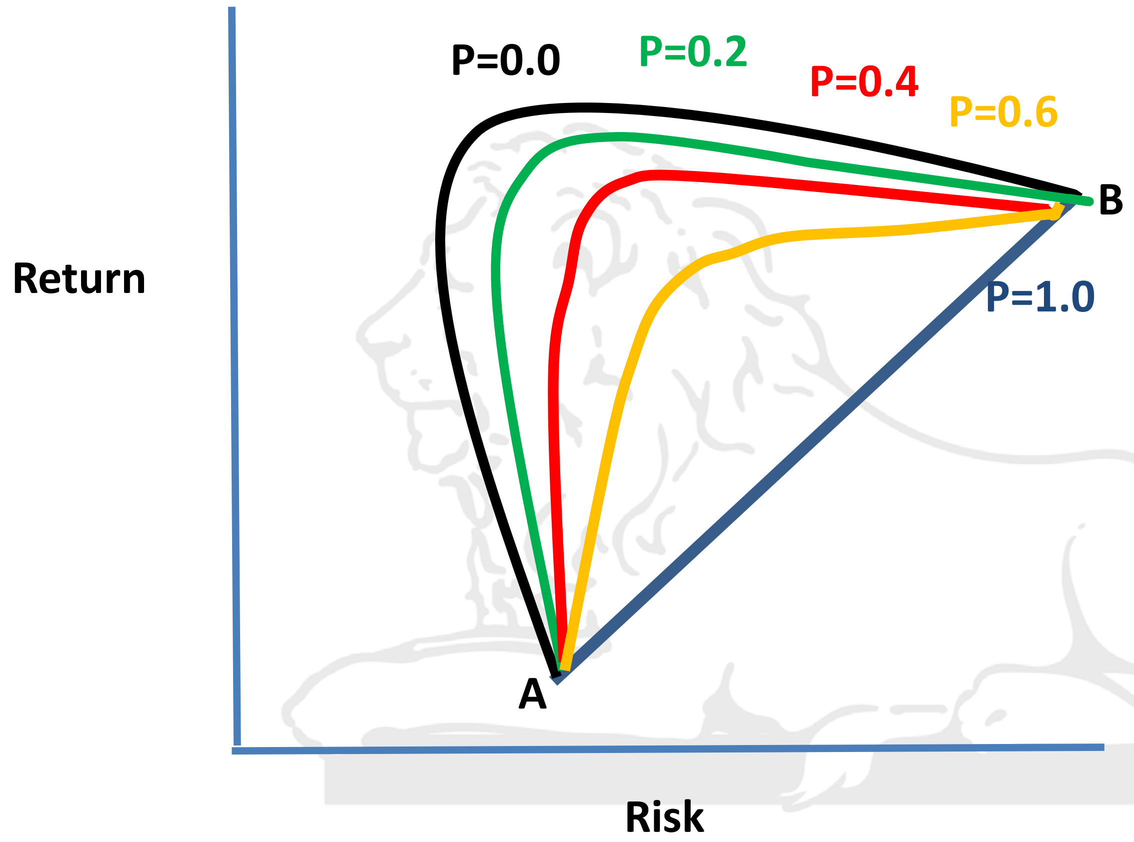
How to improve our position in this region?



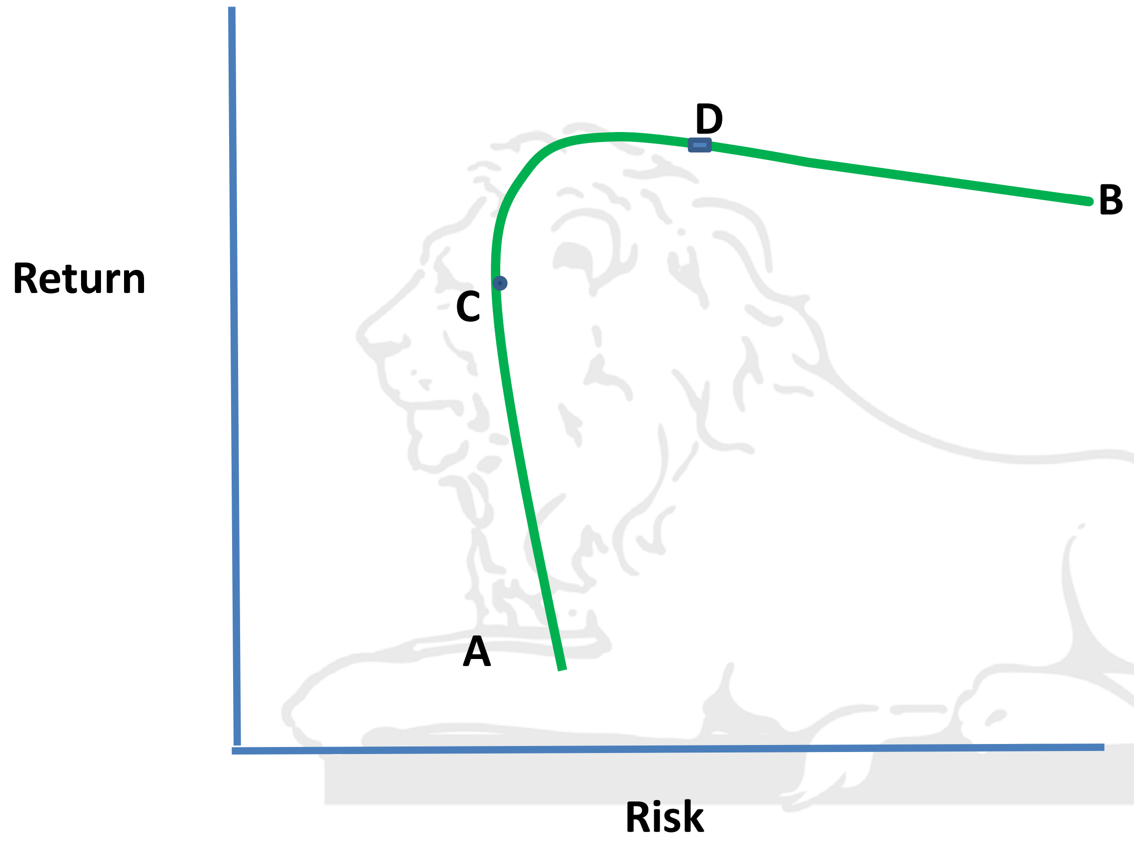
How to improve our position in this region?

- Two points on this region are particularly important for us
- Point S that has minimum risk as compared to any other point on the feasible region
- Point S’ that has maximum return as compared to any other point on the feasible region
- All the points between SS’ presents the unique and best combinations of risk and return on the feasible region

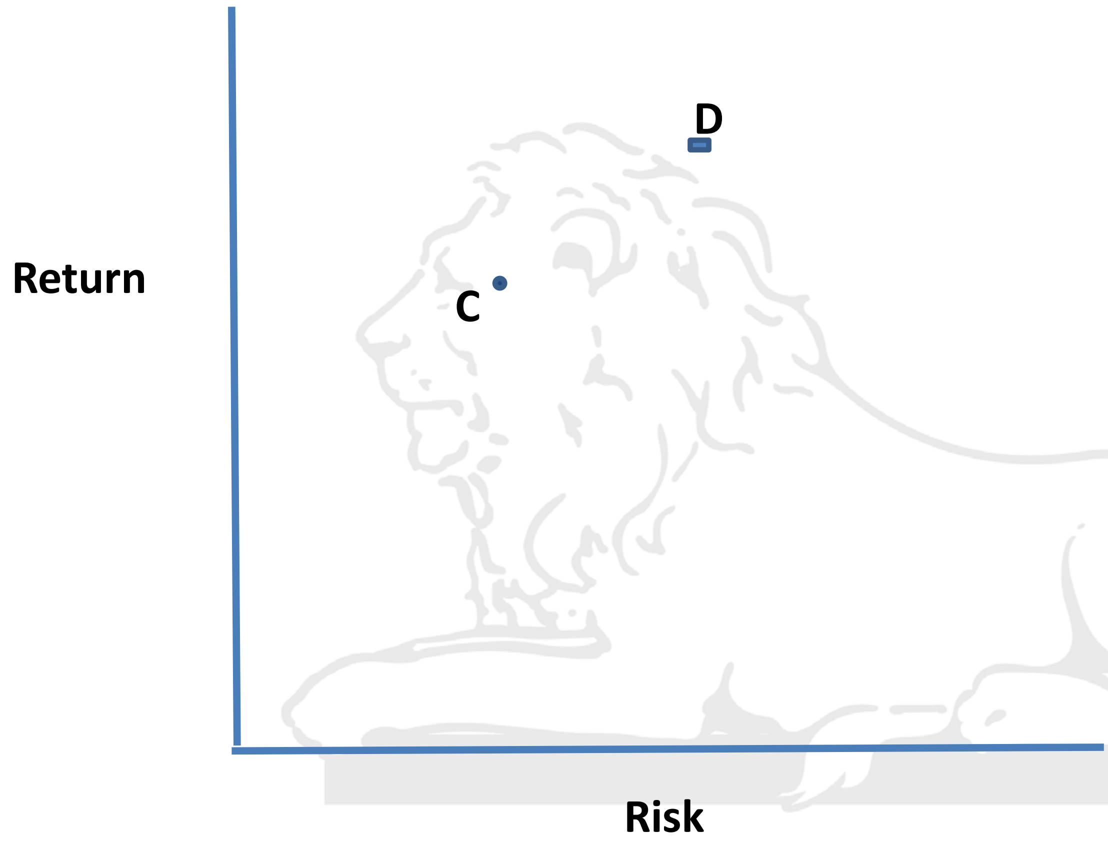
The efficient frontier



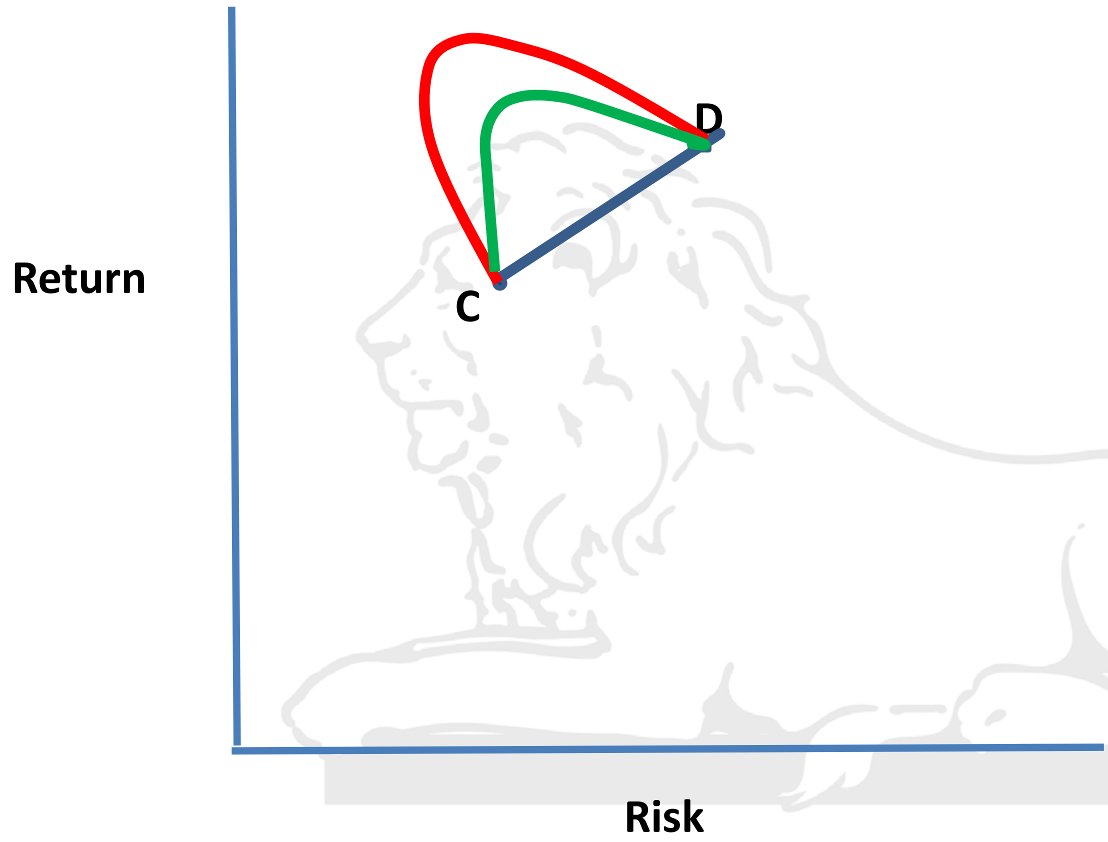
The efficient frontier



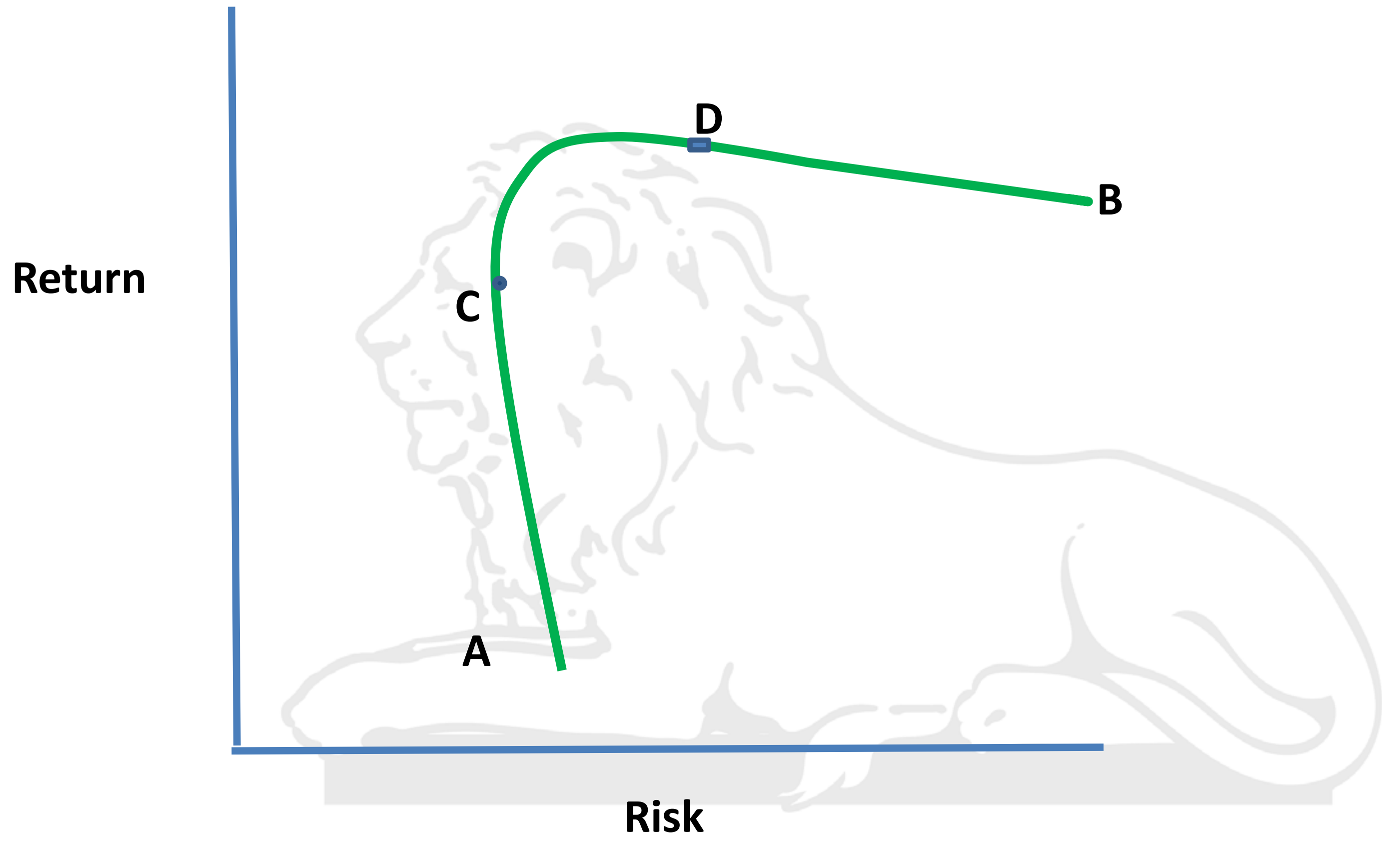
The efficient frontier



The efficient frontier



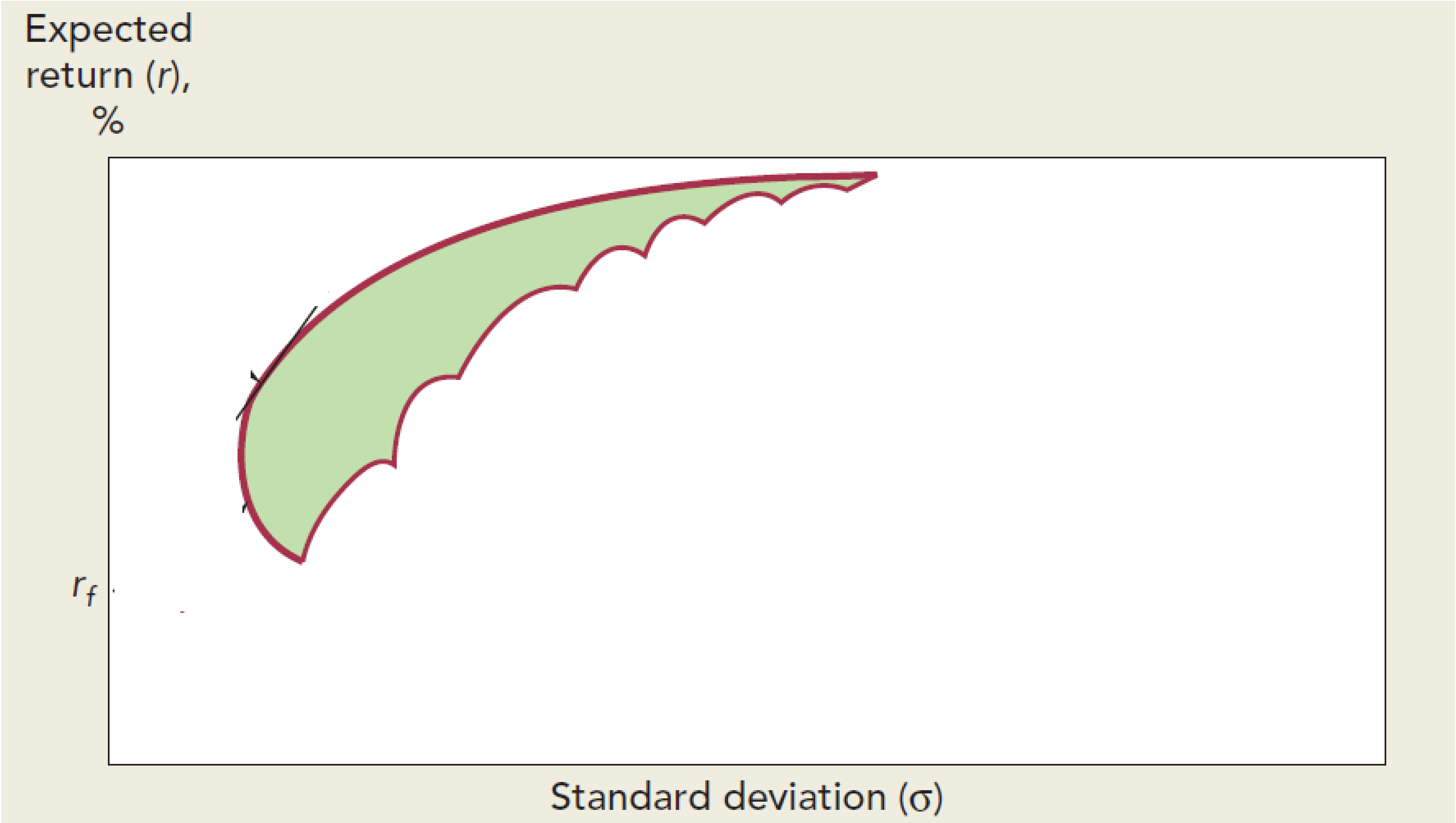
The efficient frontier



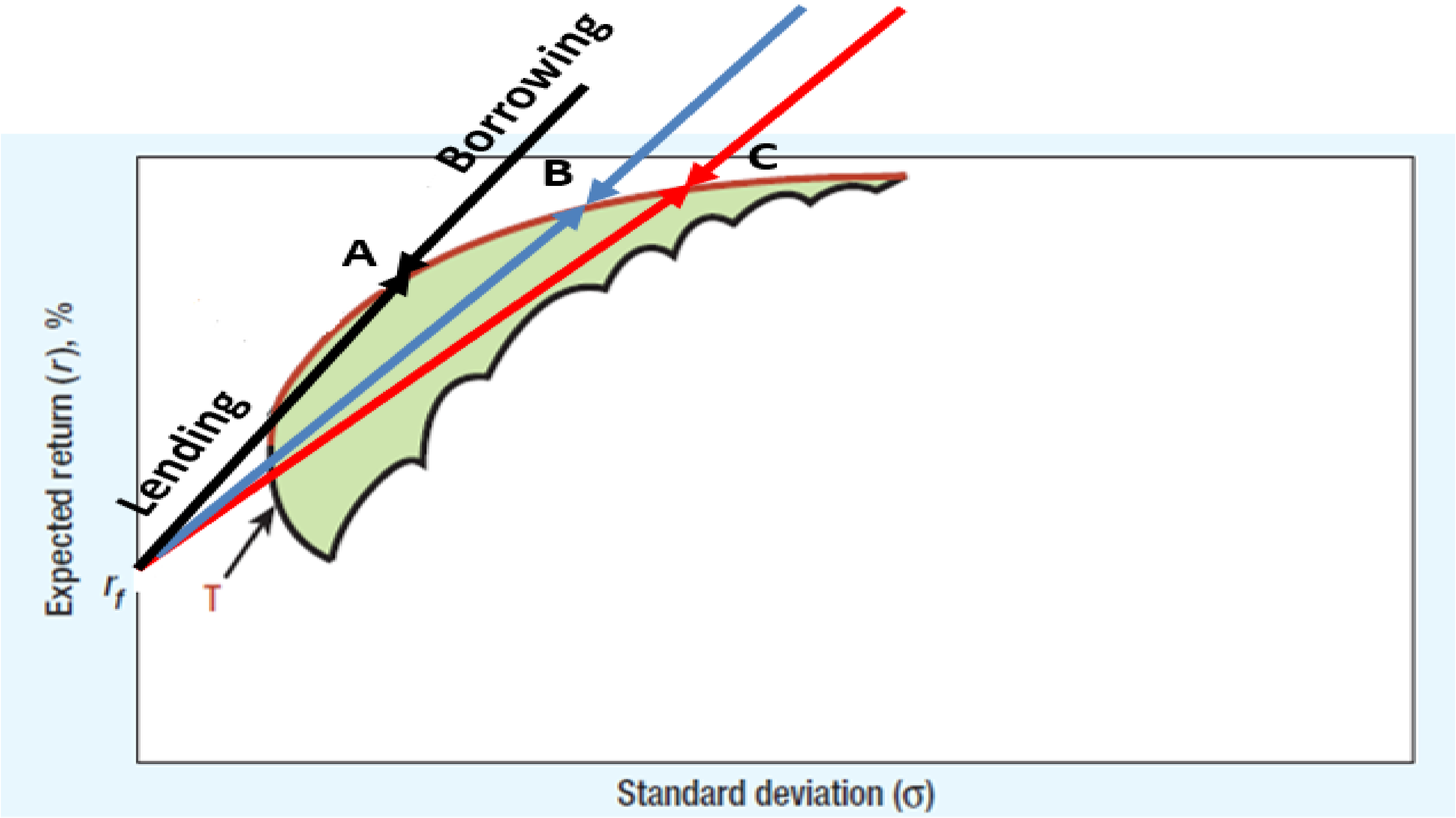
Introduction to risk-free lending and borrowing

- Let us introduce risk-free lending and borrowing at the risk-free rate of interest r_f
- What are the practical challenges with this assumption
- Can be borrow at the same rate from SBI at which we make fixed deposits with SBI
- However, this assumption has several important implications to portfolio construction
- Consider that a large number of stocks are employed to construct a feasible region of possibilities

Introduction to risk-free lending and borrowing

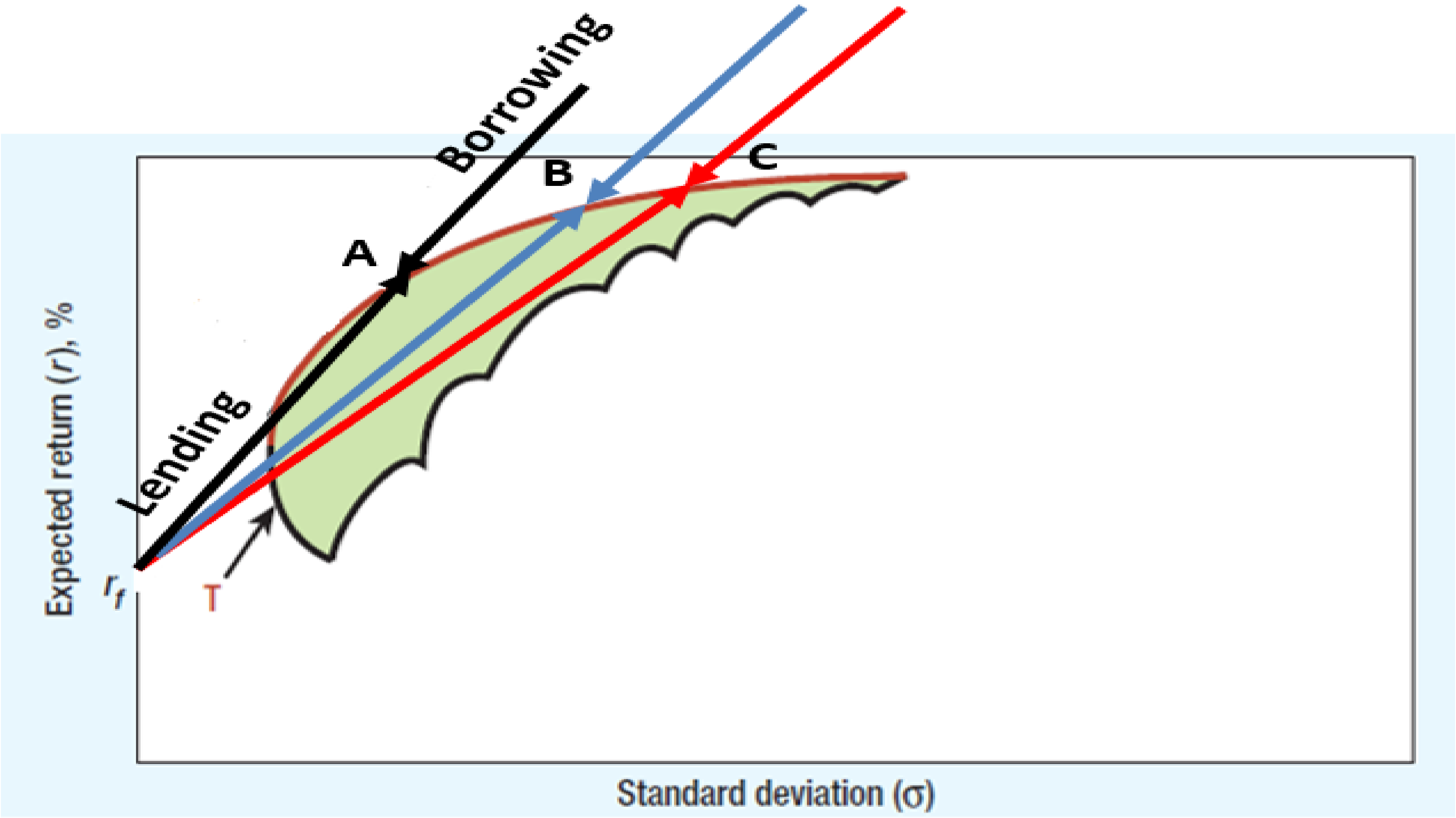


Introduction to risk-free lending and borrowing



Introduction to risk-free lending and borrowing

Y=mX+C, A: $(\bar{R}_a, \sigma_a)$, rf: $(\bar{R}_f, 0)$



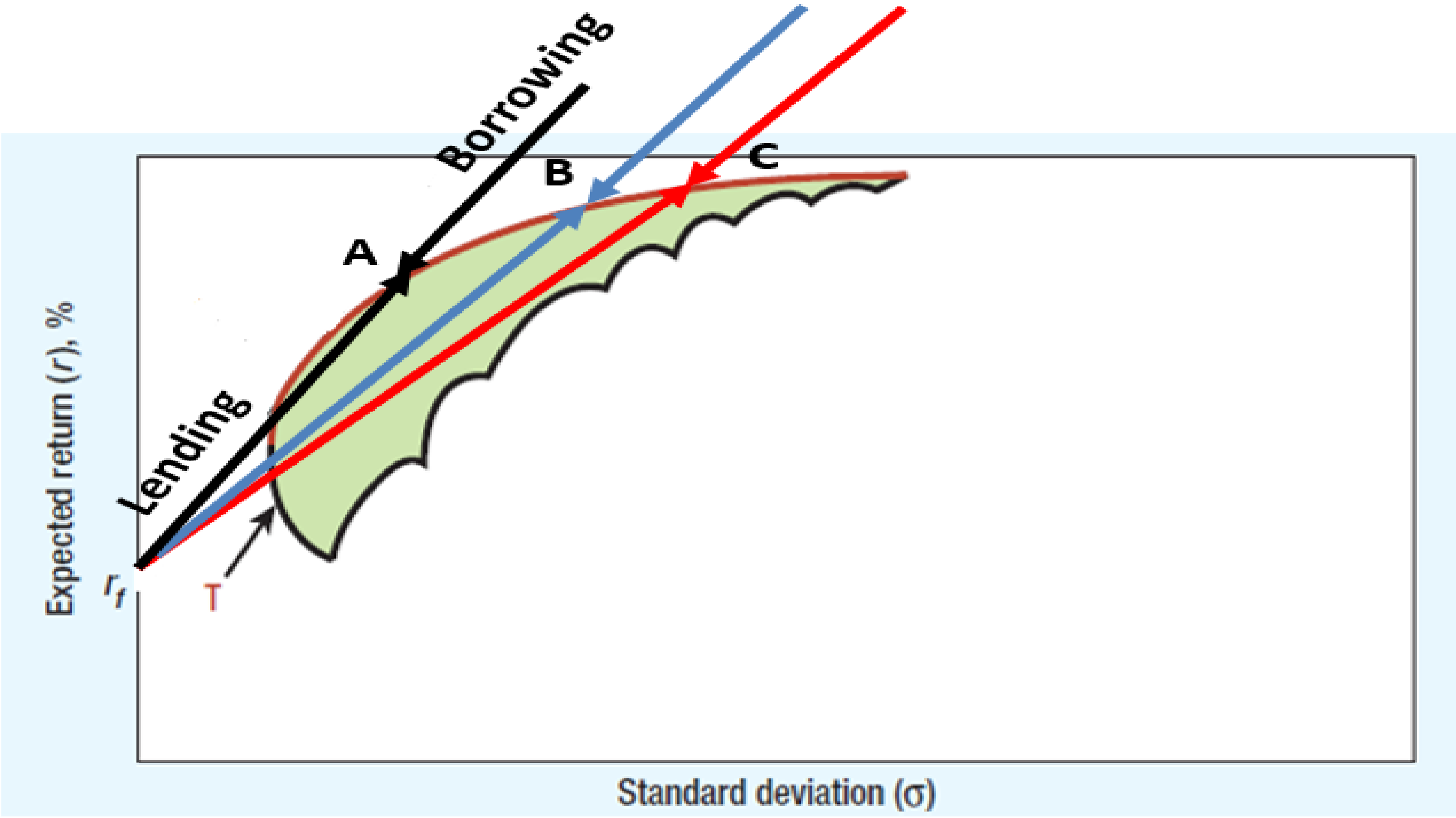
The efficient frontier with riskless lending and borrowing

- The addition of riskless securities considerably simplifies the analysis and opens new possibilities of investment
- Consider two investments (1) a portfolio of assets A that lies on the efficient frontier; and (2) One risk-free asset
- If X fraction of amount is placed in the portfolio, then 1-X fraction will be placed in the riskless asset
- The expected return on this portfolio can be expressed by the following equation.
- $\bar{R}_p = X\bar{R}_A + (1-X)R_f$ (1)
- $\sigma_p^2 = X^2\sigma_A^2 + (1-X)^2\sigma_f^2 + 2X(1-X)\rho_{Af}\sigma_A\sigma_f$ (2)

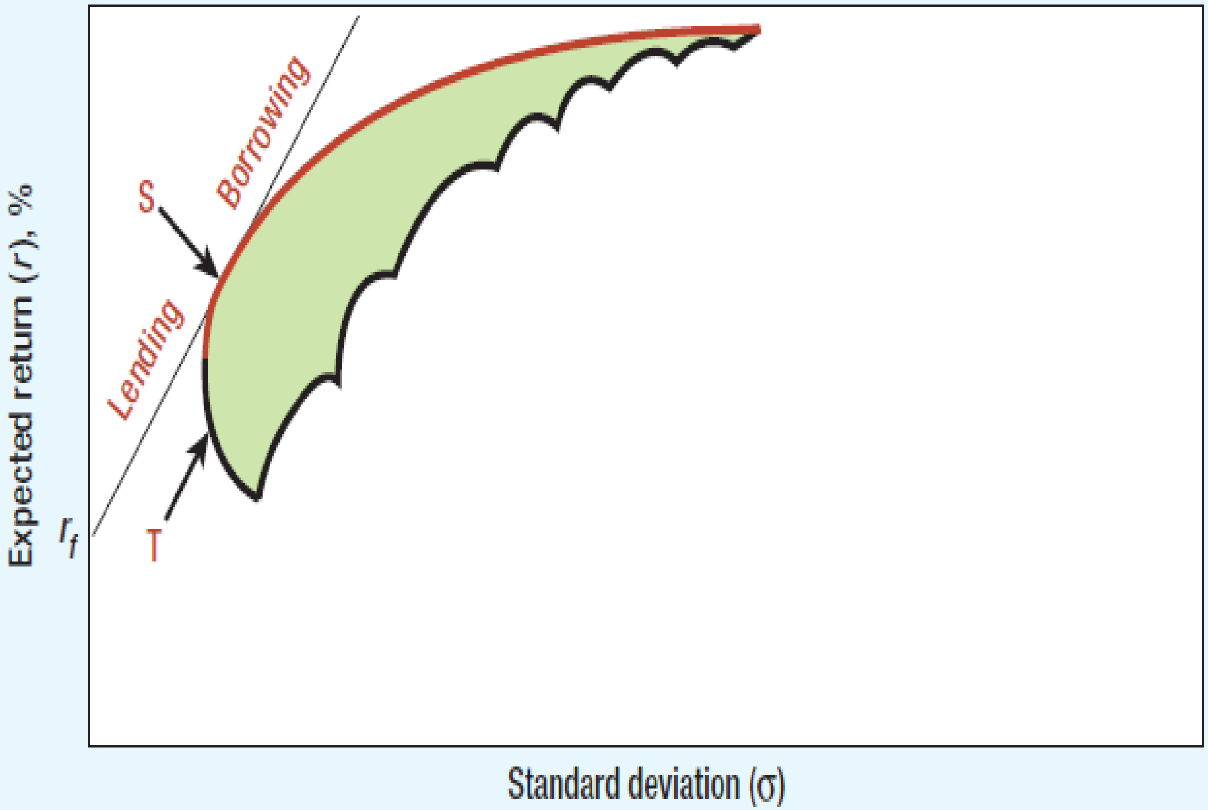
The efficient frontier with riskless lending and borrowing

- $\sigma_p^2 = X^2\sigma_A^2 + (1-X)^2\sigma_f^2 + 2X(1-X)\rho_{Af}\sigma_A\sigma_f$
- Because $\sigma_f = 0$
- The following expression of the portfolio risk is obtained.
- $\boldsymbol{\sigma_p = X\sigma_A}$ **(1)**
- $\bar{R}_p = X\bar{R}_A + (1-X)R_f$ **(2)**
- $\bar{R}_p = R_f + (\frac{\bar{R}_A - \bar{R}_f}{\sigma_A})\sigma_p$ **(3)**
- This (Eq. 3) is the equation of a straight line that passes through all the combinations of riskless lending or borrowing with portfolio A

Introduction to risk-free lending and borrowing



Introduction to risk-free lending and borrowing



Lending and borrowing extend the range of investment possibilities. If you invest in portfolio S and lend or borrow at the risk-free interest rate, r_f, you can achieve any point along the straight line from r_f through S. This gives you the highest expected return for any level of risk. There is no point in investing in a portfolio like T.

Introduction to risk-free lending and borrowing

- The brown line represents the most efficient portfolios or the efficient frontier
- Now that you have risk-free asset, you can invest a certain amount in the risk-free investment at r_f and the remaining amount on any portfolio available on the surface "S" corresponding to the efficient frontier
- Let us draw a line tangent from the point r_f to the red line curve
- The line that is the steepest among all is the tangent line
- The slope of this line is the amount of return per unit of risk. That is $\frac{r_S - r_f}{\sigma_p}$
- This means that for per unit of risk, this portfolio offers the highest return

Introduction to risk-free lending and borrowing

- Now, I have even better position, which is shown by the line going through rf and rs
- It has two segments borrowing and lending for investors with high and low risk preference
- This strategy of borrowing at r_f and investing at r_S is depicted by the line segment called borrowing
- I can invest partially at r_f and partially at r_S, and hold a portfolio on the line segment called lending
- If the portfolio S is known with reasonable certainty, everybody should hold this portfolio. And this will be called market portfolio

Introduction to risk-free lending and borrowing

- In a competitive market, everybody is expected to hold this market portfolio and the job of investment manager is expected to be fairly easy
- One has to identify the market portfolio of common stocks
- Then mix this portfolio with risk-free lending or borrowing to create a product that suits the taste and risk preference of investors

Now example

- Suppose market portfolio S here offers 15% expected returns and SD of 16%. The risk-free instrument offers a 5% uniform rate of lending and borrowing, and has a SD=0.

- You are a relatively more risk-averse investor, and therefore, you would like to invest 50% into rf and balance into S. What does your portfolio look like.

- $\boldsymbol{\sigma_p = X\sigma_A}$

- $\bar{R}_p = X\bar{R}_A + (1 - X)R_f$

Now example

- Suppose market portfolio S here offers 15% expected returns and SD of 16%. The risk-free instrument offers a 5% uniform rate of lending and borrowing, and has a SD=0.

- You are a more risk-averse investor, and therefore, you would like to invest 50% into rf and balance into S. What does your portfolio look like. The expected returns on your portfolio are: $r_f * 0.5 + r_S * 0.5 = 5\% * .5 + 15\% * 0.5 = 10\%$. The standard deviation of the portfolio will be: $\sigma_p = 0.5 * 16\% = 8\%$.
- You are standing on the lending segment of the line of investment at a point that is midway between rf and rs.

Now example

- Suppose market portfolio S here offers 15% expected returns and SD of 16%. The risk-free instrument offers a 5% uniform rate of lending and borrowing, and has a SD=0.
- Anther investor who is relatively less risk-averse in his approach, will borrow at rf almost 100% and invest 200% in the market portfolio.
- $\boldsymbol{\sigma_p = X\sigma_A}$
- $\bar{R}_p = X\bar{R}_A + (1 - X)R_f$

Now example

- Suppose market portfolio S here offers 15% expected returns and SD of 16%. The risk-free instrument offers a 5% uniform rate of lending and borrowing, and has a SD=0.
- Anther investor who is relatively less risk-averse in his approach, will borrow at rf almost 100% and invest 200% in the market portfolio. The risk return profile of this investor is shown below. His return will be: $r_f * (-1.0) + r_S * 2.0 = 5\% * -1.0 + 15\% * 2.0 = 25\%$. At the same time his risk will be $\sigma_p = 2 * 16\% = 32\%$.
- This investor has extent his possibilities and operates on the borrowing segment of the line.

In closing

- So whether it is fearful chickens or risky lions both will prefer this market portfolio as compared to any of the portfolios on the efficient frontier
- Therefore, this market portfolio is the best efficient portfolio for the entire set of investors
- And we also know how to identify this portfolio by drawing a tangent line from rf to on the surface of efficient portfolios
- This portfolio, as we discussed earlier, offers the highest risk premium to the standard deviation
- This ratio is also called the Sharpe ratio.
- Sharpe ratio = $\frac{Risk\ premium}{Standard\ Deviation} = \frac{r_S - r_f}{\sigma_p}$.

Thanks

Portfolio optimization with Multi-securities

Course: Portfolio Management

Instructor: Abhinava Tripathi

Recap: Expected returns on a portfolio

- Actual returns on the portfolio can be represented by the following
- $\mathbf{R_{Pt} = \sum_{i=1}^{N} X_i R_{it}}$ **(1)**
- Where 'i' depicts one of the 'N' securities and 'Xi' is the weight invested in the security 'i'
- Now, the expected returns of the portfolio can also be written as:
- $\bar{R}_P = E(R_{\mathrm{P}t}) = E(\sum_{i=1}^{N} X_i R_{it})$
- This can be also written as follows: $\sum_{i=1}^{N} E(X_i R_{it})\ or\ \sum_{i=1}^{N} X_i E(R_{it})$
- $\boldsymbol{\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i}$ **(2)**

Recap: Risk of a two security-portfolio

- $\sigma_p^2 = E\left(R_{pt} - \bar{R}_p\right)^2 = E[X_1 R_{1t} + X_2 R_{2t} - (X_1 \bar{R}_1 + X_2 \bar{R}_2)]^2$
- $= E[\, X_1(R_{1t} - \bar{R}_1) + X_2(R_{2t} - \bar{R}_2)]^2$
- $= E[\, X_1^2(R_{1t} - \bar{R}_1)^2 + X_2^2(R_{2t} - \bar{R}_2)^2 + 2X_1X_2(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$
- $= X_1^2 E[\,(R_{1t} - \bar{R}_1)]^2 + X_2^2 E[(R_{2t} - \bar{R}_2)^2] + 2X_1X_2 E[(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$
- The third term "$E[(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$" is called covariance and can be depicted as σ_{12} (here $\sigma_{12} = \sigma_{21}$)
- $\boldsymbol{\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + 2X_1X_2\sigma_{12}}$ **(3)**

Recap: Use the same steps to derive the 3-security formula

- $\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + X_3^2\sigma_3^2 + 2X_1X_2\sigma_{12} + 2X_1X_3\sigma_{13} + 2X_1X_3\sigma_{23}$ **(4)**

Recap: Few words on covariance

- Please note that this covariance is the product of two deviations $E[(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$
- If both the securities move together, i.e., positive deviations and negative deviations are observed for both of these securities together, then covariance is expected to be positive
- Conversely if positive deviations of one security occur together with negative deviations of the other security, then the covariance is expected to be negative

Recap: Few words on covariance

- If the securities do not move together then the covariance is expected to be low
- This covariance is standardized in the following manner to obtain the correlation coefficient, as follows.
- $\boldsymbol{\rho_{ik} = \frac{\sigma_{ik}}{\sigma_i \sigma_k}}$ (5)
- The standardized measure known as correlation coefficient
- It varies between +1 and -1

Recap: N-security case

- $\boldsymbol{\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + X_3^2\sigma_3^2 + 2X_1X_2\sigma_{12} + 2X_1X_3\sigma_{13} + 2X_1X_3\sigma_{23}}$ **(4)**
- These terms can be segregated in two segments
  - Terms like $X_1^2\sigma_1^2$, called variance terms
  - Terms like $2X_1X_2\sigma_{12}$, called covariance terms
- For 'N' securities variance, the generalized term can be simply written as: $\sum_{i=1}^{N} X_i^2\sigma_i^2$.
- The covariance [N*(N-1)] term looks like this: $\sum_{j=1}^{N}\sum_{\substack{k=1 \\ j\neq k}}^{N}(X_jX_k\sigma_{jk})$
- $\boldsymbol{\sigma_p^2 = \sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{j=1}^{N}\sum_{\substack{k=1 \\ j\neq k}}^{N}(X_jX_k\sigma_{jk})}$ **(5)**

Recap: N-security case-Variance terms

- Assume that we are investing equal amounts in each of these securities. Then, $X_1 = X_2 \ldots\ldots = X_N = \frac{1}{N}$

- This means that the variance term will become $\frac{1}{N^2}\sum_{i=1}^{N} \sigma_i^2$ or $\frac{1}{N}\sum_{i=1}^{N} \frac{\sigma_i^2}{N}$
- Assuming the average variance of average variance of $\bar{\sigma}_i^2$, the variance term can be also written as $\frac{1}{N}\bar{\sigma}_i^2$
- For a portfolio with large number of securities, this variance term will be closer to zero or very small

Recap: N-security case-Covariance terms

- What about the covariance term?
- $=\sum_{j=1}^{N}\sum_{\substack{k=1 \\ j\neq k}}^{N}(X_j X_k \sigma_{jk})$
- $=\sum_{j=1}^{N}\sum_{k=1}^{N}(\frac{1}{N^2}\sigma_{jk})$; assuming equal investment in each security
- $=\frac{N-1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}(\frac{1}{N(N-1)}\sigma_{jk})$
- The term $\sum_{j=1}^{N}\sum_{k=1}^{N}(\frac{1}{N(N-1)}\sigma_{jk})$, is the summation of covariances divided by the number of covariances: average covariance ($\bar{\sigma}_{jk}$)
- Resulting covariance term will become: $\frac{N-1}{N}\bar{\sigma}_{jk}$
- As we increase N, this term approaches to $\bar{\sigma}_{jk}$

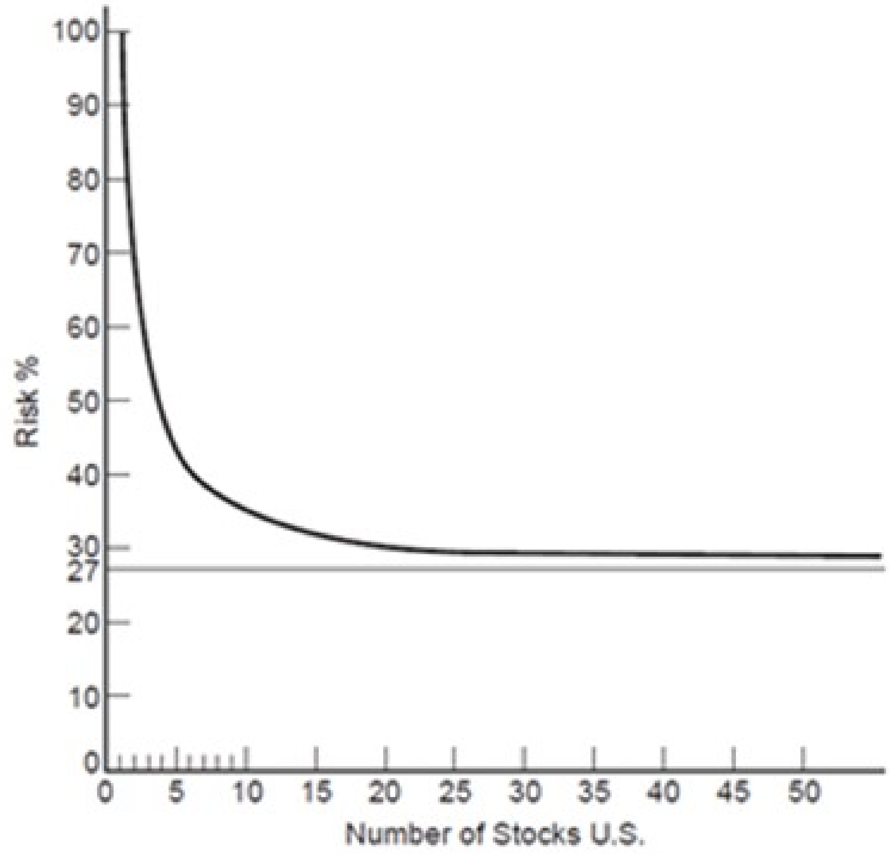
Recap: N-security case-Variance terms

- What about the covariance term?
- $=\sum_{j=1}^{N} \sum_{\substack{k=1 \\ j \neq k}}^{N} (X_j X_k \sigma_{jk})$
- $=\sum_{j=1}^{N} \sum_{k=1}^{N} (\frac{1}{N^2} \sigma_{jk})$; assuming equal investment in each security
- $=\frac{N-1}{N} \sum_{j=1}^{N} \sum_{k=1}^{N} (\frac{1}{N(N-1)} \boldsymbol{\sigma_{jk}})$
- The term $\sum_{j=1}^{N} \sum_{k=1}^{N} (\frac{1}{N(N-1)} \sigma_{jk})$, is the summation of covariances divided by the number of covariances: average covariace ($\bar{\sigma}_{jk}$)
- Resulting covariance term will become: $\frac{N-1}{N} \bar{\sigma}_{jk}$
- As we increase N, this term approaches to $\bar{\sigma}_{jk}$

Recap: N-security case-Variance terms

- Total standard deviation of the N-security portfolio converges to
- $\boldsymbol{\sigma_p^2 = \frac{1}{N}\bar{\sigma}_i^2 + \frac{N-1}{N}\bar{\sigma}_{jk}}$
- For large number of securities, this formula simplifies to
- $\boldsymbol{\sigma_p^2 \approx \bar{\sigma}_{jk}}$
- This gives us the intuition that as the number of securities are increased, the variance terms that represent the risk of individual securities is offset
- What is left is the covariance terms can not be diversified away

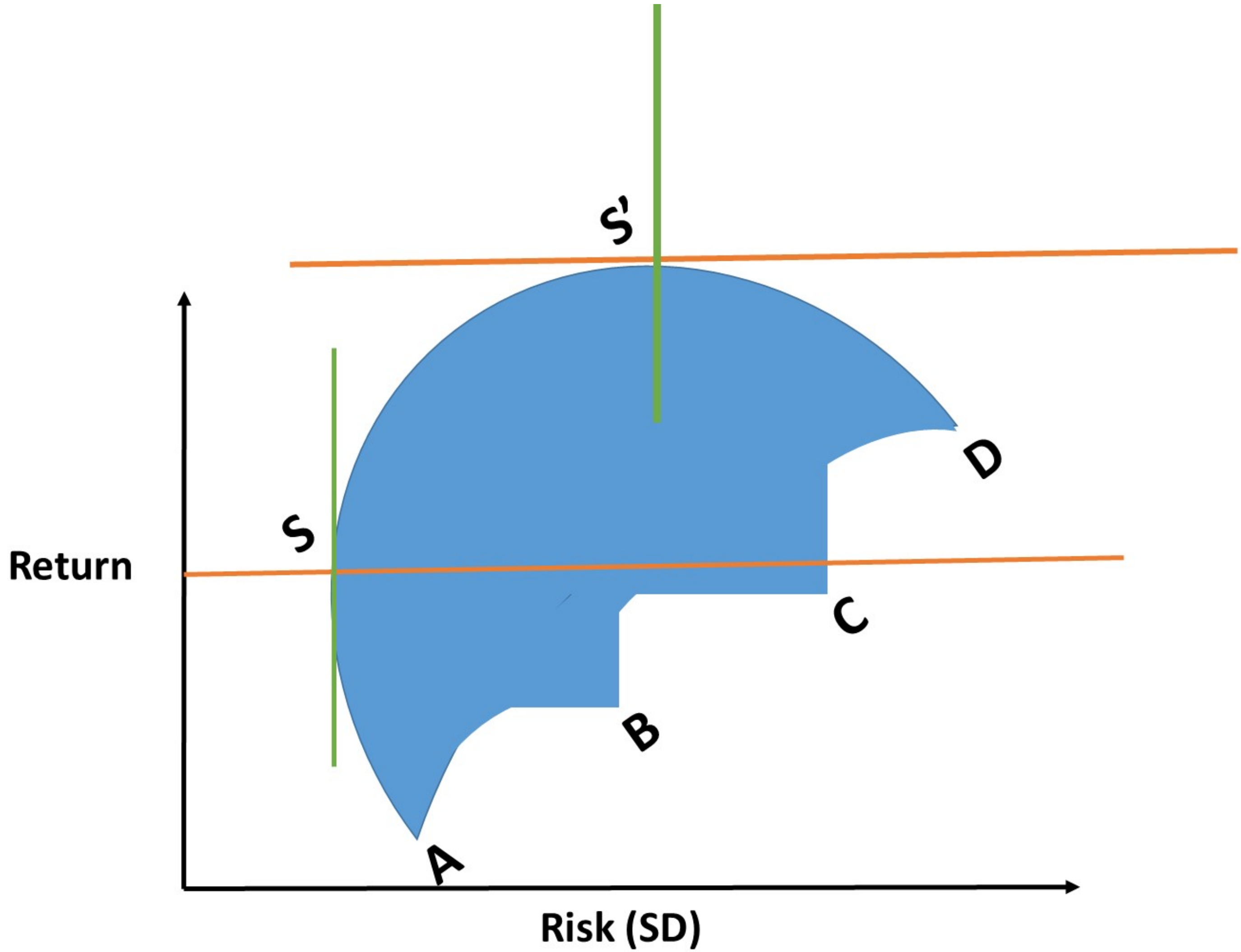
Recap: N-security case-Variance terms



The effect of number of securities on risk of the portfolio

Minimum variance portfolio

Efficient frontier in the absence of Risk-free



Minimum variance portfolio

- In the absence of short-sales, two points become extremely important on the efficient frontier
- First, the portfolio with maximum return and second the minimum variance portfolio
- In the absence of short-sales, these portfolios define the two extreme ends of the efficient frontier
- While it is easy to understand that maximum return portfolio will be the security in the portfolio that offers the maximum return
- The same is not the case for minimum variance portfolio

Minimum variance portfolio

- This portfolio is often expected to be the different from the security with minimum risk (SD) in the portfolio. How do we compute this portfolio?
- $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_B\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$ (1)
- What exactly we want to compute here?
- $\sigma_P = [X_A^2\sigma_A^2 + (1 - X_A)^2\sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$ (2)

Minimum variance portfolio

- $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_A X_B \rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$ (1)

- $\sigma_P = [X_A^2\sigma_A^2 + (1 - X_A)^2\sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$ (2)

- $\frac{d\sigma_P}{dX_A} = \frac{\frac{1}{2}[2X_A\sigma_A^2 - 2(1 - X_A)\sigma_B^2 + 2\rho_{AB}\sigma_A\sigma_B - 4X_A\rho_{AB}\sigma_A\sigma_B]}{[X_A^2\sigma_A^2 + (1 - X_A)^2\sigma_B^2 + 2X_A(1 - X_A)\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}}$ (3)

- To obtain the minima, we need to set the derivative=0, and Solving this for X_A, we get

Minimum variance portfolio

- To obtain the minima, we need to set the derivative=0, and Solving this for X_A, we get
- $2X_A\sigma_A^2 - 2(1 - X_A)\sigma_B^2 + 2\rho_{AB}\sigma_A\sigma_B - 4X_A\rho_{AB}\sigma_A\sigma_B = 0$(4)
- $X_A(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B) = (\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B)$
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)}$ (5)

Combinations of Two Risky Assets Revisited: Short-sales not allowed

Case 1: Perfect Correlation (ρ=1)

Case 1: Perfect Correlation (ρ=1)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of ρ=1, to find the relationship between expected

$$\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_B\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$$

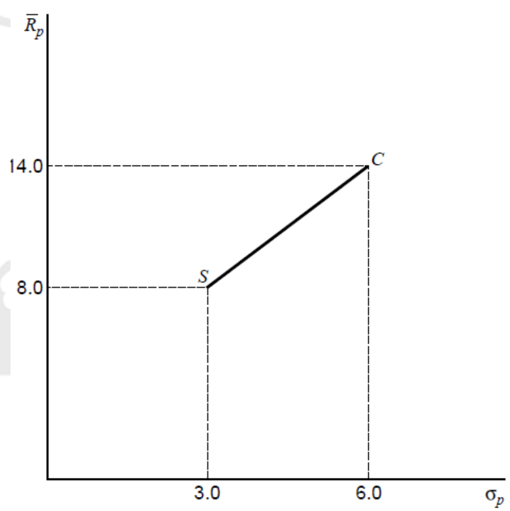
- Let us solve for σ_P and $\bar{R}_P$

Case 1: Perfect Correlation (ρ=1)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of ρ=1, to find the relationship between expected return and risk for the portfolio



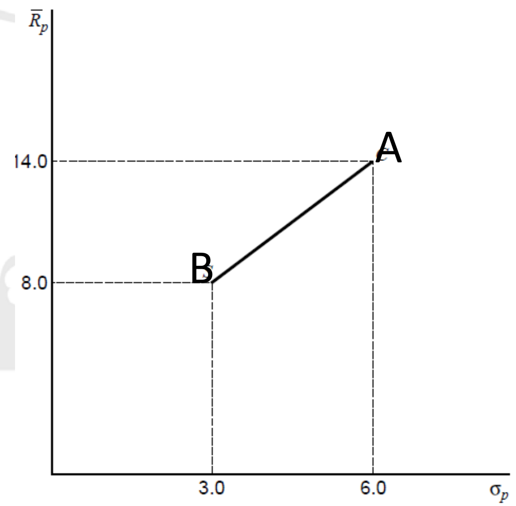
- $\sigma_P = X_A\sigma_A + (1 - X_A)\sigma_B$
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $\bar{R}_P = \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B}\bar{R}_A + (1 - \frac{\sigma_P - \sigma_B}{\sigma_A - \sigma_B})\bar{R}_B$

Case 1: Perfect Correlation (ρ=1)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of ρ=1, to find the relationship between expected return and risk for the portfolio



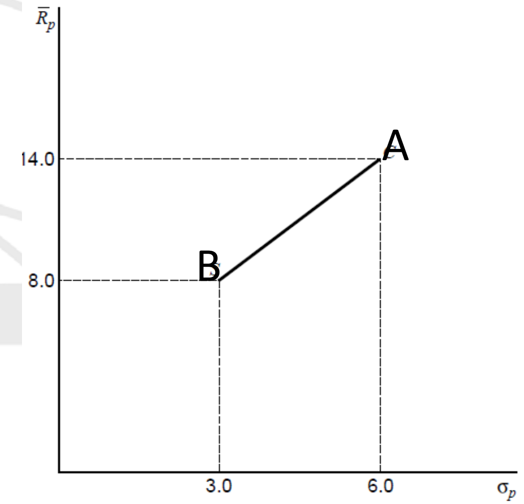
- $\sigma_P = X_A\sigma_A + (1 - X_A)\sigma_B$
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $\bar{R}_P = \left(\bar{R}_B - \frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B} * \sigma_B\right) + \left(\frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B}\right) * \sigma_P = ??$

Case 1: Perfect Correlation (ρ=1)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of ρ=1, to find the relationship between expected return and risk for the portfolio



- $\bar{R}_P = \left(\bar{R}_B - \frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B} * \sigma_B\right) + \left(\frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B}\right) * \sigma_P$

- $\bar{R}_P$=2+2σ_P

Case 1: Perfect Correlation (ρ=1)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.5	0.6	0.8	1.0
$\bar{R}_P$	??	??	??	??	??	??	??
σ_P	??	??	??	??	??	??	??

- $\sigma_P = X_A\sigma_A + (1 - X_A)\sigma_B$
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $\bar{R}_P$=2+2σ_P

Case 1: Perfect Correlation (ρ=1)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.5	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.0	11.6	12.8	14.0
σ_P	3.0	3.6	4.2	4.5	4.8	5.4	6.0

- $\sigma_P = X_A\sigma_A + (1 - X_A)\sigma_B$
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $\bar{R}_P$=2+2σ_P

Minimum variance portfolio

- Consider the example below

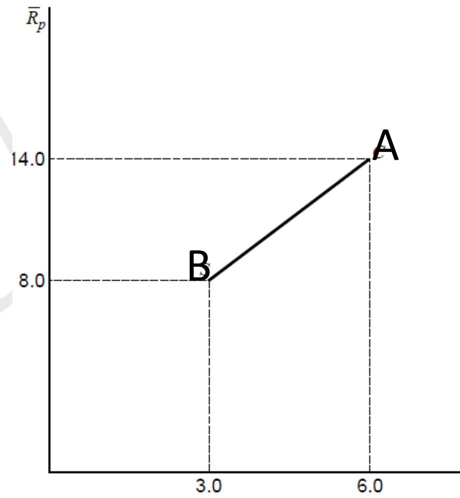
Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 1, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} = ?$

Minimum variance portfolio

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

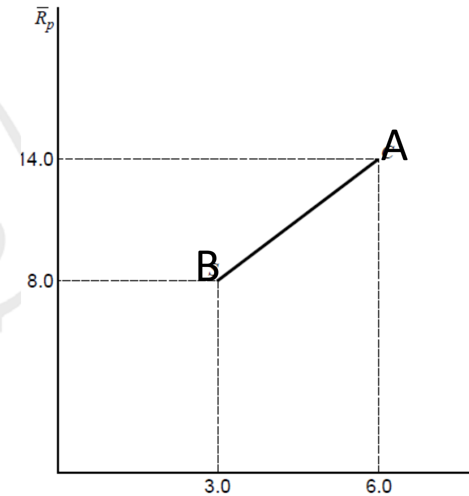
- Assume a correlation of 1, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{\left(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B\right)} = \frac{\sigma_B(\sigma_B - \sigma_A)}{(\sigma_A - \sigma_B)^2} = \frac{\sigma_B}{\sigma_B - \sigma_A} < 0$
- With limiting constraint that any weight cannot be equal to zero, this gives us $X_A = 0$
- Another way to obtain this result?



Minimum variance portfolio

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 1, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} = \frac{\sigma_B(\sigma_B - \sigma_A)}{(\sigma_A - \sigma_B)^2} = \frac{\sigma_B}{\sigma_B - \sigma_A} < 0$
- With limiting constraint that any weight cannot be equal to zero, this gives us $X_A = 0$
- Another way to obtain this result
- $\sigma_P = X_A\sigma_A + (1 - X_A)\sigma_B$

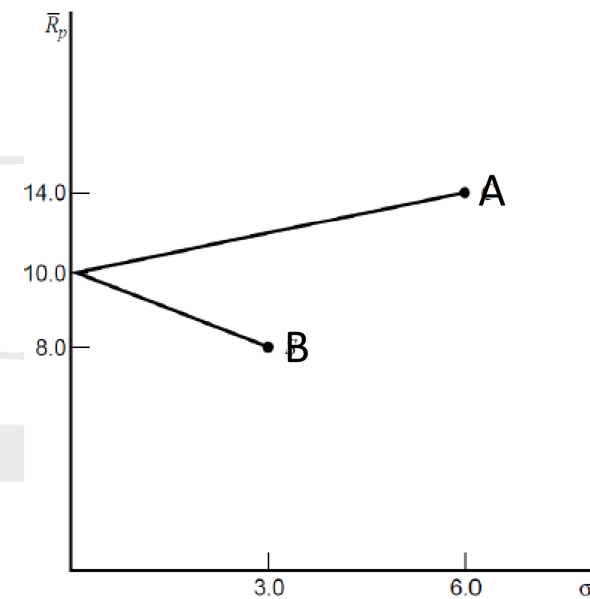


Case 2: Perfect Negative Correlation (ρ=-1)

Case 2: Perfect Negative Correlation (ρ=-1)

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

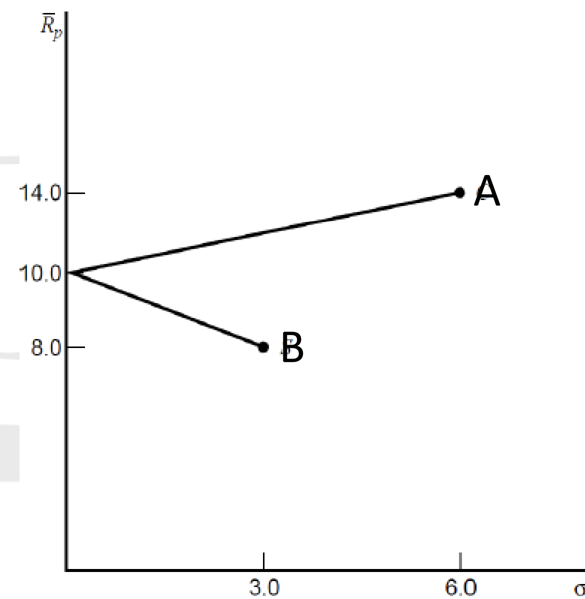
- Assume a correlation of ρ= -1, to find the relationship between expected return and risk for the portfolio
- $\sigma_P = X_A\sigma_A - (1 - X_A)\sigma_B$ or $-X_A\sigma_A + (1 - X_A)\sigma_B$
- Whichever RHS is positive
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $Equation\ of\ these\ lines?$



Case 2: Perfect Negative Correlation (ρ=-1)

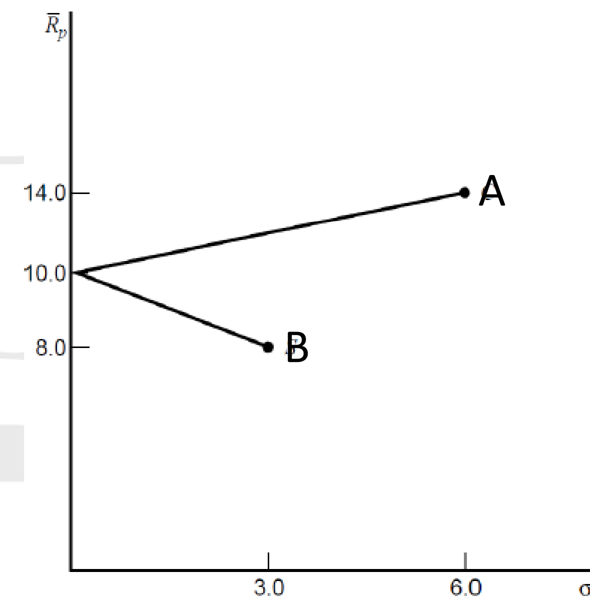
Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of ρ= -1, to find the relationship between expected return and risk for the portfolio
- $\sigma_P = X_A\sigma_A - (1 - X_A)\sigma_B$ or $-X_A\sigma_A + (1 - X_A)\sigma_B$
- Whichever RHS is positive
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $\mathit{Equation\ of\ these\ lines}$?



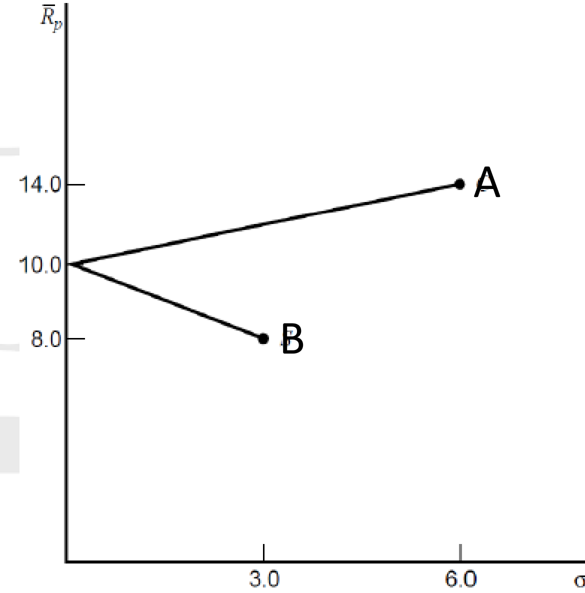
Case 2: Perfect Negative Correlation (ρ=-1)

- Assume a correlation of ρ= -1, to find the relationship between expected return and risk for the portfolio
- $\sigma_P = X_A\sigma_A - (1 - X_A)\sigma_B$ or $-X_A\sigma_A + (1 - X_A)\sigma_B$
- Whichever RHS is positive
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $\bar{R}_P = 10 + \frac{2}{3}\sigma_P$
- $\bar{R}_P = 10 - \frac{2}{3}\sigma_P$



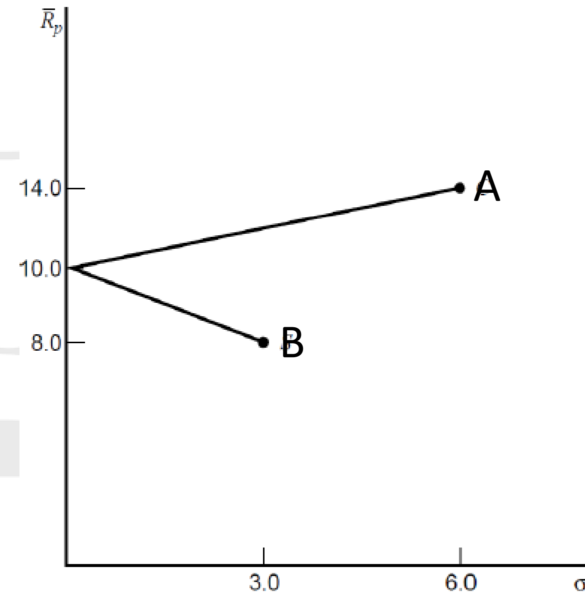
Case 2: Perfect Negative Correlation (ρ=-1)

- Assume a correlation of ρ= -1, to find the relationship between expected return and risk for the portfolio
- $\sigma_P = X_A\sigma_A - (1 - X_A)\sigma_B$ or $-X_A\sigma_A + (1 - X_A)\sigma_B$
- Whichever RHS is positive
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- At what point minimum risk will occur?



Case 2: Perfect Negative Correlation (ρ=-1)

- Assume a correlation of ρ= -1, to find the relationship between expected return and risk for the portfolio
- $\sigma_P = X_A\sigma_A - (1 - X_A)\sigma_B$ or $-X_A\sigma_A + (1 - X_A)\sigma_B$
- Whichever RHS is positive
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- Also, for $\sigma_P = 0$; $X_A = \frac{\sigma_B}{(\sigma_A+\sigma_B)} = \frac{1}{3}$
- $0 < X_A < 1$



Case 2: Perfect Negative Correlation (ρ=-1)

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	??	??	??	??	??	??
σ_P	??	??	??	??	??	??

- $\sigma_P = X_A\sigma_A - (1 - X_A)\sigma_B$ or $-X_A\sigma_A + (1 - X_A)\sigma_B$
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$

Case 2: Perfect Negative Correlation (ρ=-1)

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	3.0	1.2	0.6	2.4	5.4	6.0

- $\sigma_P = X_A\sigma_A - (1 - X_A)\sigma_B$ or $-X_A\sigma_A + (1 - X_A)\sigma_B$
- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$

Case 2: Perfect Negative Correlation (ρ=-1): Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of -1, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} =$

Case 2: Perfect Negative Correlation (ρ=-1): Minimum variance portfolio

- Consider the example below

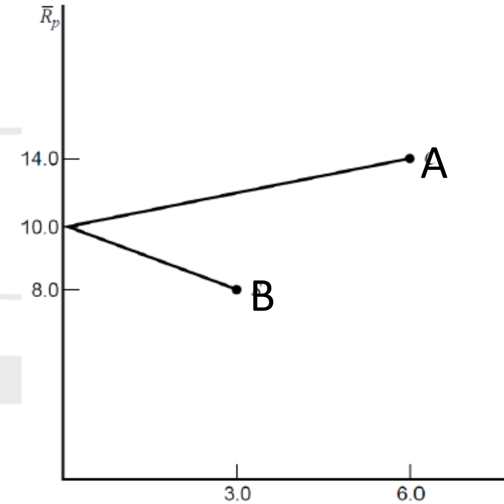
Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of -1, try to find the amount invested

- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} = \frac{\sigma_B(\sigma_B + \sigma_A)}{(\sigma_A + \sigma_B)^2} = \frac{\sigma_B}{\sigma_B + \sigma_A} = 1/3 \;,\; X_B = \frac{2}{3}$

- $\sigma_P = \left(\frac{1}{3} * 6 - \frac{2}{3} * 3\right) = 0$

- Any other way to obtain this result?

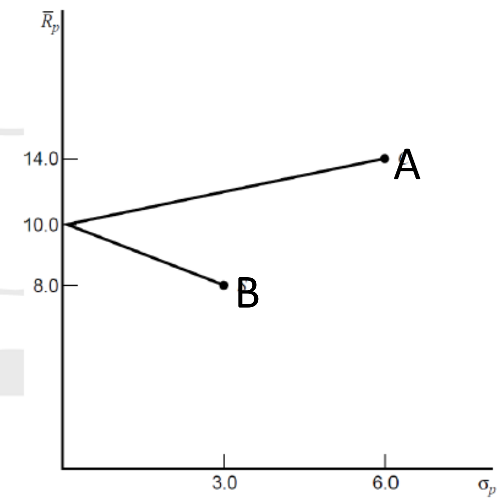


Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of -1, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2+\sigma_B^2-2\rho_{AB}\sigma_A\sigma_B)} = \frac{\sigma_B(\sigma_B+\sigma_A)}{(\sigma_A+\sigma_B)^2} = \frac{\sigma_B}{\sigma_B+\sigma_A} = 1/3$, $X_B = \frac{2}{3}$
- $\sigma_P = \left(\frac{1}{3} * 6 - \frac{2}{3} * 3\right) = 0$
- $W_A\sigma_A - \mathrm{W_B}\sigma_B$=?



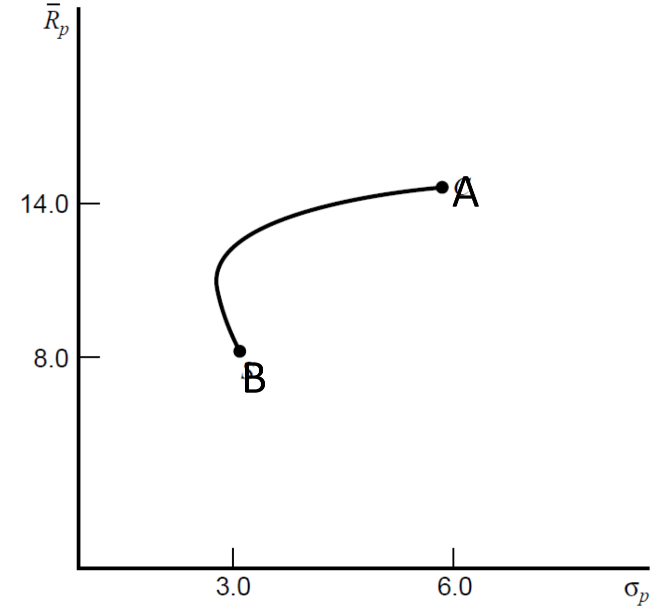
Case 3: No Correlation ($\rho=0$)

Case 3: No Correlation (ρ=0)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_B\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$

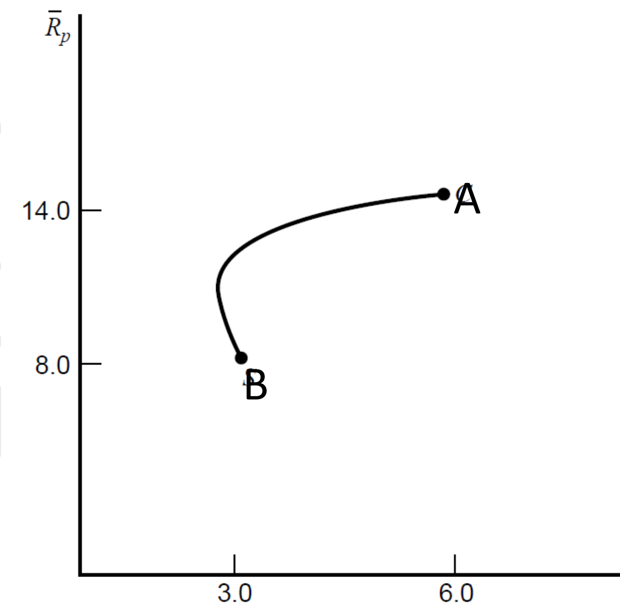


Case 3: No Correlation (ρ=0)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_B\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$
- $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2]^{\frac{1}{2}}$
- $\sigma_P = [X_A^2 6^2 + (1 - X_A^2)3^2]^{\frac{1}{2}}$
- $= [45X_A^2 - 18X_C + 9]^{\frac{1}{2}}$



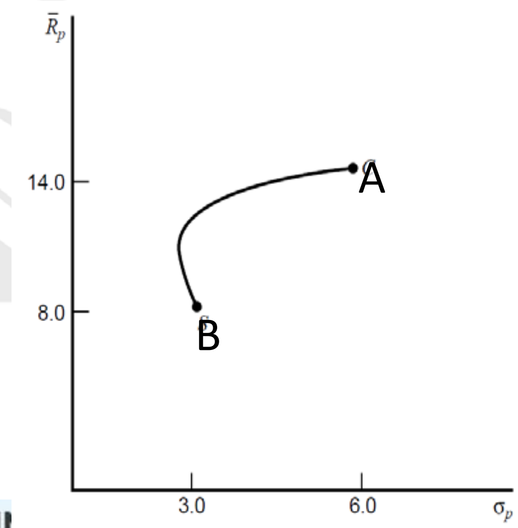
Case 3: No Correlation (ρ=0)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	??	??	??	??	??	??

- $\bar{R}_P = X_A \bar{R}_A + (1 - X_A)\bar{R}_B$
- $\sigma_P = [X_A^2 6^2 + (1 - X_A^2)3^2]^{\frac{1}{2}}$
- $= [45X_A^2 - 18X_A + 9]^{\frac{1}{2}}$



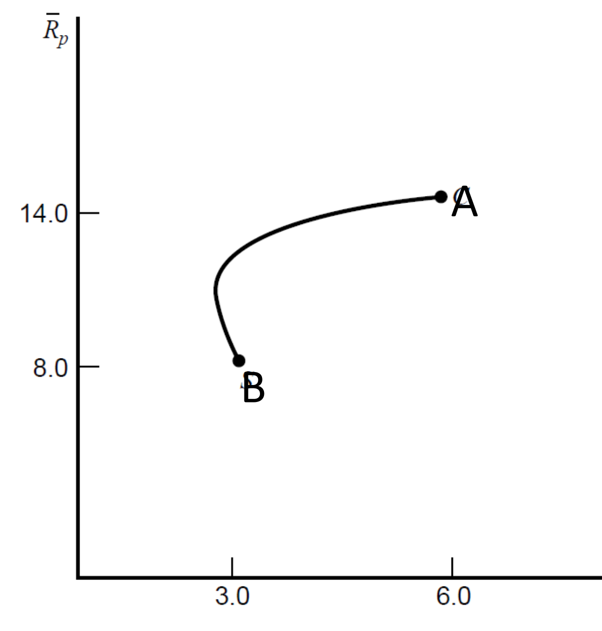
Case 3: No Correlation (ρ=0)

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	3.0	2.68	3.00	3.79	4.84	6.0

- $\bar{R}_P = X_A\bar{R}_A + (1 - X_A)\bar{R}_B$
- $\sigma_P = [X_A^2 6^2 + (1 - X_A^2)3^2]^{\frac{1}{2}}$
- $= [45X_A^2 - 18X_A + 9]^{\frac{1}{2}}$

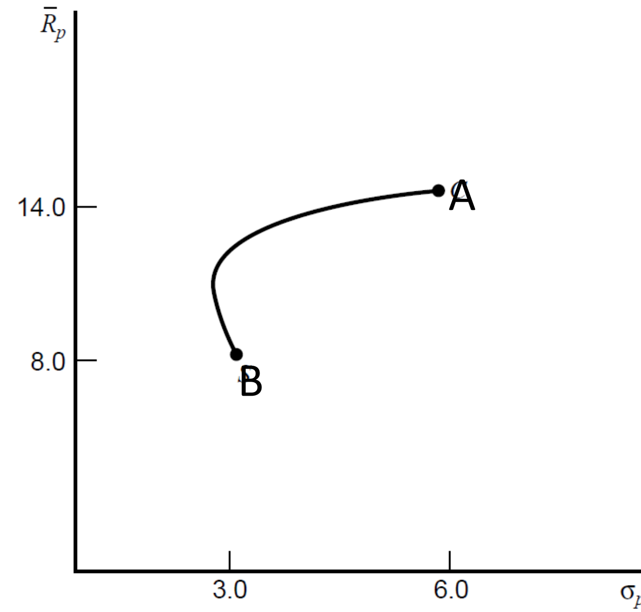


Case 3: No Correlation (ρ=0) and Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 0, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} =$
- What is the implication?



Case 3: No Correlation (ρ=0) and Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 0, try to find the amount invested in MV portfolio

- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} = \frac{\sigma_B^2}{(\sigma_A^2 + \sigma_B^2)} = ??,\ X_B = ??$
- $\sigma_P = ??$

Case 3: No Correlation (ρ=0) and Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 0, try to find the amount invested in MV portfolio
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} = \frac{\sigma_B^2}{(\sigma_A^2 + \sigma_B^2)} = \frac{3^2}{6^2 + 3^2} = \frac{1}{5} = 0.2$, $X_B = 0.8$
- $\sigma_P = (0.2^2 * 6^2 + 0.8^2 3^2)^{\frac{1}{2}} = 2.68\%$

Case 4: Mid Correlation (ρ=0.5) Minimum variance portfolio

Case 4: Mid Correlation (ρ=0.5) Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_B\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$
- Let us solve for σ_P

Case 4: Mid Correlation (ρ=0.5) Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_B\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$
- $\sigma_P = \left[6^2X_A^2 + 3^2(1 - X_A^2) + 2X_A(1 - X_A)3 * 6 * \frac{1}{2}\right]^{\frac{1}{2}}$
- $\sigma_P = [27X_A^2 + 9]^{\frac{1}{2}}$

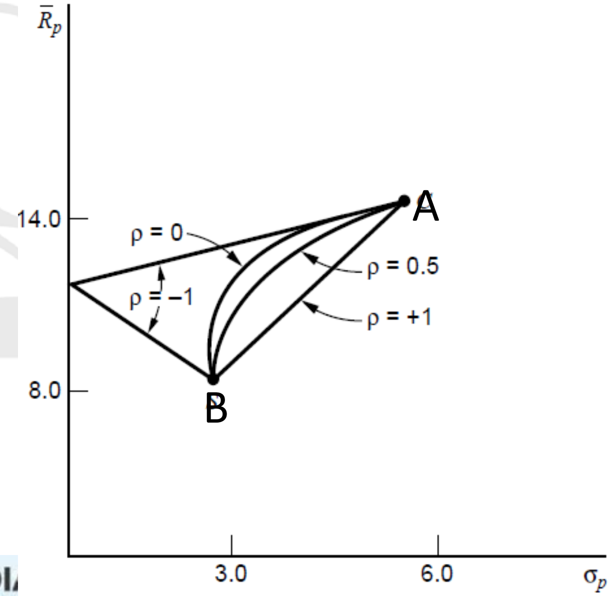
Case 4: Mid Correlation (ρ=0.5) Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	??	??	??	??	??	??

- $\sigma_P = [27X_A^2 + 9]^{\frac{1}{2}}$



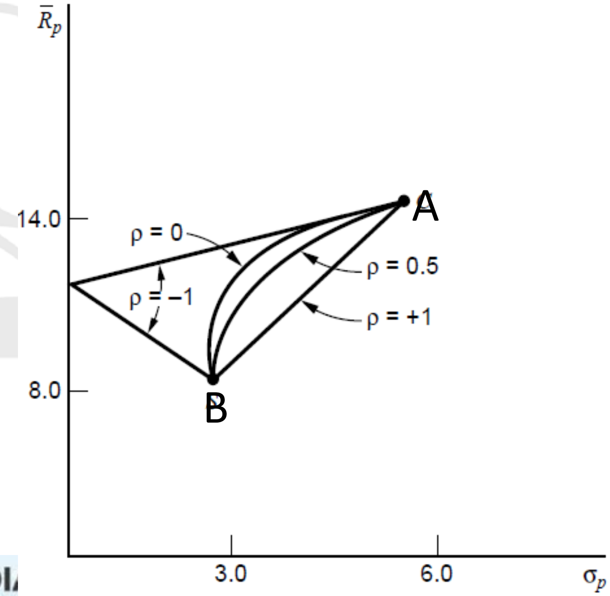
Case 4: Mid Correlation (ρ=0.5) Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	0	0.2	0.4	0.6	0.8	1.0
$\bar{R}_P$	8.0	9.2	10.4	11.6	12.8	14.0
σ_P	3.0	3.17	3.65	4.33	5.13	6.00

- $\sigma_P = [27X_A^2 + 9]^{\frac{1}{2}}$

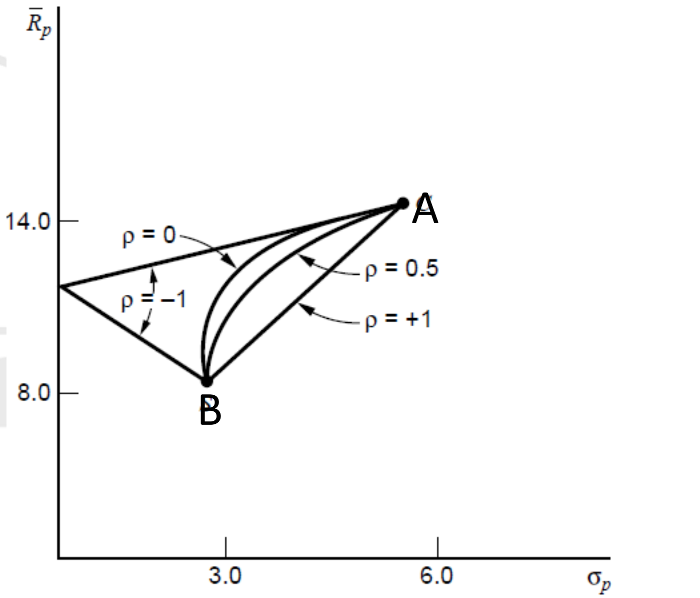


Case 4: Mid Correlation (ρ=0.5) Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 0.5, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} =$
- What is the implication?



Case 4: Correlation (ρ=0.5) and Minimum variance portfolio

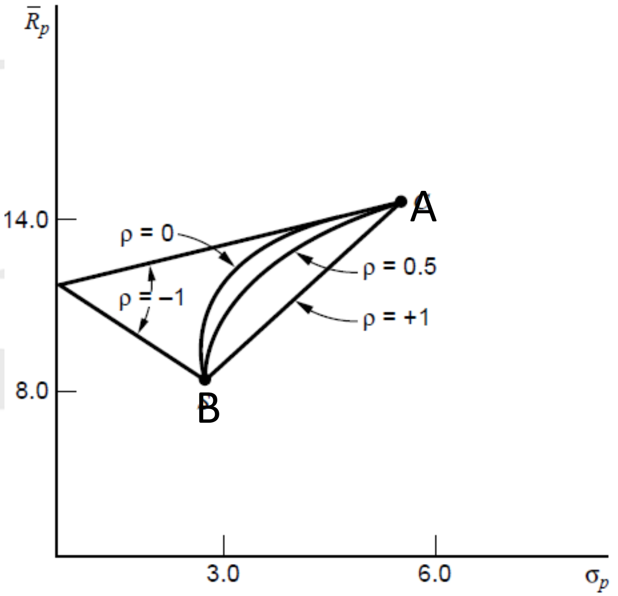
- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 0, try to find the amount invested MV portfolio

- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} =$

- What is the implication?



Case 4: Correlation (ρ=0.5) and Minimum variance portfolio

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

- Assume a correlation of 0.5, try to find the amount invested
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)} = \frac{3^2 - 0.5*6*3}{6^2 + 3^2 - 2*0.5*6*3} = 0$
- What is the implication? No combination of securities A and B has less risk than security B itself. So the minimum variance portfolio is security B itself. That also means for any correlation higher than 0.5, security B will itself B the minimum variance portfolio ($X_A = 0$)

Minimum variance portfolio

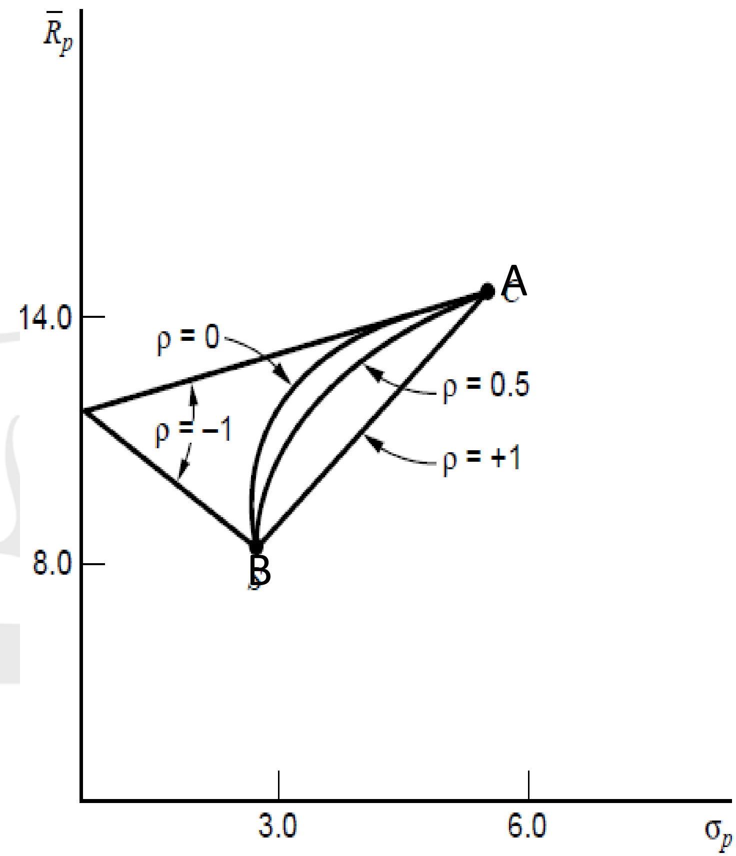
- Here, how do I find that correlation above which the minimum variance portfolio is the least risky asset?
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)}$

Minimum variance portfolio

- Here, how do I find that correlation above which the minimum variance portfolio is the least risky asset?
- $X_B = \mathbf{100\%}$
- $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)}$=0
- $\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B$=0 or
- ρ_{AB}= σ_B/ σ_A

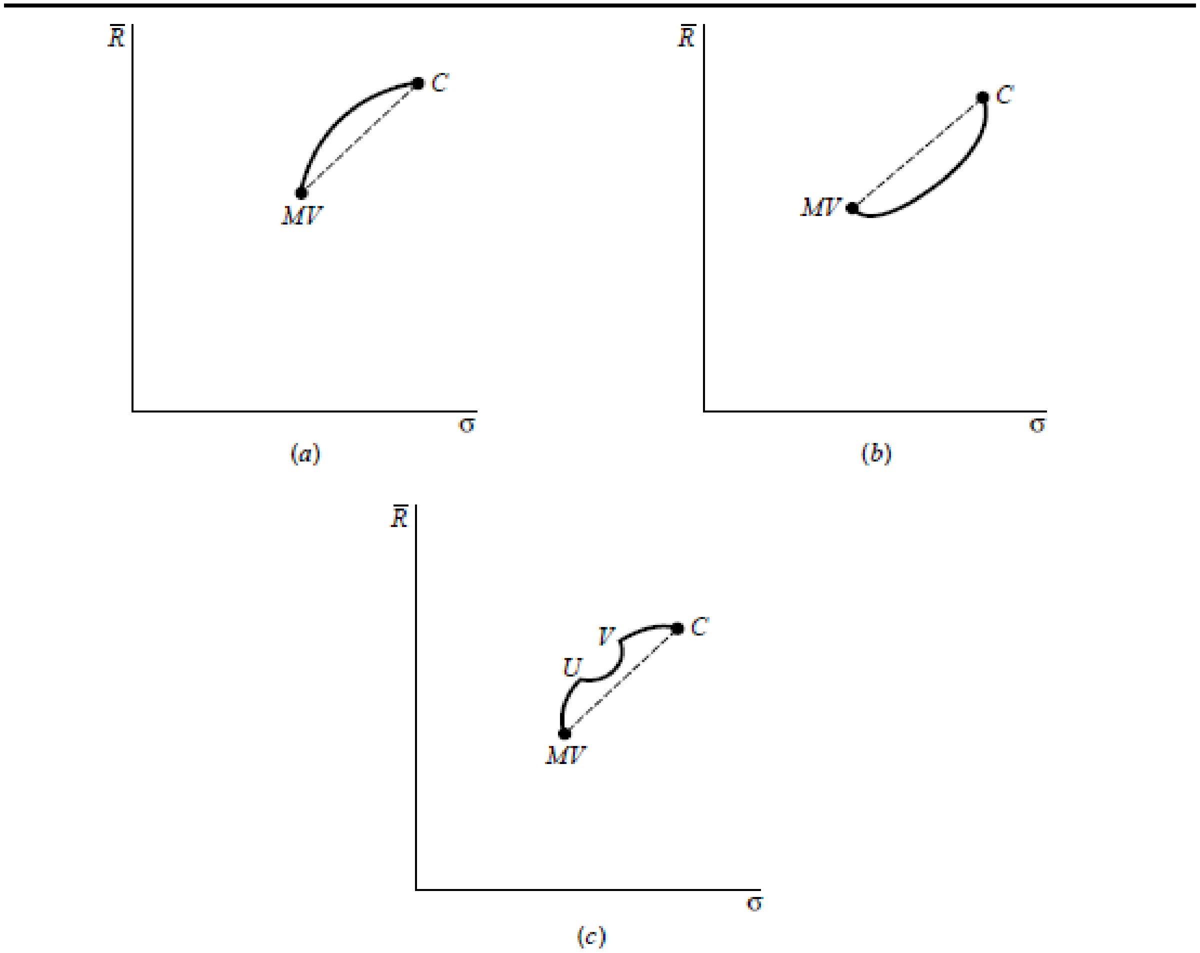
In conclusion

- For all the other correlations between +1 to -1 intermediate curves will be obtained
- Maximum diversification is achieved at ρ=-1
- Any combination of two assets can not have more risk that on the straight line (AB) connecting the two assets
- Lastly, we also developed an expression for minimum variance portfolio

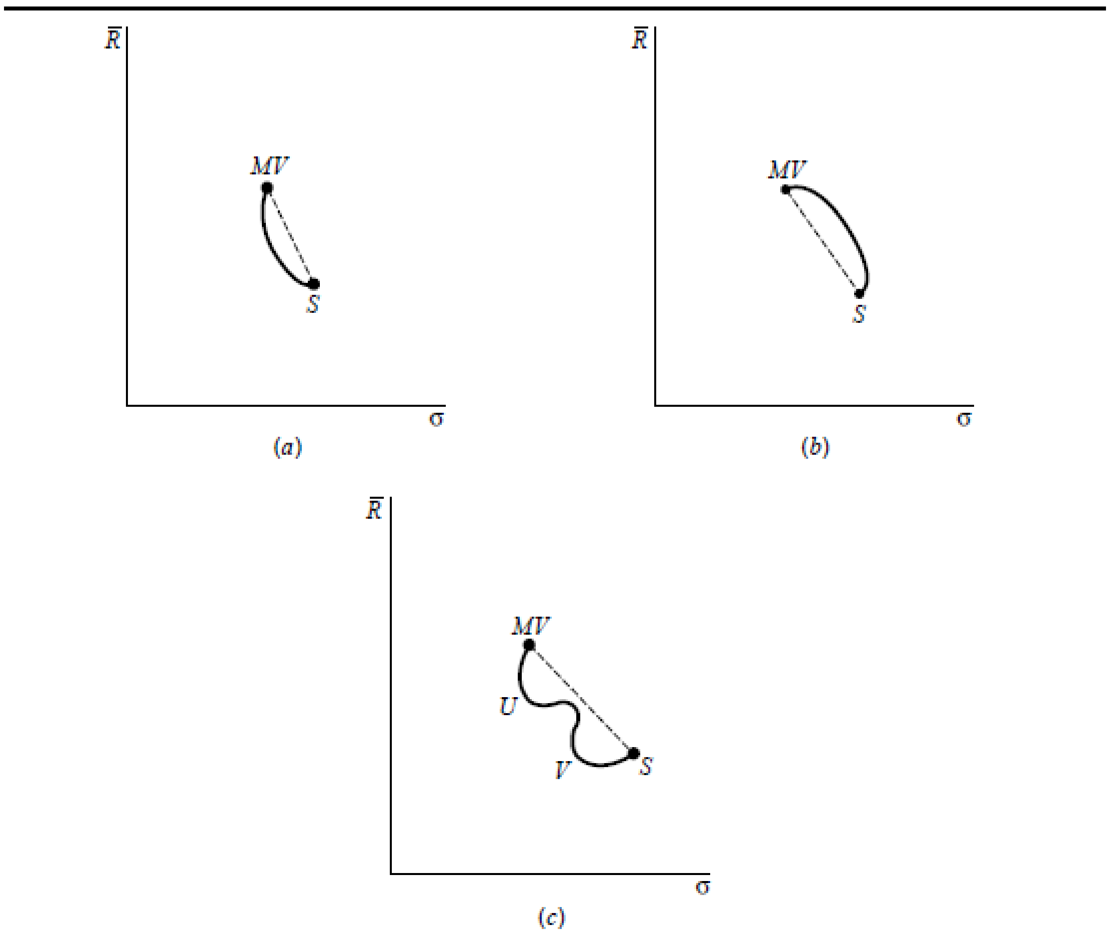


Shape of portfolio possibility curve

Which one of these are not possible



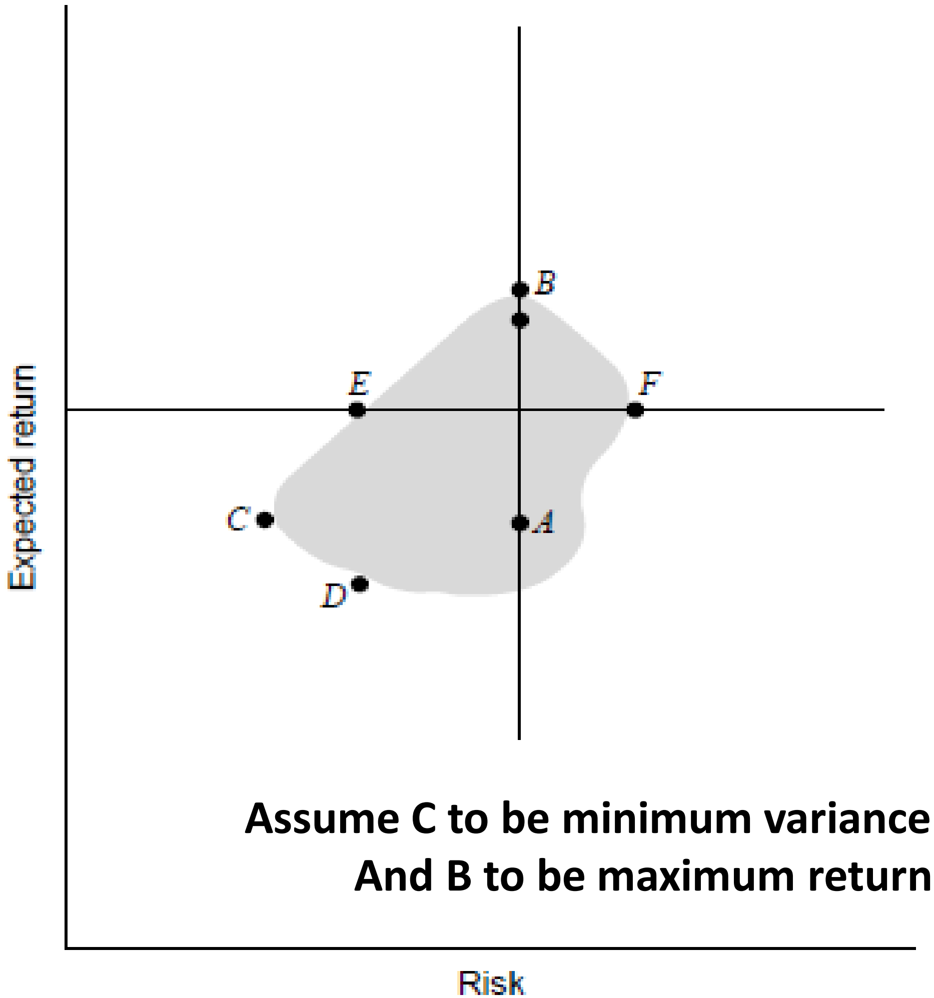
Which one of these are not possible



Efficient frontier scenarios : Multi-security case

- Efficient frontier with no-short sales
- Efficient frontier with riskless lending and borrowing
- Only riskless lending is allowed; not borrowing
- Riskless lending and borrowing at different rates
- Efficient frontier with short-sales (no risk-free lending and borrowing)

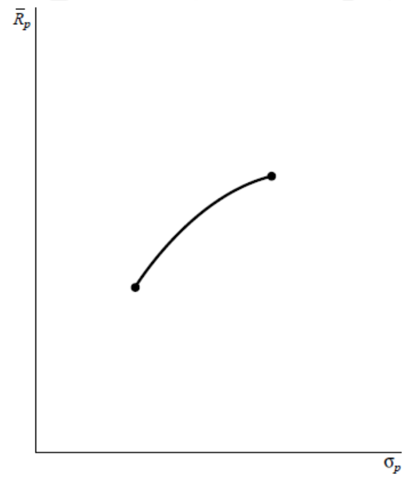
Efficient frontier with no-short sales : Multi-security case



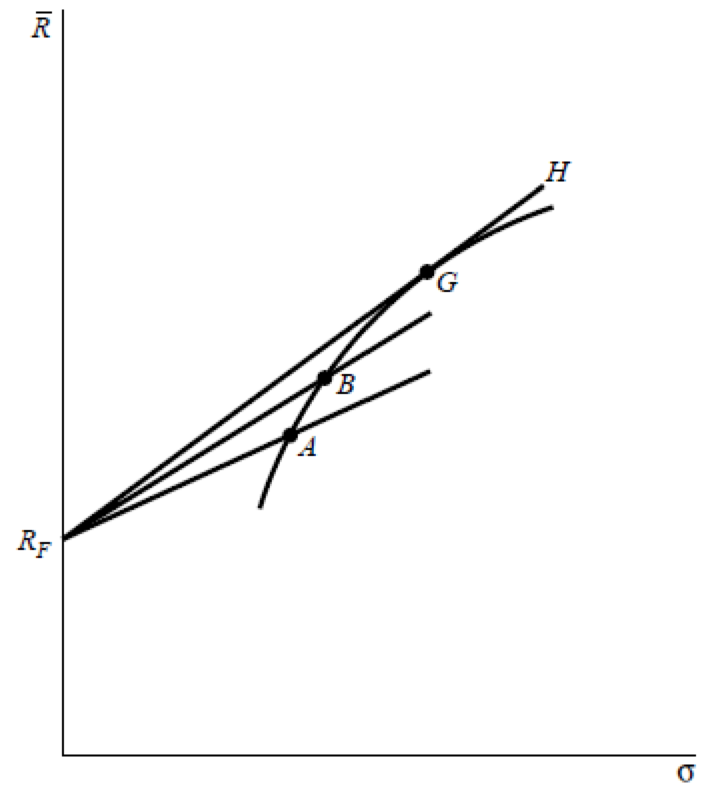
Assume C to be minimum variance (MV) portfolio
And B to be maximum return portfolio

Efficient frontier with no-short sales : Multi-security case

- Before we start our discussion, let us recollect two important goals.
  - High return for a given risk, and
  - Low risk for a given return
- Of course you know what is the efficient frontier on this diagram (CEB segment)

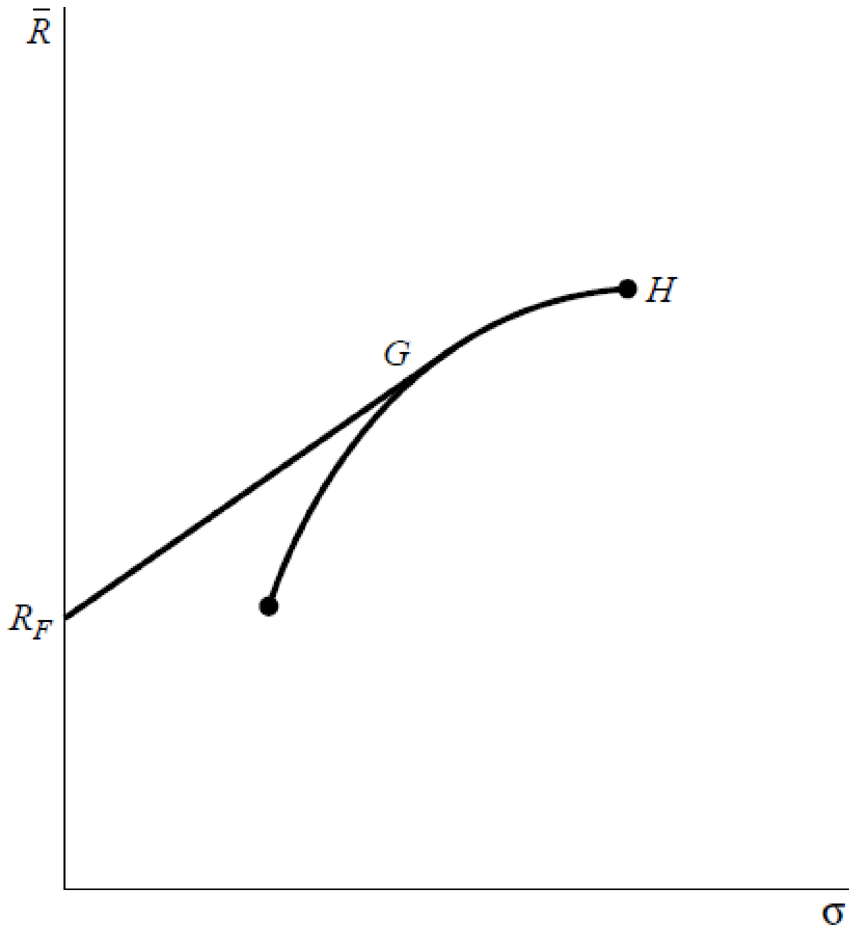


The efficient frontier with riskless lending and borrowing



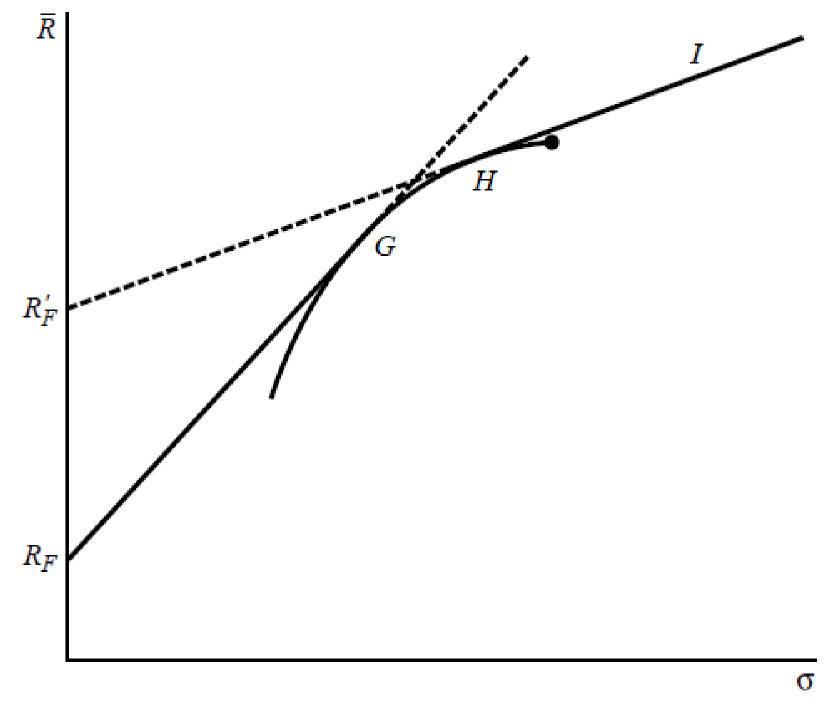
Combinations of the riskless asset and various risky portfolios.

Only riskless lending is allowed; not borrowing



The efficient frontier with lending but not borrowing at the riskless rate.

Riskless lending and borrowing at different rates



The efficient frontier with riskless lending and borrowing at different rates.

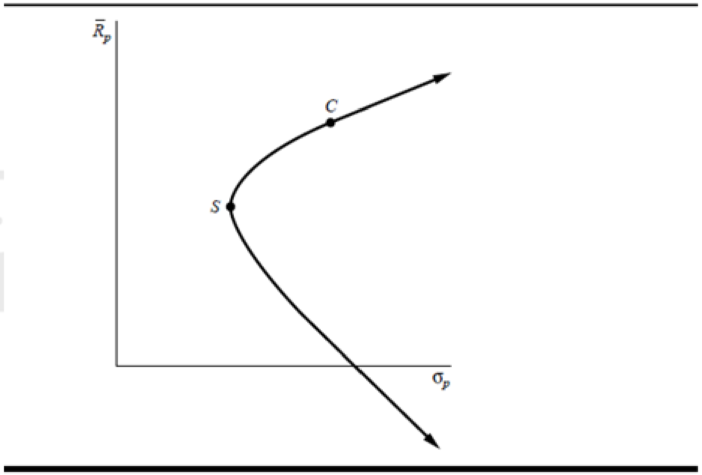
Case 4: Mid Correlation (ρ=0.5)
Short-sales allowed

- Consider the example below

Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	-1.0	-0.8	-0.6	-0.4	-0.2	1.2	1.4	1.6	1.8	2.0
$\bar{R}_P$	??	??	??	??	??	??	??	??	??	??
σ_P	??	??	??	??	??	??	??	??	??	??

- $\sigma_P = [27X_A^2 + 9]^{\frac{1}{2}}$



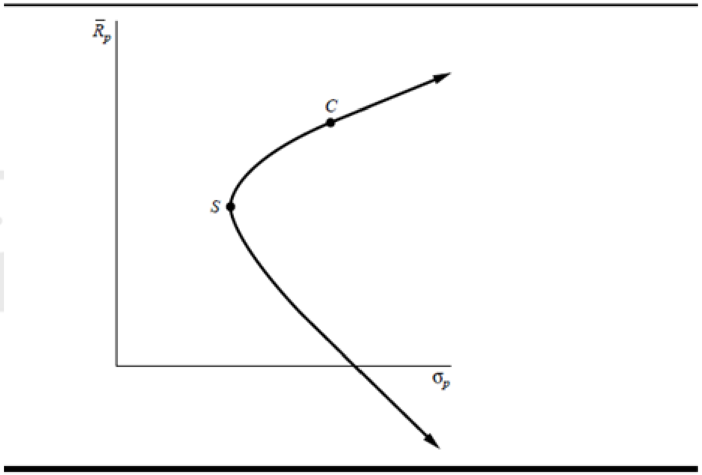
Case 4: Mid Correlation (ρ=0.5) Short-sales allowed

- Consider the example below

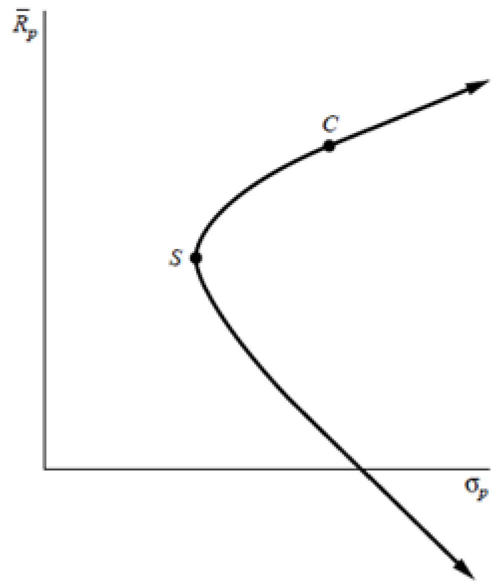
Stock	Expected Returns	SD
A	14%	6%
B	8%	3%

X_A	-1.0	-0.8	-0.6	-0.4	-0.2	1.2	1.4	1.6	1.8	2.0
$\bar{R}_P$	2.0	3.2	4.4	5.6	6.8	15.2	16.4	17.6	18.8	20.0
σ_P	6.0	5.13	4.33	3.65	3.17	6.92	7.87	8.84	9.82	10.82

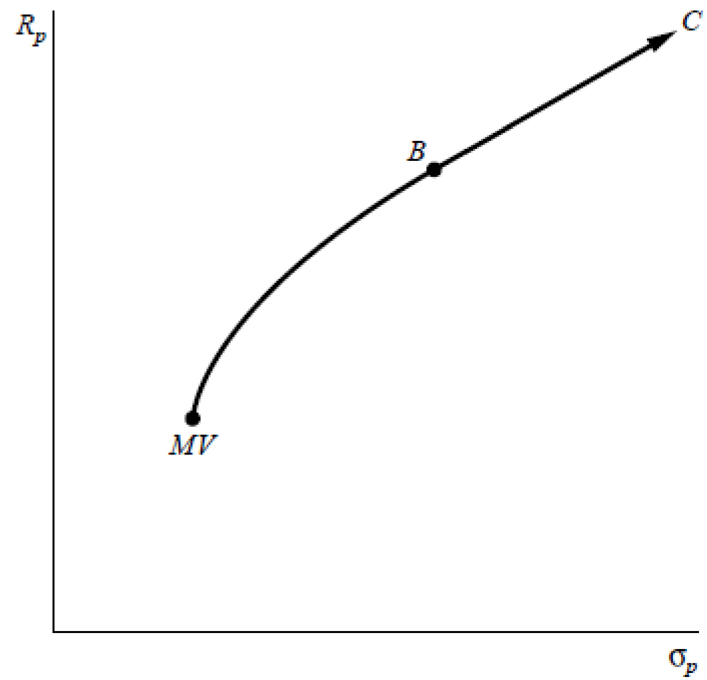
- $\sigma_P = [27X_A^2 + 9]^{\frac{1}{2}}$



Efficient frontier with short-sales with two securities (no risk-free lending and borrowing)



Efficient frontier with short-sales with two securities (no risk-free lending and borrowing)



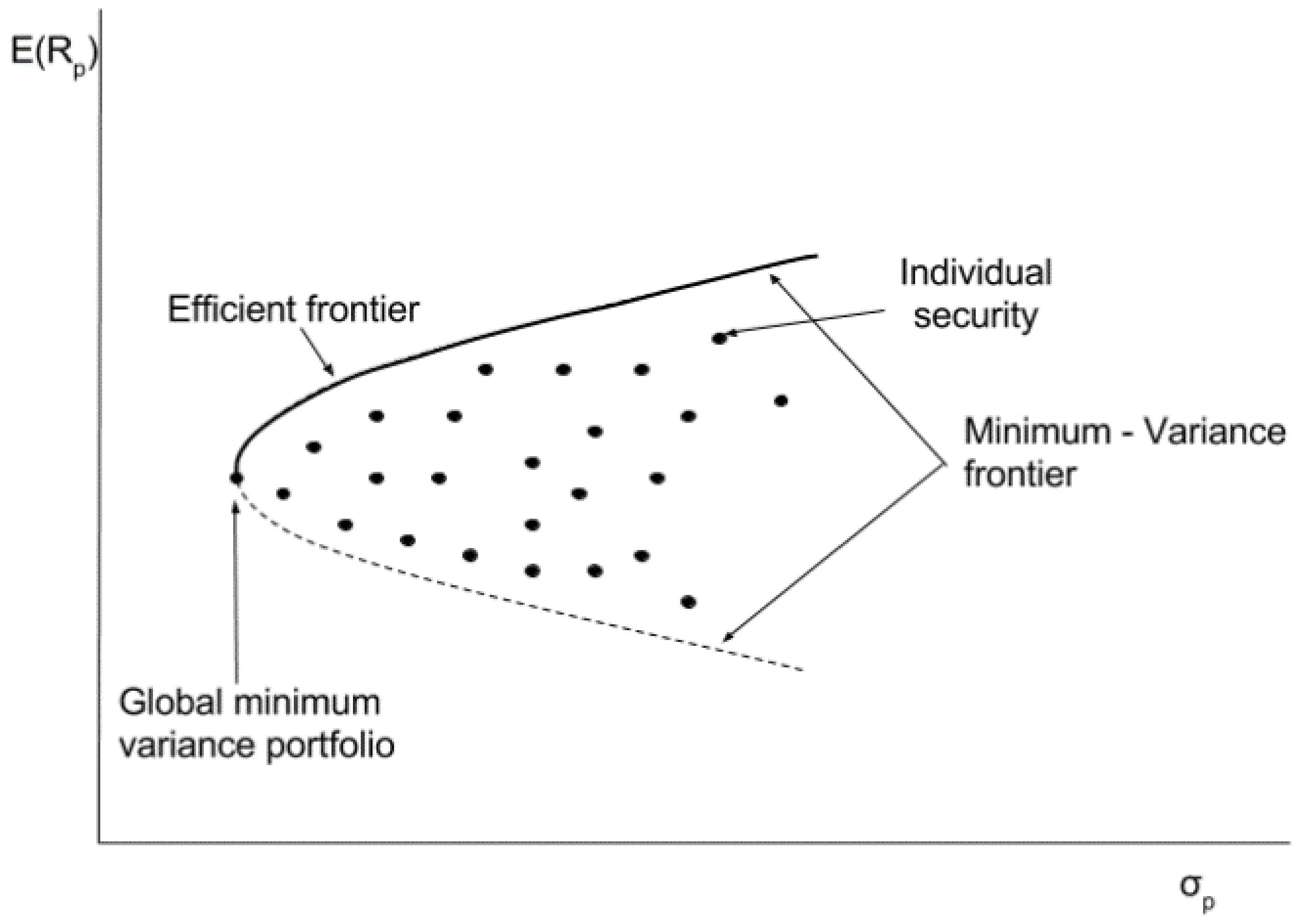
Thanks

Techniques for Computing Efficient Frontier

Course: Portfolio management

Instructor: Abhinava Tripathi

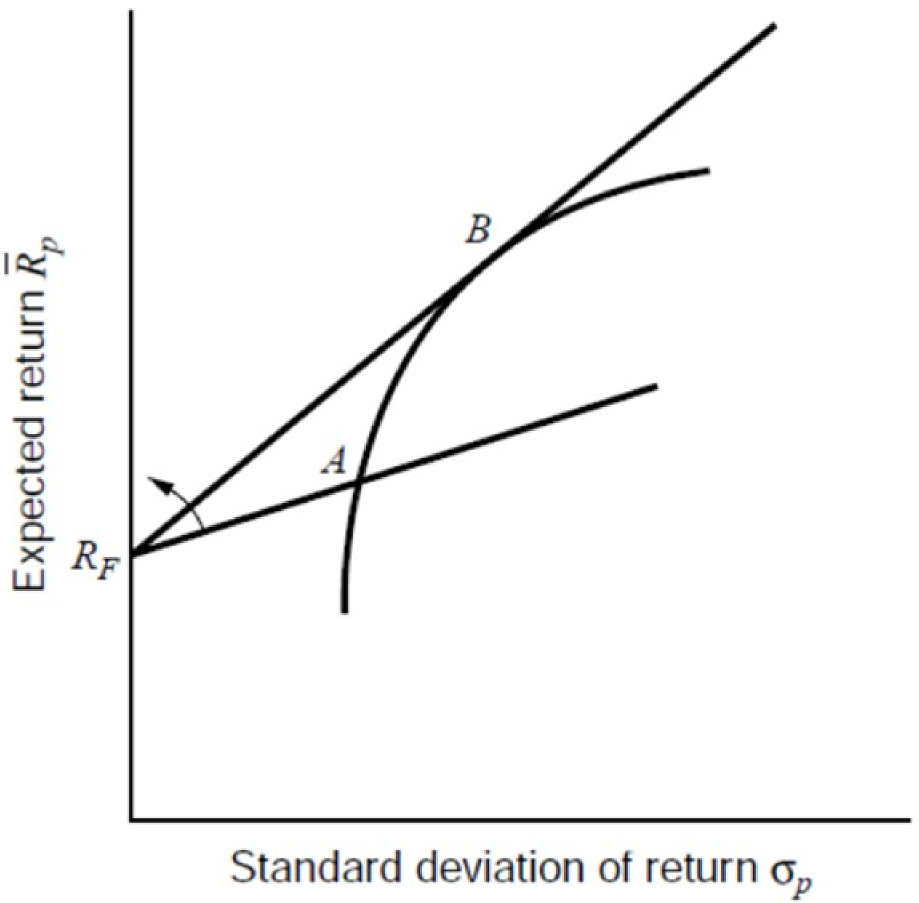
Minimum variance frontier



Efficient Frontier in Four Scenarios

- Short-sales are allowed riskless lending borrowing is also allowed
- Short-sales are allowed, but riskless lending borrowing is not allowed
- Short-sales are disallowed, but riskless lending borrowing is allowed
- Neither short-sales nor riskless lending borrowing allowed

(1) Short-sales allowed with riskless lending and borrowing



Combinations of the riskless asset in a risky portfolio.

Short-sales allowed with riskless lending and borrowing

- We know that there is a portfolio of risky assets that is best among all the portfolios
- The portfolio B is preferred to all the portfolios on the original efficient frontier
- The portfolios on the line Rf-B represent the new efficient frontier carrying the portfolios that are preferred to all the portfolios of risky assets
- This new efficient frontier is the line Rf-B extended beyond B. The slope of this line was the highest.

Short-sales allowed with riskless lending and borrowing

- **Objective function**: Maximize slope of the portfolio that lies on the efficient frontier : $\theta = \frac{\bar{R}_P - R_f}{\sigma_P}$
- **Constraint**: $\sum_{i=1}^{N} X_i = 1$
- This is similar to constrained maximization problem
- Computer programs use “Lagrangian multipliers” method to solve this
- We can substitute the constraint into objective function and then the objective function can be maximized as in an unconstrained problem

Short-sales allowed with riskless lending and borrowing

- **The constrained problem:** $\theta = \frac{\bar{R}_P - R_f}{\sigma_P}$
- $\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$
- $R_f = \sum_{i=1}^{N} X_i R_f$ (Why? $R_f = 1 * R_f = \sum_{i=1}^{N} X_i (\mathrm{R_f}) = \sum_{i=1}^{N} X_i R_f$): e.g.: 10%= .5*10%+0.3*10%+0.2*10%
- $\theta = \sum_{i=1}^{N} \frac{X_i(\bar{R}_i - R_f)}{\sigma_P}$;
- Where $\sigma_P = \left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{ij}\right]^{\left(\frac{1}{2}\right)}$
- We have to maximize this expression, with respect to all X_i's.

Short-sales allowed with riskless lending and borrowing

- This will lead to the following system of equations:
- $\frac{d\theta}{dX_1} = \frac{d\theta}{dX_2} = \frac{d\theta}{dX_3} \dots \dots \dots . = \frac{d\theta}{dX_N} = 0$
- The mathematical solution to the above set is provided below
- $\boldsymbol{\frac{d\theta}{dX_i} = -(\lambda X_1 \sigma_{1i} + \lambda X_2 \sigma_{2i} + \cdots + \lambda X_i \sigma_i^2 + \cdots + \lambda X_{N-1} \sigma_{N-1i} + \lambda X_N \sigma_{Ni}) + \overline{R_i} - R_F = 0}$
- **That is**
- $\boldsymbol{\frac{d\theta}{dX_1} = -(\lambda X_1 \sigma_1^2 + \lambda X_2 \sigma_{2i} + \cdots + \cdots + \cdots + \lambda X_{N-1} \sigma_{N-1i} + \lambda X_N \sigma_{Ni}) + \overline{R_1} - R_F = 0}$

Short-sales allowed with riskless lending and borrowing

- The optimization problem we discussed, to solve for θ with respect to X_k was required. The value of $\theta = \sum_{i=1}^{N} \frac{X_i\left(R_P - R_f\right)}{\sigma_P}$; where

- $\sigma_P = \left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{ij}\right]^{\left(\frac{1}{2}\right)}$

- F(X)=$F_1(X)F_2(X)$

- $\frac{d\left(F(X)\right)}{dX} = F_1(X)\frac{d\left(F_2(X)\right)}{dX} + \mathrm{F}_2(X)\frac{d\left(F_1(X)\right)}{dX}$

Short-sales allowed with riskless lending and borrowing

- Here $F_1(X) = \sum_{i=1}^{N} X_i(\bar{R}_i - R_F)$ and
- $F_2(X) = \left[\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{ij}\right]^{\left(-\frac{1}{2}\right)}$
- $\frac{dF_1(X)}{dX_k} = (\bar{R}_k - R_F)$
- $\frac{dF_2(X)}{dX_k} = ?$

Short-sales allowed with riskless lending and borrowing

- **Three security**
- $\sigma_p^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + X_3^2\sigma_3^2 + 2X_1X_2\sigma_{12} + 2X_1X_3\sigma_{13} + 2X_2X_3\sigma_{23}$
- $\frac{d\sigma_p^2}{dX_1} = 2X_1\sigma_1^2 + 2X_2\sigma_{12} + 2X_3\sigma_{13}$

- **N-Security**
- $\sigma_P = \sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} X_iX_j\sigma_{ij}$
- $\frac{d\sigma_p^2}{dX_k} = 2X_k\sigma_k^2 + 2\sum_{\substack{j=1 \\ j\neq k}}^{N} X_j\sigma_{jk}$

Short-sales allowed with riskless lending and borrowing

- Here $F_1(X) = \sum_{i=1}^{N} X_i(\bar{R}_i - R_F)$ and
- $F_2(X) = \left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} X_iX_j\sigma_{ij}\right]^{\left(-\frac{1}{2}\right)}$
- $\frac{dF_1(X)}{dX_k} = (\bar{R}_k - R_F)$
- $\frac{dF_2(X)}{dX_k} = \left(-\frac{1}{2}\right) * \left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} X_iX_j\sigma_{ij}\right]^{\left(-\frac{3}{2}\right)} * [2 * X_k\sigma_k^2 + 2\sum_{\substack{j=1 \\ j\neq k}}^{N} X_j\sigma_{jk}]$

Short-sales allowed with riskless lending and borrowing

- $$\frac{d\left(F(X)\right)}{dX} = F_1(X)\frac{d\left(F_2(X)\right)}{dX} + F_2(X)\frac{d\left(F_1(X)\right)}{dX}$$

- $$\frac{d\theta}{dX_K} = \left[\sum_{i=1}^{N} X_i(\bar{R}_i - R_F)\right]\left[\left(-\frac{1}{2}\right) * \left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} X_iX_j\sigma_{ij}\right]^{\left(-\frac{3}{2}\right)} * \left[2 * X_k\sigma_k^2 + 2\sum_{\substack{j=1\\ j\neq k}}^{N} X_j\sigma_{jk}\right]\right] + \left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} X_iX_j\sigma_{ij}\right]^{\left(-\frac{1}{2}\right)} * (\bar{R}_k - R_F) = 0$$

- $$(\bar{R}_k - R_F) = \left[\frac{\left(\sum_{i=1}^{N} X_i(\bar{R}_i - R_F)\right)}{\left(\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} X_iX_j\sigma_{ij}\right)}\right]\left[X_k\sigma_k^2 + \sum_{\substack{j=1\\ j\neq k}}^{N} X_j\sigma_{jk}\right]$$

Short-sales allowed with riskless lending and borrowing

- Define λ as $\left[\frac{\left(\sum_{i=1}^{N} X_i(\bar{R}_i - R_F)\right)}{\left(\sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{ij}\right)}\right]$ the resultant equation becomes

- $(\bar{R}_k - R_F) = \lambda \left[X_k \sigma_k^2 + \sum_{\substack{j=1 \\ j \neq k}}^{N} X_j \sigma_{jk} \right]$

- $\boldsymbol{\frac{d\theta}{dX_i} = \overline{R_i} - R_F = (\lambda X_1 \sigma_{1i} + \lambda X_2 \sigma_{2i} + \cdots + \lambda X_i \sigma_i^2 + \cdots + \lambda X_{N-1} \sigma_{N-1i} + \lambda X_N \sigma_{Ni}) +}$

Short-sales allowed with riskless lending and borrowing

- This will lead to the following system of equations:
- $\frac{d\theta}{dX_1} = \frac{d\theta}{dX_2} = \frac{d\theta}{dX_3} \dots \dots \dots . = \frac{d\theta}{dX_N} = 0$
- The mathematical solution to the above set is provided below
- $\boldsymbol{\overline{R_i} - R_F = (\lambda X_1 \sigma_{1i} + \lambda X_2 \sigma_{2i} + \cdots + \lambda X_i \sigma_i^2 + \cdots + \lambda X_{N-1} \sigma_{N-1i} + \lambda X_N \sigma_{Ni})}$
- There will be N such equations
- Here λ is a constant. We can substitute $\lambda X_k = Z_k$
- Where X_k's are the fractional amounts invested in the securities
- $\overline{R}_i - R_F = Z_1 \sigma_{1i} + Z_2 \sigma_{2i} + \cdots + Z_i \sigma_i^2 + \cdots + Z_{N-1} \sigma_{N-1i} + Z_N \sigma_{Ni}$

Short-sales allowed with riskless lending and borrowing

- $\bar{R}_i - R_F = Z_1\sigma_{1i} + Z_2\sigma_{2i} + \cdots + Z_i\sigma_i^2 + \cdots + Z_{N-1}\sigma_{N-1i} + Z_N\sigma_{Ni}$
- This equation is for one security 'i'. Similarly, one can formulate the equation for all the N securities, as follows
- $\overline{R_1} - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13} + \cdots + \cdots + Z_N\sigma_{1N}$
- $\overline{R_2} - R_F = Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23} + \cdots + \cdots + Z_N\sigma_{2N}$ and so on up to
- $\overline{R_N} - R_F = Z_1\sigma_{1N} + Z_1\sigma_{2N} + Z_3\sigma_{3N} + \cdots + \cdots + Z_N\sigma_N^2$

- Here, $X_k = \frac{Z_k}{\sum_{i=1}^{N} Z_i}$ is the optimum proportion to be invested in each security
- There are N equations and N unknowns (X_N's)

Example

Security	Expected Return	SD
R_A	14%	6%
R_B	8%	3%
R_C	20%	15%
R_f	5%	
Correlation		Covariance
ρ_{AB}	50%	σ_{12}=?
ρ_{AC}	20%	σ_{13}=?
ρ_{BC}	40%	σ_{23}=?

- **Compute $\boldsymbol{X_i}$s. Then $\boldsymbol{R_P}$, $\boldsymbol{\sigma_P}$, Slope.**
- The following equations will be employed
- $\overline{R_1} - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13}$
- $\overline{R_2} - R_F = Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23}$
- $\overline{R_3} - R_F = Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^3$
-

Example

Security	Expected Return	SD
R_A	14%	6%
R_B	8%	3%
R_C	20%	15%
R_f	5%	
Correlation		Covariance
ρ_{AB}	50%	σ_{12}=?
ρ_{AC}	20%	σ_{13}=?
ρ_{BC}	40%	σ_{23}=?

- **Compute X_is. Then R_P, σ_P, Slope.**
- The following equations will be employed
- $\overline{R_1} - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13}$
- $\overline{R_2} - R_F = Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23}$
- $\overline{R_3} - R_F = Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^3$
-

Example

- $\overline{R_1} - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13}$
- $\overline{R_2} - R_F = Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23}$
- $\overline{R_3} - R_F = Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^3$

- **14-5=** $\boldsymbol{Z_1}(\mathbf{6 * 6}) + \boldsymbol{Z_2}(\mathbf{0.5 * 6 * 3}) + \boldsymbol{Z_3}(\mathbf{0.2 * 6 * 15})$
- **8-5 =** $\boldsymbol{Z_1}(\mathbf{0.5 * 6 * 3}) + \boldsymbol{Z_2}(\mathbf{3 * 3}) + \boldsymbol{Z_3}(\mathbf{0.4 * 3 * 15})$
- **20-5=** $\boldsymbol{Z_1}(\mathbf{0.2 * 6 * 15}) + \boldsymbol{Z_2}(\mathbf{0.4 * 3 * 15}) + \boldsymbol{Z_3}(\mathbf{15 * 15})$

Example

- $\overline{R_1} - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13}$
- $\overline{R_2} - R_F = Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23}$
- $\overline{R_3} - R_F = Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^3$

- **14-5=** $\boldsymbol{Z_1(6*6) + Z_2(0.5*6*3) + Z_3(0.2*6*15)}$
- **8-5 =** $\boldsymbol{Z_1(0.5*6*3) + Z_2(3*3) + Z_3(0.4*3*15)}$
- **20-5=** $\boldsymbol{Z_1(0.2*6*15) + Z_2(0.4*3*15) + Z_3(15*15)}$
- $\boldsymbol{Z_1 = \frac{14}{63}; Z_2 = \frac{1}{63}; Z_3 = \frac{3}{63}}$**.** But, we are looking for X_i's.

Example

- $X_k = \frac{Z_k}{\sum_{i=1}^{N} Z_i}$

- $\bar{R}_P = X_1\bar{R}_1 + X_2\bar{R}_2 + X_3\bar{R}_3$
- $\sigma_P^2 = (X_1\sigma_1)^2 + (X_2\sigma_2)^2 + (X_3\sigma_3)^2 + 2 * \rho_{12}(X_1\sigma_1)(X_2\sigma_2) + 2 * \rho_{13}(X_1\sigma_1)(X_3\sigma_3) + 2 * \rho_{23}(X_2\sigma_2)(X_3\sigma_3)$

- **Slope=** $\boldsymbol{\frac{\bar{R}_P - R_f}{\sigma_P}}$

Solution

- $X_k = \frac{Z_k}{\sum_{i=1}^{N} Z_i}$; using this formula.

- $X_1 = \frac{Z_1}{\sum_{i=1}^{3} Z_i} = \frac{\frac{14}{63}}{(\frac{14}{63}+\frac{1}{63}+\frac{3}{63})}$=14/18

- X_2=1/18; and X_3=3/18

- $\bar{R}_P = X_1\bar{R}_1 + X_2\bar{R}_2 + X_3\bar{R}_3$

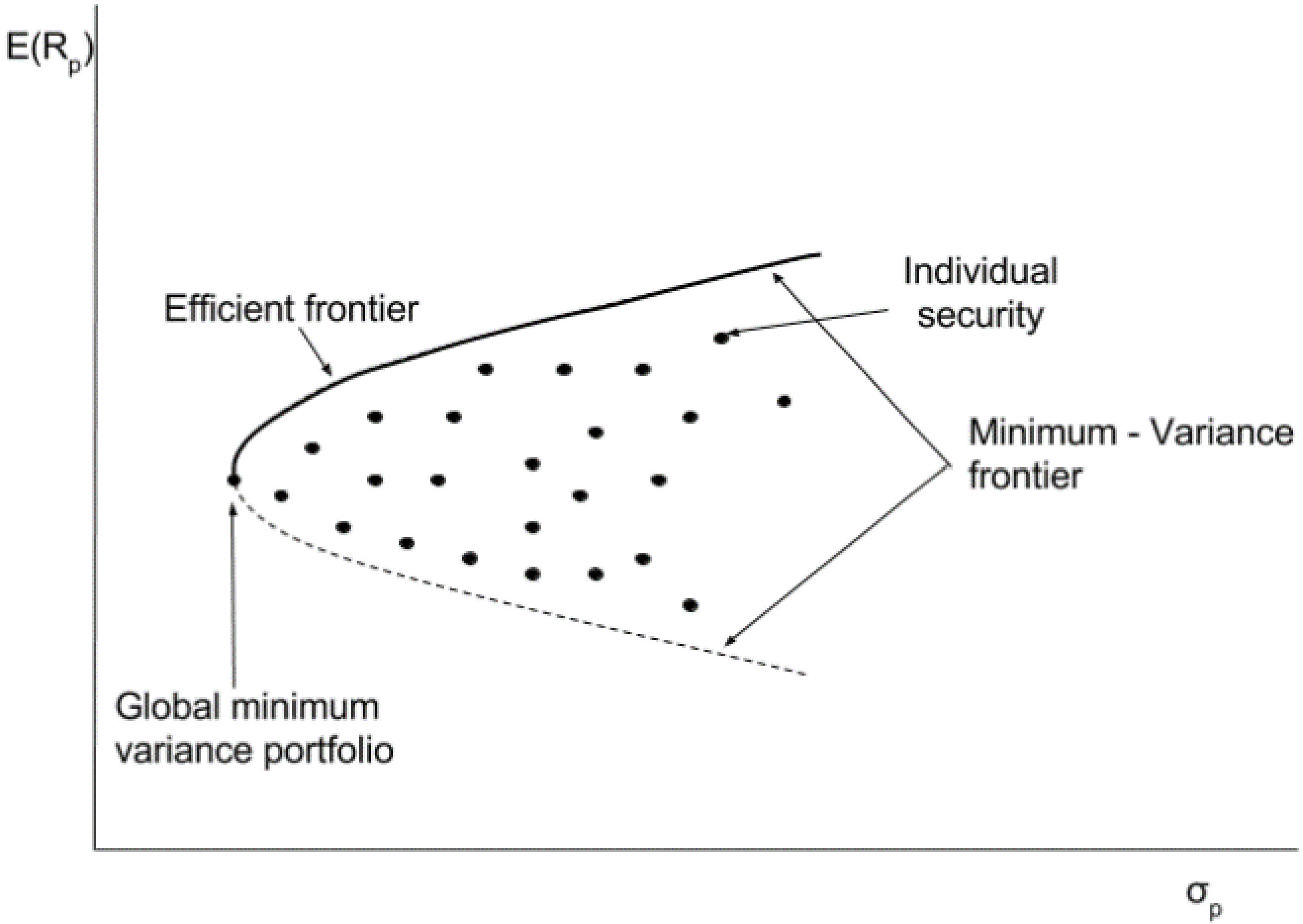
- $\sigma_P^2 = (X_1\sigma_1)^2 + (X_2\sigma_2)^2 + (X_3\sigma_3)^2 + 2 * \rho_{12}(X_1\sigma_1)(X_2\sigma_2) + 2 * \rho_{13}(X_1\sigma_1)(X_3\sigma_3) + 2 * \rho_{23}(X_2\sigma_2)(X_3\sigma_3)$

- **Slope=** $\boldsymbol{\frac{\bar{R}_P - R_f}{\sigma_P}}$

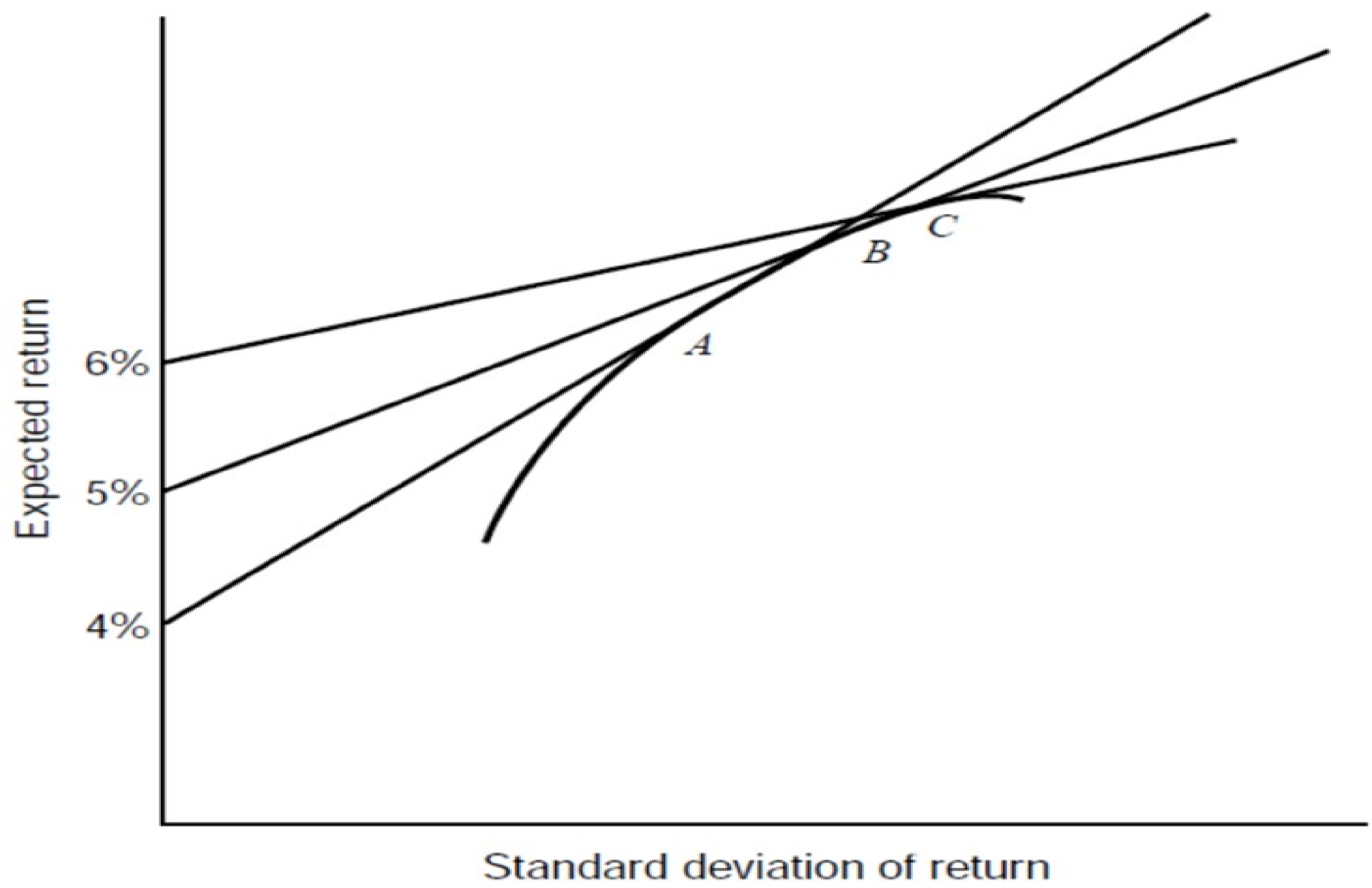
Solution

- $\bar{R}_P = X_1\bar{R}_1 + X_2\bar{R}_2 + X_3\bar{R}_3 = \frac{14}{18} * 14 + \frac{1}{18} * 8 + \frac{3}{18} * 20 = \frac{\mathbf{44}}{\mathbf{3}}\%$
- $\sigma_P^2 = (X_1\sigma_1)^2 + (X_2\sigma_2)^2 + (X_3\sigma_3)^2 + 2 * \rho_{12}(X_1\sigma_1)(X_2\sigma_2) + 2 * \rho_{13}(X_1\sigma_1)(X_3\sigma_3) + 2 * \rho_{23}(X_2\sigma_2)(X_3\sigma_3)$
- $\sigma_P^2 = \left(\frac{14}{18} * 6\right)^2 + \left(\frac{1}{18} * 3\right)^2 + \left(\frac{3}{18} * 15\right)^2 + 2 * (0.5)\left(\frac{14}{18} * 6\right)\left(\frac{1}{18} * 3\right) + 2 * (0.2)\left(\frac{14}{18} * 6\right)\left(\frac{3}{18} * 15\right) + 2 * (0.4)\left(\frac{1}{18} * 3\right)\left(\frac{3}{18} * 15\right)$
- $\sigma_P^2 = \frac{203}{6}$;
- **Slope=** $\frac{\bar{R}_P - R_f}{\sigma_P} = \frac{\left(\frac{44}{3} - 5\right)}{\left(\frac{203}{6}\right)^{\frac{1}{2}}}$ **= 1.66**

(2) Short-sales allowed: No riskless lending and borrowing



(2) Short-sales allowed: No riskless lending and borrowing



Tangency portfolios for different riskless rates.

(2) Short-sales allowed: No riskless lending and borrowing

- Here assuming different riskless rates (hypothetically) leads us to 2 portfolios that are tangent
- Thus, one can compute the efficient frontier by assuming a riskless rate and obtaining a tangent portfolio
- By varying the riskless rate, the entire efficient frontier can be traced on the feasible region

- $\overline{R_1} - R_F = Z_1\sigma_1^2 + Z_2\sigma_{12} + Z_3\sigma_{13}$
- $\overline{R_2} - R_F = Z_1\sigma_{12} + Z_2\sigma_2^2 + Z_3\sigma_{23}$
- $\overline{R_3} - R_F = Z_1\sigma_{13} + Z_2\sigma_{23} + Z_3\sigma_3^3$

Short-sales allowed: No riskless lending and borrowing

- Here assuming different riskless rates (hypothetically) leads us to 2 portfolios that are tangent
- Thus, one can compute the efficient frontier by assuming a riskless rate and obtaining a tangent portfolio
- By varying the riskless rate, the entire efficient frontier can be traced on the feasible region
- $Z_k = C_{0k} + C_{1k}R_F$; here C_{0k} and C_{1k} are constants
- These constants vary for each security but not with the value of R_F
- Thus, once the value of Z_k is determined as a function of R_F, we need only two points to identify these constants

Short-sales allowed: No riskless lending and borrowing

- Thus, once the value of Z_k is determined as a function of R_F
- **14-$\boldsymbol{R_F}$= $\boldsymbol{Z_1(6*6)+Z_2(0.5*6*3)+Z_3(0.2*6*15)}$**
- **8-$\boldsymbol{R_F}$ = $\boldsymbol{Z_1(0.5*6*3)+Z_2(3*3)+Z_3(0.4*3*15)}$**
- **20-$\boldsymbol{R_F}$= $\boldsymbol{Z_1(0.2*6*15)+Z_2(0.4*3*15)+Z_3(15*15)}$**

- **Simplifying this, we get the following equations.**

- 14-R_F= $Z_1(36)+Z_2(9)+Z_3(18)$

- 8-R_F = $Z_1(9)+Z_2(9)+Z_3(18)$

- 20-R_F= $Z_1(18)+Z_2(18)+Z_3(225)$

Short-sales allowed: No riskless lending and borrowing

- $Z_1 = \frac{42}{189}$
- $Z_2 = \frac{118}{189} - \frac{23}{189} R_F$
- $Z_3 = \frac{4}{189} + \frac{1}{189} R_F$
- We can change the values of Rf to obtain Zi's
- That is identify the entire efficient frontier, however, that is tedious
- We only need two values of Risk-free asset are needed to trace any two points on the efficient frontier
- Once, we have identified these two optimum portfolios on the efficient frontier, we can trace the entire efficient frontier
-

Short-sales allowed: No riskless lending and borrowing

- We can change the values of Rf to obtain Zk’s
- That is identify the entire efficient frontier, however, that is tedious
- But we do know that the relationship between R_F and Z_k is of the following form
- $Z_k = C_{0k} + C_{1k}R_F$; here C_{0k} and C_{1k} are constants
- I only need to identify two points on the efficient frontier to solve for these two constants
- And for that, we only need two values of Risk-free asset are needed to trace any two points on the efficient frontier
- Once, we have identified these two optimum portfolios on the efficient frontier, we can trace the entire efficient frontier

Short-sales allowed: No riskless lending and borrowing

- We only need two values of Risk-free asset are needed to trace any two points on the efficient frontier
- Once, we have identified these two optimum portfolios on the efficient frontier, we can trace the entire efficient frontier
- $R_F = 2$, the following values are obtained:
- 14-R_F= $Z_1(36) + Z_2(9) + Z_3(18)$
- 8-R_F = $Z_1(9) + Z_2(9) + Z_3(18)$
- 20-R_F= $Z_1(18) + Z_2(18) + Z_3(225)$
- $X_1 =?\,; X_2 =?\,; X_3 =?$

Short-sales allowed: No riskless lending and borrowing

- We only need two values of Risk-free asset are needed to trace any two points on the efficient frontier
- Once, we have identified these two optimum portfolios on the efficient frontier, we can trace the entire efficient frontier
- $R_F = 2$, the following values are obtained: $X_1 = \frac{7}{20}; X_2 = \frac{12}{20}; X_3 = \frac{1}{20}$
- $\bar{R}_P = ?$
- $\sigma_P^2 = ?$

Short-sales allowed: No riskless lending and borrowing

- We only need two values of Risk-free asset are needed to trace any two points on the efficient frontier
- Once, we have identified these two optimum portfolios on the efficient frontier, we can trace the entire efficient frontier
- $R_F = 2$, the following values are obtained: $X_1 = \frac{7}{20}; X_2 = \frac{12}{20}; X_3 = \frac{1}{20}$
- $\bar{R}_P = \frac{7}{20} * 14 + \frac{12}{20} * 8 + \frac{1}{20} * 20 = \frac{107}{10}$
- $\sigma_P^2 = \left(\frac{7}{20} * 6\right)^2 + \left(\frac{12}{20} * 3\right)^2 + \left(\frac{1}{20} * 15\right)^2 + 2\left(\frac{7}{20} * 6\right)\left(\frac{12}{20} * 3\right) * 0.5 +$
  $2\left(\frac{7}{20} * 6\right)\left(\frac{1}{20} * 15\right) * 0.2 + 2\left(\frac{12}{20} * 3\right)\left(\frac{1}{20} * 15\right) * 0.4 = \frac{5481}{400}$

Short-sales allowed: No riskless lending and borrowing

- Now that we have identified two portfolios (for Rf=2%, Rf=5%).

- P1= [$\bar{R}_{P1}$=$\frac{\mathbf{44}}{\mathbf{3}}$%; σ^2_{P1}=$\frac{203}{6}$]; X_1 =14/18 ;X_2=1/18; and X_3=3/18

- P2= [$\bar{R}_{P2}$=$\frac{107}{10}$%; σ^2_{P1}=$\frac{5481}{400}$]; $X_1 = \frac{7}{20}$; $X_2 = \frac{12}{20}$; $X_3 = \frac{1}{20}$

- What is the missing piece to identify the complete efficient frontier

Short-sales allowed: No riskless lending and borrowing

- Now that we have identified two portfolios (for Rf=2%, Rf=5%). **We only require covariance between these two portfolios to construct the efficient frontier.** Consider a portfolio that involves these two portfolios in equal proportion (50% investment).

- P1= [$\bar{R}_{P1}$=$\frac{44}{3}$%; σ_{P1}^{2}=$\frac{203}{6}$]; X_1 =14/18 ;X_2=1/18; and X_3=3/18

- P2= [$\bar{R}_{P2}$=$\frac{107}{10}$%; σ_{P1}^{2}=$\frac{5481}{400}$]; $X_1 = \frac{7}{20}; X_2 = \frac{12}{20}; X_3 = \frac{1}{20}$

- Compute the new weights: $X_1 =?; X_2 =?; X_3 =?$

Short-sales allowed: No riskless lending and borrowing

- P1= [$\bar{R}_{P1}$=$\frac{\mathbf{44}}{\mathbf{3}}$ %; σ_{P1}^{2}=$\frac{203}{6}$]; X_1 =14/18 ;X_2=1/18; and X_3=3/18

- P2= [$\bar{R}_{P2}$=$\frac{107}{10}$%; σ_{P1}^{2}=$\frac{5481}{400}$]; $X_1 = \frac{7}{20}$; $X_2 = \frac{12}{20}$; $X_3 = \frac{1}{20}$

- Compute the new weights: $X_1 = ?$; $X_2 = ?$; $X_3 = ?$

- What else do we need, and how to compute that

Short-sales allowed: No riskless lending and borrowing

- **Hint: Covariance**
- Compute the variance of this new portfolio using the weights from the three securities
- $\sigma_P^2 =?$

Security	Expected Return	SD
R_A	14%	6%
R_B	8%	3%
R_C	20%	15%
R_f	5%	
Correlation		Covariance
ρ_{AB}	50%	σ_{12}=?
ρ_{AC}	20%	σ_{13}=?
ρ_{BC}	40%	σ_{23}=?

Short-sales allowed: No riskless lending and borrowing

- $\sigma_P^2 = \left(\frac{203}{360} * 6\right)^2 + \left(\frac{118}{360} * 3\right)^2 + \left(\frac{39}{360} * 15\right)^2 + 2\left(\frac{203}{360} * 6\right)\left(\frac{118}{360} * 3\right) * 0.5 +$

  $2\left(\frac{203}{360} * 6\right)\left(\frac{39}{360} * 15\right) * 0.2 + 2\left(\frac{118}{360} * 3\right)\left(\frac{39}{360} * 15\right) * 0.4 = 21.859$

- But we also know that this portfolio is a combination of the two other portfolios already identified on the efficient frontier. The variance of the combination of these two portfolios can be written as:
- $\sigma_P^2 = (X_1\sigma_1)^2 + (X_2\sigma_2)^2 + 2\,(X_1\sigma_1)(X_2\sigma_2)(\rho_{12})$ =?

Short-sales allowed: No riskless lending and borrowing

- But we also know that this portfolio is a combination of the two other portfolios already identified on the efficient frontier. The variance of the combination of these two portfolios can be written as:
- $\sigma_P^2 = (X_1\sigma_1)^2 + (X_2\sigma_2)^2 + 2\,(X_1\sigma_1)(X_2\sigma_2)(\rho_{12})$
- $\sigma_P^2 = \left(\frac{1}{2}\right)^2 * \left(\frac{203}{6}\right) + \left(\frac{1}{2}\right)^2 * \left(\frac{5481}{400}\right) + 2\,\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(\sigma_{12})$
- But we know that $\sigma_P^2 = 21.859$; therefore, $\sigma_{12} = 19.95$.
- Now that we know the variances and covariance of the two portfolios on the efficient frontier, we can traceback the entire efficient frontier.

(3) Riskless lending and borrowing with short-sales not allowed

- With riskless lending and borrowing, one portfolio is optimal
- This portfolio can be identified by maximizing the slope of the line connecting the riskless asset and the efficient frontier
- **Objective function**:

- **Constraint 1**:
- **Constraint 2**:

(3) Riskless lending and borrowing with short-sales not allowed

- With riskless lending and borrowing, one portfolio is optimal
- This portfolio can be identified by maximizing the slope of the line connecting the riskless asset and the efficient frontier
- **Objective function**: Maximize slope of the portfolio that lies on the efficient frontier : $\theta = \frac{\bar{R}_P - R_f}{\sigma_P}$
- **Constraint 1**: $\sum_{i=1}^{N} X_i = 1$
- **Constraint 2**: ?
- Since, the objective function is non-linear (involves variance terms), it is called quadratic programming problem. This can be solved using computer programs

(3) Riskless lending and borrowing with short-sales not allowed

- With riskless lending and borrowing, one portfolio is optimal
- This portfolio can be identified by maximizing the slope of the line connecting the riskless asset and the efficient frontier
- **Objective function**: Maximize slope of the portfolio that lies on the efficient frontier : $\theta = \frac{\bar{R}_P - R_f}{\sigma_P}$
- **Constraint 1**: $\sum_{i=1}^{N} X_i = 1$
- **Constraint 2**: $X_i > 0 \; for \; all \; i's$
- Since, the objective function is non-linear (involves variance terms), it is called quadratic programming problem. This can be solved using computer programs

(4) Both Riskless lending and borrowing and short-sales not allowed

- We obtained the efficient frontier by minimizing the risk level for the given level of returns
- That is, once we specify the level of return, and then minimize the risk, then we get one point of the efficient frontier
- Subject to **Constraint 1**:
- Minimize the **object function**:
- Subject to **Constraint 2**:
- Subject to **Constraint 3**:
- .

(4) Both Riskless lending and borrowing and short-sales not allowed

- We obtained the efficient frontier by minimizing the risk level for the given level of returns

- That is, once we specify the level of return, and then minimize the risk, then we get one point of the efficient frontier

- Subject to **Constraint 1**: $\sum_{i=1}^{N}(X_i\bar{R}_i) = \bar{R}_P$; fix the return
- Minimize the **object function**: $\sigma_P^2 = \left[\sum_{i=1}^{N} X_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N} X_iX_j\sigma_{ij}\right]$
- Subject to **Constraint 2**: $\sum_{i=1}^{N} X_i = 1$
- Subject to **Constraint 3**: $X_i > 0 \; for \; all \; i's$
- One can vary $\bar{R}_P$ between the return on minimum variance portfolio and that on the maximum return portfolio traces out the efficient-set. This is also a quadratic programming problem.

Thanks

Single and Multi Index Models

Course: Portfolio Management

Instructor: Abhinava Tripathi

Single index models and correlation structure

- These are the equations corresponding to portfolio returns and standard deviation
- $\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$ **(1)**
- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{j=1}^{N} \sum_{k=1}^{N} (X_j X_k \sigma_{jk})$, **where i≠j** **(2)**
- In order to draw efficient frontier, three key inputs were required
  - Expected returns from each security
  - Standard deviations from each security
  - Correlations between each possible pair of security

Single index models and correlation structure

- An analyst follows 150 stocks so how many estimates she requires
- 150 estimates of expected returns, 150 estimates of standard deviation but, in addition, she also needs 150*149/2=11,175 estimates of covariance (or correlations)
- What if one factor or index affected all these 150 securities
- That means, the observed covariances essentially reflected the correlation structure between that index and these securities
- This leads to the genesis of single index models

Single index models and correlation structure

- The model assumes a single common influence that affects a large number of securities in a similar manner
- This is a more data driven model
- Researchers in the early days realized that market movements affect a large number of stocks in a similar manner
- Indices like Nifty affect the returns on a large number of securities
- $R_i = a_i + \beta_i R_m + e_i$ (3)

Single index models and correlation structure

- $R_i = a_i + \beta_i R_m + e_i$
- Both R_m and e_i are both random variables
- Random variables are defined by a probability distribution with a mean and standard deviation
- Mean of R_m and e_i are $\bar{R}_m$ and 0
- Standard deviation of R_m and e_i as σ_m and σ_{ei}
- Here, by definition and to some extent by construction R_m and e_i are uncorrelated
- $\boldsymbol{Cov(e_i, R_m) = E[(e_i - 0)(R_m - \bar{R}_m)] = 0}$
- The model is generally estimated using regression analysis

Single index models and correlation structure

- Single index model also assumes that e_i is independent of all the e_j's
- **More formally E($e_i e_j$)=0**
- This means that the only reason two stocks commove is because of market. No other effects (industry etc.)
- This is not ensured by the regression analysis
- Thus, the performance of the model depends how good this assumption is
- $R_i = a_i + \beta_i R_m + e_i$ under the assumption of single index model is assumed to represent the return dynamics for all the stocks, i=1,2,3,…N.

Single index models and correlation structure

- $\boldsymbol{R_i = \ a_i + \beta_i(R_m) + e_i}$:

- Under the assumption of single index model, this equation is assumed to represent the return dynamics for all the stocks, i=1,2,3,…N

  – **By the design** (or construction) of the regression model. Mean of e_i, i.e., E(e_i)=0.

  – **By assumption and construction** Index (market) is unrelated to the idiosyncratic specific component (e_i), that is $E[e_i(R_m - R_m)] = 0$

  – **By assumption s**ecurities are only related to each other through the index (market). That is $E\left[\left(e_i e_j\right)\right] = 0$

  – **By definition** Variance of $e_i = E(e_i)^2 = \sigma_{ei}^2$

  – **By definition** Variance of $R_m = E(R_m - \bar{R}_m)^2 = \sigma_m^2$

Single index models and correlation structure

- Now that we have boundary conditions, we can derive the expressions for expected return, standard deviation, and covariance
- **Expected returns**
- $E(R_i) = E[a_i + \beta_i R_m + e_i]$
- $E(R_i) = E(a_i) + E(\beta_i R_m) + E(e_i)$,
- $E(e_i)$=0, and that a_i and β_i are constants
- $\boldsymbol{E(R_i) = \bar{R}_i = a_i + \beta_i \bar{R}_m}$

Single index models and correlation structure

- **Standard Deviation ($\boldsymbol{\sigma_i^2}$)**
- $\sigma_i^2 = E(R_i - \bar{R}_i)^2$
- $\sigma_i^2 = E[\big((a_i + \beta_i R_m + e_i) - (a_i + \beta_i \bar{R}_m)\big)]^2$
- $\sigma_i^2 = E[\beta_i (R_m - \bar{R}_m) + e_i]^2$
- $\sigma_i^2 = \beta_i^2 E([(R_m - \bar{R}_m)]^2 + E(e_i)^2 + 2\beta_i E[e_i(R_m - \bar{R}_m)]$
- $\sigma_i^2 = \beta_i^2 E([(R_m - \bar{R}_m)]^2 + E(e_i)^2; because\ E[e_i(R_m - \bar{R}_m)] = 0$
- $\boldsymbol{\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2}$

Single index models and correlation structure

- **Covariance** (σ_{ij})
- $\sigma_{ij} = E\left[(R_i - \bar{R}_i)\left(R_j - \bar{R}_j\right)\right]$, substituting expressions for $\bar{R}_i$ and $\bar{R}_j$ as done earlier
- $\sigma_{ij} = E[\left((a_i + \beta_i R_m + e_i) - (a_i + \beta_i \bar{R}_m)\right)\left(\left(a_j + \beta_j R_m + e_j\right) - \left(a_j + \beta_j \bar{R}_m\right)\right)]$
- $\sigma_{ij} = E[(\beta_i(R_m - \bar{R}_m) + e_i)(\beta_j(R_m - \bar{R}_m) + e_j)]$
- $\sigma_{ij} = \beta_i\beta_j E[(R_m - \bar{R}_m)^2] + \beta_j E[e_i(R_m - \bar{R}_m)] + \beta_i E\left[e_j(R_m - \bar{R}_m)\right] + E\left(e_i e_j\right)$
- $\boldsymbol{\sigma_{ij} = \beta_i\beta_j E\left[(R_m - \bar{R}_m)^2\right] = \beta_i\beta_j\sigma_m^2}$

Example

Period	R_{it}	R_{mt}	$(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})$	Value
1	10	4	??	??
2	3	2		?
3	15	8		?
4	9	6		?
5	3	0		?
Average	?	A	Total	B
Variance	?	σ_m^2=?	Covariance (σ_{im})	=B/5

- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$?

Example

Period	R_{it}	R_{mt}	$(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})$	Value
1	10	4	(10-8)*(4-4)	0
2	3	2		?
3	15	8		?
4	9	6		?
5	3	0		?
Average	?	A	Total	B
Variance	?	σ_m^2=?	Covariance (σ_{im})	=B/5

- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$?

Example

Period	R_{it}	R_{mt}	$(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})$	Value
1	10	4	(10-8)*(4-4)	0
2	3	2		10
3	15	8		28
4	9	6		2
5	3	0		20
Average			Total	?
Variance		σ_m^2=?	Covariance (σ_{im})	=12/5

- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}?$

Example

Period	R_{it}	R_{mt}	$(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})$	Value
1	10	4	(10-8)*(4-4)	0
2	3	2	(3-8)*(2-4)	10
3	15	8	(15-8)*(8-4)	28
4	9	6	(9-8)*(6-4)	2
5	3	0	(3-8)*(0-4)	20
Average	8	4	Total	60
Variance	20.8	σ_m^2=8	Covariance (σ_{im})	=60/5=12

- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$=1.5,

Example

Period	Ri ($\beta_i = 1.5$)	Rm	$e_i = R_i - a_i - \beta_i R_m$
1	**10**	**4**	?
2	**3**	**2**	?
3	**15**	**8**	?
4	**9**	**6**	?
5	**3**	**0**	?
Average	?	?	
Variance	σ_i^2=?	σ_m^2=?	σ_{ei}^2=?

- $\bar{R}_i = a_i + \beta_i \bar{R}_m$ **to get a_i=?**
- $R_i = a_i + \beta_i R_m + e_i$ to get e_i
- $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$ =? also

Example

Period	R_{it}	R_{mt}	$(R_{it}-\bar{R}_{it})(R_{mt}-\bar{R}_{mt})$	Value
1	10	4	(10-8)*(4-4)	0
2	3	2	(3-8)*(2-4)	10
3	15	8	(15-8)*(8-4)	28
4	9	6	(9-8)*(6-4)	2
5	3	0	(3-8)*(0-4)	20
Average	8	4	Total	60
Variance	20.8	σ_m^2=8	Covariance (σ_{im})	=60/5=12

- $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$=1.5, now $a_i = 8 - 1.5 * 4 = 2$

Example

Period	Ri ($\beta_i = 1.5$)	Rm	$e_i = R_i - a_i - \beta_i R_m$
1	10	4	2
2	3	2	-2
3	15	8	1
4	9	6	-2
5	3	0	1
Average	8	4	
Variance	20.8	σ_m^2=8	σ_{ei}^2=2.8

- $\bar{R}_i = a_i + \beta_i \bar{R}_m$ **to get a_i=2**
- $R_i = a_i + \beta_i R_m + e_i$ to get e_i
- $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 = 20.8$ also

Example

Period	Ri and $\beta_i = 1.5$	Rm	$e_i = R_i - a_i - \beta_i R_m$
1	10	4	10-2-1.5*4=2
2	3	2	3-2-1.5*2=-2
3	15	8	15-2-1.5*8=1
4	9	6	9-2-1.5*6=-2
5	3	0	3-2-1.5*0-3=1
Average	8	4	0
Variance	20.8	8	2.8

- : $\boldsymbol{\bar{R}_i = a_i + \beta_i \bar{R}_m}$, we can estimate $\boldsymbol{a_i}$. 8= a_i+1.5*4, i.e.,a_i=2. Now that we have a_i, we can estimate the values of e_i for each period.

- Here, one can confirm that $E(e_i) = 0$. Also, note that $\sigma_{ei}^2 = 14$. Using the $\boldsymbol{\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2}$, we get the value σ_i^2= 1.5*1.5*8+2.8=20.8. This variance is same as that directly calculated from the table.

Single index models with portfolios

- With the assumption that single-index model holds, let us examine its impact on portfolio returns and standard deviation

Expected return

- $\bar{R}_p = \sum_{i=1}^{N} X_i \bar{R}_i$; substitute the single index model $\bar{R}_i = a_i + \beta_i \bar{R}_m$
-
- $\bar{R}_p = \sum_{i=1}^{N} X_i a_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$
- $\beta_p = \sum_{i=1}^{N} X_i \beta_i$; $a_p = \sum_{i=1}^{N} X_i a_i$
- $\bar{R}_p = a_p + \beta_p \bar{R}_m$

Single index models with portfolios

Standard deviation

- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \sigma_{ij}$; substituting the expression for variance and covariance

- $\boldsymbol{\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2}$

- $\boldsymbol{\sigma_{ij} = \beta_i \beta_j \sigma_m^2}$

- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$

Single index models with portfolios

Expected return

- $\bar{R}_p = \sum_{i=1}^{N} X_i a_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$
- $\bar{R}_p = a_p + \beta_p \bar{R}_m$

Standard deviation

- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$

- Assume we have a portfolio of 150 stocks, How many estimates we require??

Single index models with portfolios

Expected return

- $\bar{R}_p = a_p + \beta_p \bar{R}_m$

Standard deviation

- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$

- Assume we have a portfolio of 150 stocks, we require estimates of
  - (1) a_i, β_i, and σ_{ei} for each of the stock
  - (2) $\bar{R}_m$ and σ_m^2 for the market
- That is 150*3+2=452 estimates are needed (as compared to 11,485 estimates in the absence of single index model)

Characteristics of single index model

- **Portfolio expected return**
- $\bar{R}_p = \sum_{i=1}^{N} X_i a_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$
- $\beta_p = \sum_{i=1}^{N} X_i \beta_i$; $a_p = \sum_{i=1}^{N} X_i a_i$
- $\bar{R}_p = a_p + \beta_p \bar{R}_m$
- Please note that if the portfolio under consideration is the market portfolio then a_p=0 and β_p=1. And then $\bar{R}_p$=$\bar{R}_m$. Thus portfolios with β_p>1 are said to be more risky than the market and portfolios with β_p<1 are said to be less risky than the market

Characteristics of single index model

- **Portfolio standard deviation**
- $\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$
- $\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$
- $\sigma_p^2 = (\sum_{i=1}^{N} X_i \beta_i)(\sum_{j=1}^{N} X_j \beta_j) \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2$
- $But\ (\sum_{j=1}^{N} X_i \beta_i) = (\sum_{j=1}^{N} X_j \beta_j) = \beta_p$
- $\boldsymbol{\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2}$
- Consider equal investments in the securities so that $X_1 = X_2 = \cdots .. =$

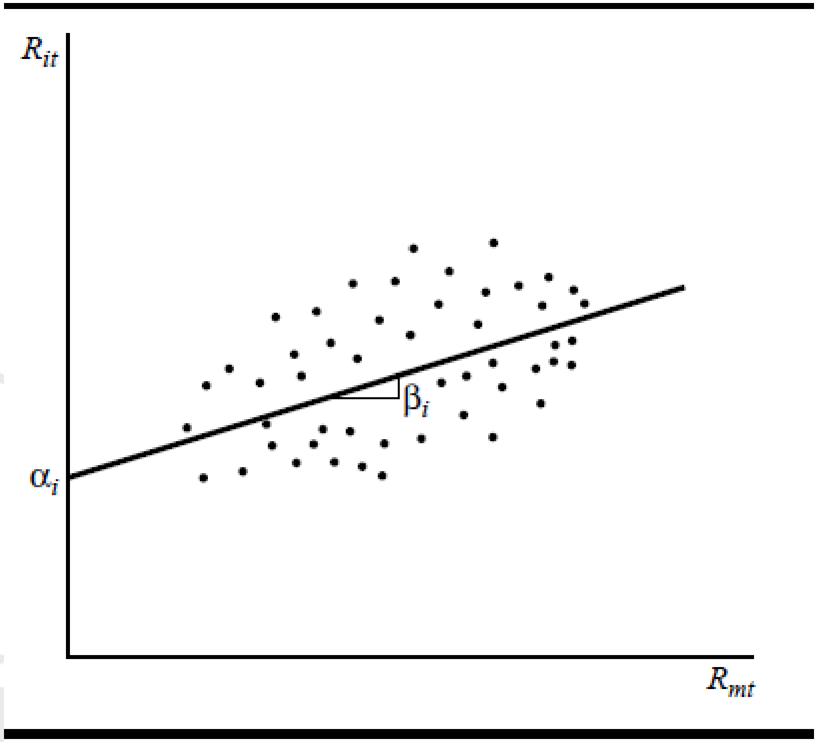
$X_N = \frac{1}{N}$

Characteristics of single index model

- $\boldsymbol{\sigma_p^2 = \beta_p^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\, \sigma_{ei}^2}$
- Consider equal investments in the securities so that $X_1 = X_2 = \cdots .. = X_N = \frac{1}{N}$
- If there are large number of securities then the term $\frac{\sigma_{ei}^2}{N}$, which represents the residual (or specific risk) approaches to zero
- $\boldsymbol{\sigma_p^2 = \beta_p^2\sigma_m^2}$ (For fairly diversified portfolios)

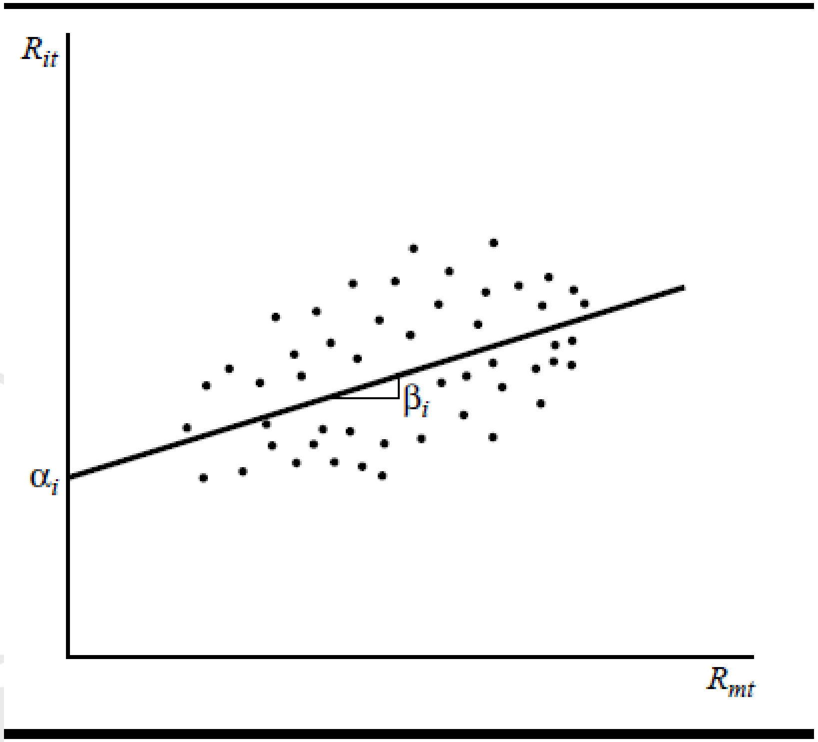
Estimation of Betas

- We had discussion about estimation of betas!
- $R_{it} = \ a_i + \beta_i R_{mt} + e_{it}$
- Here, we are fitting a line across the scatter points of R_i and R_m observations, available over time
- Slope of this line is the best estimate of the beta over the period of examination



Estimation of Betas

- $R_i = a_i + \beta_i R_m + e_i$
- Here, we are fitting a line across the scatter points of R_i and R_m observations, available over time
- Slope of this line is the best estimate of the beta over the period of examination
- $\boldsymbol{\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\sum_{t=1}^{T}[(R_{it}-\bar{R}_{it})(R_{mt}-\bar{R}_{mt})]}{\sum_{t=1}^{T}(R_{mt}-\bar{R}_{mt})^2}}$



Estimation of Betas

- But beta estimates are also subject to estimation errors
- Also, firm betas change overtime (changes in capital structure, industry, etc.)
- Therefore, analysts estimate betas of industry portfolios
- These are less noisy and more reliable estimates
- The random variation in one security (upwards) and the other security (downwards) tend to cancel out each other

Fundamental estimate of beta

- Beta is a risk measure that is estimated from the relationship between the return of a security and that of the market
- Some of the well-known fundamental variables that affect the risk of a stock are dividend payout, asset growth, leverage, liquidity, asset size, earnings variability
- **Firm beta and dividends**
- **Firm beta and growth**
- **Firm beta and liquidity**
- **Size and beta**
- **Earnings variability and beta**

Fundamental estimate of beta

- **Firm beta and dividends:** Firms that pay more dividends have positive future expectations and are considered to be less risky: **low beta**
- **Firm beta and growth:** High-growth firms are generally young firms with high capital requirements, and are considered to be more risk**: high beta**
- **Firm beta and liquidity:** Firms with high-liquidity are considered to be less risky: **Low beta**
- **Size and beta:** Large firms are considered to be less risky than smaller firms: **Large firms low beta**
- **Earnings variability and beta:** A firm with high earnings variability (earnings beta) is considered as riskier: **positive beta**

Fundamental estimate of beta

- The next step is to develop the estimate of fundamental beta using the following model
- $\beta = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_N X_N + e_i$
- Here, X_i is one of the hypothesized variable affecting beta
- **Fundamental vs Historical**
  - Betas based on historical data measure the response of each stock to market movements; however, this measure reflects beta only with a certain lag
  - Conversely, fundamental betas respond quickly to a change in the companies' characteristics because they are computed directly from these characteristics
  - However, estimates of fundamental betas assume that responsiveness to a particular fundamental variable is the same for all firms

Market model

- $\bar{R}_i = a_i + \beta_i \bar{R}_m$ index model, when the assumption of $Cov(e_i e_j) = 0$ is waived, then it becomes the market model
- This allows for comovement across securities because of factors other than the market
- This means, it is a less restrictive form of index model family
- It suggests that, in addition to systematic marketwide factors, other systematic influences can also affect the individual securities.
- What can be these factors?

Introduction to Multi-index models

- An improvement over single-models is multi-index model
- These models aim to capture the non-market influences that may cause securities to move together
- These multi-index models aim to capture the economic factors or structural groups (e.g., industrial effects)
- The generalized multi-index models can be written in following form
- $R_i = a_i^* + b_{i1}^* I_1^* + b_{i2}^* I_2^* + b_{i3}^* I_3^* + \cdots + b_{iL}^* I_L^* + c_i$
- Interpretation of a_i^*, b_{i1}^*, c_i

Introduction to Multi-index models

- The indices (I_j^*'s) would capture the systematic marketwide influence of market returns, level of interest rate, and various industry effects, etc.
- However, this model faces one major challenge
- Some of the indices employed in the model may be correlated
- This vitiates the estimation, as the regression estimations of this kind require the independent variables to be uncorrelated
- When the variables are correlated it is difficult to segregate their respective effects (b_{ij}^*'s) on the security

Introduction to Multi-index models

- Researchers often perform a procedure called orthogonalization to remove the correlated portion from the respective indices and create orthogonalized indices
- The new transformed equation is provided below
- $R_i = a_i + b_{i1}\, I_1 + b_{i2}\, I_2 + b_{i3}\, I_3 + \cdots + b_{iL}\, I_L + c_i$
- The new indices are so constructed as they have no correlation
- Also, that the error term (c_i) is not correlated with indices, i.e., $E[c_i(I_j - \bar{I_j})]$=0
- However, the economic interpretation of new indices is slightly difficult

Introduction to Multi-index models: Basic equation

- $R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + b_{i3} I_3 + \cdots + b_{iL} I_L + c_i$ **; for all stocks i= 1, 2, 3 ..N and indices j=1,2,3,….L**
- **By definition**
  - Residual variance of stock *i* equals σ_{ci}^2
  - Variance of index Ij equals σ_{Ij}^2
- **By Construction**
  - Mean of c_i equals E(c_i)=0
  - Covariance between indexes j and k equals $E[(I_j - \bar{I}_j)\,(I_k - \bar{I}_k)]$=0
  - Covariance between residuals for stock i and index j equals $E[c_i(I_j - \bar{I}_j)]$=0
- **By assumption**
  - Covariance between c_i and c_j is zero, i.e., $E[c_i c_j]$=0

Introduction to Multi-index models: Basic equation

- **By assumption**
  - **Covariance between c_i and c_j is zero, i.e., $E[c_i c_j]$=0**
- This last assumption suggests that the only reason stocks vary together is because of their common relationship with the indexes specified in the model
- There is no other reason that two stocks (i,j) should have a correlation
- However, there is nothing in the model estimation that forces this to be true
- This is only an approximation, and the performance of the model will be as good as the approximation

Introduction to Multi-index models: Expected return and risk

- **Expected return**
- $\bar{R}_i = a_i + b_{i1}\bar{I}_1 + b_{i2}\bar{I}_2 + \cdots + b_{iL}\bar{I}_L$
- **Variance of return**
- $\sigma_i^2 = b_{i1}^2\sigma_{I1}^2 + b_{i2}^2\sigma_{I2}^2 + \cdots + b_{iL}^2\sigma_{IL}^2 + \sigma_{ci}^2$
- **Covariance between security i and j**
- $\sigma_{ij} = b_{i1}b_{j1}\sigma_{I1}^2 + b_{i2}b_{j2}\sigma_{I2}^2 + \cdots + b_{iL}b_{jL}\sigma_{IL}^2$

Introduction to Multi-index models

- To estimate the expected return and risk, the following estimates are required
  - a_i and σ_{ci}^2 for each stock
  - b_{ik} between each stock and index
  - an estimate of index mean ($\bar{I}_{\mathrm{j}}$) and variance σ_{Ij}^2 of each index
- Assuming N securities and L indices, this is a total 2N+LN+2L estimates
- An analyst following 150 stocks having 10 indices, this means 1820 inputs
- This structure, though more complex than single-index models, is still less complex when no simplifying correlation-structure is assumed

Industry Index models

- These models argue that correlation across securities are driven by industry effects in addition to market, as specified below
- $R_i = a_i + b_{iM}\, I_M + b_{i1}\, I_1 + b_{i2}\, I_2 + b_{i3}\, I_3 + \cdots + b_{iL}\, I_L + c_i$
- Here I_M is the market index and I_j's are industry effects that are uncorrelated with market and each other as well
- A more simplified version argues that one firm is affected by only one industry; thus for firm 'i' in industry 'j', the equation can be written as
- $R_i = a_i + b_{iM}\, I_M + b_{ij}\, I_j + c_i$
- The covariance between any two securities 'i' and 'k' that are in the same industry: $b_{im}b_{jm}\sigma_{Im}^2 + b_{ij}b_{kj}\sigma_{Ij}^2$ and from the different industry: $b_{im}b_{jm}\sigma_{Im}^2$

Industry Index models

- These models argue that correlation across securities are driven by industry effects in addition to market, as specified below
- $R_i = a_i + b_{iM}\, I_M + b_{ij}\, I_j + c_i$
- Here I_M is the market index and I_j's are industry effects that are uncorrelated with market and each other as well
  - Now we need 4N inputs for the a_i, b_{iM}, b_{ij}, and σ_{ci}^2
  - 2L estimate of index mean ($\bar{I}_j$) and variance σ_{Ij}^2 of each index
  - 2 estimates for mean and variance of market
- A total of 4N+2L+2 estimates

Introduction to Multi-index models

- Researchers often derive indices from the available data using quantitative techniques (e.g., Principal Component Analysis, Factor Analysis, etc.)
- One can add more indices to increase the explanatory power of the model
- However, with more indices model becomes less efficient and more complex
- Therefore, it is a sort of trade-off between complexity, efficiency, and explanatory power of the model
- Other method is **average correlation models** using historical correlation matrix to forecast future correlations

Multi-index models: 3 Factor Fama-French Model

- $\bar{R}_i = a_i + b_{iM}(\bar{R}_M - \bar{R}_f) + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML}$
- **Or**
- $\bar{R}_i - \bar{R}_f = a_i^* + b_{iM}(\bar{R}_M - \bar{R}_f) + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML}$
- $(\bar{R}_M - \bar{R}_f)\boldsymbol{Market}$: is the market index indicating the excess returns over risk-free rate
- $\bar{R}_{SMB}$ **(Small minus big):** indicates the excess return on a portfolio of small stocks over large stocks. The excess returns by small stocks captures the fact that they are riskier than large stocks
- $\bar{R}_{HML}$ **(High minus low):** indicates the excess return on a portfolio of high book-to-market (BTM) stocks (value stocks) over that of low (BTM) stocks (growth stocks) stocks

Multi-index models: 4 Factor Carhart Model

- $\bar{R}_i = a_i + b_{iM}(\bar{R}_M - \bar{R}_f) + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML} + b_{iMOM}\bar{R}_{MOM}$
- **Or**
- $\bar{R}_i - \bar{R}_f = a_i^* + b_{iM}(\bar{R}_M - \bar{R}_f) + b_{iSMB}\bar{R}_{SMB} + b_{iHML}\bar{R}_{HML} + b_{iMOM}\bar{R}_{MOM}$
- $\bar{R}_{MOM}$ **(Winners minus Losers):** indicates the excess return on a portfolio of winner stocks over that of loser stocks.

- Momentum (MOM) factor shows the difference in return between a portfolio of past 12 months winners and a portfolio of past 12month losers.
- The momentum factor is based on the idea that stocks that have performed well in the past are more likely to continue to perform well in the future.

Multi-index models: 5 Factor Fama-French Model

- $\overline{R}_i = a_i + b_{iM}(\overline{R}_M - \overline{R}_f) + b_{iSMB}\overline{R}_{SMB} + b_{iHML}\overline{R}_{HML} + b_{iRMW}\overline{R}_{RMW} + b_{iCMA}\overline{R}_{CMA}$
- **Or**
- $\overline{R}_i - \overline{R}_f = a_i^* + b_{iM}(\overline{R}_M - \overline{R}_f) + b_{iSMB}\overline{R}_{SMB} + b_{iHML}\overline{R}_{HML} + b_{iRMW}\overline{R}_{RMW} + b_{iCMA}\overline{R}_{CMA}$
- $\overline{R}_{RMW}$ **(Robust minus Weak):** indicates the difference between the returns for a diversified portfolio of robust and weak profitability assets.

- $\overline{R}_{CMA}$ **(Conservative minus Aggressive):** indicates the difference between conservative investment portfolios (stocks with a low investment) and aggressive investment portfolios (stocks with a high investment).

Value vs. Growth investing

- We often hear investment management firms define themselves as value vs. growth firms
- For example, growth firms focus on the earnings (EPS) part of the P/E ratio. He expects the earnings to grow which will lead prices to rise (constant P/E) ratio
- Growth stocks are not necessarily cheap based on the current earnings' levels, in fact they may be costly
- But the investor believes that the earnings will rise significantly and lead to price rise in near future

Value vs. Growth investing

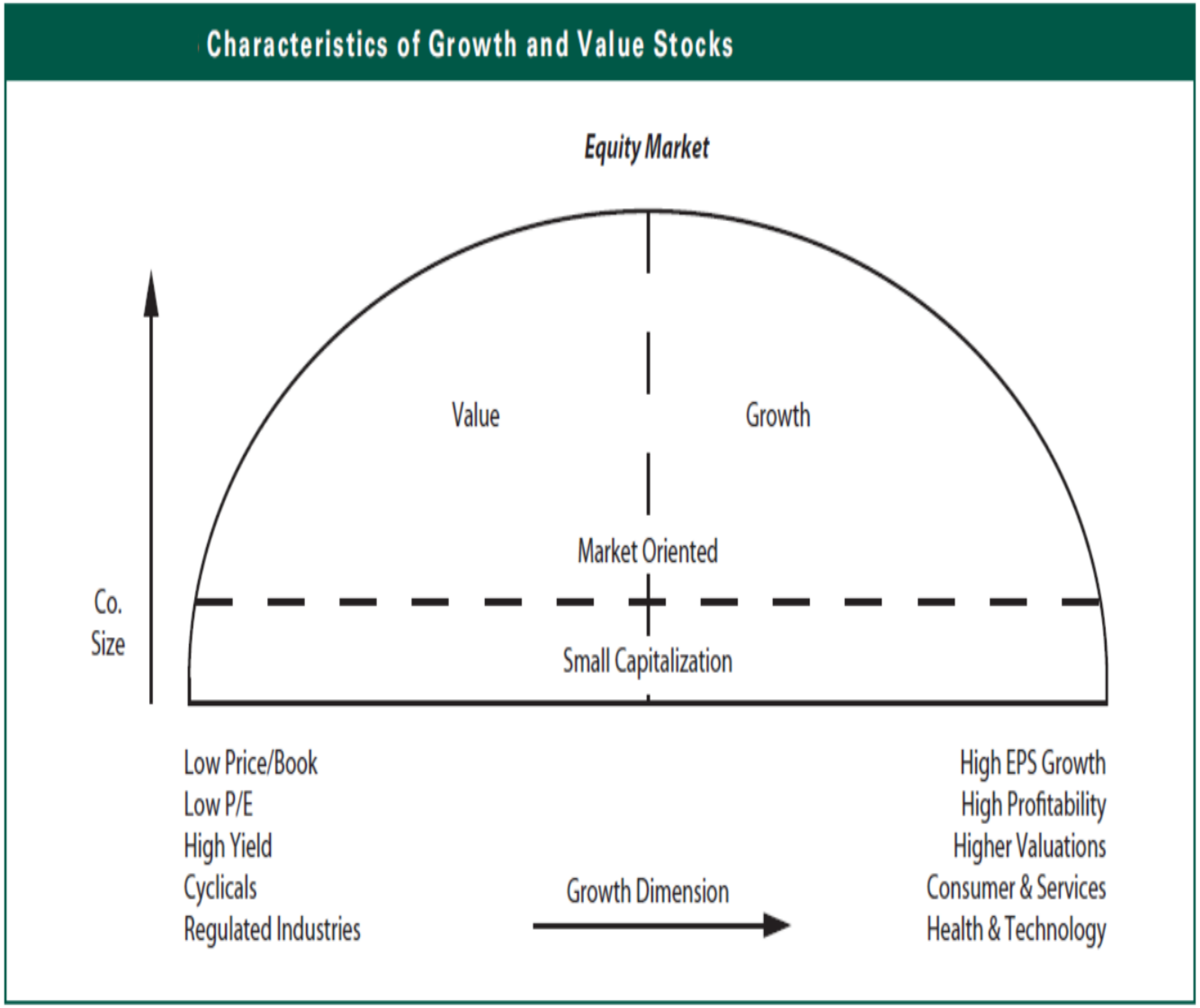
- In contrast, the **value investor** defines the price (P) component of P/E ratio. The value investor believes that the given the current level of earnings, prices are low (cheap) as compared to the other stocks in the same industry with similar profile
- P/E level is below the level based on some comparison, and the fact that market will correct itself in the near term
- And the prices will rise, thus Value stocks are cheap given their current earning levels

Value vs. Growth investing

- To summarize, growth investor focuses on the current and future economic “story” of the firm, with less regard to share valuation
- The value investor focusses on share prices in anticipation of market correction, possibly on account of improving company fundamentals

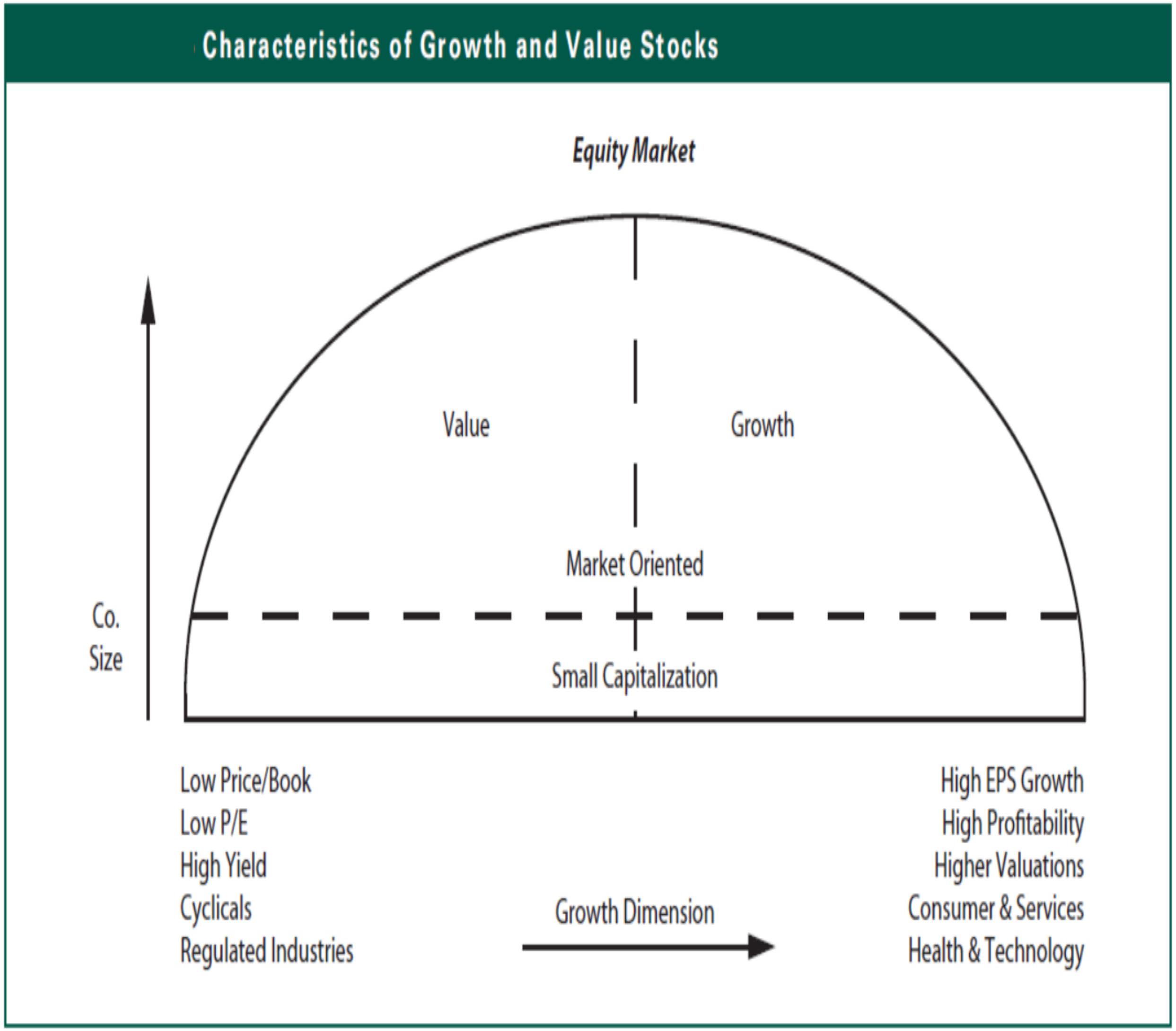
Value vs. Growth investing

- Notice the characteristics of the value and growth stocks shown in Figure below
- The figure shows one approach to classify securities according to style and market capitalization



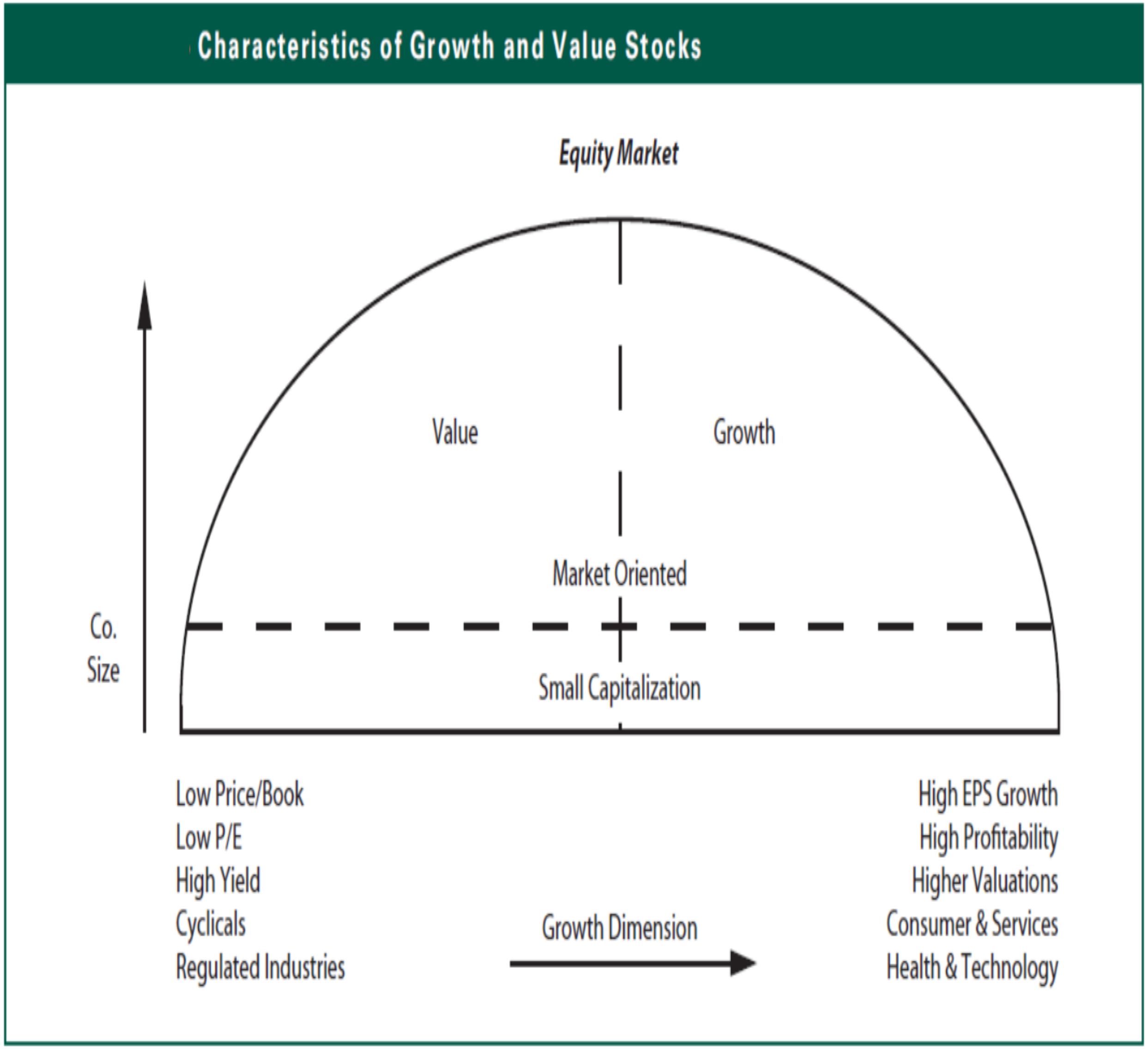
Value vs. Growth investing

- We can see that value stocks are cheap (i.e., low P/BV, high yield) and have modest growth opportunities
- In contrast, growth stocks are expensive, reflecting their high future earning potential



Value vs. Growth investing

- It is tempting to say that value style appears to be more tempting than growth, and in fact, studies show that value style indeed produces higher average returns that growth investing
- However, both strategies have their clientele



Multi-index models: Chen, Roll, and Ross Model

- $\boldsymbol{R_i = a_i + b_{i1}\, I_1 + b_{i2}\, I_2 + b_{i3}\, I_3 + \cdots + b_{iL}\, I_L + c_i}$
- This model tries to capture the influence of fundamental factors that affect future cash flows
- Value of a security is a function of future cash flows and discount rates; any influence that affect these can affect security prices
- Only unexpected changes in these influences and affect security returns (or prices)

Multi-index models: Chen, Roll, and Ross Model

- $R_i = a_i + b_{i1}\, I_1 + b_{i2}\, I_2 + b_{i3}\, I_3 + \cdots + b_{iL}\, I_L + c_i$
- Five variables are found to be most useful
  - Risk-premia on risky vs safe instruments: Unexpected differences in returns of long-term (20 year) corporate bonds and government bonds
  - Term structure or the shape of yield curve: Returns of long-term government bond minus that one month treasury bill one month in the future
  - Unexpected inflation: Rate of inflation at the beginning of the month minus actual rate realized at the end of the month
  - Unexpected changes in the long-run profits in the economy: Expected long-run growth rate in real final sales at the beginning of the month minus expected long-run growth rate in the real final sales at the end of the month
  - Influence of market: Expected excess returns (Rm-Rf) after accounting for any changes of the previous four variables (how?)

Average Correlation Models and Mixed Models

- **Overall mean model**
  - Apply Historical correlation matrix with averaging techniques
  - Compute all the possible pairwise correlation coefficients over some period. For each pair, use the average correlation to forecast
  - This assumes that past average correlation remains constant. Provides a sort of Naïve model to compare against more advanced models
- **Traditional Mean Model (or Disaggregate averaging model)**
  - Assume that the correlation between stocks from the same industry I (e.g., steel) is the same as any two other stocks from the same industry and this is equivalent to historical average in that industry
  - Compute correlation between all the pairwise stocks in same industry I (e.g., Steel) over some period and use the average of this to forecast the correlation across industry I

Average Correlation Models and Mixed Models

- **Traditional Mean Model (or Disaggregate averaging model)**
  - Assume that the correlation between stocks from the same industry I (e.g., steel) is the same as any two other stocks from the same industry and this is equivalent to historical average in that industry
  - Compute correlation between all the pairwise stocks in same industry I (e.g., Steel) over some period and use the average of this to forecast the correlation across industry I
  - Assume that the correlation between industry I and J remain the same for any two stocks from industries I and J
  - Compute all the possible pairwise correlations from industries I and J over a given period, and consider the average of this as your forecast for the correlation between all the stocks in industry I and J

Average Correlation Models and Mixed Models

- **Mixed models**
  - Single index, multi-index and averaging techniques are employed
  - First, single index model is constructed and error terms are extracted for each stock
  - The correlation of this error with other stock errors is also called extra market covariance
  - Additional indices based on economic and industry influences are constructed to explain this extramarket covariance
  - Some models use averaging techniques to explain this extra market covariance

Numerical: Question 1

- **Que1. In a two stock market, the capitalization of stock A is twice that of stock B. The standard deviation of stock A is (σ_a) is 30% and that of B (σ_b) is 50%. The correlation coefficient (ρ_{ab}) between the returns in 0.7. Assume that the single-index is the right model**

- The following single index model is assumed

- $R_{it} = R_f + \beta_i\left(R_{mt} - R_f\right) + e_{it}$

- **What is the standard deviation of the market portfolio (σ_m)**

Numerical: Question 1

- **What is the standard deviation of the market portfolio (σ_m)**

- $w_a + w_b = 1$; $\sigma_m^2 = What\ is\ market\ portfolio??$

Numerical: Question 1

- **What is the standard deviation of the market portfolio ($\boldsymbol{\sigma_m}$)**

- $w_a + w_b = 1$; $w_a = ??\, w_b$; $w_a = ??$; $w_b = ??$

- $\sigma_m^2 = (w_a\sigma_\mathrm{a})^2 + (w_b\sigma_\mathrm{b})^2 + 2\rho_{ab}w_a w_b\sigma_\mathrm{a}\sigma_\mathrm{b}$

- $\sigma_m^2 = ??$

Numerical: Question 1

- **What is the standard deviation of the market portfolio (σ_m)**

- $w_a + w_b = 1$; $w_a = w_b$; $w_a = \frac{2}{3}$; $w_b = \frac{1}{3}$

- $\sigma_m^2 = (w_a\sigma_a)^2 + (w_b\sigma_b)^2 + 2\rho_{ab}w_a w_b\sigma_a\sigma_b$

- $\sigma_m^2 = \left(\frac{2}{3} * 0.3\right)^2 + \left(\frac{1}{3} * 0.5\right)^2 + 2 * 0.7 * \frac{1}{3} * \frac{2}{3} * 0.3 * 0.5 = 0.1144$

- $\sigma_m = 0.3383\ or\ 33.83\%$

Numerical: Question 1

- **Que1. In a two stock market, the capitalization of stock A is twice that of stock B. The standard deviation of stock A is (σ_a) is 30% and that of B (σ_b) is 50%. The correlation coefficient (ρ_{ab}) between the returns in 0.7. Assume that the single-index is the right model**
- The following single index model is assumed
- $R_{it} = R_f + \beta_i\left(R_{mt} - R_f\right) + e_{it}$
- **What is the beta of each stock**
- $\beta_a = \frac{cov(r_a r_m)}{variance(r_m)} = \frac{\sigma_{am}}{\sigma_m^2}$

Numerical: Question 1

- **What is the beta of each stock**
- $\beta_a = \frac{cov(r_a r_m)}{variance(r_m)} = \frac{\sigma_{am}}{\sigma_m^2}$
- $\sigma_{am} = E((r_a - \bar{r}_a)(r_m - \bar{r}_m)) = E\left((r_a - \bar{r}_a)\left((\frac{2}{3}r_a + \frac{1}{3}r_b) - (\frac{2}{3}\bar{r}_a + \frac{1}{3}\bar{r}_b)\right)\right)$
- $= E\left((r_a - \bar{r}_a)\left(\frac{2}{3}(r_a - \bar{r}_a) + \frac{1}{3}(r_b - \bar{r}_b)\right)\right) = \frac{2}{3}[E(r_a - \bar{r}_a)^2] + \frac{1}{3}\left[E\left((r_a - \bar{r}_a)(r_b - \bar{r}_b)\right)\right]$
- $= \frac{2}{3}\sigma_a^2 + \frac{1}{3}\sigma_{ab}$; where $\sigma_{ab} = \rho_{ab}\sigma_a\sigma_b$
- $=??$
- $\beta_a = \frac{cov(r_a r_m)}{variance(r_m)} = \frac{\sigma_{am}}{\sigma_m^2} = ??$
- $\beta_m = w_a\beta_a + w_b\beta_b = 1$
- $\beta_b = ??$

Numerical: Question 1

- **What is the beta of each stock**
- $\beta_a = \frac{cov(r_a r_m)}{variance(r_m)} = \frac{\sigma_{am}}{\sigma_m^2}$
- $\sigma_{am} = E((r_a - \bar{r}_a)(r_m - \bar{r}_m)) = E\left((r_a - \bar{r}_a)\left((\frac{2}{3}r_a + \frac{1}{3}r_b) - (\frac{2}{3}\bar{r}_a + \frac{1}{3}\bar{r}_b)\right)\right)$
- $= E\left((r_a - \bar{r}_a)\left(\frac{2}{3}(r_a - \bar{r}_a) + \frac{1}{3}(r_b - \bar{r}_b)\right)\right) = \frac{2}{3}[E(r_a - \bar{r}_a)^2] + \frac{1}{3}[E\left((r_a - \bar{r}_a)(r_b - \bar{r}_b)\right)]$
- $= \frac{2}{3}\sigma_{\mathrm{a}}^2 + \frac{1}{3}\sigma_{ab}$; where $\sigma_{ab} = \rho_{ab}\sigma_{\mathrm{a}}\sigma_b$
- $=\frac{2}{3}0.3^2 + \frac{1}{3}0.7 * 0.3 * 0.5 = 0.095$
- $\beta_a = \frac{cov(r_a r_m)}{variance(r_m)} = \frac{\sigma_{am}}{\sigma_m^2} = \frac{.095}{.1144} = 0.8304$
- $\beta_m = w_a\beta_a + w_b\beta_b = 1$
- $1 = \frac{2}{3} * 0.8304 + \frac{1}{3} * \beta_b$; $\beta_b = 1.3392$

Numerical: Question 1

- **Que1. In a two stock market, the capitalization of stock A is twice that of stock B. The standard deviation of stock A is (σ_a) is 30% and that of B (σ_b) is 50%. The correlation coefficient (ρ_{ab}) between the returns in 0.7. Assume that the single-index is the right model**

- The following single index model is assumed

- $R_{it} = R_f + \beta_i\left(R_{mt} - R_f\right) + e_{it}$

- **What is the residual variance of each stock**

$\sigma_i^2 = \sigma_{ei}^2 + (\beta_i \sigma_m)^2$;

Numerical: Question 1

a) **What is the residual variance of each stock**

- $\sigma_i^2 = \sigma_{ei}^2 + (\beta_i \sigma_m)^2$;
- $\sigma_{ea}^2 = \sigma_a^2 - (\beta_a \sigma_m)^2 = ??$
- $\sigma_{eb}^2 = \sigma_b^2 - (\beta_b \sigma_m)^2 = ??$

Numerical: Question 1

a) **What is the residual variance of each stock**

- $\sigma_i^2 = \sigma_{ei}^2 + (\beta_i \sigma_m)^2$; $\sigma_{ei}^2 = \sigma_i^2 - (\beta_i \sigma_m)^2$
- $\sigma_{ea}^2 = \sigma_a^2 - (\beta_a \sigma_m)^2 = 0.3^2 - (0.8304 * 0.3383)^2$=0.0111
- $\sigma_{eb}^2 = \sigma_b^2 - (\beta_b \sigma_m)^2 = 0.5^2 - (1.3392 * 0.3383)^2$=0.0447

Numerical: Question 1

- **Que1. In a two stock market, the capitalization of stock A is twice that of stock B. The standard deviation of stock A is (σ_a) is 30% and that of B (σ_b) is 50%. The correlation coefficient (ρ_{ab}) between the returns in 0.7. Assume that the single-index is the right model**

- The following single index model is assumed
- $R_{it} = R_f + \beta_i(R_{mt} - R_f) + e_{it}$

a) **If the Single-index model holds and stock A is expected to earn 11% in excess of the risk-free rate, what must be the risk premium on the market portfolio**

- $\bar{R}_i = R_f + \beta_i(\bar{R}_m - R_f)$

Numerical: Question 1

a) **If the Single-index model holds and stock A is expected to earn 11% in excess of the risk-free rate, what must be the risk premium on the market portfolio**

- $\bar{R}_i = R_f + \beta_i(\bar{R}_m - R_f)$

- For stock A we know that

- $\bar{R}_a - R_F = \beta_a(\bar{R}_m - R_f)$ =11%; $(\bar{R}_m - R_f) = \frac{.11}{0.8304} = 0.1325\ or\ 13.25\%$

Numerical: Question 2

- **Que. 2 A portfolio management organization analyses 60 securities and constructs the mean-variance efficient portfolios, using these 60 securities**

❖ How many estimates of expected returns, variances, and covariances are needed to optimize this portfolio

Ans: ??

❖ If you believe in the single-index model, then how many estimates are required

Ans: ??

❖ If you believe in a multi-index model with three indices, then how many estimates are required

Ans: ??

Numerical: Question 2

- **Que. 2 A portfolio management organization analyses 60 securities and constructs the mean-variance efficient portfolios, using these 60 securities**

❖ How many estimates of expected returns, variances, and covariances are needed to optimize this portfolio

Ans: 60 estimates of expected returns, 60 estimates of variances, and 60*59/2= 1770 estimates of covariances. That is a total of 1890 estimates are needed

❖ If you believe in the single-index model, then how many estimates are required

60 estimates of expected returns, 60 estimates of variances, and 60 estimates of beta (correlation with the single index, i.e., market index) are needed. Also, two further estimates, one for the expected returns on the market and one for the variance of the market are required. That is a total of 182 estimates.

❖ If you believe in a multi-index model with three indices, then how many estimates are required

Ans: That is a total $\alpha_i \; and \; \sigma_i$=60*2 estimates, $b_{ij} = 60 * 3 \; estimates$and $I_j \; and \sigma_{I_j}$=6 estimates. A total of 2*60+60*3+3*2 =306 (i.e., 2N+LN+2L) estimates

Numerical: Question 3

- **Ques 3) The following information is provided**

Stock	Expected returns (%)	Beta	Firm-specific Standard Deviation σ_{ei}
A	13	0.8	30%
B	18	1.2	40%

- Also, the market index has a standard deviation of 22% and the risk-free rate of 8%

- **What is the standard deviation of stocks A and B**

Numerical: Question 3

Stock	Expected returns (%)	Beta	Firm-specific Standard Deviation σ_{ei}
A	13	0.8	30%
B	18	1.2	40%

- Also, the market index has a standard deviation of 22% and the risk-free rate of 8%
- **What is the standard deviation of stocks A and B**
- $\sigma_A^2 = \sigma_{eA}^2 + \beta_A^2 \sigma_m^2 = ??$
- $\sigma_B^2 = \sigma_{eB}^2 + \beta_B^2 \sigma_m^2 = ??$
- Find in % terms

Numerical: Question 3

Stock	Expected returns (%)	Beta	Firm-specific Standard Deviation σ_{ei}
A	13	0.8	30%
B	18	1.2	40%

- Also, the market index has a standard deviation of 22% and the risk-free rate of 8%
- **What is the standard deviation of stocks A and B**
- $\sigma_A^2 = \sigma_{eA}^2 + \beta_A^2\sigma_m^2 = 0.3^2 + (0.8 * 0.22)^2$ =0.120976
- $\sigma_A = 0.3478\ or\ 34.78\%$
- $\sigma_B^2 = \sigma_{eB}^2 + \beta_B^2\sigma_m^2 = 0.4^2 + (1.2 * 0.22)^2$ =0.2297
- $\sigma_B = 0.4793\ or\ 47.93\%$

Numerical: Question 3

- **Ques 3) The following information is provided**

Stock	Expected returns (%)	Beta	Firm-specific Standard Deviation σ_{ei}
A	13	0.8	30%
B	18	1.2	40%

- Also, the market index has a standard deviation of 22% and the risk-free rate of 8%
- **If we were to construct the portfolio with the following proportions**

Stock A	30%
Stock B	45%
Government Security	25%

Compute the expected returns, standard deviations, beta, and the non-systematic component of the standard deviation of the portfolio

Numerical: Question 3

Stock	Expected returns (%)	Beta	Firm-specific Standard Deviation σ_{ei}
A	13	0.8	30%
B	18	1.2	40%

- Also, the market index has a standard deviation of 22% and the risk-free rate of 8%
- **If we were to construct the portfolio with the following proportions**

Stock A	30%
Stock B	45%
Government Security	25%

Expected returns $(\bar{R}_p)= w_A\bar{R}_A + w_B\bar{R}_B + w_G\bar{R}_G$ =??

Idiosyncratic Standard deviation $\sigma^2_{eP} = (w_A\sigma_{eA})^2+(w_B\sigma_{eB})^2$

Standard deviation (σ^2_P)=$\sigma^2_{eP} + \beta^2_P\sigma^2_m$=??

$\beta_P = w_A\beta_A + w_B\beta_B + w_G\beta_G$

Numerical: Question 3

Stock	Expected returns (%)	Beta	Firm-specific Standard Deviation σ_{ei}
A	13	0.8	30%
B	18	1.2	40%

Stock A	30%
Stock B	45%
Government Security	25%

$\beta_{AP} = w_A\beta_A + w_B\beta_B + w_G\beta_G$

=0.3*0.8+0.45*1.2+0.25*0=0.78

(σ_P^2)=$\sigma_{eP}^2 + \beta_P^2\sigma_m^2$=$0.0405 + 0.78^2 0.22^2 = 0.07$

$\sigma_P = 0.2645 \; or \; 26.45\%$

Expected returns $(\bar{R}_p)$= $w_A\bar{R}_A + w_B\bar{R}_B + w_G\bar{R}_G = 0.30 * 13 + 0.45 * 18 + 0.25 * 8 = 14$ (%)

Idiosyncratic Standard deviation $\sigma_{eP}^2 = (w_A\sigma_{eA})^2 + (w_B\sigma_{eB})^2 = (0.3 * 0.3)^2 + (0.45 * 0.4)^2$ =0.0405

Thanks

Capital Asset Pricing Model (CAPM)

Course: Portfolio management (PM)

Instructor: Abhinava Tripathi

Capital Asset Pricing Model (CAPM)

- The standard form of CAPM equilibrium relation is first shown by Sharpe, Lintner, and Mossin. Hence it is also referred to as Sharpe-Lintner-Mossin model of CAPM (1960s)
- This is the simplest and most widely employed model of asset pricing
- It has been documented to be extremely efficient in explaining the observed prices
- It involves some important assumptions

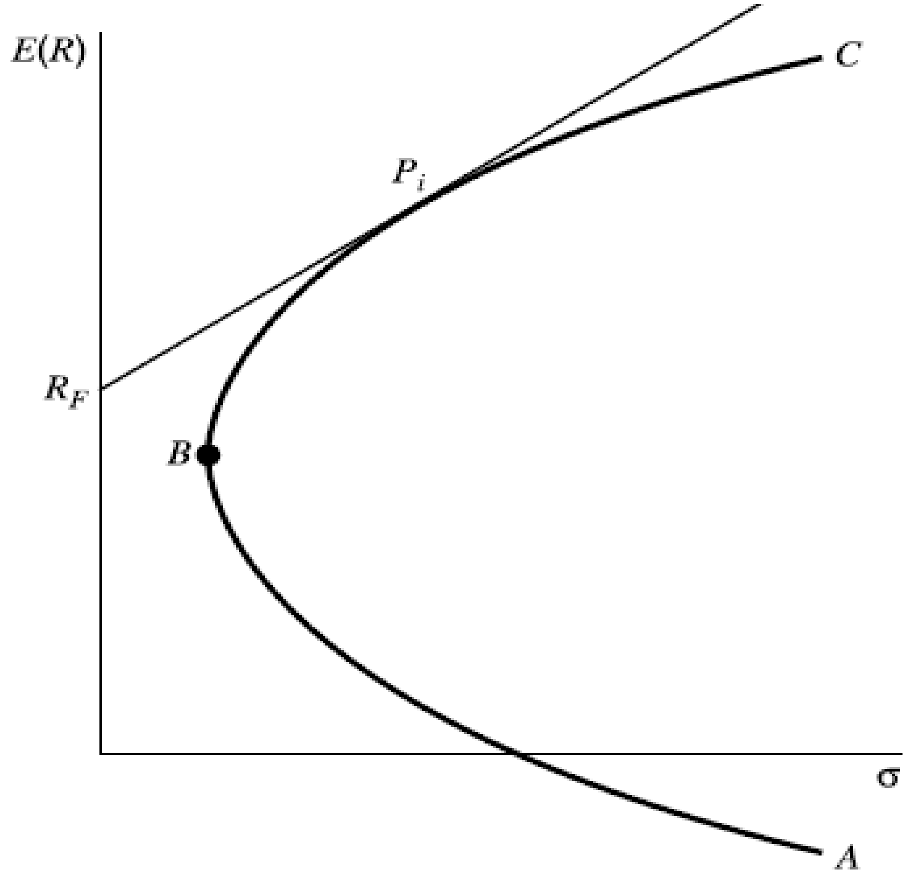
CAPM: Assumptions

- **No transaction costs:** What are these transaction costs?
- **Securities are infinitely divisible:** One can take as small a position as INR 1.
- **Prices are given:** Traders can not affect prices
- **Investors are rational**: They understand the return distributions, risk, and also process all the available information

CAPM: Assumptions

- **Unlimited short-sales are allowed**
- **Unlimited lending and borrowing is allowed**
- **Uniform expectations:** At equilibrium, all the investors have the same expectation of a security's return distribution (i.e., expected return, risk, correlation structure across securities); they define the period of equilibrium in a similar manner
- **All the assets are marketable**

A simple approach to understanding the CAPM



The efficient frontier with lending and borrowing.

A simple approach to understanding the CAPM

- Our old story of one risky asset (market portfolio) in the presence of risk-free lending and borrowing
- We said (under the assumptions specified) that all the investors will hold this portfolio along with the risk-free asset (investing or borrowing)
- This line is called capital market line (CML)

A simple approach to understanding the CAPM

- If all the investors have the same expectations and they face the same lending and borrowing curve, then this tangency portfolio will have the same composition for all of them
- That is, the proportion of any stock in a portfolio held by an individual investor will be the same as that in the market portfolio
- All the investors will hold a market portfolio and riskless security (two mutual fund theorem)

A simple approach to understanding the CAPM

- The equation of this line is as follows

- $\bar{R}_e = R_F + \frac{(\bar{R}_M - R_F)}{\sigma_M} \sigma_e$, here subscript 'e' denotes an efficient portfolio

- The term $\frac{(\bar{R}_M - R_F)}{\sigma_M}$ indicates the price of risk, i.e., excess returns per unit of risk

- The combined term $[\frac{(\bar{R}_M - R_F)}{\sigma_M} \sigma_e]$ is the total reward for taking on σ_e risk

- R_F is simply the risk-free rate that is the price of time, that is delaying the consumption (time-value of money)

A simple approach to understanding the CAPM

- The combined term $[\frac{(\bar{R}_M - R_F)}{\sigma_M}\sigma_e]$ is the total reward for taking on σ_e risk

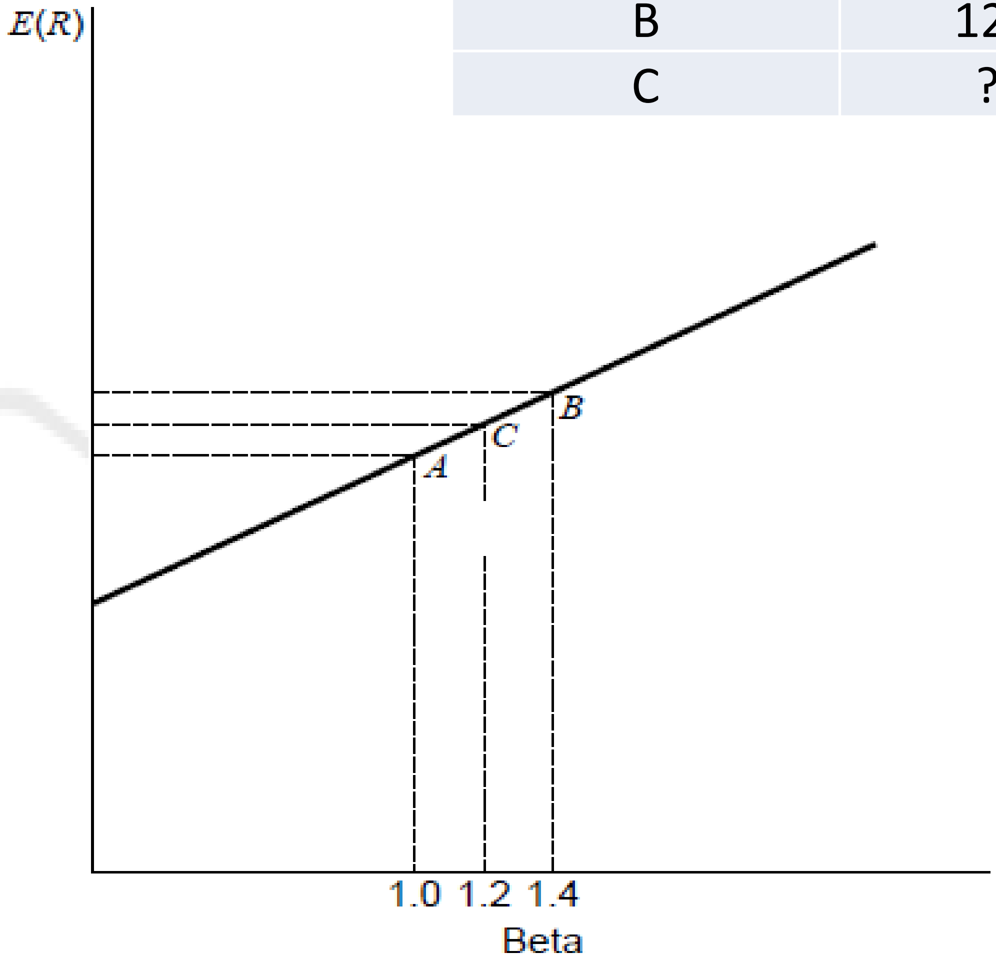
- R_F is simply the risk-free rate that is the price of time, that is delaying the consumption (time-value of money)

- Therefore, the equation can be simply written as

- Expected return= Price of time + Price of risk * Risk

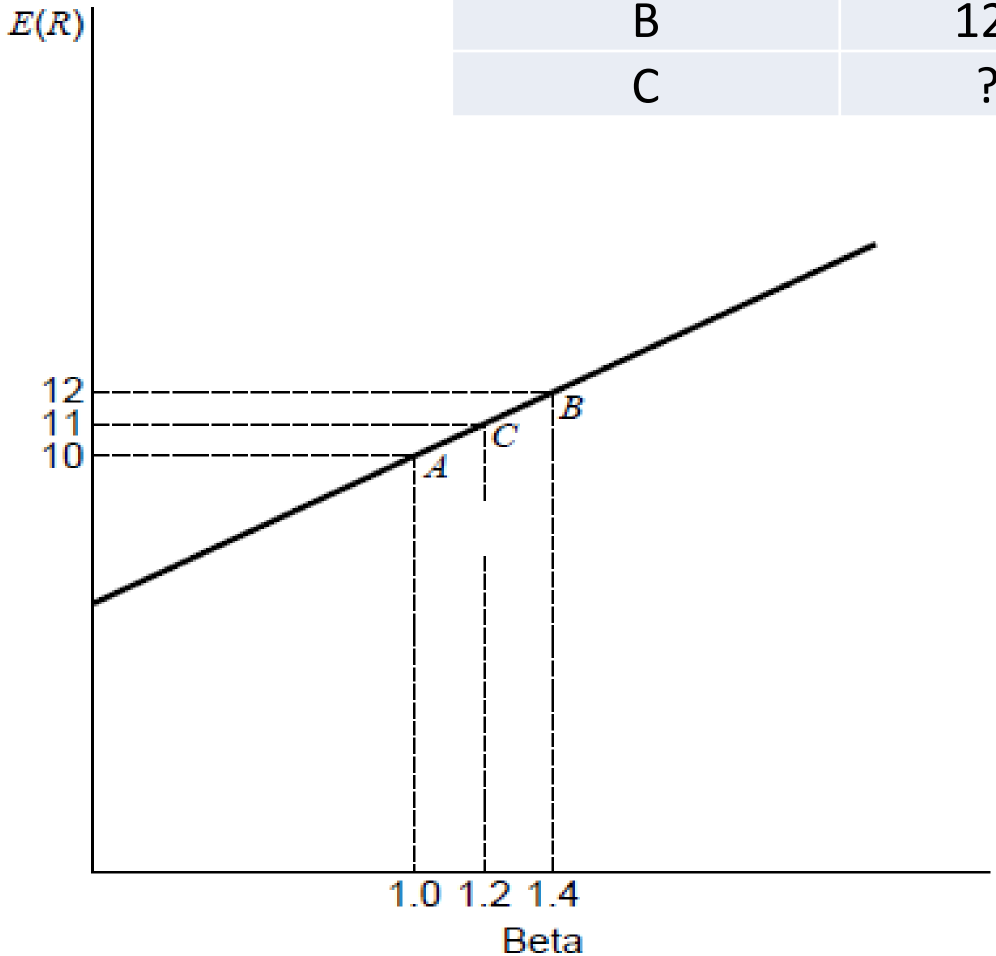
Security market line

Investment	Expected return	Beta	Portfolio
A	10%	1	Correctly priced
B	12%	1.4	Correctly priced
C	??	1.2	Correctly priced



Security market line

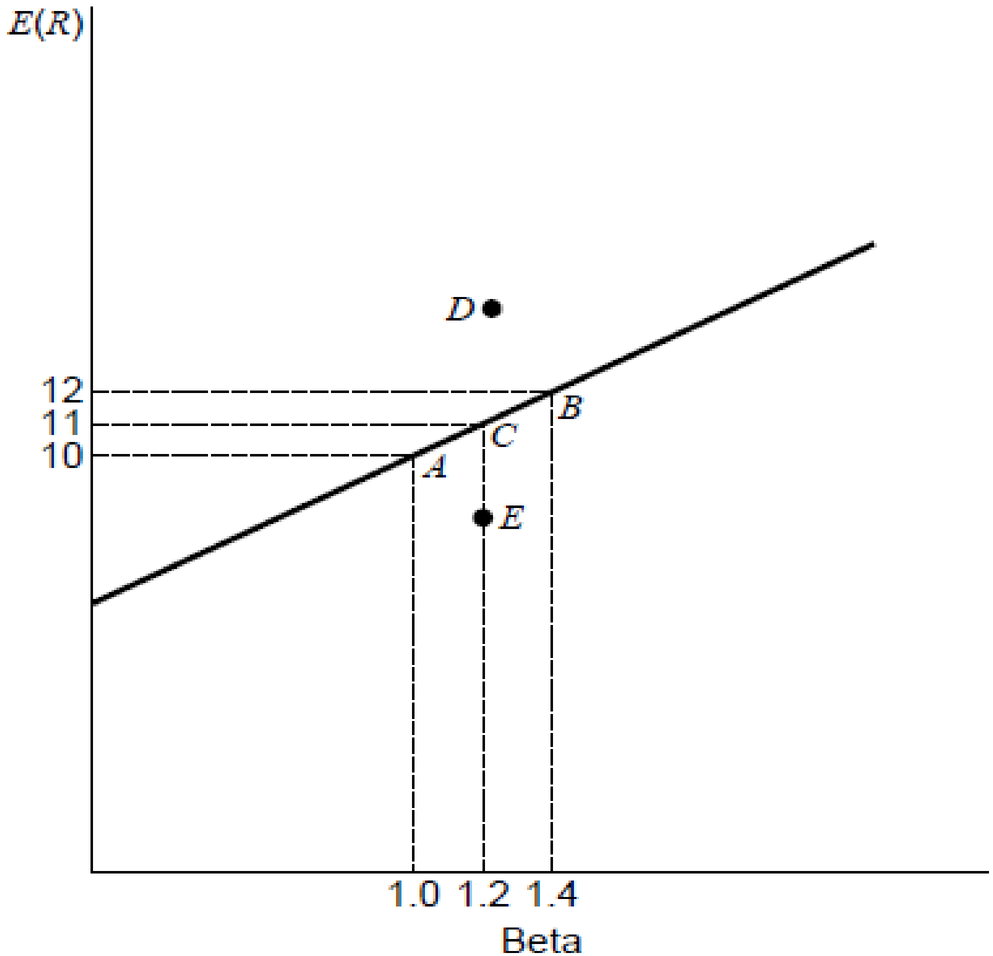
Investment	Expected return	Beta	Portfolio
A	10%	1	Correctly priced
B	12%	1.4	Correctly priced
C	??	1.2	Correctly priced



Arbitrage portfolio

Investment	Expected return	Beta	Portfolio
A	10%	1	Correctly priced
B	12%	1.4	Correctly priced
D	13%	1.2	?
E	8%	1.2	?

Security market line



Arbitrage portfolio

Investment	Expected return	Beta	Portfolio
A	10%	1	Correctly priced
B	12%	1.4	Correctly priced
D	13%	1.2	
E	8%	1.2	
Arbitrage portfolio			
Buy D	13%	1.2	
Sell E	-8%	-1.2	
Expected Return	5%	0	

Imagine the expected return and risk of a portfolio with 50% amount in A and B

Arbitrage portfolio

Investment	Expected return	Beta	Portfolio
A	10%	1	Correctly priced
B	12%	1.4	Correctly priced
D	13%	1.2	?
E	8%	1.2	?
C (Average of A and B)	**11%**	**1.2**	
Arbitrage portfolio			
Sell C	-11%	-1.2	
Buy D	13%	1.2	
Expected Return	2%	0	
Arbitrage portfolio			
Buy C	11%	1.2	
Sell E	-8%	-1.2	
Expected Return	3%	0	

Arbitrage portfolio

Investment	Expected return	Beta	Portfolio
A	10%	1	Correctly priced
B	12%	1.4	Correctly priced
D	13%	1.2	Under priced
E	8%	1.2	Overpriced

Imagine the expected return and risk of a portfolio with 50% amount in A and B

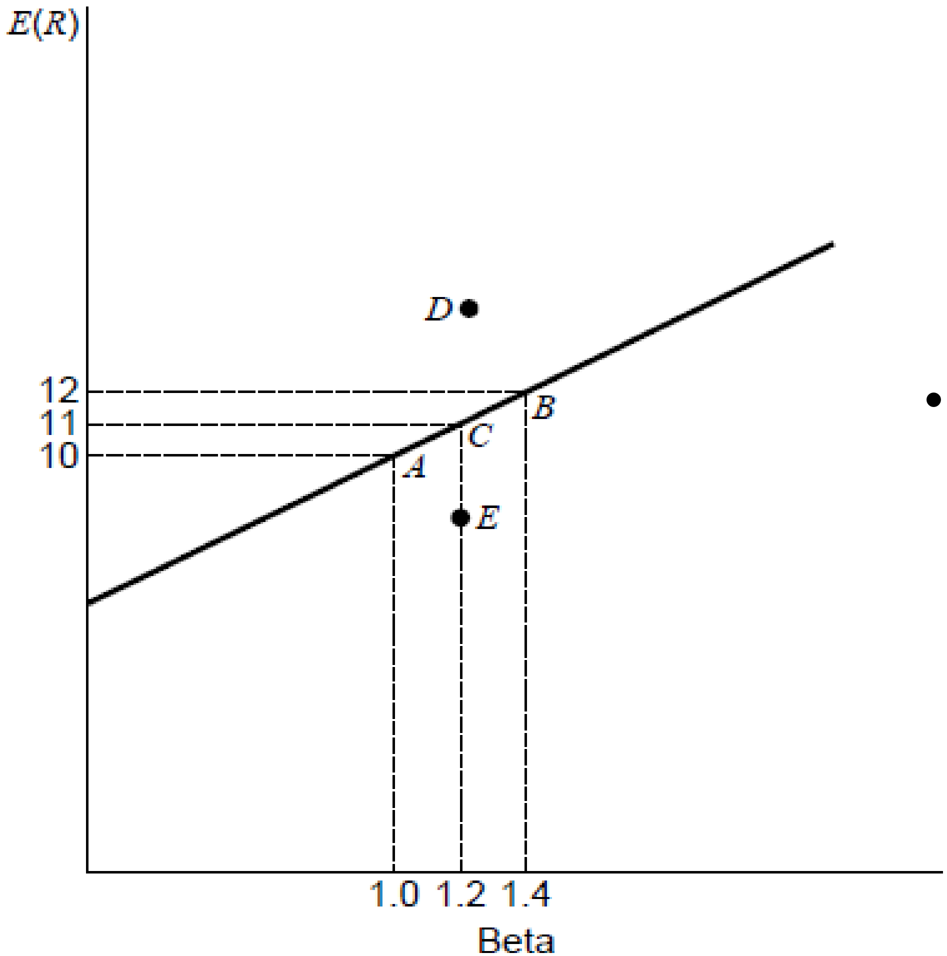
Security market line (SML)

- For a well-diversified portfolio, non-systematic risk tends to go to zero
- Market risk is the only relevant risk, measured by beta
- The SML shown here plots 5 portfolios (A, B, C, D, E)
- Here portfolio D and E are anomalous, i.e., their expected return is not aligned to the systematic-risk (beta)
- E is overpriced, and therefore, offers a lower expected return
- D is underpriced, and therefore, offers a higher expected return

Security market line (SML)

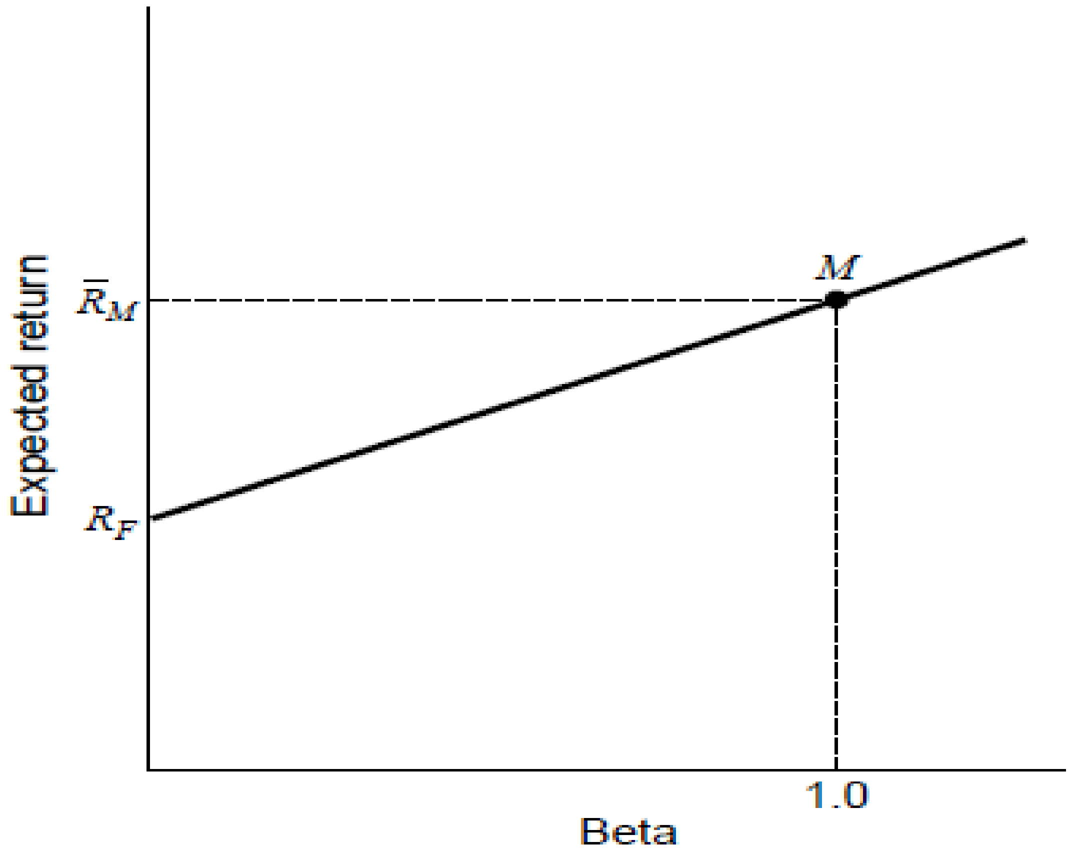
- Here portfolio D and E are anomalous, i.e., their expected return is not aligned to the systematic-risk (beta)- i.e., not in equilibrium
- E is overpriced, and therefore, offers a lower expected return
- D is underpriced, and therefore, offers a higher expected return
- There is a (partially) riskless arbitrage opportunity by selling E and buying D and make profits
- This will bring securities back to the SML

Security market line



- If I want to identify the equation of this straight line, what information is required?
- What are those two portfolios that are easily identifiable on this line?

Security market line (SML)



The security market line.

SML and CAPM

- SML can be identified using the two points through which it passes

- One, the risk-free investment (beta=0 and interest rate of R_F) and market portfolio (beta=1 and interest rate of $\bar{R}_M$)

- Using these points we can write down the equation of SML below

- $\boldsymbol{\bar{R}_i = R_F + \beta_i(\bar{R}_M - R_F)}$ **-------------→ CAPM**

- Here $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$

Derivation of CAPM a rigorous proof

- Recall the analysis in the previous discussions while estimating the efficient frontier
- The following function was maximized: $\theta = \frac{\bar{R}_P - R_f}{\sigma_P}$
- Derivative of θ was taken for all the securities and each set of equation ($\frac{\mathrm{d}\theta}{dx_i} = 0$) was made zero
- The following set of simultaneous equations was derived for all *i's*
- $\lambda(X_1\sigma_{1i} + X_2\sigma_{2i} + \cdots + X_i\sigma_i^2 + \cdots + X_{N-1}\sigma_{N-1i} + X_N\sigma_{Ni}) = \bar{R_i} - R_F$

Derivation of CAPM a rigorous proof

- One important assumption in CAPM was that all the securities held in the market portfolio are exactly in the same proportion as held in the actual market
- $R_M = \sum_{i=1}^{N} R_i X_i'$ and $\bar{R}_M = \sum_{i=1}^{N} \bar{R}_i X_i'$; here X_i' is ratio of market capitalization of the security to that of aggregate market capitalization
- $Cov(R_k R_M) = E[(R_k - \bar{R}_k)(R_M - \bar{R}_M)]$
- $Cov(R_k R_M) = E[(R_k - \bar{R}_k)\left(\sum_{i=1}^{N} R_i X_i' - \sum_{i=1}^{N} \bar{R}_i X_i'\right)]$

Derivation of CAPM a rigorous proof

- $Cov(R_k R_M) = E[(R_k - \bar{R}_k)\left(\sum_{i=1}^{N} R_i X_i' - \sum_{i=1}^{N} \bar{R}_i X_i'\right)]$

- $Cov(R_k R_M) = E[(R_k - \bar{R}_k)\left(\sum_{i=1}^{N} X_i'(R_i - \bar{R}_i)\right)]$

- $Cov(R_k R_M) = E[X_1'(R_k - \bar{R}_k)(R_1 - \bar{R}_1) + X_2'(R_k - \bar{R}_k)(R_2 - \bar{R}_2) + \cdots +$

  $X_k'(R_k - \bar{R}_k)(R_k - \bar{R}_k) + \cdots + X_N'(R_k - \bar{R}_k)(R_N - \bar{R}_N)]$

- $Cov(R_k R_M) = X_1' E[(R_k - \bar{R}_k)(R_1 - \bar{R}_1)] + X_2' E[(R_k - \bar{R}_k)(R_2 - \bar{R}_2)] + \cdots +$

  $X_k' E[(R_k - \bar{R}_k)(R_k - \bar{R}_k)] + \cdots + X_N' E[(R_k - \bar{R}_k)(R_N - \bar{R}_N)]$

Derivation of CAPM a rigorous proof

- $Cov(R_kR_M) = X_1'E[(R_k - \bar{R}_k)(R_1 - \bar{R}_1)] + X_2'E[(R_k - \bar{R}_k)(R_2 - \bar{R}_2)] + \cdots + X_k'E[(R_k - \bar{R}_k)(R_k - \bar{R}_k)] + \cdots + X_N'E[(R_k - \bar{R}_k)(R_N - \bar{R}_N)]$

- $\boldsymbol{Cov(R_kR_M) = X_1'\sigma_{1k} + X_2'\sigma_{2k} + \cdots + X_k'\sigma_k^2 + \cdots + X_N'\sigma_{Nk}}$

- $\boldsymbol{\lambda(X_1\sigma_{1k} + X_2\sigma_{2k} + \cdots + X_k\sigma_k^2 + \cdots + X_{N-1}\sigma_{N-1k} + X_N\sigma_{Nk}) = \overline{R_k} - R_F}$

Derivation of CAPM a rigorous proof

- $\boldsymbol{Cov(R_k R_M) = X'_1\sigma_{1k} + X'_2\sigma_{2k} + \cdots + X'_k\sigma_k^2 + \cdots + X'_N\sigma_{Nk}}$
- $\boldsymbol{\overline{R_k} - R_F = \lambda(X_1\sigma_{1k} + X_2\sigma_{2k} + \cdots + X_k\sigma_k^2 + \cdots + X_{N-1}\sigma_{N-1k} + X_N\sigma_{Nk})}$
- Compare the RHS of these equations
- With the assumption that Xk's are actually the market proportions, that is $X_k = X'_k$
- $\overline{R_k} - R_F = \lambda Cov(R_k R_M)$;
- What is the value of λ?
- Hint: value of λ will hold for the market as well (i.e., R_k=R_M)

Derivation of CAPM a rigorous proof

- $\boldsymbol{Cov(R_k R_M) = X_1'\sigma_{1k} + X_2'\sigma_{2k} + \cdots + X_k'\sigma_k^2 + \cdots + X_N'\sigma_{Nk}}$
- $\boldsymbol{\overline{R_k} - R_F = \lambda(X_1\sigma_{1k} + X_2\sigma_{2k} + \cdots + X_k\sigma_k^2 + \cdots + X_{N-1}\sigma_{N-1k} + X_N\sigma_{Nk})}$
- Compare the RHS of these equations
- With the assumption that Xk’s are actually the market proportions, that is $X_k = X_k'$
- $\overline{R_k} - R_F = \lambda Cov(R_k R_M)$; also $Cov(R_M R_M) = \sigma_m^2$
- $\overline{R_M} - R_F = \lambda\sigma_m^2$; or $\lambda = \frac{\overline{R_M} - R_F}{\sigma_m^2}$

Derivation of CAPM a rigorous proof

- $\overline{R_k} - R_F = \lambda Cov(R_k R_M)$; also $Cov(R_M R_M) = \sigma_m^2$
- $\overline{R_M} - R_F = \lambda \sigma_m^2$; or $\lambda = \frac{\overline{R_M} - R_F}{\sigma_m^2}$
- Substituting the value of λ
- $\overline{R_k} - R_F = \frac{\overline{R_M} - R_F}{\sigma_m^2} Cov(R_k R_M)$ or
- $\boldsymbol{\overline{R_k} = R_F + \beta_k(\overline{R_M} - R_F)}$
- **Where** $\boldsymbol{\beta_k = \frac{Cov(R_k R_M)}{\sigma_m^2} = \frac{\sigma_{km}}{\sigma_m^2}}$

Few last words on CAPM

- Under the assumptions of CAPM, the only portfolio of risky assets that investors will hold, will be the market portfolio
- In this market portfolio, any security has a proportion that is same as the ratio of the market capitalization of that security to the total market capitalization of that market
- Investors depending upon their risk tolerance, will adjust the proportions of the market portfolio and risk-free asset
- **However, we know that individual investors do hold non-market, smaller portfolios**

Thanks

Alternative forms of CAPM

Course: Portfolio management

Instructor: Abhinava Tripathi

Fallings of CAPM

- CAPM appears to hold at aggregate level
- However, individual investors do hold smaller portfolios, not similar to market portfolios
- Many CAPM assumptions violate the real-world conditions
- **No transaction costs**
- **Securities are infinitely divisible**
- **Prices are given**

CAPM: Assumptions

- **Investors are rational**
- **Unlimited short-sales are allowed**
- **Unlimited lending and borrowing is allowed**
- **Uniform expectations**
- **All the assets are marketable**
- However, there are certain assumptions that can be relaxed and alternative variants can derived

Alternative forms of CAPM

- **Short-sales disallowed**
- **Riskless lending and borrowing**
- **Personal taxes**
- **Non-marketable assets**
- **Heterogenous expectations**
- **Non-price taking behavior**
- **Multi-period /Multi-beta /Consumption CAPM**

Short-sales not allowed

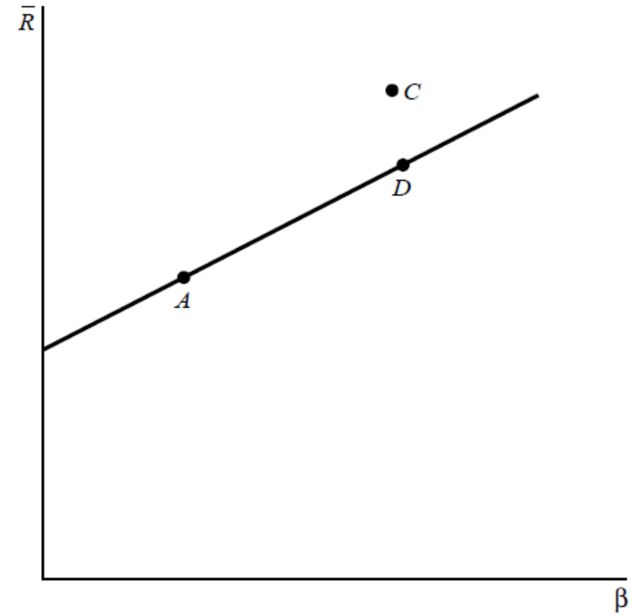
Short-sales not allowed

- One assumption in CAPM is that unlimited short-sales are allowed in the broadest terms
- Investor was allowed to sell any security (whether owned or not) and to use the proceeds to buy any other security
- Graphical interpretation of the case suggests that even if short-sale is disallowed, the same result would be obtained
- In the CAPM framework, all the investors hold market portfolio in the equilibrium, no investor sells any security short
- Thus, irrespective of the fact whether short-sales is allowed or not, CAPM would be derived

No riskless lending or borrowing

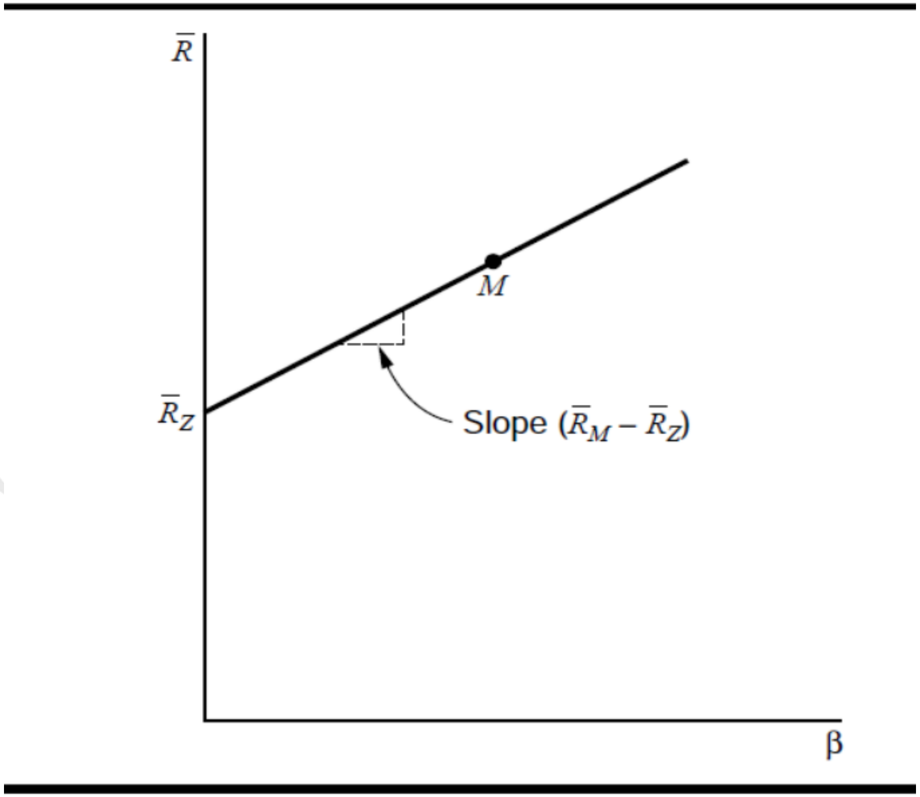
No riskless lending or borrowing- Simple proof

- Only systematic risk, i.e., beta matters
- In equilibrium, All securities will fall on this straight line, else arbitrage opportunities may arise
- Else two assets would exist, with the same systematic risk and different return
- One convenient point is the market portfolio, which is observed also (Nifty-500), but what is the intercept
- At intercept beta is zero



No riskless lending or borrowing- Simple proof

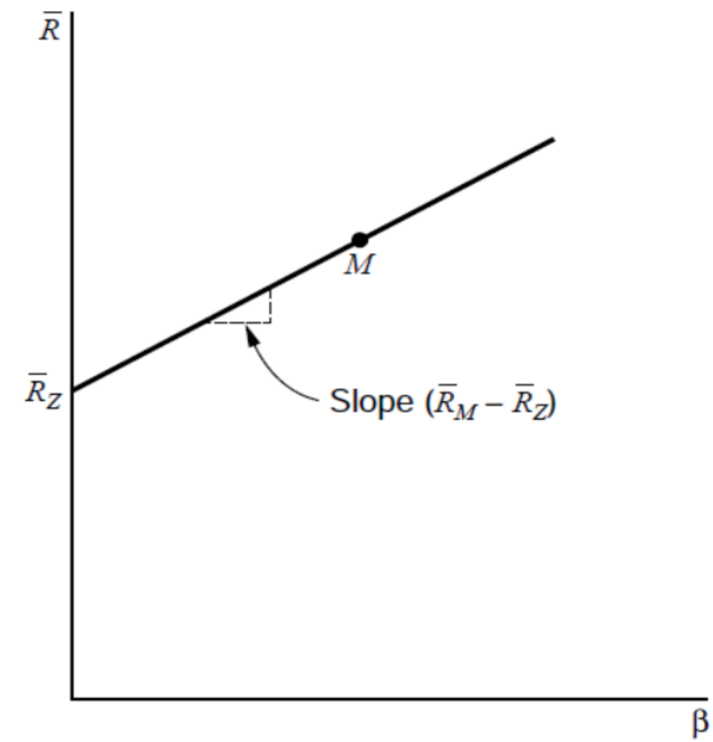
- With this assumption, the SML will be defined in a slightly different manner
- Here, we introduce zero-beta portfolio with interest rate of R_z
- R_z has a beta of zero (no correlation with market). But its total risk (σ) is not zero



The zero-beta capital asset pricing line.

No riskless lending or borrowing- Simple proof

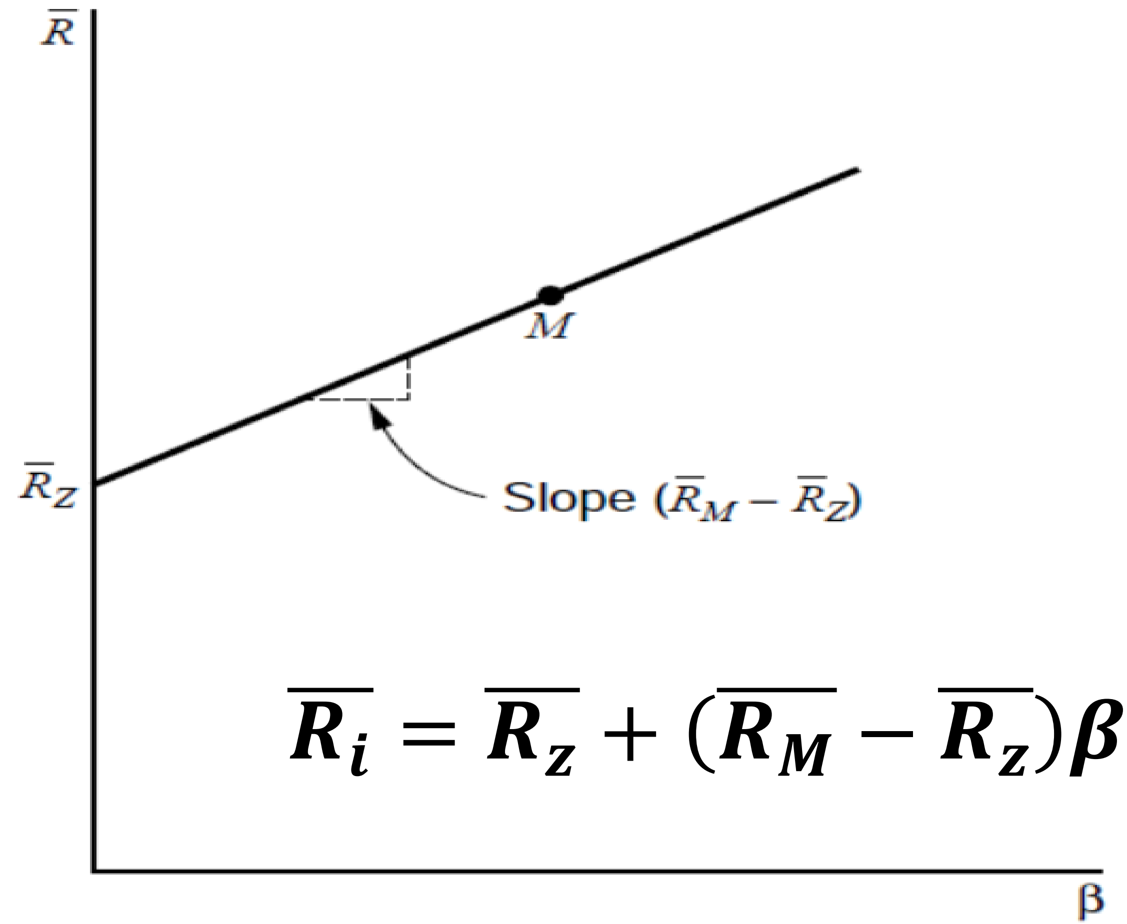
- The equation of new SML will be
- $Expected\ return = a + b(Beta)$
- For a portfolio with zero beta (expected return= $\overline{R_z}$)
- $\overline{R_z} = a + b(0)$ or $\overline{R_z} = a$
- This equation must also hold for the market portfolio.
- $\overline{R_M} = \overline{R_z} + b(1)$ or b=$\overline{R_M} - \overline{R_z}$



The zero-beta capital asset pricing line.

- Now consider the relation between any security with expected return $\bar{R}_i$ and β_i
- $\bar{R}_i = \overline{R_z} + (\overline{R_M} - \overline{R_z})\beta_i$

No riskless lending or borrowing



$$\overline{R_i} = \overline{R_z} + (\overline{R_M} - \overline{R_z})\beta_i$$

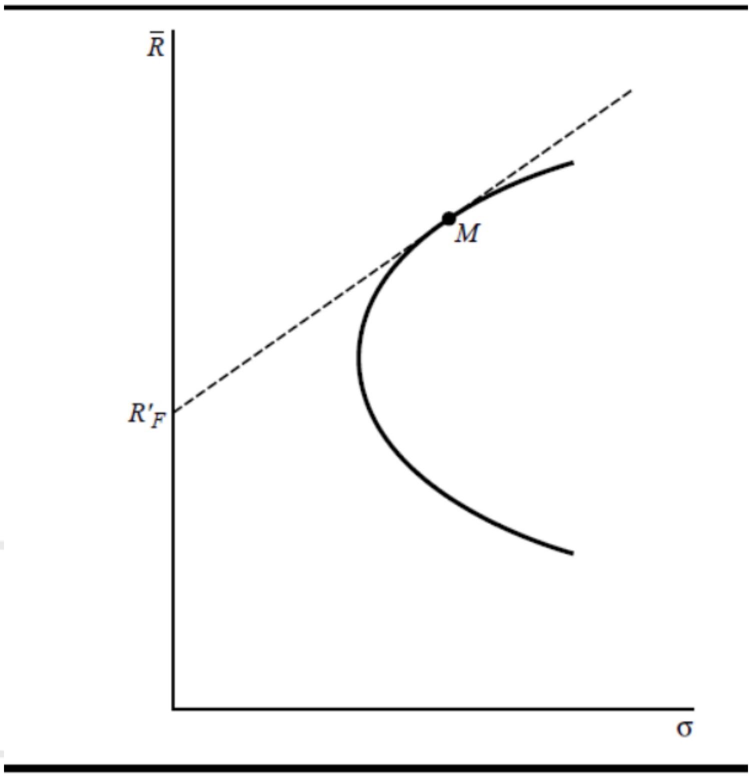
The zero-beta capital asset pricing line.

No riskless lending or borrowing: rigorous derivation

- Define R_F' as the rate at which if investors can lend and borrow an unlimited amount of money they will hold market portfolio

- If such a tangent portfolio exists then one can obtain optimum proportions of the individual securities in it

- The same procedure of slope maximization $\frac{d\theta}{dx}$ can be employed these optimum proportions



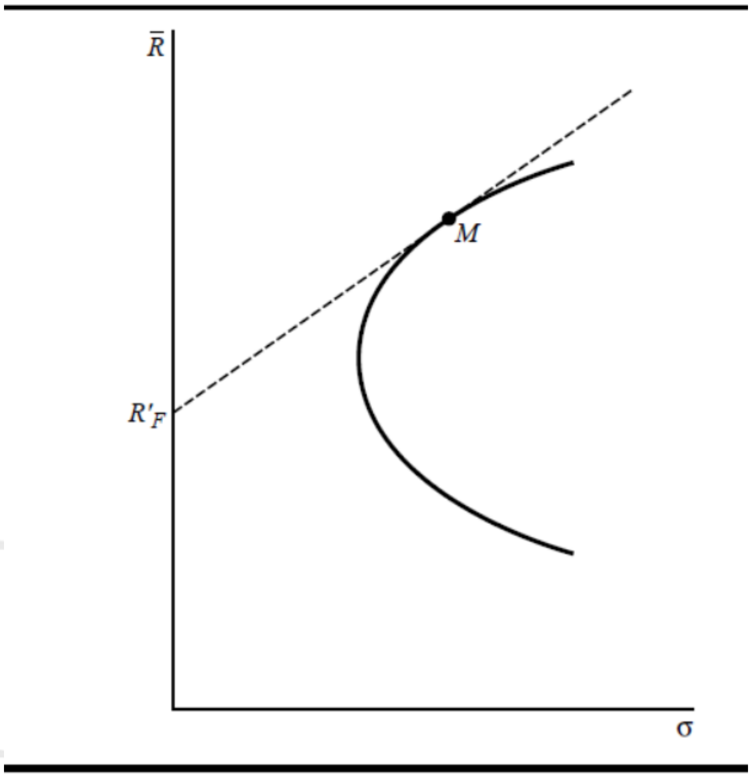
The opportunity set with rate R_F.

No riskless lending or borrowing: rigorous derivation

- The same procedure of slope maximization $\frac{d\theta}{dx}$ can be employed to obtain the following Eqs

- $\lambda\left(X_1\sigma_{1j} + X_2\sigma_{2j} + \cdots + X_j\sigma_j^2 + \cdots + X_N\sigma_{Nj}\right) = \bar{R}_j - R'_F$; Also

- $\left(X_1\sigma_{1j} + X_2\sigma_{2j} + \cdots + X_j\sigma_j^2 + \cdots + X_N\sigma_{Nj}\right) = Cov\left(R_j R_M\right)$ (Why?)



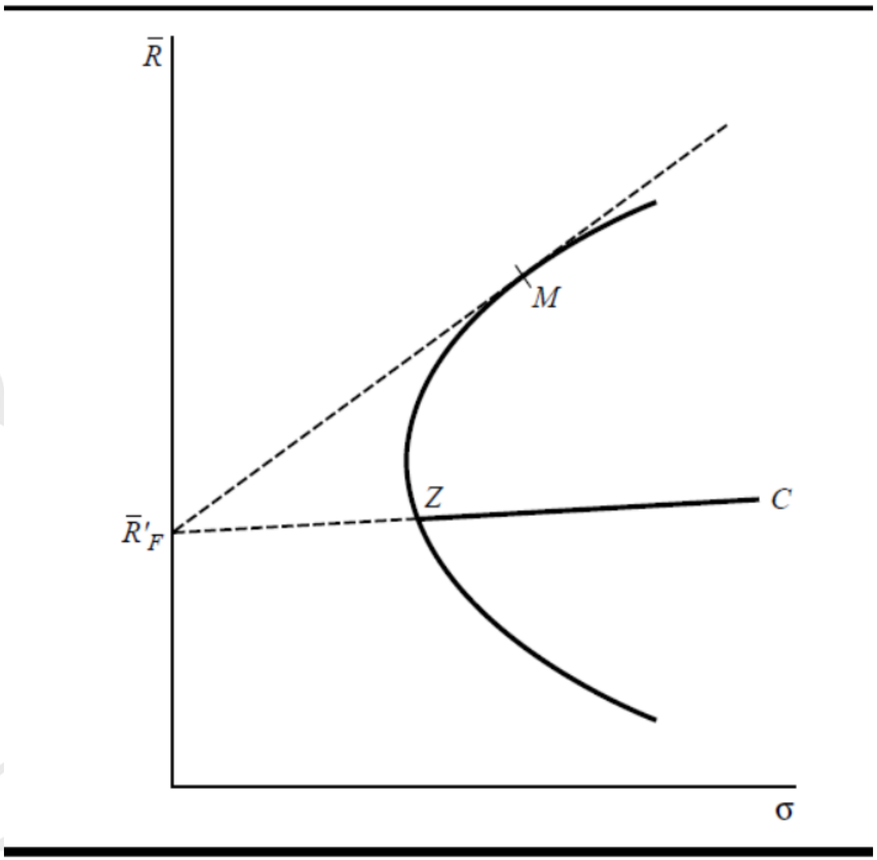
The opportunity set with rate R_F.

No riskless lending or borrowing: rigorous derivation

- $\lambda\left(X_1\sigma_{1j} + X_2\sigma_{2j} + \cdots + X_j\sigma_j^2 + \cdots + X_N\sigma_{Nj}\right) = \bar{R}_j - R_F'$; Also
- $\left(X_1\sigma_{1j} + X_2\sigma_{2j} + \cdots + X_j\sigma_j^2 + \cdots + X_N\sigma_{Nj}\right) = Cov\left(R_j R_M\right)$
- Using the identical steps that were employed in CAPM derivation, we can obtain
- $\bar{R}_j = R_F' + \beta_j(\bar{R}_M - R_F')$
- However, R_F' is not the riskless asset
- If investors have unlimited lending and borrowing available, they will hold the market portfolio
- Its key property is that its beta (or correlation/covariance with market) is zero

No riskless lending or borrowing: rigorous derivation

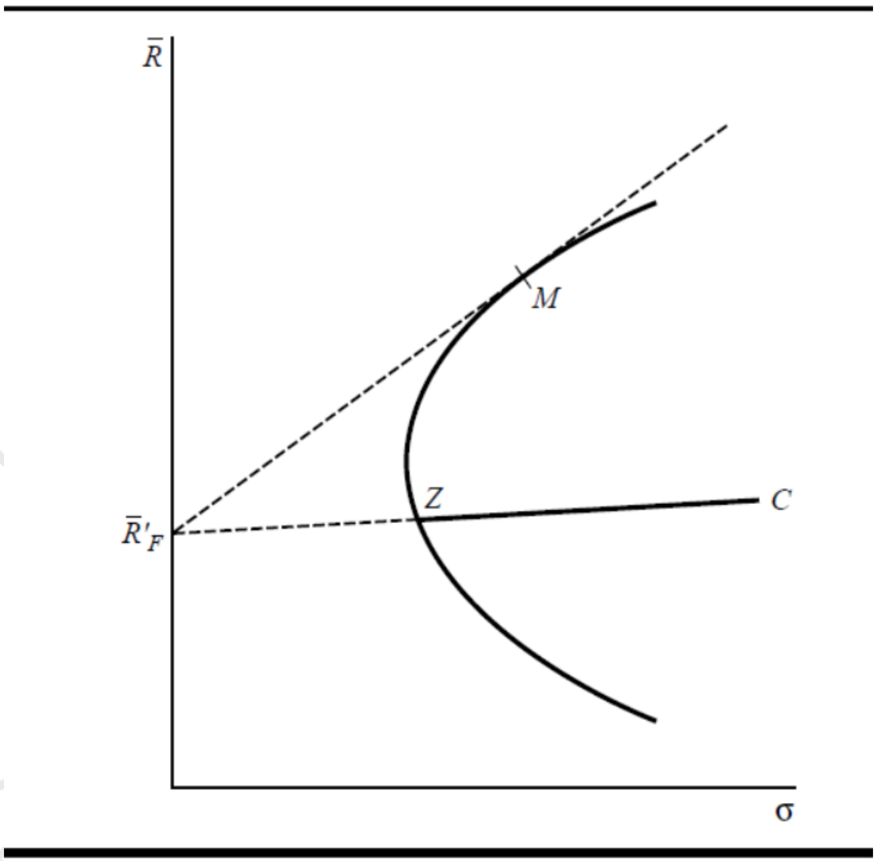
- However, R_F' is not the riskless asset
- Its key property is that its beta (or correlation/covariance with market) is zero
- On the line segment $R_F' - C$, we choose the zero beta portfolio that has the least amount of risk



The location of portfolios with return R_F'.

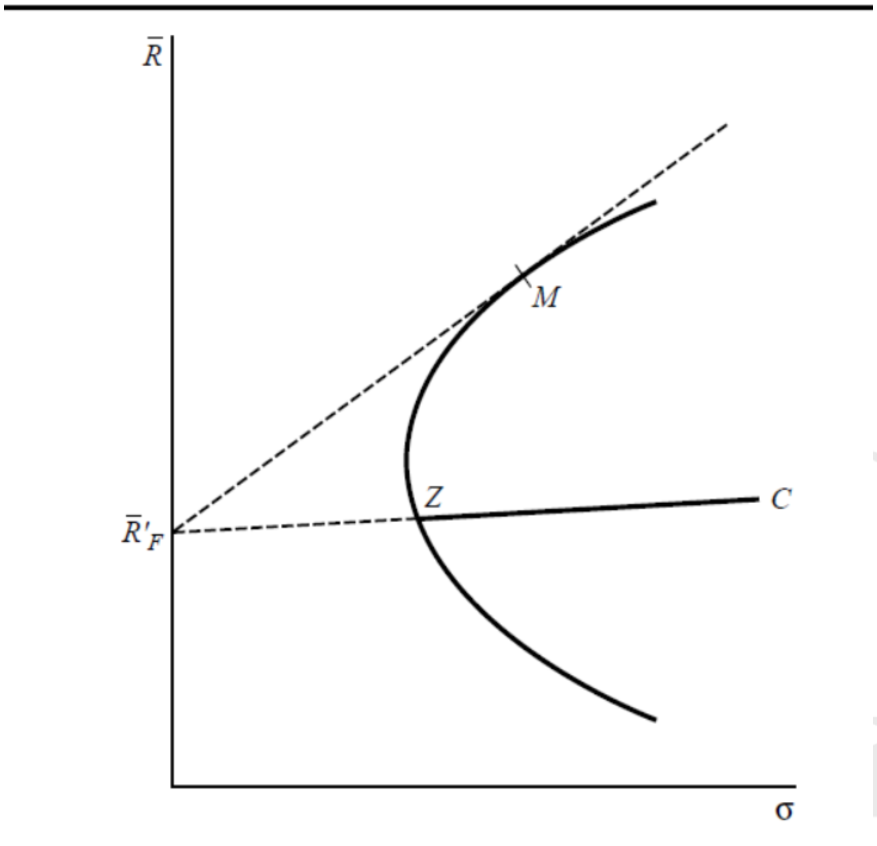
No riskless lending or borrowing: rigorous derivation

- This portfolio can be designated as Z and its expected return as $\bar{R}_Z$. Here $\bar{R}_Z = R'_F$
- Our new equation of security market line can be written as
- $\bar{R}_j = \bar{R}_Z + \beta_j(\bar{R}_M - \bar{R}_Z\ )$



The location of portfolios with return R'_F.

No riskless lending or borrowing: rigorous derivation

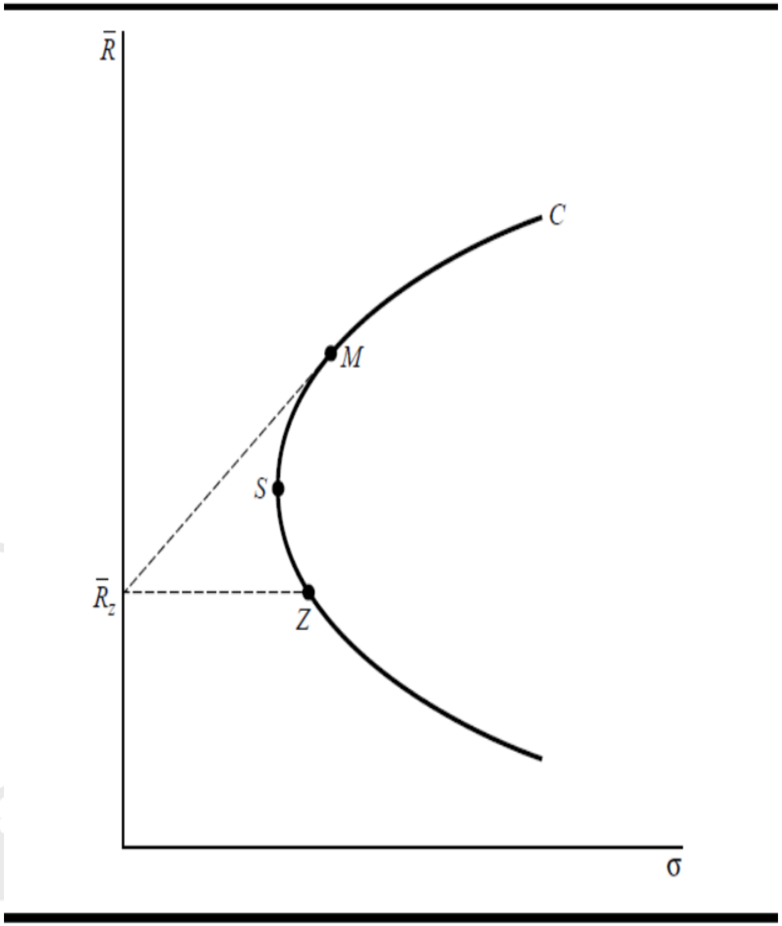


The location of portfolios with return R'_F.

$$\bar{R}_j = \bar{R}_Z + \beta_j(\bar{R}_M - \bar{R}_Z)$$

No riskless lending or borrowing: rigorous derivation

- A few important details about security z are as follows
- The risk of this portfolio can be written as σ_Z
- Let us try and write the risk of minimum variance portfolio (σ_s) as the combination of zero-beta portfolio and market portfolio
- $\sigma_s^2 = X_Z^2 \sigma_Z^2 + (1 - X_Z)^2 \sigma_M^2$

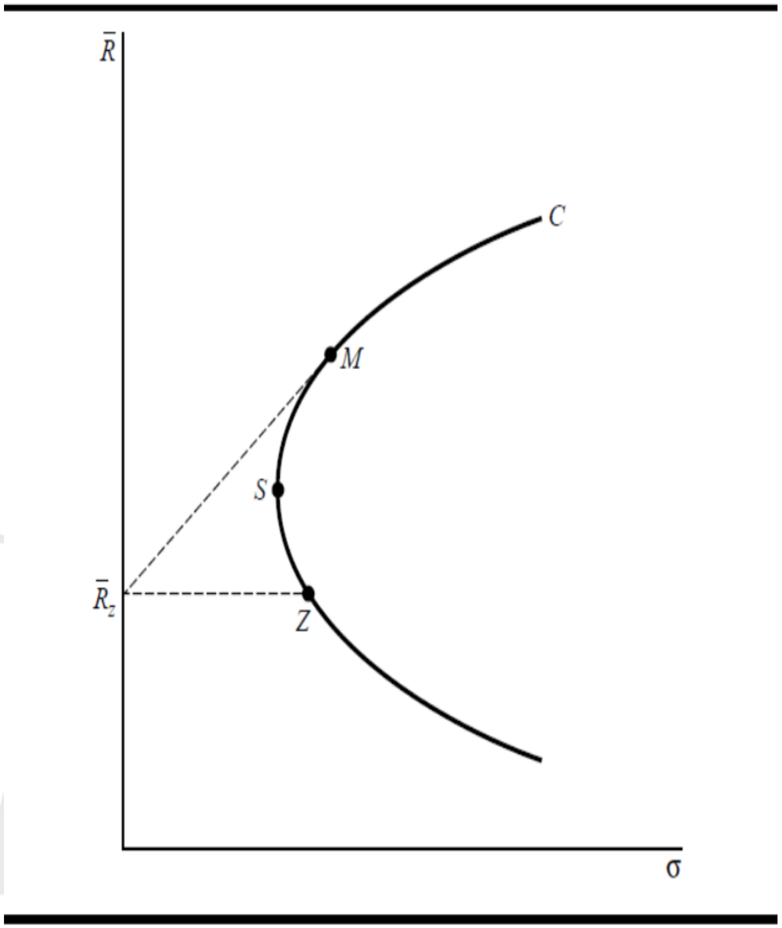


The minimum-variance frontier.

No riskless lending or borrowing: rigorous derivation

- The covariance between the zero-beta portfolio and market portfolio is zero
- To minimize the variance, we take the first derivative and set it equal to zero.
- $\frac{d\sigma_S^2}{dX_Z} = 2X_Z\sigma_Z^2 - 2\sigma_M^2 + 2X_Z\sigma_M^2$ =0;

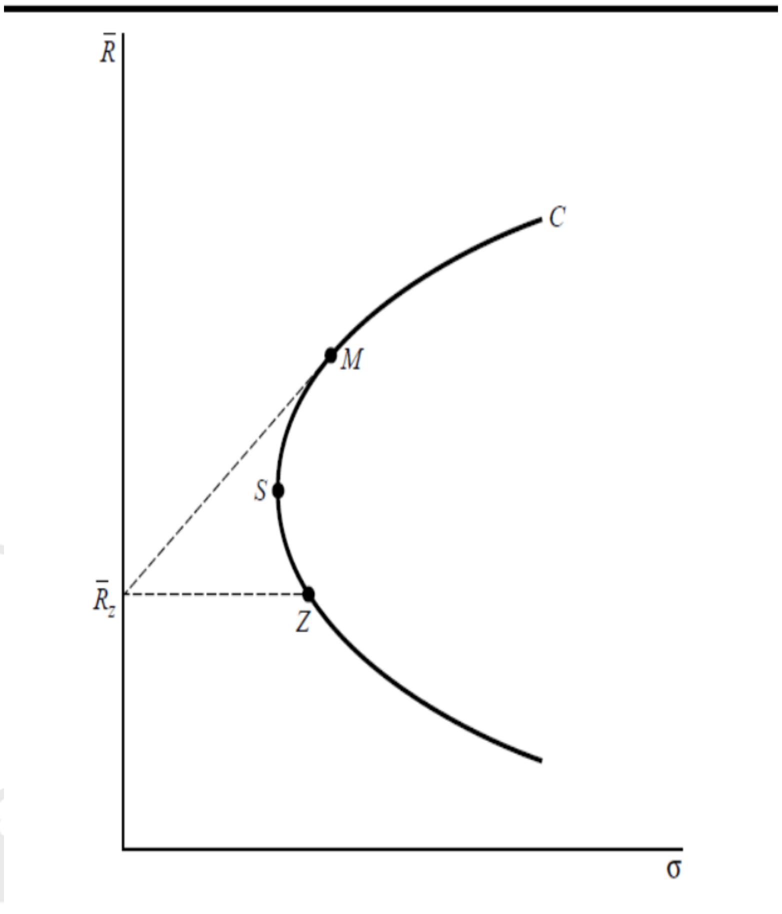
  solving for X_Z
- $X_Z = \frac{\sigma_M^2}{(\sigma_M^2+\sigma_Z^2)}$



The minimum-variance frontier.

No riskless lending or borrowing: rigorous derivation

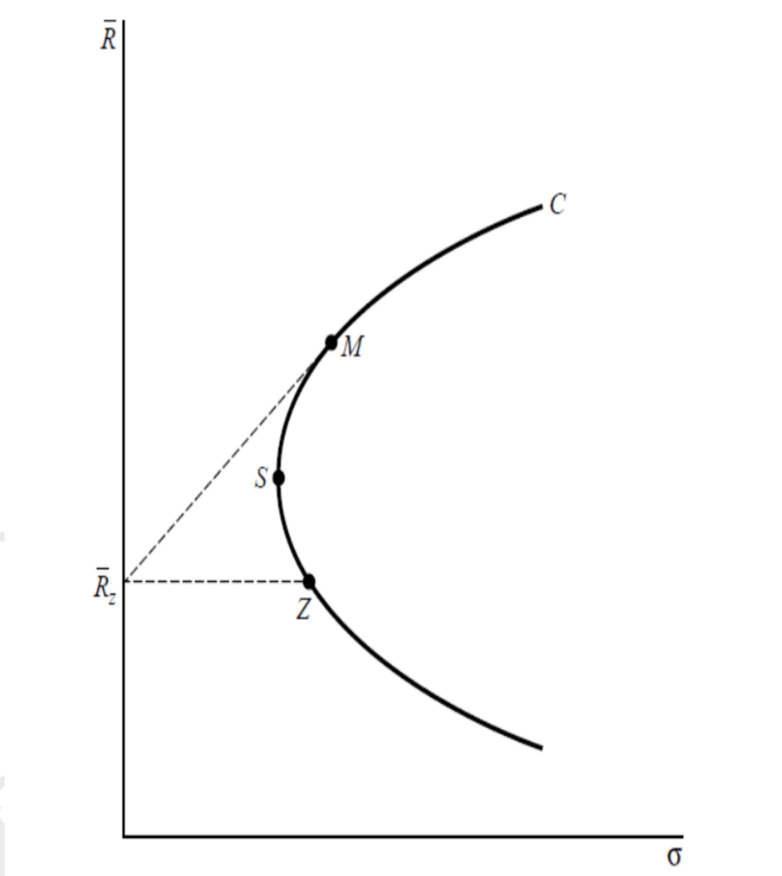
- $X_Z = \frac{\sigma_M^2}{(\sigma_M^2 + \sigma_Z^2)}$
- X_Z is a positive fraction
- S will be above Z portfolio
- Since Z is below S, Z is not an efficient portfolio
- Its overall risk (σ) is more than S
- No investor will select Z on standalone basis
- All investors will hold efficient portfolio that lie along SMC curve



The minimum-variance frontier.

No riskless lending or borrowing: rigorous derivation

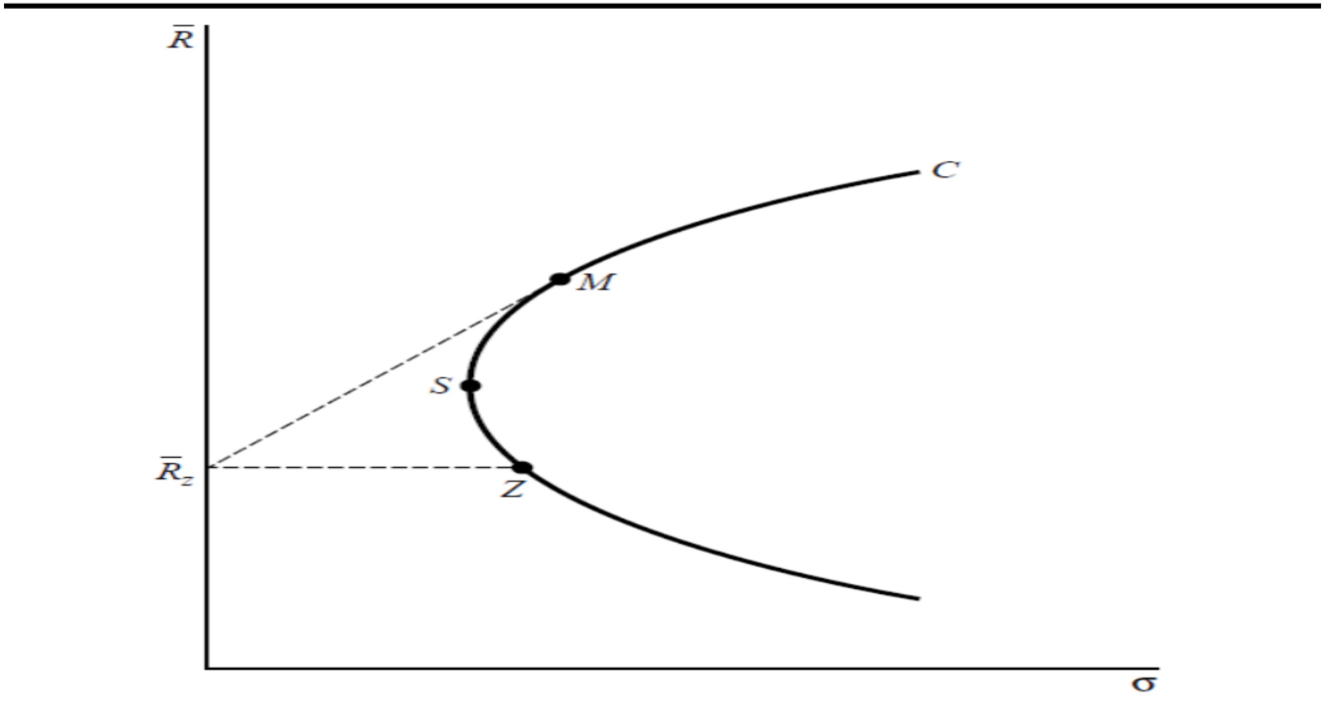
- Some of the investors may hold zero-beta portfolio (s) and market portfolio (R_M)
- They will lie between the segment (S−M)
- Portfolio that are to the right of market portfolio are constructed by shorting the Z portfolio and buying the market portfolio
- Since, on the aggregate investors hold market portfolio, the aggregate holdings of Z will be zero



The minimum-variance frontier.

No riskless lending or borrowing: rigorous derivation

- Overall, the investors will only hold two portfolio, the market portfolio and the minimum-variance zero-beta portfolio

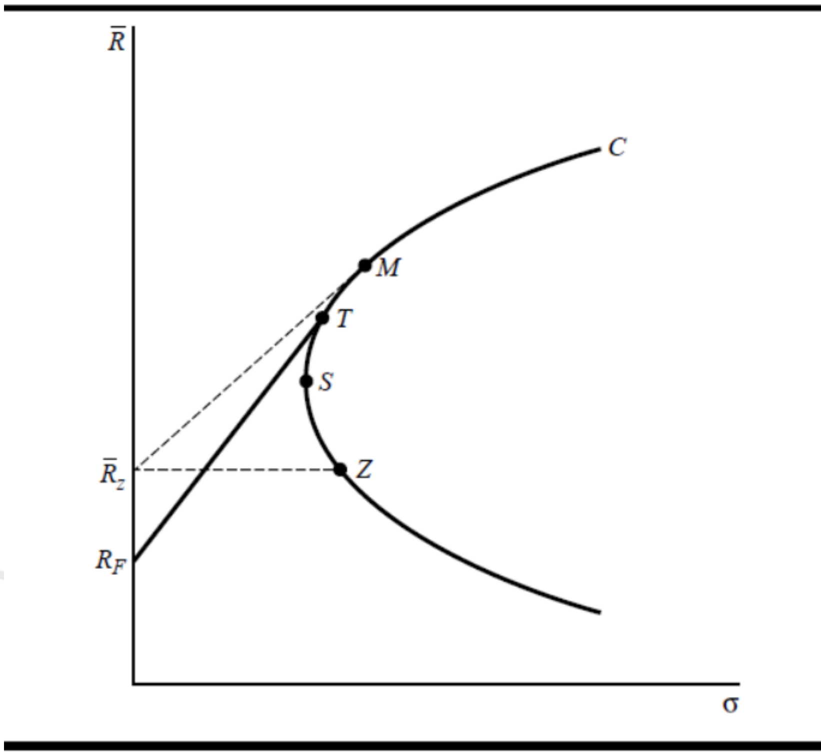


The minimum-variance frontier.

Riskless lending but no riskless borrowing

Riskless lending but no riskless borrowing

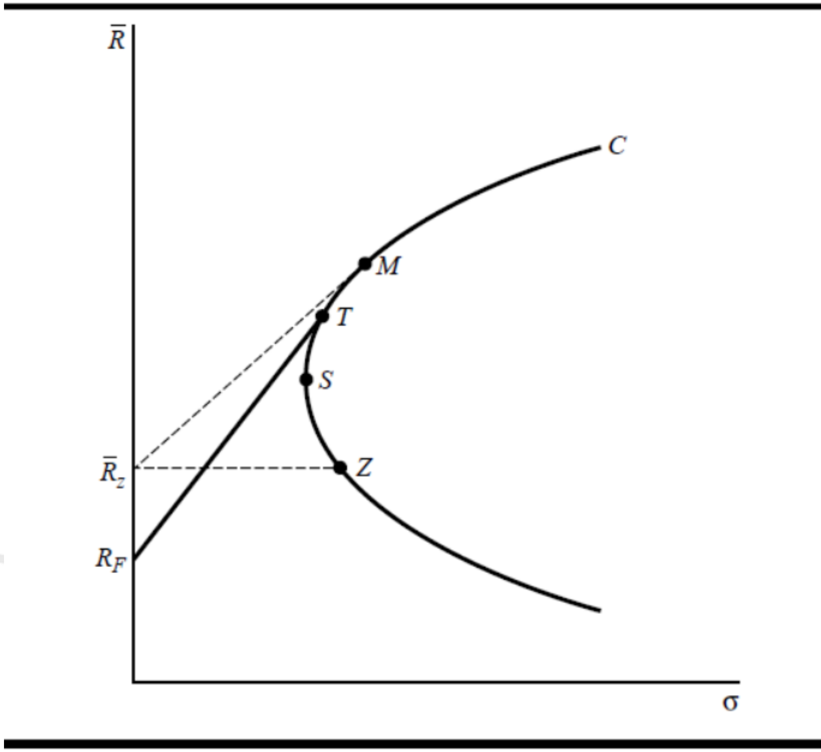
- It is realistic to assume that investors can lend in a riskless manner
- One set of efficient portfolios will lie on the line segment $R_F T$
- Here R_F is the riskless lending rate
- Rz, which is zero-beta portfolio, the tangent goes through the market portfolio
- Originally, SMC was the efficient frontier



The opportunity set with riskless lending.

Riskless lending but no riskless borrowing

- If the investor wants higher returns than T, he can hold any portfolio on the segment TC
- This also means that market portfolio will be above T (why?)
- This also means that Rz will be above R_F



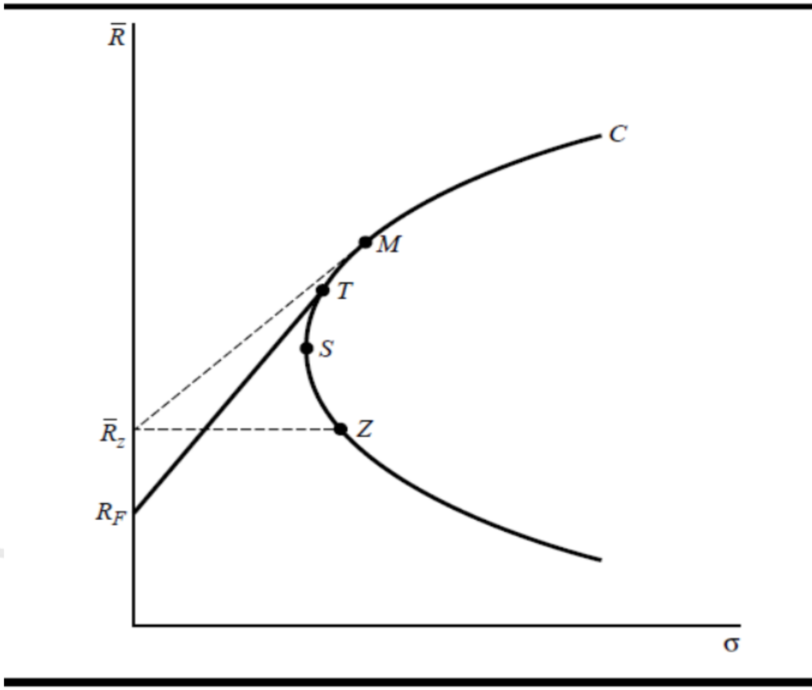
The opportunity set with riskless lending.

- Thus, the overall efficient frontier becomes the line segment $R_F - T$ and TMC

Riskless lending but no riskless borrowing

- Thus, the overall efficient frontier becomes the line segment $R_F - T$ and TMC

- Portfolio T can be obtained by combining market portfolio and zero-beta portfolio



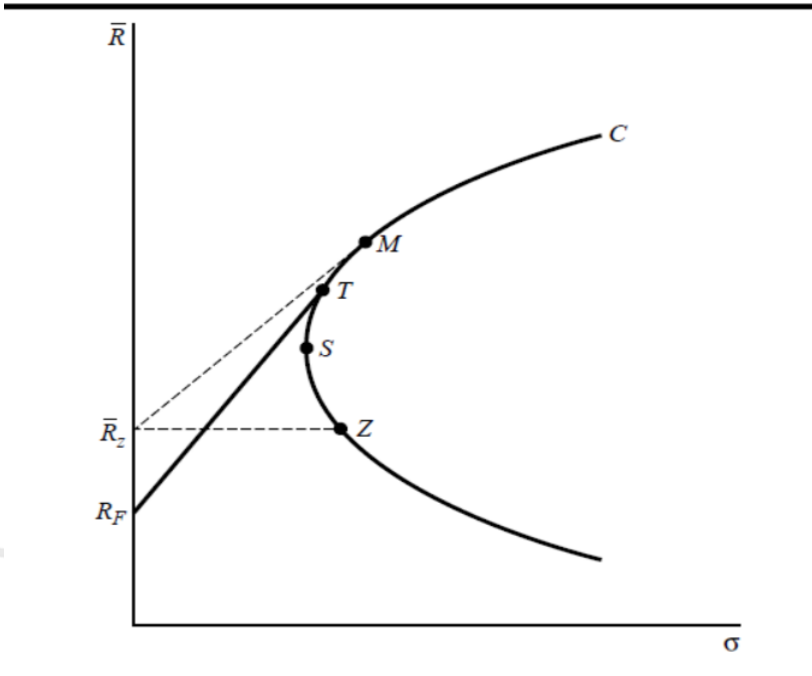
The opportunity set with riskless lending.

- All the investors on the line segment $R_F - T$ will use one riskfree asset and T portfolio of risky securities

- Those who are on TM, they are effectively putting their money in Z and M

Riskless lending but no riskless borrowing

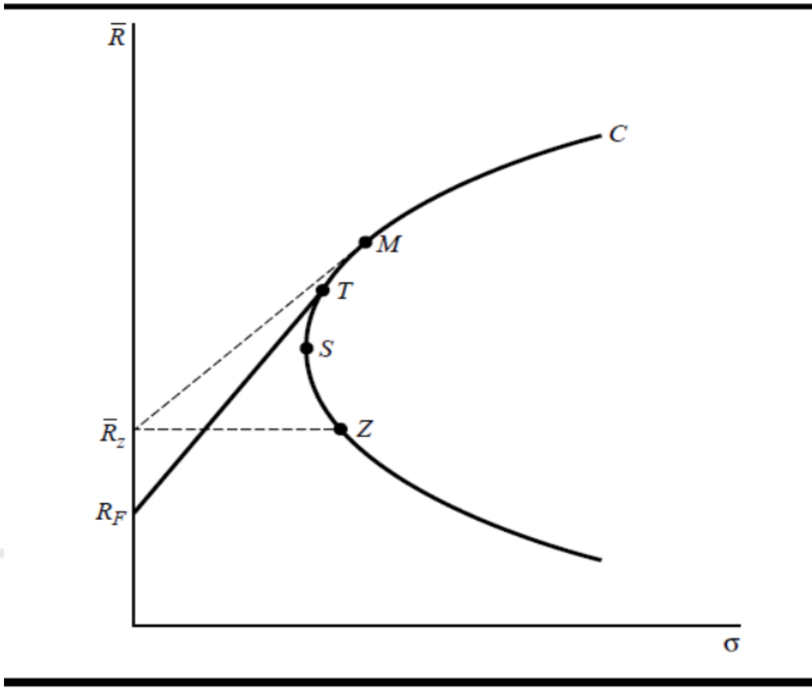
- Those who are selecting a position on the segment MC, they will be selling portfolio Z and investing in M
- In this special case, everybody will hold three portfolios:
  - Market portfolio,
  - Zero-beta portfolio
  - Riskless asset
- Combination of efficient portfolios Rf, T, and M are inefficient! This is in contrast to what we have said till now



The opportunity set with riskless lending.

Riskless lending but no riskless borrowing

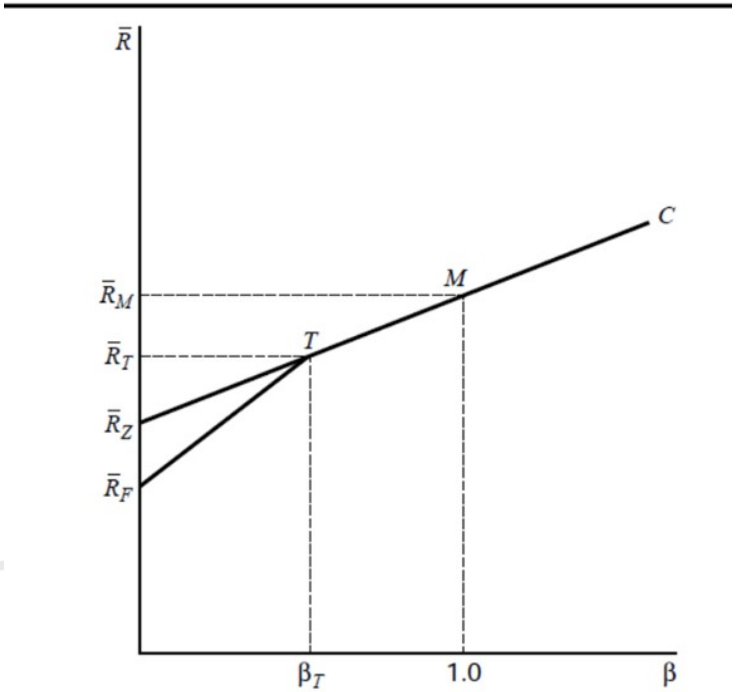
- Combination of efficient portfolios Rf, T, and M are inefficient!
- This is in contrast to what we have said till now , i.e., combination of efficient portfolios were efficient
- Why?
-



The opportunity set with riskless lending.

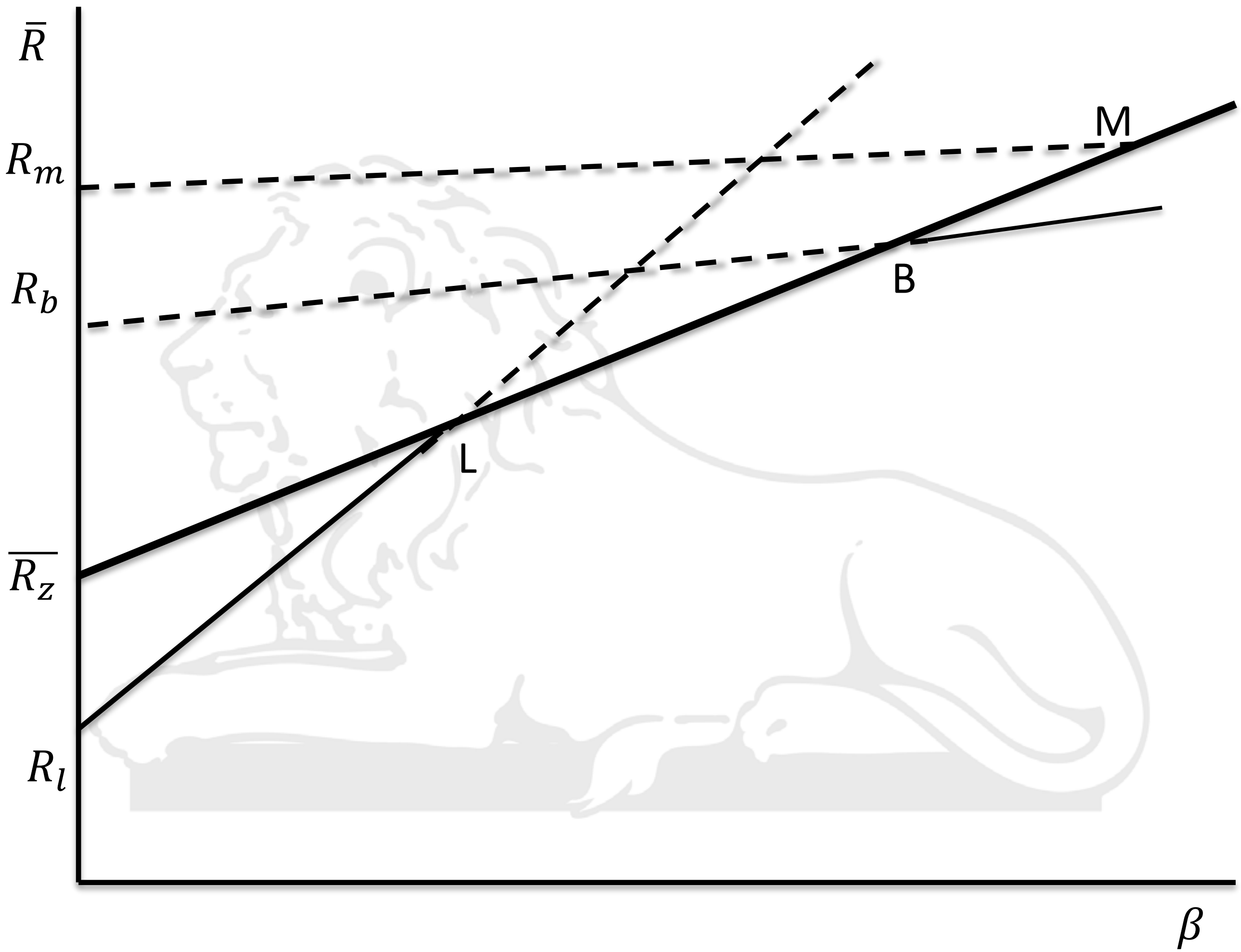
Riskless lending but no riskless borrowing

- Let us discuss SML
- Market portfolio M remains an efficient portfolio
- The expected return of all the securities contained by M are given by the following equation
- $\bar{R}_j = \bar{R}_Z + \beta_j(\bar{R}_M - \bar{R}_Z)$
- This line RzTMC describes only pure-risky assets
- The portfolios that combine riskless asset, the relevant line is R_F-T



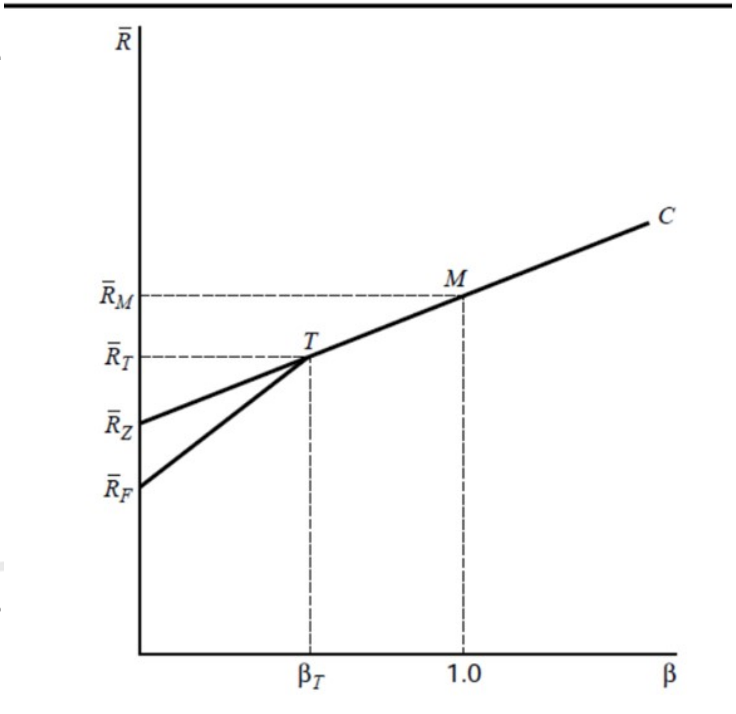
The location of investments in expected return beta space.

Riskless lending but no riskless borrowing: SML



Riskless lending but no riskless borrowing

- Overall, the efficient portfolios can be described by the line segments R_FT and TC
- This is because, the investors who will lend, will be holding portfolio T
- This means that many risky-securities and assets (on R_ZT) may have higher returns than the efficient portfolios (R_FT), for a given level of beta



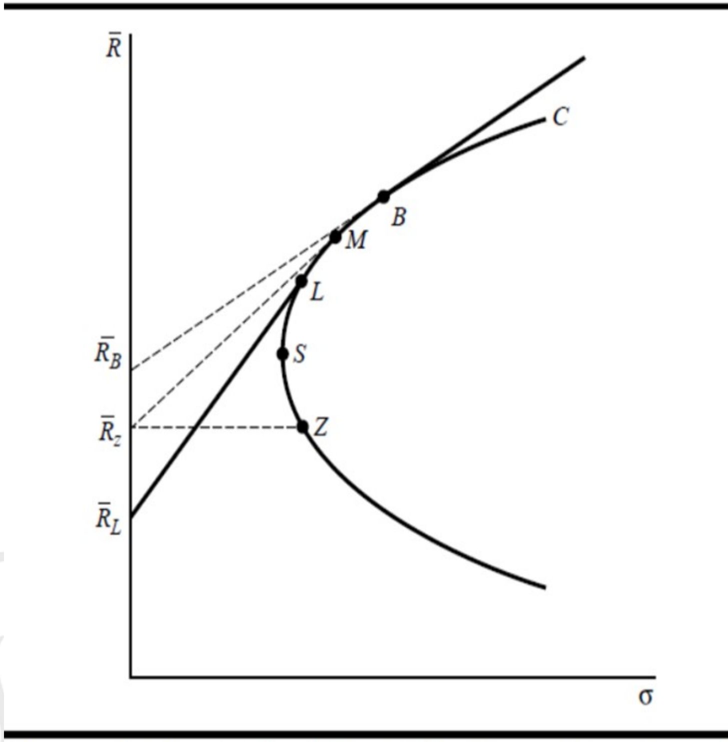
The location of investments in expected return beta space.

- But please note that R_FT portfolios have lower standard deviation than R_ZT
- Important to note about these models is that they allow investors to hold other securities (in addition to the market portfolio)

Different riskfree lending and borrowing rates

Different riskfree lending and borrowing rates

- L is the portfolio of risky securities held by investors who lend money
- B is the portfolio of securities held by the investor who borrow money
- Market portfolio lies on the efficient frontier somewhere between the portfolios L & B
- The only risky securities/portfolios that are held by the investor are L and B

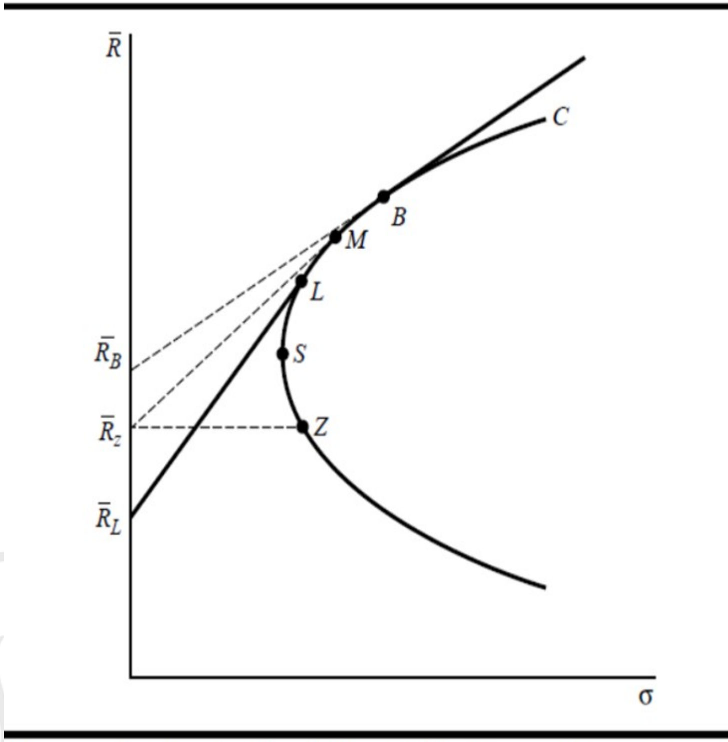


The opportunity set with a differential lending and borrowing rat

- Since, market portfolio is a weighted average of all the portfolios and no security is shorted in the equilibrium it will lie in between the portfolios L and B

Different riskfree lending and borrowing rates

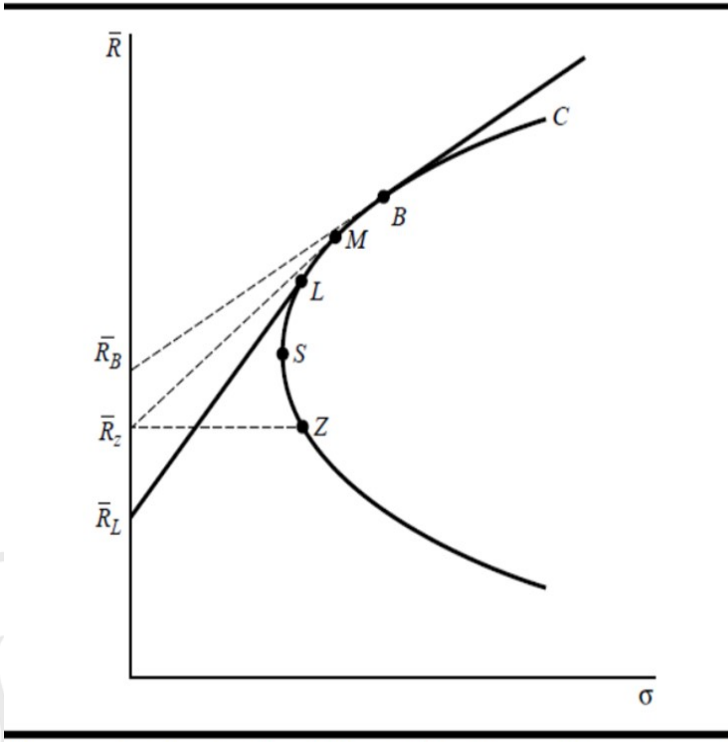
- L and B are efficient portfolios
- All the risky portfolios between L and B, that lie on the efficient frontier (including the market portfolio), can be created by combining L and B
- Since, we are able to identify an efficient market portfolio the following equation holds
- $\bar{R}_j = \bar{R}_Z + \beta_j(\bar{R}_M - \bar{R}_Z)$
- However, now this equation only defines those securities and portfolios that do not include riskfree assets



The opportunity set with a differential lending and borrowing rat

Different riskfree lending and borrowing rates

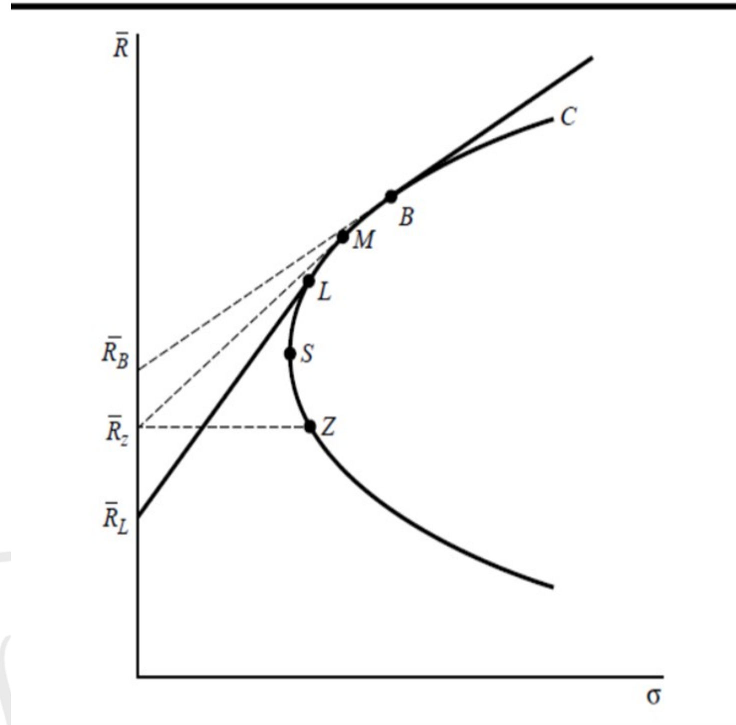
- However, now this equation only defines those securities and portfolios that do not include riskfree assets
- Those efficient portfolios that lie on the line segment R_LL (lending) and R_BB (Borrowing), indicating the combination of risky and riskfree assets will not be described by this security market line



The opportunity set with a differential lending and borrowing rat

Different riskfree lending and borrowing rates

- However, now this equation only defines those securities and portfolios that do not include riskfree assets
- Those efficient portfolios that lie on the line segment R_LL (lending) and R_BB (Borrowing), indicating the combination of risky and riskfree assets will not be described by this security market line



The opportunity set with a differential lending and borrowing rat

Personal taxes

Personal taxes

- CAPM ignores personal taxes
- That is, investors are indifferent to receiving income from capital gains or dividends
- Real-life examples suggest that capital gains (long term) are often charged at lower rates than the tax rates on dividend income
- This should impact the equilibrium prices, as investors judge their portfolio returns after taxes
- The general equilibrium relationship in the presence of taxes is provided below
- $E(R_i) = R_F + \beta_i[(E(R_M) - R_F) - \tau(\delta_M - R_F)] + \tau(\delta_i - R_F)$

Personal taxes

- $E(R_i) = R_F + \beta_i[(E(R_M) - R_F) - \tau(\delta_M - R_F)] + \tau(\delta_i - R_F)$
- Where
- δ_M =the dividend yield (dividends divided by price) of the market portfolio
- δ_i =the dividend yield for stock i
- τ = a complex weighted average of capital gains tax and dividend tax, investor wealth and other taxes
- τ is a positive number
- The expected return is an increasing function of dividend yields
- If a larger fraction of returns are paid in the form of dividends, then investors will pay more taxes, and in turn, may expect higher returns

Personal taxes

- $E(R_i) = R_F + \beta_i[(E(R_M) - R_F) - \tau(\delta_M - R_F)] + \tau(\delta_i - R_F)$
- Where
- The negative signed R_F along with dividend yield indicates the interest on lending and borrowing
- There are two ways to think of the negative sign of R_F
- First, R_F is essentially the opportunity cost of investing in stocks
- Second, in addition to investing in stock one can also borrow to the extent that the interest on borrowed capital is equivalent to the dividend payment
- Essentially, now our investment is converted from a dividend paying stock to a capital gains stock

Personal taxes

- As compared to the original CAPM, there is a new variable in our security market line- the **dividend yield on the security *i*,** $(\boldsymbol{\delta_i})$
- The equilibrium must be described in a three dimensional space (R_i, β_i, δ_i), instead of two-dimensional space of return-beta
- For a given beta, the expected return will go up as dividend yield goes up
- For any given yield, the expected return goes up and the beta goes up
- One can derive a set of efficient and optimal portfolios that are a function of tax rates on dividends, capital gains, and interest income

Personal taxes

- The investor in low marginal tax bracket, will prefer high-dividend paying stocks as compared to the percentage (or proportion) of these stocks in the market portfolio
- Because, the disadvantage of taxation on dividends is less harmful to these investors
- Vice-versa analysis will apply for the investors in high marginal tax bracket

Non-Marketable assets

Non-Marketable assets

- CAPM assumes that all assets are readily marketable
- The implication is that each investor can freely adjust her portfolio to optimum
- Is it? Regulations may prohibit investment into Tobacco stocks
- In reality there may be securities that may not be marketable
- Assuming that non-marketable assets are a substantial portion of overall market portfolio
- We can consider non-marketability as a sort of additional risk

Non-Marketable assets

- If the world can be divided into non-marketable and marketable assets, then equilibrium can be defined as

$$E(R_j) = R_F + \frac{E(R_M) - R_F}{\sigma_M^2 + P_H / P_M \operatorname{cov}(R_M R_H)}\left[\operatorname{cov}(R_j R_M) + \frac{P_H}{P_M}\operatorname{cov}(R_j R_H)\right]$$

- R_H is the return on non-marketable assets, P_H is the total value of non-marketable assets, P_M is the value of the marketable assets
- Contrast this with simple CAPM below

$$E(R_j) = R_F + \frac{E(R_M) - R_F}{\sigma_M^2}\left[\operatorname{cov}(R_j R_M)\right]$$

- With the inclusion of non-marketable asset the risk-return trade-off has changed

Non-Marketable assets

- The new risk-return trade-off function is $$\frac{E(R_M)-R_F}{\sigma_M^2+\frac{P_H}{P_M}\operatorname{cov}(R_M R_H)}$$
- Instead of $$\frac{E(R_M)-R_F}{\sigma_M^2}$$
- It seems a lower return-risk trade-off is suggested as compared to the simple CAPM
- It depends upon covariance of the non-marketable asset with the marketable assets
- Also depends upon the relative value of the non-marketable asset vis-à-vis the marketable assets
- For example, if the relative value is very less or the covariance is very less then there is no harm in using the standard CAPM

Non-Marketable assets

- Also the definition of the risk has been changed $\left[\operatorname{cov}\left(R_j R_M\right)+\frac{P_H}{P_M}\operatorname{cov}\left(R_j R_H\right)\right]$
- In addition to the covariance of asset with the marketable assets, it also depends upon the covariance of the asset with non-marketable assets and the proportion of all the marketable assets relative to the non-marketable assets
- Any asset that has a higher positive correlation with non-marketable asset will be risker than that implied by the standard CAPM
- Equilibrium returns can be higher or lower (than the standard CAPM)
- If the asset has a negative covariance with the non-marketable asset then its equilibrium return will be lower

Non-Marketable assets

$$E(R_j) = R_F + \frac{E(R_M) - R_F}{\sigma_M^2 + P_H / P_M \operatorname{cov}(R_M R_H)} \left[\operatorname{cov}(R_j R_M) + \frac{P_H}{P_M} \operatorname{cov}(R_j R_H) \right]$$

- However, if the return is positively correlated with the return on nonmarketable assets then it will depend upon the following of which dominates the overall return
  - Increased risk of the asset
  - or the decreased price of risk

Non-Marketable assets

- First, each investor will hold a mutual fund that has equal and opposite covariance with each marketable asset, to the covariance between each of these marketable assets and his nonmarketable assets
- This portfolio has different composition depending upon their holdings of the non-marketable assets
- This portfolio aims to diversify the risk of non-marketability with the nonmarketable asset
- The second asset is risk-free asset and third is a fund which is difference between the market portfolio and aggregate of all the investments in the first fund
- The second and third asset are the same across all investors

Non-Marketable assets

- In this model, they also hold three portfolios: (a) **a portfolio that has an equal and opposite correlation with each marketable asset to that of his non-marketable portfolio with each marketable asset,** (b) a riskless asset, (c) Market portfolio minus aggregate portfolio (b) with all the investors
- For all the investors, the first portfolio is different (and depends upon their holding of the non-marketable asset)
- This provides useful insights into missing asset problem. Empirical tests of equilibrium models are often conducted with market defined as something less than the full set of assets in economy. The real holdings of assets can tell us the presence of hidden/non-marketable assets

Heterogeneous expectations

Heterogeneous expectations

- A very important assumption of CAPM is homogenous expectations
- This means that all individuals have the same expectations of risk and returns (and correlation) for any given security
- The implication is that all the investors are expected to arrive at the same efficient frontier and same market portfolio
- Thus, without this assumption, for each individual an equilibrium return equation is drawn (from expected returns, variances, and covariances) factoring in his individual utility functions

Heterogeneous expectations

- If one were to aggregate all the individual return expectations, it will become a complex weighted average of expected return, variances, and covariances of individual investor estimates
- This is problematic because these complex weightings involve information about individual investor utility functions
- This, in turn, requires information about risk-return trade-off (marginal rate of substitution between return and variance), which eventually is a function of wealth (or prices)

Heterogeneous expectations

- That is prices are required to determine risk-return trade-off that in turn will determine utility functions and therefore the minimum variance frontier, then equilibrium pricing model, and then prices
- Some researchers do create testable models under heterogeneous expectations, by defining the (or putting restrictions) on the form of heterogeneity
- One approach is to consider a utility function using constant risk aversion (independent of wealth), this leads to a form of CAPM very similar to Standard CAPM

Heterogeneous expectations

- Second, for example, Gonedes (1976) assumes that each firm is a combination of economic activities. If there is disagreement about these economic activities that represent the firm, it may be a major source of heterogeneous expectations
- Under this condition he finds that minimum-variance frontier is the same for all the individual investors
- There is a market portfolio that is efficient for each investor. Also that there is a linear return-beta relationship similar to the original CAPM.

Non-Price taking behaviour

Non-Price taking behaviour

- In the derivation of CAPM, we also assumed that individual investors are simply price takers
- The evidence suggests that large mutual funds, index funds, and pension funds do affect prices through their trades
- The price affector optimizes her trading decisions to maximize her utility
- She evaluates the equilibrium prices that will result from his or her own action
- Theory suggests that all the investors including the price affector should hold market portfolio along with the riskless asset

Non-Price taking behaviour

- However, these price affectors with their knowledge and resources acquire more information about securities and hold large amount of risky (market) portfolio and less of riskless asset
- She, thus, increases her utility of the portfolio. Since she still holds the riskless asset along the market portfolio, we again get the same simple form of CAPM
- Only that, the proportion of risky asset in the aggregate holding is relatively higher and the price of risk is relatively lower (expected returns per unit of risk)
- This analysis also gives the reason for the existence of large institutional traders and portfolio investors

Multi-period CAPM

Multi-period CAPM

- CAPM assumes that all the investment and consumption decisions have a single period horizon
- However, in reality, any investor holds a series of portfolios overtime to maximize her utility of lifetime consumption
- Thus, decision in a single period can not be taken altogether in an isolation
- We try to answer the following question: Are there any conditions over which the single-period CAPM models can adequately describe multi-period decision making

Multi-period CAPM

- The following assumptions (Fama, 1970) can reduce the multiperiod investment consumption decisions into the problem of maximizing a single period utility function
  - Investors act as if their tastes and preferences for consumption (of goods and services) are independent of future events and conditions
  - Investors act as if consumption opportunities of goods and their prices are known at the beginning and are not state dependent
  - The investor acts as if distribution of security returns are known at the beginning of the period (are not state dependent)

Multi-period CAPM

- In addition, if the assumptions of preferring more to less and risk aversion hold with respect to each period consumption, then the one period utility function has the same properties as one period consumption
- This helps obtain one-period utility function
- With these assumptions, single period CAPM models become appropriate for investors with multiperiod horizons
- If the additional assumptions are made as per the zero-beta CAPM version, then the zero-beta model becomes the appropriate version for investors with multiperiod horizon

Multi-Beta CAPM

Multi-Beta CAPM

- Investors solve the problem of lifetime consumption as they face multiple sources of uncertainty
- Investors form portfolio of securities to hedge away these risks
- Thus, these sources of risks also affect the expected returns on securities

$$\overline{R}_i - R_F = \beta_{iM}\left(\overline{R}_M - R_F\right) + \beta_{iI1}\left(\overline{R}_{I1} - R_F\right) + \beta_{iI2}\left(\overline{R}_{I2} - R_F\right) + \cdots$$

- The generalized model shown above represents the expected returns R_{I1} on a set of risk factors Ij's (proxied by portfolios)
- The model doesn't tell us about these factors

Multi-Beta CAPM

- In addition to market, there can be a number of risk sources that can be priced
- Then these risks, like beta, will also affect the expected returns on securities, a simple generalized model for multiple risks is provided below
- For example, default risk, inflation risk, profit risk, term-structure risk

Consumption CAPM

Consumption CAPM

- The model aims to capture the fact that investors may optimize their consumption over time (now and in future)
- The discount factors (Ri's) represent the cost of delaying the consumption to some point in future
- It is somewhat similar to multi-period CAPM
- The model provides the framework to value intertemporal financial claims
- Security prices are determined so that the expected value of the growth of a dollar invested discounted by the stochastic discount factor equals the value of a dollar today

Consumption CAPM

- The discount factors are observed from the security price behavior (observed returns)
- Often aggregate consumption is obtained with a lot of delay, this requires certain assumption about vector of factors affecting stock prices, which affect the consumption growth. These vector of factors can be used to proxy aggregate consumption on more timely basis

In conclusion

- Overall, it appears that single period CAPM is a fairly robust model
- Modifying some assumptions do not change the original form of the model
- While modifying other few assumptions lead to appearance of new terms or modification of old terms.
- More importantly, many of the conclusions/implications of the original Standard CAPM even hold with changes in CAPM

Thanks

Empirical Tests of CAPM

Course: Portfolio management

Instructor: Abhinava Tripathi

Testing the CAPM: Expectations approach

- Most of the standard testing methods entail standard CAPM or zero beta version of CAPM
- $E(R_i) = R_F + \beta_i[E(R_M) - R_F]$
- $E(R_i) = R_Z + \beta_i[E(R_M) - R_Z]$
- Here beta and expected returns indicate future expectations
- How can we measure and test future expectations
- Application of historical averages in measuring expectations

Testing the CAPM: Realized returns approach

- The second set starts by assuming a liner relationship between market and security returns
- This helps in **using realized returns** instead of expected returns
- We start by measuring our beta of the portfolio using single-index/market model shown below
- $R_{it} = \alpha_i + \beta_i(R_{mt}) + e_{it}$
- This requires three assumptions
  a) The market model holds in every period;
  b) The CAPM model holds in every period;
  c) The beta is stable over time

$$R_{it} = R_F + \beta_i(R_{mt} - R_F) + e_{it}$$

Testing the CAPM: Realized returns approach

- This requires three assumptions

  a) The market model holds in every period;

  b) The CAPM model holds in every period;

  c) The beta is stable over time

  $R_{it} = R_F + \beta_i(R_{mt} - R_F) + e_{it}$

  – A test of this model is simultaneous test of all these three hypothesis

  – Instead of standard CAPM, if we had assumed two-factor zero beta mode: $R_{it} = R_{Zt} + \beta_i(R_{mt} - R_{Zt}) + e_{it}$ would be obtained

Testing the CAPM: Hypothesis Building

- Depending on whether you believe in the simple CAPM or the two-factor general equilibrium model, the following hypothesis should hold

    (a) Beta (or higher risk) should be associated with a higher level of returns

    (b) Returns are linearly related to beta

    (c) Markets do not reward for bearing non-market risk

- In addition, Whatever the form of the general equilibrium model holds, investing is fair game
- That is, deviations from the model are purely random
- In addition, one can formulate specific hypothesis that applies and differentiates between standard CAPM and zero-beta versions (slope and intercept)

Testing the CAPM: Exploratory testing

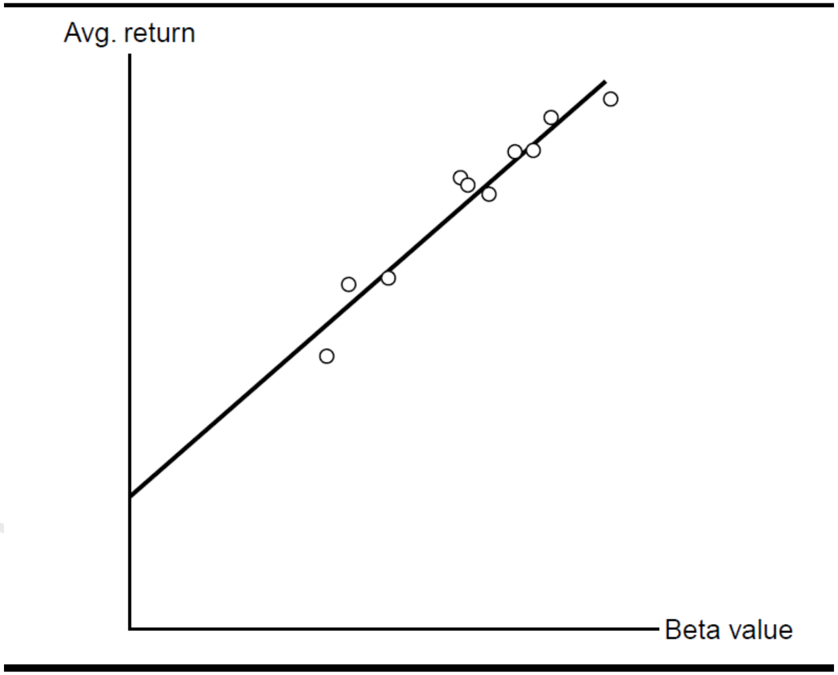
- **Whether higher return is associated with higher risk [Sharpe and Cooper (1972)]?**
- We start by measuring our beta of the portfolio using single-index/market model: $R_{it} = \alpha_i + \beta_i(R_{mt} - R_F) + e_{it}$
- If riskless rate is not available then Rz is employed
- **Sharpe and Cooper (1972) constructed** ten beta sorted portfolios by estimating betas for the previous 60 months (period 1931 to 1967, all the NYSE stocks)
- The regression model is employed on portfolios ranked according to their beta
- These 10 stock portfolios (equally weighted manner) were held for that one year and average returns for each of these portfolios were computed

Testing the CAPM:
Average returns on 1931-1967

Average Return	Portfolio Beta
22.67	1.42
20.45	1.18
19.116	1.14
21.77	1.24
18.49	1.06
19.13	0.98
18.88	1.00
14.99	0.76
14.63	0.65
11.58	0.58

Testing the CAPM

- They found a linear relationship shown below
- $E(R_i) = 5.54 + 12.75\beta_i$ to explain 95% variation in returns
- This early version also confirmed the CAPM postulation that returns are positively related to beta
- The intercept of 5.54% was higher than the risk-free rate (2%) and indicated that zero-beta version was more appropriate



Estimated security market line.

Testing the CAPM: Early Test

- More advanced tests of CAPM included the following format [Douglas and Lintner]
- A time-series regress called first pass regression was employed to estimate beta as follows
- $R_{it} = \alpha_i + b_i(R_{mt}) + e_{it}$ (For each security separately)
- Once b_i was estimated the following cross-sectional regression model was employed
- $\bar{R}_i = \alpha_1 + a_2 b_i + a_3 S_{ei}^2 + n_{it=T}$ (in a cross-section for all the securities)
- Here S_{ei}^2 is the variance of residual e_{it}
- What are the hypothesis for α_1, α_2, α_3?

Testing the CAPM: Early Test

- $R_{it} = \alpha_i + b_i(R_{mt}) + e_{it}$ (For each security separately)
- Once b_i was estimated the following cross-sectional regression model was employed
- $\bar{R}_i = \alpha_1 + a_2 b_i + a_3 S_{ei}^2 + n_{it}$ (in a cross-section for all the securities)
- Here S_{ei}^2 is the variance of residual e_{it}
- Here theory suggests that
- α_1 can be Rf or Rz; a_2 can be $\bar{\text{R}}\text{m}$-Rf or $\bar{\text{R}}\text{m}$-Rz; a_3=0
- Most of the results concur with the zero-beta version of the model
- Problem of error in beta estimates: use po

Testing the CAPM: Tests of Fama and McBeth

- Fama and McBeth formed 20 portfolios of securities to estimate betas from a first-pass regression
- $R_{it} = \alpha_i + b_i(R_{mt}) + e_{it}$ (For each portfolio separately)
- Once b_i was estimated the following cross-sectional regression model was employed (1935–1968) [Second pass each month]
- $R_{it} = \gamma_{0t} + \gamma_{1t}\beta_i + \gamma_{2t}\beta_i^2 + \gamma_{3t} * S_{ei} + n_{it}$
- Here $S_{ei}$is the variance of residual e_{it}

Testing the CAPM

- $R_{it} = \alpha_i + b_i(R_{mt}) + e_{it}$ (For each security separately)
- Once b_i was estimated the following cross-sectional regression model was employed
- $R_{it} = \gamma_{0t} + \gamma_{1t}\beta_i + \gamma_{2t}\beta_i^2 + \gamma_{3t} * S_{ei} + n_{it}$ (in a cross-section for all the securities)
- This form of the equation allows the test of a series of hypotheses regarding the CAPM. The tests are as follows:
- $E(\gamma_{3t}) = 0$, or residual risk does not affect return
- $E(\gamma_{2t}) = 0$, or there are no nonlinearities in the security market line.
- $E(\gamma_{1t}) > 0$, that is, there is a positive price of risk in the capital markets

Testing the CAPM

- Only if both (1) and (2) are true, then one can examine both $E(\gamma_{0t})$ and $E(\gamma_{1t})$ to see whether standard CAPM or zero-beta model is a better description of market returns
- We can examine all of the coefficients and the residual term to see if the market operates as a fair game
- For example, if the standard CAPM or the zero-beta model holds, then, regardless of the prior values of γ_{2t} and γ_{3t}, each of their expected values at time t+1 should be zero, regardless of their values at t or earlier periods

Testing the CAPM

- Furthermore, if the zero-beta model is the best description of general equilibrium, then deviations of γ_{0t} from its mean $E(R_Z)$ and γ_{1t} from its mean $E(R_M)$- $E(R_Z)$ are random
- Also, when beta is estimated in individual securities, the errors are high and some of the beta related risk is loaded on the error terms and appears as a relationship between the residual and returns
- Many studies account for this problem by estimating betas in portfolios

Thanks

Arbitrage Pricing Theory (APT)

Course: Portfolio management (PM)

Instructor: Abhinava Tripathi

Arbitrage pricing theory (APT)

- CAPM had its genesis in the mean-variance analysis
- Investors choose the optimum diversified portfolio on efficient frontier based on the expected return and variance analysis
- The Arbitrage pricing theory (APT) of Ross (1966, 1977) employs multifactor (alternatively called multi-index) approach to explain the pricing of assets
- It relies on the Single/multi-index approach to provide the return generating process

Arbitrage pricing theory (APT)

- Using that return generating process, APT derives the definition of expected returns in equilibrium with certain assumptions
- At the heart of this approach is the arbitrage argument (and thus the name), similar to that employed in the CAPM
- Two items with the same risk cannot sell at different prices
- APT is more generic than CAPM, in the sense that it does not assume that only expected return and risk affect the security prices

Arbitrage pricing theory (APT)

- Though the assumption of homogenous expectations remains
- Instead of mean-variance framework, we make assumptions about the return generating process
- APT argues that returns on any stocks are linearly related to a set of indices

Arbitrage pricing theory (APT)

- APT argues that returns on any stocks are linearly related to a set of indices
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ \ldots\ldots + b_{ij}I_j + e_i$
- Here,
- a_i is the expected level of return on the stock 'i' if all indices have a value of zero.
- I_j is the value of jth index that affects the return on stock i.
- b_{ij} is the sensitivity of stock i's return the to jth index.
- e_i is a random error term with a mean of zero and variance equal to σ_{ei}^2
- Essentially, the above equation describes the process that generates security returns

Arbitrage pricing theory (APT)

- APT argues that returns on any stocks are linearly related to a set of indices
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots\ldots + b_{ij}I_j + e_i$
- For the above model to be more accurate the following assumptions are made
- $E(e_i e_k) = 0$; for all i and k where i≠k.
- $E\left[e_i\left(I_j - \bar{I}_j\right)\right] = 0$ for all the stocks and indices.
- It is an extension of multi-index family of models

A simple proof of APT

- Suppose the following two-index model describes the returns
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$; also consider that $E(e_i e_j) = 0$
- Here, each index represents certain systematic risk
- Now if, the investor holds A well diversified portfolio, only the systematic risk – represented by the indices I_1 and I_2 will matter
- The residual risk captured by σ_{ei}^2 will be close to zero
- The sensitivity of the portfolio to these two components of the systematic risk is represented by b_{i1} and b_{i2}.

A simple proof of APT

- Consider the three well-diversified portfolios shown below

Portfolio	Expected return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- The returns are provided at equilibrium: No arbitrage
- Remember our discussion of CAPM with sensitivity towards a single index (market portfolio), where all the securities in equilibrium were lying on a straight line (two axes: R, b_1).
- Here, since we have two sensitivities (two betas with respect to each axis), we can safely assume that these three portfolios will lie on a plane (three axes: R, b_{i1}, b_{i2}).

A simple proof of APT

- Consider the three well-diversified portfolios shown below

Portfolio	Expected return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- Here, since we have two sensitivities (two betas with respect to each axis), we can safely assume that these three portfolios will lie on a plane (three axes: R, b_{i1}, b_{i2})
- The generic equation for a plane is as follows: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- Can we solve for the values of $\lambda_0, \lambda_1, and\ \lambda_2$ using the values provided in the table

A simple proof of APT

- We get the following equation
- $\bar{R}_i = 7.75 + 5b_{i1} + 3.75b_{i2}$
- Now consider a third portfolio E with expected returns of 15%, b_{i1}=0.6 and b_{i2}=0.6

Portfolio	Expected return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- Compare E with another portfolio D that places 1/3rd in A, B, C
- What is my expected return and sensitivities of D, and are there arbitrage opportunities

A simple proof of APT

- For D we get the following Figures
- $b_{p1} = \frac{1}{3} * (1.0) + \frac{1}{3}(0.5) + \frac{1}{3}(0.3)$=0.6
- $b_{p2} = \frac{1}{3} * (0.6) + \frac{1}{3}(1.0) + \frac{1}{3}(0.2)$=0.6
- $\bar{R}_D = \frac{1}{3}(15) + \frac{1}{3}(14) + \frac{1}{3}(10)$=13
- D has an identical risk profile as E, but offers a lower return
- We could have also computed the expected return on $\bar{R}_D$ using the equation of plane
- $\bar{R}_D = 7.75 + 5b_{D1} + 3.75b_{D2}$=$7.75 + 5 * 0.6 + 3.75 * 0.6 = 13$

A simple proof of APT

- By the arbitrage argument (or law of one price), two portfolios with the same risk, can not sell at different prices (or have different expected returns)
- Arbitrageurs (or investors in general) would buy E and sell D short
- This would guarantee riskless profit (2%)
- This will continue until E falls back on the plane defined by A, B, C
- The plane that we draw on expected return, b_{i1}, b_{i2} space
- If any security (like E) is undervalued/overvalued it will be above or below this plane
- This would lead to arbitrage opportunity, and such securities will converge back to this plane

A simple proof of APT

- The general equation of the plane in the return, b_{i1}, and b_{i2} space is shown below
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- This is the equilibrium model provided by APT when the returns are generated by a two-index model
- Here λ_1 and λ_2 are the increase in returns for one unit increase b_{i1} and b_{i2}
- Essentially, λ_1 and λ_2 reflect the returns for bearing the risks associated with the indices I_1 and I_2
- Consider a zero b_{ij} portfolio with no sensitivity to either indices

A simple proof of APT

- Consider a zero b_{ij} portfolio with no sensitivity to either indices (something similar to zero beta portfolio in CAPM)
- If it has no risk then it should offer riskfree return $\lambda_0 = R_F$
- In case, the riskless rates are not available then instead of R_F, we denote it by $\bar{R}_Z$, i.e., the return on the zero beta portfolio
- Imagine a portfolio that mimics index 1, and therefore, has a b_{i1}=1
- Also that it is not sensitive to I_2, and therefore, has a b_{i2}=0

A simple proof of APT

- For this portfolio, the Eq [$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$] becomes
- $\bar{R}_1 = R_F + \lambda_1$; here $\lambda_1 = \bar{R}_1 - R_F$
- Similarly λ_2 =$\bar{R}_2 - R_F$
- The above analysis can be generalized to a J index case shown below
- $\boldsymbol{\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}}$
- $\lambda_0 = R_F\ and\ \lambda_j = \bar{R}_j - R_F$; where the return generating process can be described as
- $\boldsymbol{R_i = a_i + b_{i1} I_1 + b_{i2} I_2 + \ \ldots \ldots + b_{ij} I_j + e_i}$

A simple proof of APT

- The above analysis can be generalized to a J index case shown below
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- $\lambda_0 = R_F \ and \ \lambda_j = \bar{R}_j - R_F$
- The derivation assumes here that both the indices are orthogonal
- In practical situations there are always correlations between the risk-factors represented by two indices
- Researchers orthogonalize both the indices to remove any common component. In that case the new indices may not be well defined

A simple proof of APT

- The straight line in the CAPM had two axes: Return and beta axes
- Thus two coordinates corresponding to each point that denotes a portfolio
- In the context of two-index APT model we have one return and two beta axes (for each index)
- Thus three coordinates that define the plane that is efficient frontier
- If a point is above (or below) this plane, this means that the security is under (or over) priced with respect to one or both of these indices
- Thus, it violates the law of one price

A simple proof of APT

- Arbitrageurs may conduct risk-less arbitrage by selling (or buying) the under (or over) priced portfolio and taking a counter position in the portfolios that are fairly priced

- This will drive the prices of inefficient portfolio towards this plane that is efficient frontier or efficient plane

- The implication of this riskless arbitrage is that all portfolios in the equilibrium would lie on this plane that is efficient frontier

- That is, in the space defined by three coordinates: expected return, b_{i1}, and b_{i2}

A few important points about APT

- In the context of CAPM, it was needed to identify the “market portfolio”, and therefore, all the risky assets
- While testing CAPM, one can always question whether all the securities are truly captured in the risky assets
- And therefore have we achieved the true market portfolio
- However, in the context of APT, arbitrage conditions can be applied to any security or portfolio
- Thus, it is not necessary to identify all the risky securities and market portfolio

A few important points about APT

- APT can very well be tested for small number of stocks for example, all the 50 stocks making-up the “Nifty-50” index
- Given this advantage with APT, many studies argue that the tests designed for CAPM, are actually, the tests of single-factor APT
- Since, they utilized a limited number of securities, which arguably may not capture the entire market
- The only caution need here is that the systematic influences (or indices/factors) affecting these set of stocks that are tested for APT should be adequately described
- This can be an issue when we have a large set-of securities. Then finding the adequate number of indices (or systematic influences) may become a challenge

A few important points about APT

- APT is extremely general
- It allows us to describe the equilibrium in terms of a Single/multi index model
- However, it does not define what would be the most appropriate multi-index model
- We do not know $\lambda's$ or I's
- They are generated from the data available (e.g., through factor analysis)
- For example, what risk factor a given index (I) indicates (inflation risk, market risk, etc.) that is not provided by the model
- So one does not have the direct specific economic rationale for the given factor

A more rigorous proof of APT

- Now we will try to derive the expected equilibrium return for two-index return generating process
- This derivation can be easily extended to multi-index return generating process
- Consider the two-index process below
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$ (1)
- Taking expectations of eq.
- $\bar{R}_i = a_i + b_{i1}\bar{I}_1 + b_{i2}\bar{I}_2$ (2)
- Subtracting Eq. (2) from Eq. (1) and with slight rearrangement
- $R_i = \bar{R}_i + b_{i1}(I_1 - \bar{I}_1) + b_{i2}(I_2 - \bar{I}_2) + e_i$

A more rigorous proof of APT

- Please remember the original fundamentals of APT here
- APT assumes a riskless arbitrage. That is no investment, no risk, and excess risk-adjusted return
- This leads to the following four equations for a 'N' security portfolio with two index case
- $\sum_{i=1}^{N} X_i * 1 = 0$ (3a: Indicating that long and short positions entail same investment)
- $\sum_{i=1}^{N} X_i b_{i1} = 0$ and $\sum_{i=1}^{N} X_i b_{i2} = 0$ (3b-c: Indicating that overall sensitivity on any index is zero , i.e., no systematic risk. If the portfolio is well diversified, this would entail almost no risk: riskless arbitrage).

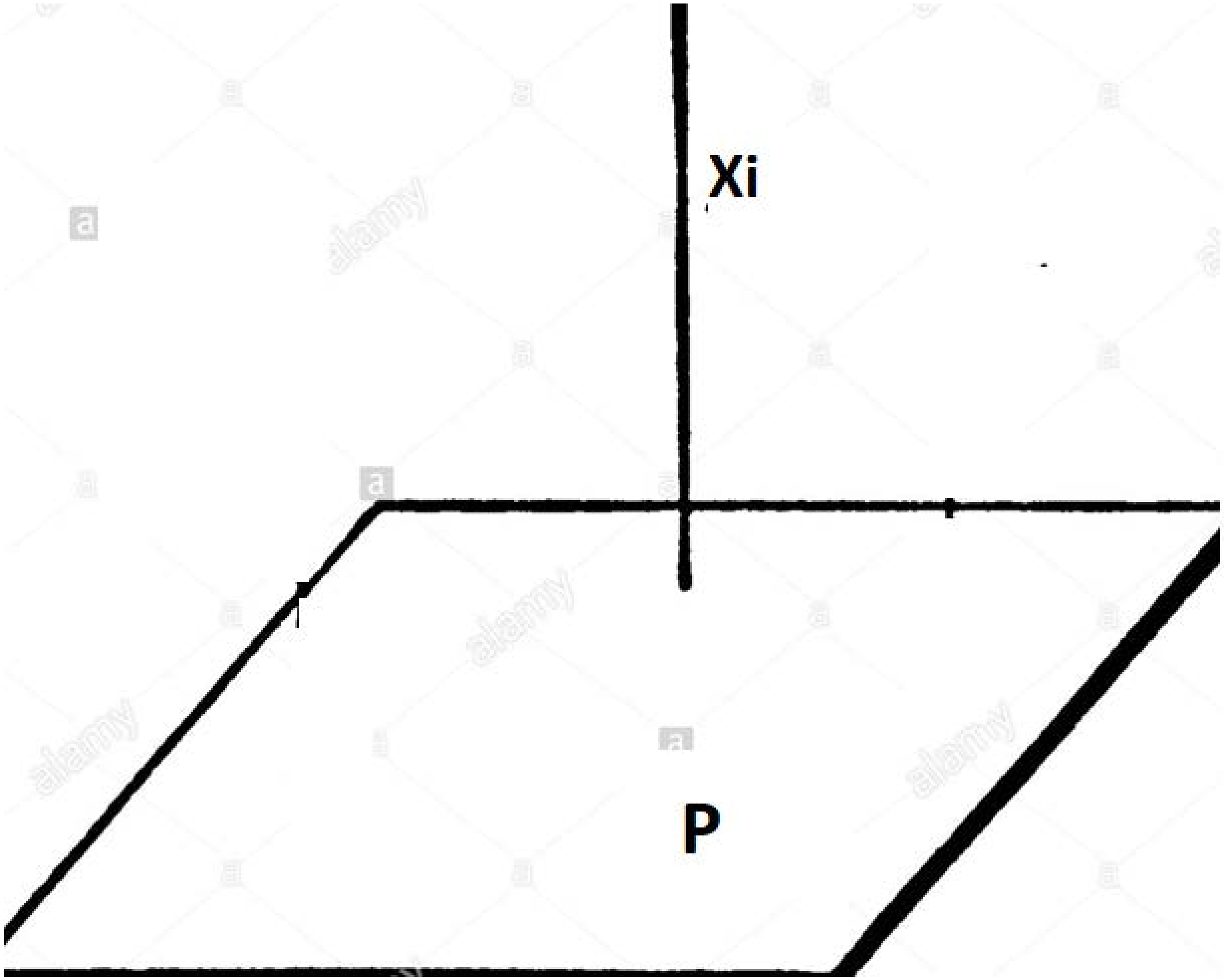
A more rigorous proof of APT

- $\sum_{i=1}^{N} X_i e_i = 0$ (Indicating that the portfolio is fairly diversified)
- Since this portfolio involves no risk and no investment, then it should produce an expected return of Zero. This will lead to the following Equation
- $\sum_{i=1}^{N} X_i \bar{R}_i = 0$ (3d)
- The mathematical interpretation of Equations (3a-d) suggests that the vector of X_i is orthogonal (at 90^0 or uncorrelated) to the vectors of ones, $b_{i1}'s$, $b_{i2}'s$, and $\bar{R}_i$

A more rigorous proof of APT

- A well-known theorem in linear algebra indicates that if N vectors are orthogonal to a given vector, then any of these vectors can expressed as a linear combination of the other N-1 vectors
- Xi is orthogonal to vector of ones, expected returns, and b_i's
- $\bar{R}_i = \lambda_0 * 1 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- Please note that λ_0 is multiplied to the vector of ones
- This is an equation of a plane on expected return ($\bar{R}_i$), b_{i1}, and b_{i2} space

A more rigorous proof of APT



A more rigorous proof of APT

- $\boldsymbol{\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}}$
- This is an equation of a plane on expected return ($\bar{R}_i$), b_{i1}, and b_{i2} space
- Since APT applies to all the securities, it should apply to portfolios as well
- The $\lambda's$ here can be interpreted by employing the following three portfolios
- Portfolio 1: b_{p1}=0 and b_{p2}=0
- Portfolio 2: b_{p1}=1 and b_{p2}=0
- Portfolio 3: b_{p1}=0 and b_{p2}=1

A more rigorous proof of APT

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- Portfolio 1: b_{p1}=0 and b_{p2}=0
- Portfolio 2: b_{p1}=1 and b_{p2}=0
- Portfolio 3: b_{p1}=0 and b_{p2}=1
- For portfolio 1, there is no sensitivity to any index. Thus it should offer only risk-free return (R_F). Putting the values in Eq., we get: $\lambda_0 = R_F$
- Putting the values of Portfolio 2 in Eq. (9) and using $\lambda_0 = R_F$, we get, $\lambda_1 = \bar{R}_1 - R_F$
- Similarly for Portfolio 3, we get $\lambda_2 = \bar{R}_2 - R_F$

A more rigorous proof of APT

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- The resulting equation becomes
- $\bar{R}_i = R_F + (\bar{R}_1 - R_F)b_{i1} + (\bar{R}_2 - R_F)b_{i2}$
- For a more general case, this can be generalized to
- $\boldsymbol{\bar{R}_i = R_F + (\bar{R}_1 - R_F)b_{i1} + (\bar{R}_2 - R_F)b_{i2} + \cdots + (\bar{R}_J - R_F)b_{iJ}}$
- Here λ_0 is defined as R_F, and the generic term λ_j is defined as $\bar{R}_J - R_F$
- The equation is essentially an alternative form of Eq. (11), as shown below
- $\boldsymbol{\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_J b_{iJ}}$

Testing the APT

- The multifactor return generating process is provided below
- $\boldsymbol{R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ \dots\dots + b_{ij}I_j + e_i}$
- The corresponding APT model is shown below
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- In order to test the APT, one has to identify I's, that is risk factors
- Subsequently, one can define the sensitivity of a given security b_{ij} to this risk factor
- Unfortunately, APT does not offer a direct economic rationale or description of I's
- What do we know abut $\boldsymbol{b_{ij}}$, $\boldsymbol{I_j}$, and λ_j

Testing the APT

- Each firm has unique sensitivity $\boldsymbol{b_{ij}}$ for each Index $\boldsymbol{I_j}$
- Thus $\boldsymbol{b_{ij}}$ is a security specific attribute (such as dividend yield) or security specific sensitivity to an Index
- The value of $\boldsymbol{I_j}$ is same for all the securities
- These $\boldsymbol{I_j}$'s are systematic influences, affecting a large number of securities, and therefore, are the source of covariance between those securities
- λ_j is the extra-expected return required because of the sensitivity of a security to the jth attribute of the security

Testing the APT

- For CAPM $\boldsymbol{b_{ij}}$ (sensitivity to market, beta), $\boldsymbol{I_j}$ (market index), and λ_j (Rm-Rf) were well defined
- For APT these are not defined in the model
- One has to test the model below with the observed returns
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- This requires estimates of $\boldsymbol{b_{ij}}$ and λ_j
- Most of the APT tests use the following equation on a set of predefined indices to obtain $\boldsymbol{b_{ij}}$
- $\boldsymbol{R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots\ldots + b_{ij}I_j + e_i}$

Testing the APT- Factor analysis

- A slightly purer and advanced method calls for factor analysis of the security returns

- The analysis determines a specific set of I_j's and b_{ij}'s, and also aims to reduce the covariance of the residual returns to as low as possible

- In the Factor analysis terminology, I_j's become the factors and b_{ij}'s become the factor loadings

- One can keep adding factors in the model, when the ability of the additional factor to explain the covariance matrix drops below a certain level

Testing the APT - Factor analysis

- Then the following equation is used to obtain the estimates of $\lambda_j's$ and thus the APT model
- $\boldsymbol{\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}}$
- The challenges with the factor analysis are discussed as follows
- Like any similar analysis, the estimates of I_j's and b_{ij}'s are subject to the error of the estimate
- The factors produced in the analysis have no meanings
- For example, the signs of factors and their betas (and therefore, the lambda's) can be reversed with no change in the resulting expected return

Testing the APT - Factor analysis

- Most of the APT tests use the following equation on a set of predefined indices to obtain $\boldsymbol{b_{ij}}$
- $\boldsymbol{R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots\ldots + b_{ij}I_j + e_i}$
- Then the following equation is used to obtain the estimates of $\lambda_j's$ and thus the APT model
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- In this manner, one can keep identifying risk factors until a sizable portion of expected returns are identified
- Effectively, these are joint tests of APT as well as the factors/influences/portfolios considered in the model

Testing the APT: Two Step Procedures

- Since, there is no generalizable theory that explains all the factors, three methods are used to provide broad set of factors in the APT model
- **1) Specifying the attributes of the security**
- If we can establish, a priori, that a certain set of attributes of security that affect the return
- Then the extra return required on account of these attributes can be measured through the following equation
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$

Testing the APT

- **1) Specifying the attributes of the security**
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- Here, b_{ij}'s would represent the level of attribute (j) associated with the security 'i' associated with each characteristics
- λ_j would represent the extra-return because of the sensitivity to that characteristics
- "n% increase in dividend of the portfolio is associated with Δ% increase in the expected returns."

Testing the APT

- **1) Specifying the attributes of the security**

- Once these $\boldsymbol{b_{ij}}$'s are directly obtained, risk-premium for these attributes are computed using the APT model

- These attributes directly affect the expected returns

- Once the major firm attributes and the corresponding risk premiums (λ s) identified, Eq. [$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$] can be estimated to define the APT

Testing the APT

- **2) Specifying the influences (factors) affecting the return generating process**
- Another alternative is to determine and pre-decide the set of risk factors (influences) that affect the return generating process
- A set of economic variables that affect the cash flows associated with the security
- For example, inflation, term structure of interest rates, risk-premia, industrial production

Testing the APT

- **2) Specifying the influences (factors) affecting the return generating process**
- These set of tests involve time series regressions of the individual portfolios to examine their sensitivities (b_{ij}) towards these macroeconomic variables
- $\boldsymbol{R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \ldots\ldots + b_{ij}I_j + e_i}$
- In the second stage, cross sectional regressions are performed using all the portfolios to determine the market price of risk (λ_j).
- $\boldsymbol{\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}}$

Testing the APT

- **2) Specifying the influences (factors) affecting the return generating process**

- For example, ONGC will be definitely affected by the crude-oil prices

- So a crude oil price index or any broad energy index can provide one risk factor, that is I_j

- Using these indices the return generating process can be employed to estimate the betas ($\boldsymbol{b_{ij}}$)

- Once the betas are obtained, APT model can be used to obtain risk-premiums (λ_j: Rj-Rf)

Testing the APT

- **3) Specifying a set of portfolios that capture the return generating process**

- Another option is to construct a set of portfolios that capture the influence of risk factors affecting the return generating process. For example:
  - Difference in the returns on small and large stock portfolios.
  - Difference in returns on the high book to market and low book to market stocks
  - Difference in the returns on long term corporate and long term government bonds

APT and CAPM

- Does CAPM become inconsistent in the presence of APT?
- We start with a simple single-index case, where this index is market portfolio (or market index like Nifty-50)
- The return generating process is of the following form
- $R_i = a_i + \beta_i R_m + e_i$
- Now refer to our earlier discussions on APT, where we said that above return generating process can be written in terms of sensitivities of the securities to the index and the price of risk in the following form
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$; with $\lambda_0 = R_F \; and \; \lambda_j = \bar{R}_j - R_F$

APT and CAPM

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$; with $\lambda_0 = R_F \ and\ \lambda_j = \bar{R}_j - R_F$
- For a single index case, that is market index and in the presence of risk-free rate, the above expression becomes
- $\bar{R}_i = R_F + \beta_i(\bar{R}_m - R_F)$: this is the expected return form provided by CAPM
- This suggests when single-index return generating process is true depiction of returns, the CAPM is clearly consistent
- But what about multi-indices?

APT and CAPM

- The return generating process in the context of two indices becomes
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$
- The equilibrium model for this return generating process with a risk-less asset becomes
- $\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$; λ_1=$\bar{R}_1$-R_F
- But if you believe in CAPM then

APT and CAPM

- The return generating process in the context of two indices becomes
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$
- The equilibrium model for this return generating process with a risk-less asset becomes
- $\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$;
- Recall that λ_j is the price of risk for a portfolio that has b_{ij}=1 for one index and zero for all the other indices: λ_j=$\bar{R}_j$-R_F
- If we say that CAPM holds, it holds for all the securities as well as portfolios

APT and CAPM

- If we say that CAPM holds, it holds for all the securities as well as portfolios
- Therefore, this industry portfolio may have some sensitivity to the market portfolio that is β_{λ_j}
- Recall that the risk premium was $\beta_i(\bar{R}_m - R_F)$, when the sensitivity to market was β_i
- Then the effective risk premium, for this index λ_j becomes $\beta_{\lambda_j}\,(\bar{R}_m - R_F)$

APT and CAPM

- The return generating process in the context of two indices becomes
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$
- The equilibrium model for this return generating process with a risk-less asset becomes
- $\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$; λ_1=$\bar{R}_1$-R_F
- But if you believe in CAPM then
- $\bar{R}_1$-R_F= β_{λ_1} $(\bar{R}_m - R_F)$ for Index I_1
- $\bar{R}_2$-R_F= β_{λ_2} $(\bar{R}_m - R_F)$ for Index I_2

APT and CAPM

- $\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$ this can be effectively written as
- $\bar{R}_i = R_F + b_{i1}\beta_{\lambda_1}(\bar{R}_m - R_F) + b_{i2}\beta_{\lambda_2}(\bar{R}_m - R_F)$
- $\bar{R}_i = R_F + (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})(\bar{R}_m - R_F)$
- Define $\beta_i = (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})$
- Then we obtain the CAPM form below
- $\bar{R}_i = R_F + \beta_i(\bar{R}_m - R_F)$

APT and CAPM

- Then we obtain the CAPM form below
- $\bar{R}_i = R_F + \beta_i(\bar{R}_m - R_F)$
- This can be extended to multiple factors (indices) as well
- Therefore, APT solution, even with multiple factors is consistent with CAPM
- This means, that despite the fact that multiple indices (risk factors) explain the covariance between the returns, but the CAPM holds

Another approach to APT

- $\boldsymbol{R_{it} = a_i + b_{i1}I_{1t} + b_{i2}I_{2t} + \ldots\ldots + b_{ij}I_{jt} + e_i}$; recall this model
- It is often convenient to construct the indices to have a mean zero, in the following form
- $\boldsymbol{R_i = \bar{R}_i + b_{i1}(I_{1t} - \bar{I_1}) + b_{i2}(I_{2t} - \bar{I_2}) + \ldots\ldots + b_{ij}(I_{jt} - \bar{I_j}) + e_i}$;
- Thus, here Constant = $\boldsymbol{\bar{R}_i}$ (in the mean zero form);
- But we also know from APT that: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij}$
- A more comprehensive return generating process form emerges
- $\boldsymbol{R_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_j b_{ij} + b_{i1}(I_{1t} - \bar{I_1}) + b_{i2}(I_{2t} - \bar{I_2}) + \ldots\ldots + b_{ij}(I_{jt} - \bar{I_j}) + e_i}$;

Application of APT (or multi-indices) in active and passive management

A) Passive management

- A simple application of APT is to construct a portfolio of stocks that closely tracks an index
- Index that represents a risk-factor (Bank-Nifty, represents the risk of banking stocks)
- The attempt is made to use a rather lesser number of stocks
- A large number of stocks would incur significant transaction costs
- In order to track the market index (market portfolio) one cannot hold all the stocks in the markets

Application of APT (or multi-indices) in active and passive management

- **A) Passive management**
- One attempts to hold only to the extent the diversifiable risk can be offset
- Those indices for which corresponding stocks are unavailable, if receive unexpected shocks (like Oil price shock) may appear as the residual risk in the model
- That is, our index may be exposed to these changes

Application of APT (or multi-indices) in active and passive management

- **A) Passive management**
- The benefit of using multi-indices instead of a single market index can be explained here as follows
- Consider five indices, including the energy portfolio, banking, inflation, cyclical stocks, and government bond portfolio
- Compare this to holding only the market portfolio (Nifty)
- Both of these strategies will capture the sensitivity to market risk, as all the portfolios (except government bond) may reflect, to some extent, the risk of market

Application of APT (or multi-indices) in active and passive management

- **A) Passive management**
- However, if there is certain oil price shock or unexpected changes in inflation, the market portfolio with its sensitivity matched to Nifty may not be very efficient in tracking the index.
- This is because, one is indifferent to holding stock from different industries (e.g., Oil stocks) in constructing the Nifty, as long as she is able to replicate market portfolio with no diversifiable risk
- However, this portfolio's sensitivity to Oil shocks can be very different to that of a multi-index (APT) model that is explicitly matched to the sensitivity of Oil price index

Application of APT (or multi-indices) in active and passive management

- **B) Active management**
- In active management one continuously holds on the market portfolio and makes calculated bets on different risk factors
- For example, if one believes that Oil prices can go up – this means that currently the stocks that are sensitive to this risk are under priced and will go up in future
- Then one can increase the sensitivity of her portfolio by adding additional stocks from Oil companies to the extent that increases the sensitivity to this risk index
- Once the price increase has materialized, one can go back to holding the market portfolio by selling the additional stocks and realizing the gains

Factor Investing

- We know that APT can be used to take on more risk, or conversely hedge the risk
- Factor investing rests on this ability of APT, without getting in the forecasting of future, prediction, or timing the market
- In factor investing, the investor has a certain appetite for different risk factors
- And she holds portfolios that matche this risk appetite
- Factor investing is particularly important for long horizon investors that are not much concerned about short term fluctuations

Factor Investing

- These factors, over long horizons consistently generate risk premia
- Some of the widely documented factors include
  - **a) Term structure factor:** Long term bonds have provided higher yields. This portfolio is constructed by taking long position in long-term government bonds and a short position in short-term treasuries
  - **(b) Default risk factor**: The factor is formed by taking long position in corporate debt and short position in equal maturity government. The idea is that default risk associated with corporate debt is a priced factor
  - **(c) Value factor**: This factor is constructed by taking long position in high book to market and short position in low book to market securities.

Factor Investing

- Some of the widely documented factors include
  - **(d) Size factor:** This factor is constructed by taking long position in small stocks and short position in large stocks
  - **(e) Momentum factor**: This factor is created by taking long position in past winners (high returns) and short position in past losers (low returns). The idea is that the momentum of the past winners and conversely the past losers is expected to sustain for a reasonable amount of time.
- Other factors include, volatility, liquidity, inflation, GDP, etc.
- One key limitation is that these factors, that deliver excess returns, many times do not have clear economic interpretations
- That is, the associated fundamental risks, on account of one is getting excess returns, may not be exactly identified with each factor.

Thanks

Portfolio Management Strategies

Course: Portfolio Management

Instructor: Abhinava Tripathi

Portfolio management strategies

- Equity portfolio management strategies can be placed into either a passive or active category
- The passive category portfolios aim to replicate some index (e.g., Nifty)
- Since not much effort is put in terms of time and resources in acquisition of information, these strategies involve very less management fees
- In contrast, active portfolio management involves continuous accumulation of information to achieve higher risk-adjusted returns as compared to the market or some other benchmark
- Given this effort of management, management charges excess fees

Portfolio management strategies-Passive strategy

- The total returns from passive strategy are decomposed into two components: Risk free return + risk premium
- Passive funds follow the approach called indexing
- It’s a long term buy-and-hold strategy, except the occasional rebalancing of the portfolio that is required due to changes in the index
- The deviation between the passive funds and index returns is called ‘tracking error”
- The portfolio is judged by its ability to minimize this tracking error

Portfolio management strategies-Active strategy

- The active funds, in contrast, attempt to beat the market/benchmark and claim to offer some risk adjusted excess abnormal return, often denoted as ‘Alpha’
- That is, outperform some benchmark (usually an Index) on risk-adjusted basis
- This Alpha is the difference between the actual and expected returns
- Essentially this alpha is the value a manager had added or subtracted from the investment process

Portfolio management strategies- Active vs Passive tradeoffs

- Investor faces certain trade-offs while selecting between these two active and passive strategies
- Indexing is a low cost (because of low management fee) strategy but assured returns
- The active strategy may offer at times lucrative returns but with higher management costs
- At times these higher management costs make net-returns inferior to investors.

Passive strategies

Passive strategies

- Stock markets world-over are said to be considerably efficient
- The implication is that it is extremely difficult for active fund managers to beat the market, and justify the active management fee (1-2%) charged
- Passive funds don't charge this management fee
- However, passive management strategies also require buying and selling of portfolios overtime. This leads to a slight under performance by the fund amounting to 0.05 to 0.25%

Passive strategies

- Often three techniques are employed to construct a passive index portfolio
- **(a) Full replication**: all the securities in the index are purchased in proportion to their weights in the index
- While this strategy ensures extremely efficient tracking, but the need to purchase/sale many securities will reduce the returns by transaction costs
- Also, for such a large number of securities, considerable amount of dividends are paid.

Passive strategies

- **(b) Sampling:** In this technique only a limited sample of stocks are employed that broadly represents all the industry sector classification, as captured by the benchmark index
- This solves the problem of buying large number of stocks
- In particular, the stocks with large weights are purchased according to their weight in the index
- The small stocks are purchased to approximate/mimic their aggregate characteristics in the index (e.g., beta, industry, dividend yield)
- While this will decrease the transaction cost, the efficiency of tracking and therefore, the returns of the portfolio may differ from the benchmark.

Passive strategies

- **(c) Quadratic programing:** in this case, sampling technique differs from sampling
- That is, for sampling, rather than matching the characteristics of the security, historical information about the security returns and correlations are employed to construct a portfolio that can minimize the return deviations from the benchmark
- One challenge is that this technique draws heavily from the past information of the securities, and therefore, if the security characteristics change from that observed in the past then the portfolio may not be efficient in tracking the returns.

Tracking error and index portfolio construction

- The main objective of a passive portfolio is to replicate a particular benchmark index
- It does not aim to achieve higher returns but to match the performance of that portfolio
- Therefore, a manager is judged by his performance relative to the performance of benchmark, using a measure called tracking error
- Consider a period t return on the portfolio is:
- $R_{pt} = \sum_{i=1}^{N} w_i R_{it}$, where N is the number of assets in the portfolio
- The difference between the period t benchmark portfolio and index:
- $\Delta_t = R_{pt} - R_{bt}$.

Tracking error and index portfolio construction

- Consider a period t return on the portfolio is:
- $R_{pt} = \sum_{i=1}^{N} w_i R_{it}$, where N is the number of assets in the portfolio
- The difference between the period t benchmark portfolio and index:
- $\Delta_t = R_{pt} - R_{bt}$; Generally, Δ_t is a function of the portfolio weights
- Also, since all the assets (mostly the small ones) may not be included in the managed portfolio, weight (w)=0 for those assets
- For a sample of T return observations, the variance of Δ_t can be calculated as : $\sigma_{\Delta}^2 = \frac{\sum_{t=1}^{T} (\Delta_t - \overline{\Delta})^2}{(T-1)}$

Tracking error and index portfolio construction

- For a sample of T return observations, the variance of Δ_t can be calculated as : $\sigma_\Delta^2 = \frac{\sum_{t=1}^{T}(\Delta_t - \bar{\Delta})^2}{(T-1)}$
- If σ_Δ is calculated for daily period then annualized tracking error TE= $\sigma_\Delta\sqrt{252}$
- For monthly period the error will be TE= $\sigma_\Delta\sqrt{12}$
- Basically , TE (Annualized)= $\sigma_\Delta\sqrt{t}$. Where t are the number of returns periods in the year.

Tracking error and index portfolio construction

Period	Return on Portfolio (%)	Return on Index (%)	Difference (%)
1	2.3	2.7%	-0.4%
2	-3.6	-4.6	
3	11.2	10.1	
4	1.2	2.2	
5	1.5	0.4	
6	3.2	2.8	
7	8.9	8.1	
8	-0.8	0.6	
		Average	

- $\sigma_{\Delta}^{2} = \frac{\sum_{t=1}^{T}(\Delta_t - \bar{\Delta})^2}{(T-1)}$ $=?;\ \sigma_{\Delta} =?\,\%$ quarterly
- TE= $\sigma_{\Delta} * \sqrt{t}$=? annual

Tracking error and index portfolio construction

Period	Return on Portfolio (%)	Return on Index (%)	Difference (%)
1	2.3	2.7%	-0.4%
2	-3.6	-4.6	1.0
3	11.2	10.1	1.1
4	1.2	2.2	-1.0
5	1.5	0.4	1.1
6	3.2	2.8	0.4
7	8.9	8.1	0.8
8	-0.8	0.6	-1.4
		Average	0.20%

- $\sigma_\Delta^2 = \frac{\sum_{t=1}^{T}(\Delta_t - \bar{\Delta})^2}{(T-1)} = \frac{[(-0.4-0.2)^2+(1.0-.2)^2+\cdots+(-1.4-0.2)^2]}{(8-1)}$ =1.0; $\sigma_\Delta =$ 1.0% quarterly
- TE= $\sigma_\Delta * \sqrt{4}$=2.0%.

Active strategies

Active investment strategies

- Active equity management strategies aim to earn return that exceed market (benchmark) returns, net of transaction costs
- These strategies aim to increase exposure to those stocks/sectors that the fund considers undervalued
- It may be noted that increasing exposure to a certain sector may lead to additional risk
- However, the fund management believes that actual returns will be higher (net of transaction costs) than those justified by the risk premium associated with the risk of investment
- These strategies are classified in three buckets **(a) Fundamental, (b) Technical, (c) Market anomalies and security attributes.**

Active investment strategies

- **(A) Fundamental Strategies:** The fundamental strategies are of two kinds (a) top-down, and (b) bottom-up
- In the top-down investment process, one start with the broad country level and sector level analysis. And then move towards asset class to security specific allocation
- The bottom up approach straight away focusses on the individual security rather than the market-sector analysis. Then if found good the analysis moves from asset class to sector, and then to the country level
- The end objective in both the approaches is to identify the securities that are undervalued given their fundamentals

Active investment strategies

- **(A) Fundamental Strategies:** For example, a fund manager may identify the asset class that is undervalued, e.g., stocks, bonds, government securities
- They may increase the exposure to that asset class as a whole
- Secondly, they may invest (increase exposure) in certain industry-sectors or the investment styles (large cap, small cap, value, growth)
- Lastly, funds can identify and add undervalued stocks to their portfolios
- Another strategy recently developed, called as "130/30". Funds take long positions up to 130% of the original capital. And then they take short positions of 30%

Active investment strategies

- **(B) Technical strategies:** Technical strategies rely on two aspects of past price performance (a) the past trends will continue, and (b) the past trends will reverse
- For example, a contrarian strategy will suggest that the best time to buy a stock when everybody is acting bearish (selling), and vice-versa
- This strategy relies on the overreaction hypothesis. The hypothesis suggests that investors overreact to the information leading to excess movements in the prices
- As the prices correct in the short to medium term there is a reversal
- The contrarian investor will purchase the stock when the price is low and falling, and sell it when the price is high and rising

Active investment strategies

- **(B) Technical strategies:** In contrast, the momentum trading strategy assumes that the momentum will continue
- This strategy relies on the underreaction hypothesis.
- That is, investors have limited capacity to absorb information. As the information arrives in the market, investors gradually absorb this information
- The investor following the momentum strategy buys the stock when the prices start rising and holds in expectation of further increase, and vice versa

Active investment strategies

- **(C) Anomalies and attributes:** These strategies rely on anomalies or firm attributes
- It has been observed that firms with small capitalizations produce bigger risk adjusted returns that those with large market capitalizations
- Similarly, firms with low P/E and P/BV ratios produce risk-adjusted returns than those with higher levels these ratios
- It appears that market, at times, favors some attributes more than others
- In this context, sector rotation involves increasing (overweighing) stocks with certain attributes and decreasing the stocks with opposite attributes

Style Analysis

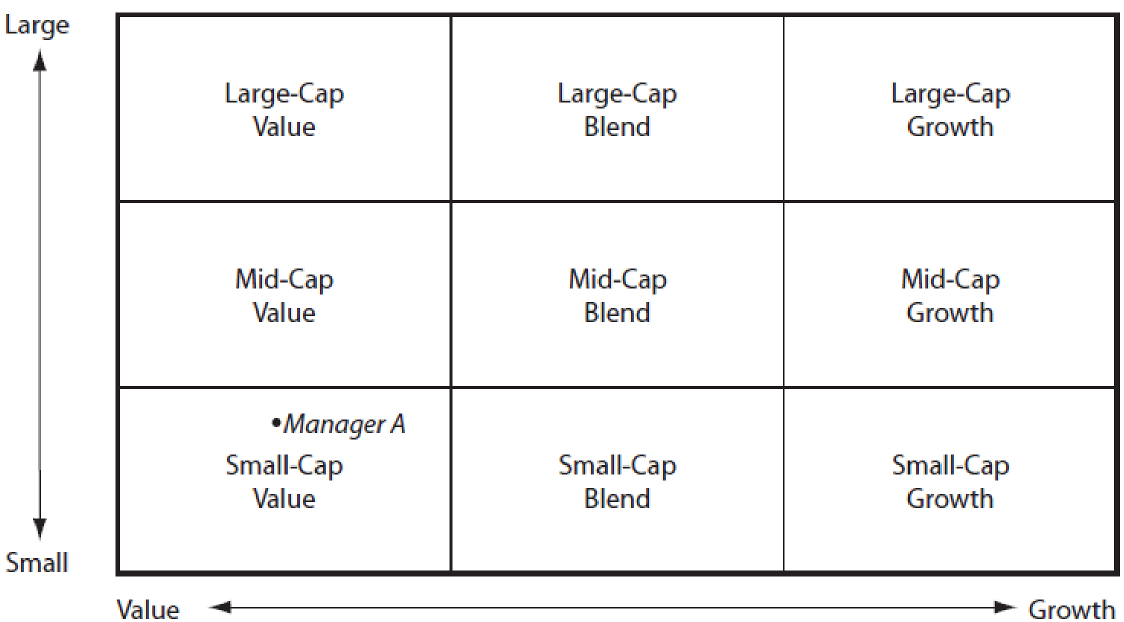
Investing styles

- Various equity investment styles are available to investors
- These include forming portfolios with stock characteristics including market capitalization, leverage, industry sector, relative valuation and growth potential
- Essentially, style analysis defines benchmark portfolios (index) based on these characteristics
- Securities are chosen depending upon their sensitivity to this portfolio
- The relationship between a funds return to that with various indices is examined
- The higher the correlation of the fund with a portfolio associated with certain characteristics, it is said that the portfolio manager gives a higher weight to that investment style

Investing styles: Style Grid Analysis

- Consider the style grid below. Here, we are trying to capture the performance of the manager along two dimensions: firm size (large-, mid-, and small-cap) and relative value (value, growth, and blend)

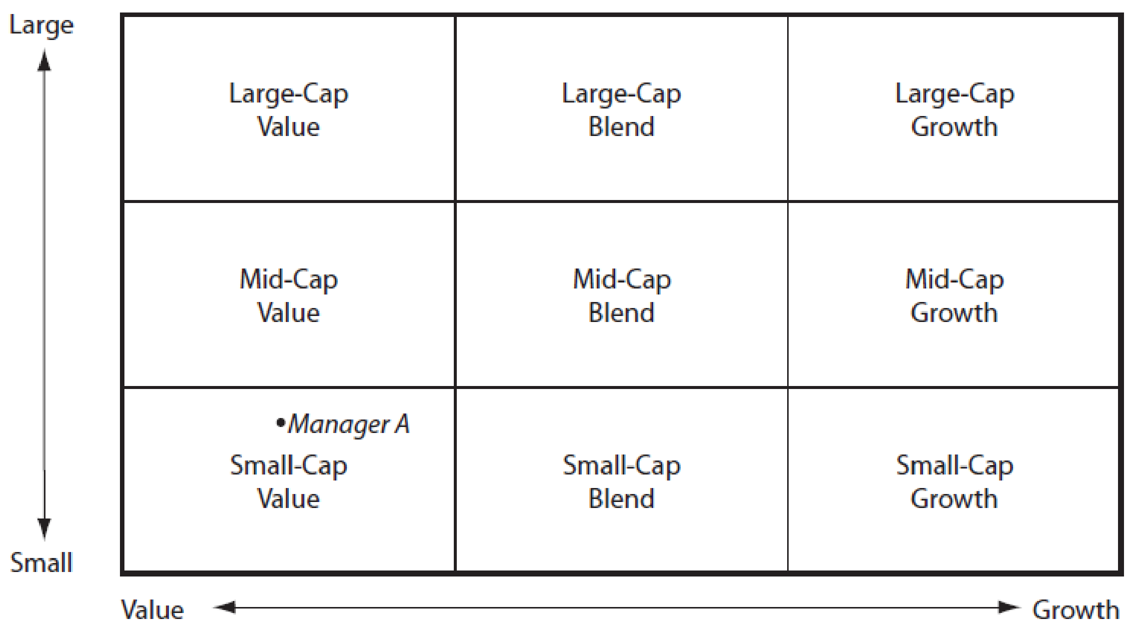
- The manager A's performance is best captured by the small-cap value style



Investing styles: Style Grid Analysis

- These style grids can be formed to classify funds, indices, or other portfolios

- Only those portfolios that are found similar in styles, can be compared for their return performances.



Investing styles- a formal approach

- A more formal constrained least square approach to style analysis is discussed below

- Only those portfolios that are found similar in styles, can be compared for their return performances.

- The return from the manager's portfolio 'R_{pt}' are regressed on the returns on different style (j) factor 'F_{jt}' for the same period

- The following form of regression model is employed

- $R_{pt} - Rf = a_0 + [b_{p1}F_{1t} + b_{p2}F_{2t} + \cdots + b_{pn}F_{nt}] + e_{pt}$

- Here b_{pj} is the sensitivity of the portfolio to style j. e_{pt} is the portion of the returns not explained by the variability in the set of employed factors

Investing styles- a formal approach

- $R_{pt} - Rf = a_0 + \left[b_{p1}F_{1t} + b_{p2}F_{2t} + \cdots + b_{pn}F_{nt}\right] + e_{pt}$
- Here b_{pj} is the sensitivity of the portfolio to style j. e_{pt} is the portion of the returns not explained by the variability in the set of employed factors
- The regression R^2 is interpreted as the percentage of return variability due to style
- a_0 is ascribed to the manager's selection skills
- The styles are measured through benchmark portfolios
- The coefficients must sum to one,; Here, $R^2 = 1 - [\frac{\sigma^2(e_p)}{\sigma^2(R_p)}]$.
- Higher the value of 1- R^2 (assuming diversified portfolio) missing style factors

Investing styles- a formal approach

- It may often that a manager may profess a different style while following another style

- This analysis clearly brings forth the true picture

- The analysis also helps in finding out if there has been a style drift.

Value-Growth Style

Value vs. Growth investing

- We often hear investment management firms define themselves as value vs. growth firms
- For example, growth firms focus on the earnings (EPS) part of the P/E ratio. He expects the earnings to grow which will lead prices to rise (constant P/E) ratio
- In contrast, the value investor defines the price (P) component of P/E ratio. The value investor believes that the given the current level of earnings, prices are low (cheap) as compared to the other stocks in the same industry with similar profile

Value vs. Growth investing

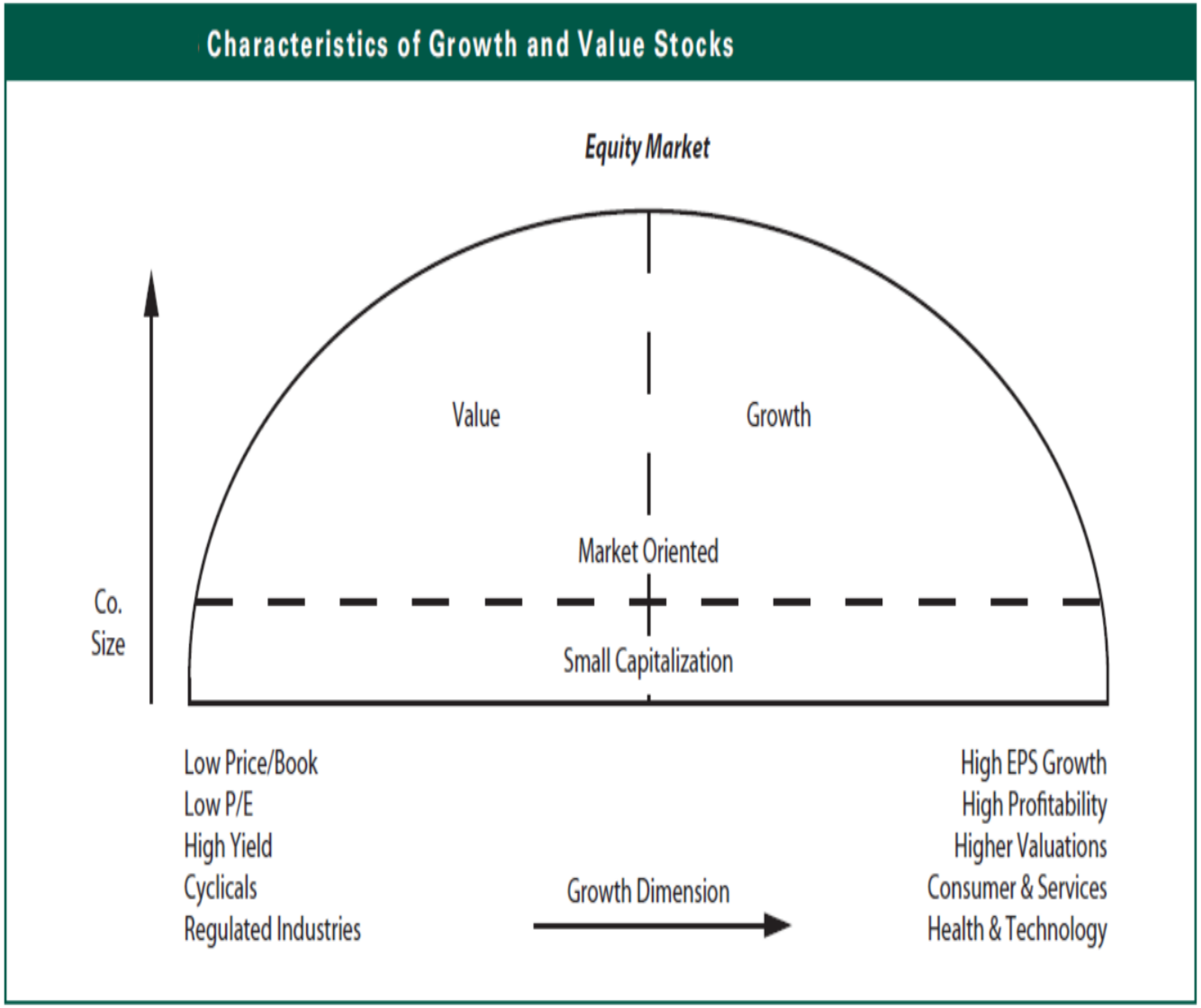
- P/E level is below the level based on some comparison, and the fact that market will correct itself in the near term
- And the prices will rise, thus Value stocks are cheap given their current earning levels
- **Growth firms** focus on the earnings (EPS) part of the P/E ratio. He expects the earnings to grow which will lead prices to rise (constant P/E) ratio
- Growth stocks are not necessarily cheap based on the current earnings' levels, in fact they may be costly
- But the investor believes that the earnings will rise significantly and lead to price rise in near future

Value vs. Growth investing

- To summarize, growth investor focuses on the current and future economic "story" of the firm, with less regard to share valuation
- The value investor focusses on share prices in anticipation of market correction, possibly on account of improving company fundamentals

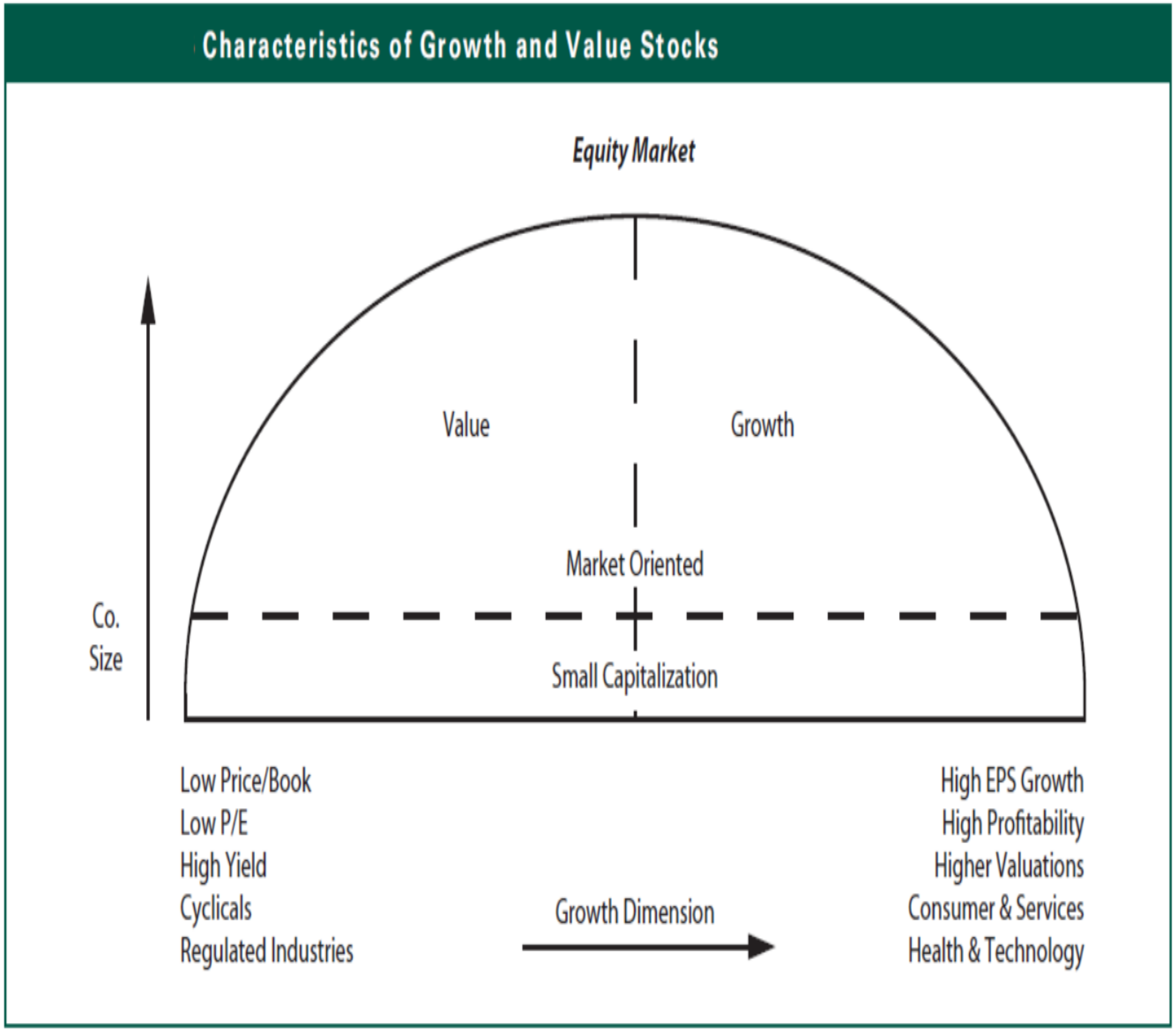
Value vs. Growth investing

- Notice the characteristics of the value and growth stocks shown in Figure below
- The figure shows one approach to classify securities according to style and market capitalization



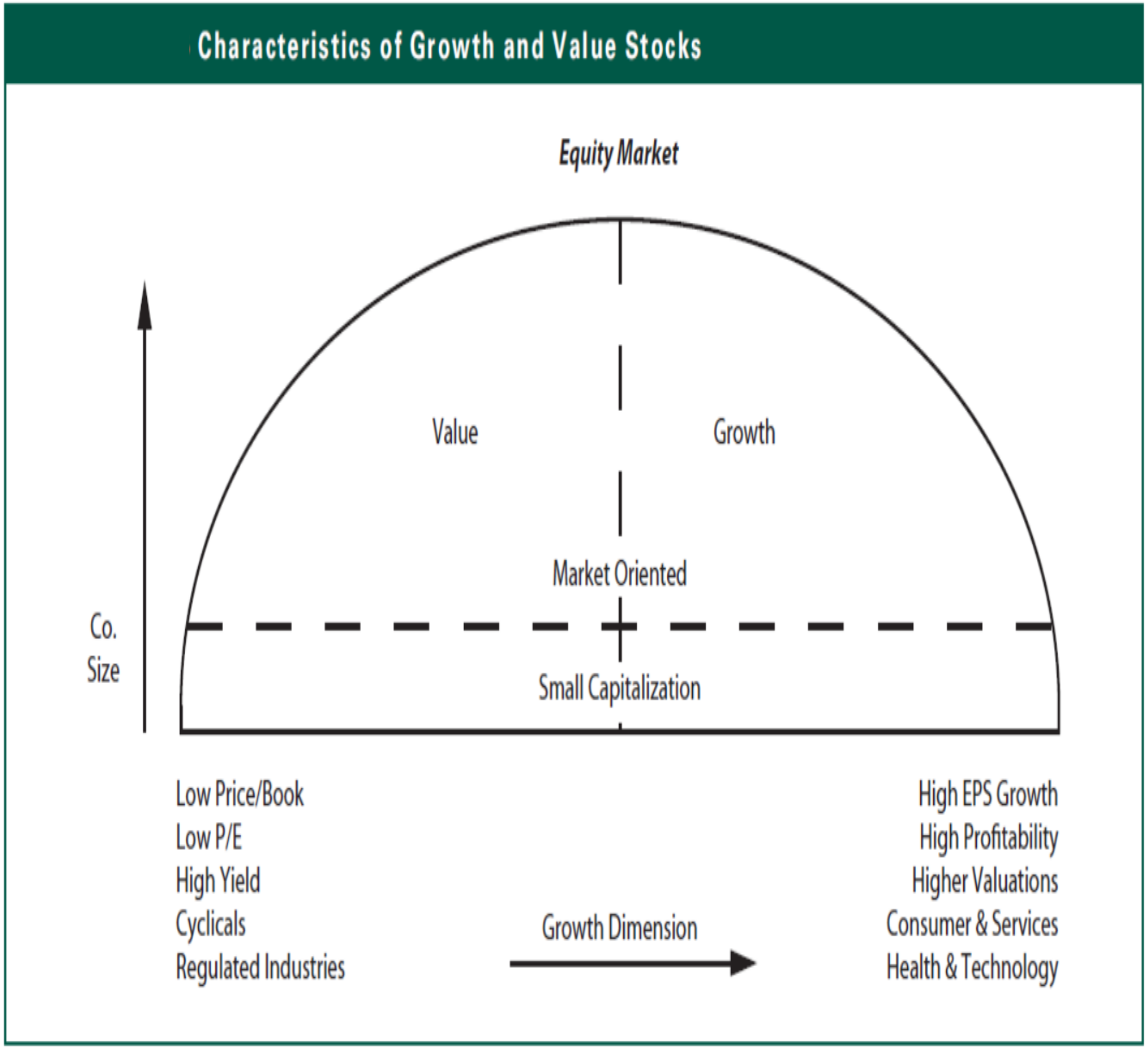
Value vs. Growth investing

- We can see that value stocks are cheap (i.e., low P/BV, high yield) and have modest growth opportunities
- In contrast, growth stocks are expensive, reflecting their high future earning potential



Value vs. Growth investing

- It is tempting to say that value style appears to be more tempting than growth, and in fact, studies show that value style indeed produces higher average returns that growth investing
- However, both strategies have their clientele



Thanks

Portfolio Performance Evaluation: One-parameter measures

Course: Portfolio Management

Instructor: Abhinava Tripathi

Portfolio Performance Evaluation

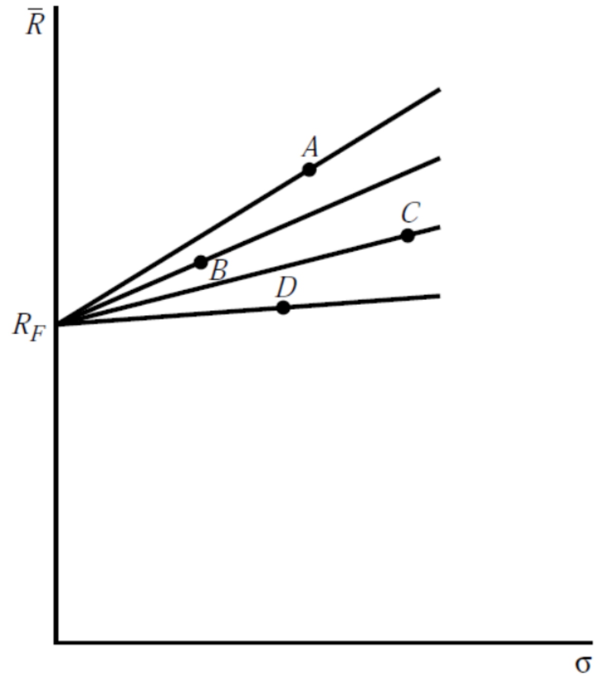
- While evaluating the performance of a portfolio, the following questions are asked
  - What are the policies that the fund has pronounced for itself, and how well those policies are followed
  - How diversified is the fund
  - What is the asset allocation
- The portfolios being evaluated must be comparable
- For example, if a fund has restricted that its managers should invest only in AA rated instruments or better should not compare with those funds that invest in funds that have no such restrictions

Portfolio Performance Evaluation

- Therefore, the return earned is directly linked to the amount of risk borne by the fund
- But problems arise where the funds that are compared have different risk levels
- In the ensuing discussions, we will focus on one-parameter measures that are most commonly employed in the literature

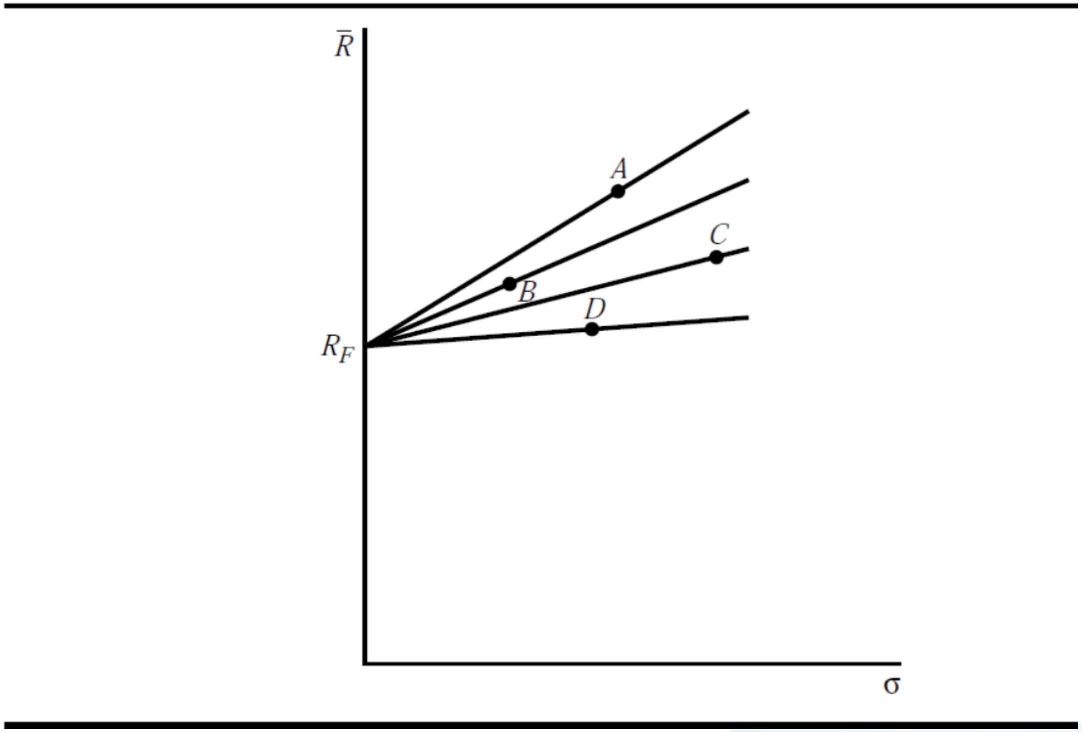
Sharpe ratio

- **Sharpe ratio of point A=** $\dfrac{\overline{R_A} - R_f}{\sigma_A}$
- The ratio measures excess return over risk-free rate against the risk borne by the fund



Sharpe ratio

- The portfolios on line joining the investment A and Rf offer the highest slope and therefore, the best Sharpe ratio measure of performance



Sharpe ratio

- Compare the examples of three portfolios that follow Sharpe measure
- **Sharpe ratio of point A=** $\frac{\overline{R_A} - R_f}{\sigma_A}$

Portfolio	Average annual rate of return	SD	Sharpe measure
D	13%	0.18	(.13-.08)/.18=.278
E	17%	0.22	?
F	16%	0.23	?
Market	14%	0.20	?
Risk-free	8%		

- Here portfolio D performs the worst, even as compared to the market portfolio
- Portfolio E performs best

Sharpe ratio

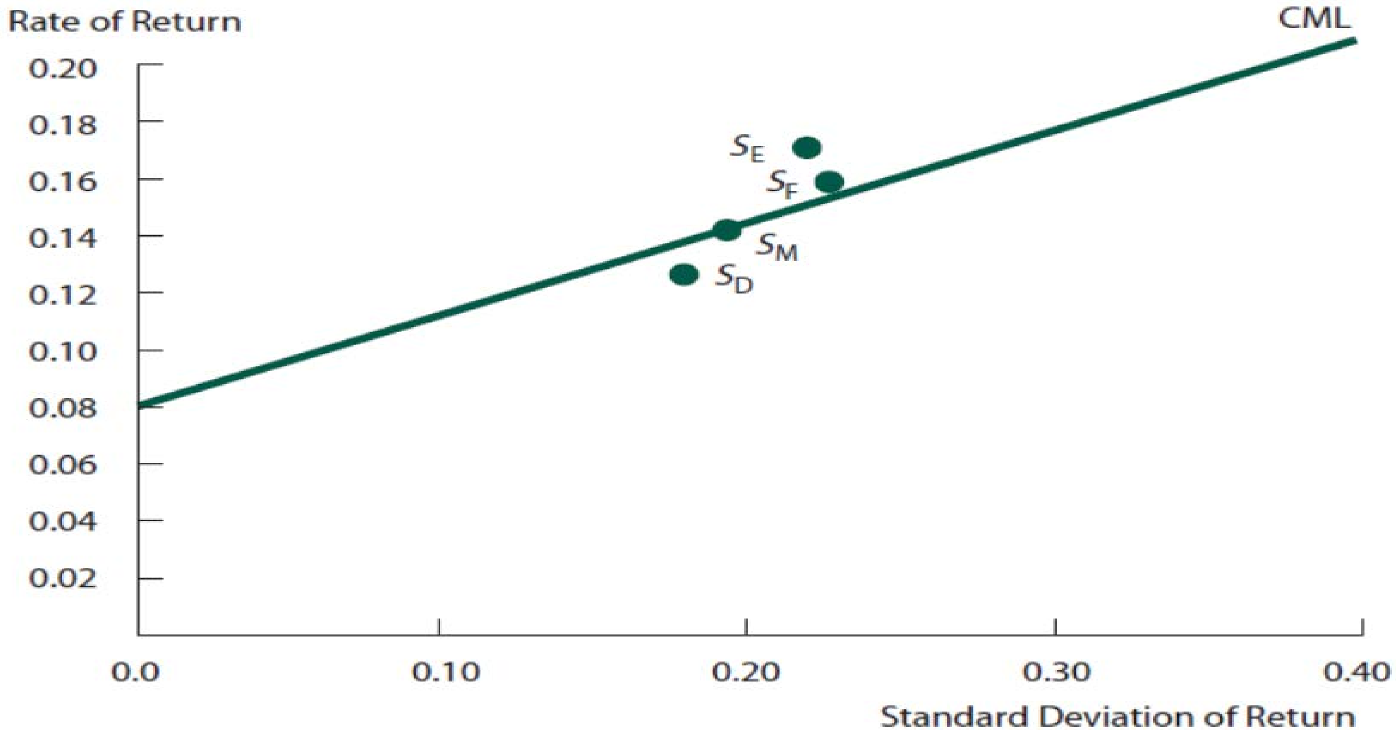
- Compare the examples of three portfolios that follow Sharpe measure

Portfolio	Average annual rate of return	SD	Sharpe measure
D	13%	0.18	(.13-.08)/.18=.278
E	17%	0.22	(.17-.08)/.22=.409
F	16%	0.23	(.16-.08)/.23=.348
Market	14%	0.20	(.14-.08)/.20=.300
Risk-free	8%		

- Here portfolio D performs the worst, even as compared to the market portfolio
- Portfolio E performs best

Sharpe ratio

- The performance of these portfolios can be plotted on Capital Market Line (CML)
- Portfolios E and F are above the CML line, indicating better risk-adjusted performance



Sharpe ratio

- The measure of risk considered here is the standard-deviation, i.e., total risk of the fund
- This includes the market risk (systematic risk) and stock specific risk
- Please note that if the fund is well diversified then most of the fund's risk will be systematic risk
- In most of the situations, the investors invested in the fund are small retail investors, who invest a sizable portion of their risk in the fund
- For the investor, the entire risk of the fund is important, not only the market risk part of it
- Since these investors rely precisely on the ability of the fund to diversify on behalf of them

Sharpe ratio

- The Sharpe measure looks at the decision from the point of view of an investor choosing a mutual fund to represent the majority of his investment
- An investor choosing a mutual fund to represent a large part of her wealth would likely be concerned with the full risk of the fund, and standard deviation is a measure of that risk
- The measure computes risk-premium earned per unit of total risk
- This measure uses capital market line (CML) to compare portfolios

Treynor’s measure

- **Treynor’s measure=** $\frac{\overline{R_p}-R_f}{\beta}$
- The Treynor’s measure examines excess return with risk measure being beta
- For the diversified investors who only consider the systematic risk for performance evaluation, the Treynor’s measure is the appropriate measure
- Treynor’s measure is applicable to majority of the investors irrespective of their risk preferences
- Treynor argues that rational, risk-averse investors would always prefer the portfolios on security market line

Treynor's measure

- **Treynor's measure=** $\frac{\overline{R_p}-R_f}{\beta}$
- Treynor argues that rational, risk-averse investors would always prefer the portfolios on security market line
- That is risk-free asset combined with risky portfolios with the largest slope, in order to achieve the highest indifference curve
- The slope of this curve is the Treynor's measure
- The risk measure here is the systematic risk component, beta
- The measure assumes a diversified portfolio, and that all investors are risk-averse and would like to maximize this value

Treynor's measure

- The measure for the standard market portfolio will be $\frac{\overline{R_M} - R_f}{\beta_M}$, **where** $\boldsymbol{\beta_M = 1}$
- For any portfolio in general: $\frac{\overline{R_P} - R_f}{\beta_P} = (\overline{R_M} - R_f)$ from SML
- Equation of SML is shown below
- $\overline{R_P} = R_f + \beta_p * (\overline{R_M} - R_f)$
- Combined with Rf, this portfolio will generate the security market line (SML)
- Any higher value would indicate that the portfolio offers excess risk-adjusted returns and plots above SML

Treynor’s measure

- Consider the information about three investment managers below (1, 2, and 3)
- In addition, we are given the market rate of return and risk-free rate

Investment Manager	Average annual rate of return	Beta	Treynor’s measure
w	12%	0.90	(.12-.08)/0.90=.044
x	16%	1.05	(.16-.08)/1.05=.076
y	18%	1.20	(.18-.08)/1.20=.083
Market	14%	1.00	(.14-.08)/1.00=.060
Risk-free	8%	0.00	

- These results indicate that manager 1 not only performed worst across the three managers, but performed worse than the market as well, on risk-adjusted basis

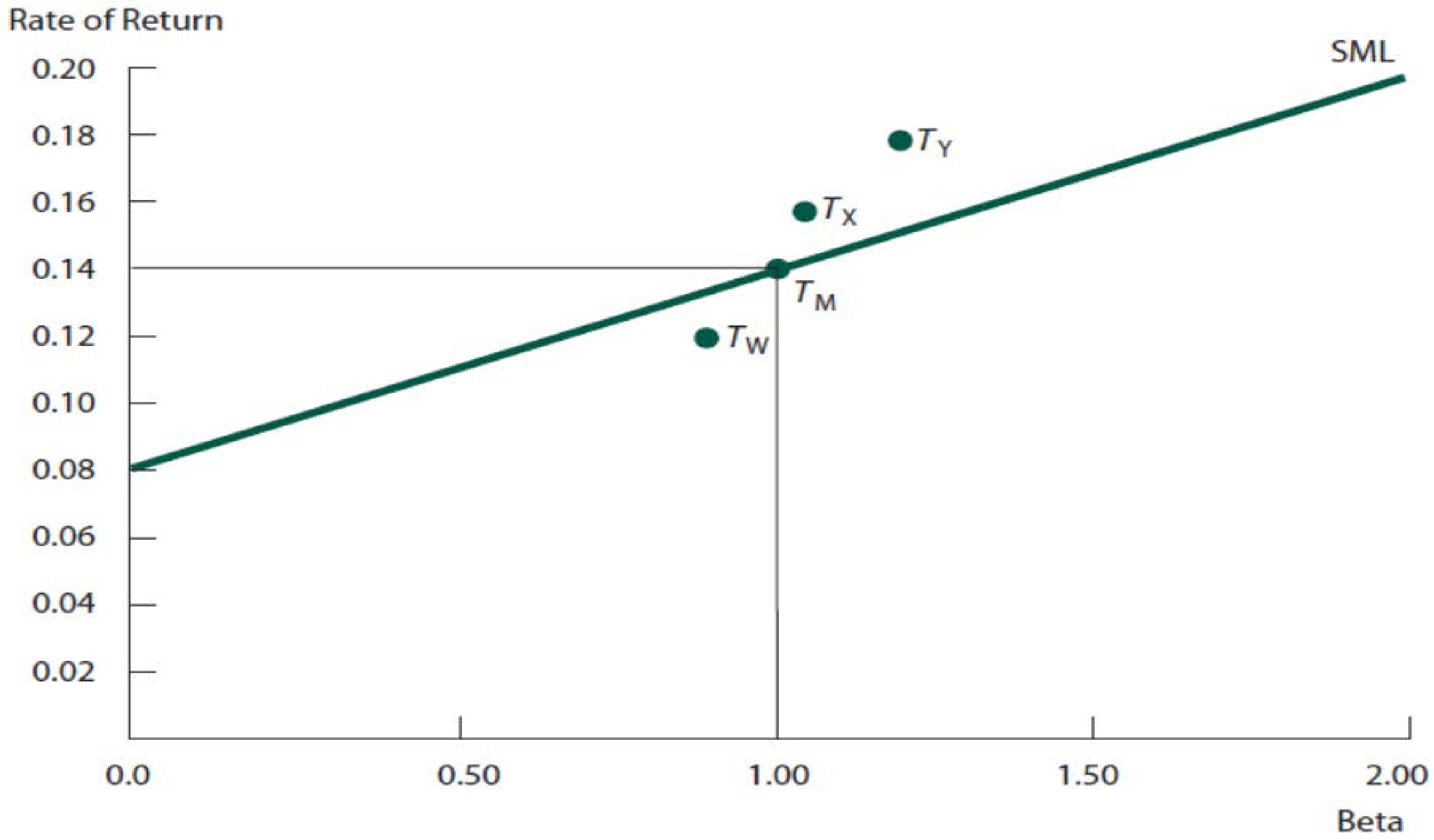
Treynor’s measure

- These results indicate that manager 1 not only performed worst across the three managers, but performed worse than the market as well, on risk-adjusted basis
- While 2 and 3 performed better than market, 3 performed best

Investment Manager	Average annual rate of return	Beta	Treynor’s measure
w	12%	0.90	(.12-.08)/0.90=.044
x	16%	1.05	(.16-.08)/1.05=.076
y	18%	1.20	(.18-.08)/1.20=.083
Market	14%	1.00	(.14-.08)/1.00=.060
Risk-free	8%	0.00	

Treynor's measure

- Their performance on SML can be plotted as follows



Treynor’s measure

- What is the challenge with this measure
- Consider two portfolios: One, which offers a return that is below risk-free (though with positive beta)
- The negative measure would indicate poor performance
- Even when plotted on SML, this point would indicate a very poor performance
- Second, consider a security with negative beta that offers a very high return above the risk-free rate
- This would also offer a negative measure despite good performance

Treynor's measure

- For example, a portfolio of gold mining stocks with a beta of -.2 performs well and offers 10% return
- Then the measure would be (.10-.08)/-0.2= -0.10
- However, if plotted on SML, this will be above SML and indicate exceptional returns
- See for example. $E\left(R_{gold}\right) = R_f + \beta_{gold}(R_M - Rf)$= 0.08+(-0.2)*(.14-.08)=6.8% expected returns, which is lower than the actual return of 10%; Thus the point will be above SML

Jensen's measure (α)

- Jensen's measure is the differential in the return as predicted by the CAPM model

- $R_p = \alpha_p + \bar{R}_p = \alpha_p + R_f + (\bar{R}_M - R_f)\beta_p$

- $\bar{R}_p$ is the expected return. Then $R_p - \bar{R}_p$, this differential return is called the Jensen's measure of performance

- Key assumption here is that CAPM is the guiding model

Jensen's measure (α)

- $R_{pt} - R_f = \alpha_p + \beta_j\left[R_{mt} - R_f\right] + e_{jt}$

- In this model, we expected α_p=0

- Presence of positive intercept (constant term) α_j would indicate the ability of security selection or or predicting the market performance by a portfolio manager

- A negative alpha would indicate poor performance

Information ratio measure (IR)

- **Information ratio measure (IR):** $\frac{\overline{R_P}-\overline{R_b}}{\sigma_{ER}} = \frac{\overline{ER_B}}{\sigma_{ER}}$
- Here, $\overline{R_P}$ is the return on portfolio, $\overline{R_b}$ is the return on the benchmark portfolio
- $\overline{ER_B}$ is the excess return. σ_{ER} is the standard deviation of excess returns
- The numerator here measures the ability of the portfolio manager to perform better than a given benchmark (e.g., Nifty)
- The denominator measures the residual (or incremental) risk that the manager took to obtain these excess returns
- Thus, IR can be interpreted as benefit to cost ratio
- It evaluates the quality of information with the manager (or stock selection ability) adjusted by the non-systematic taken by the investor

Information ratio measure (IR)

- Consider a set of quarterly returns below
- Compute IR= $\frac{\overline{R_P}-\overline{R_b}}{\sigma_{ER}} = \frac{\overline{ER_B}}{\sigma_{ER}}$

Quarter	Portfolio returns	Benchmark returns	Difference
1	2.30%	2.70%	-0.40%
2	-3.60%	-4.60%	?
3	11.20%	10.10%	?
4	1.20%	2.20%	?
5	1.50%	0.40%	?
6	3.20%	2.80%	?
7	8.90%	8.10%	?
8	-0.80%	0.60%	?
Average	**?**	**?**	$\overline{R_P} - \overline{R_b}$=?
SD			σ_{ER}=?

Information ratio measure (IR)

- Consider a set of quarterly returns below

Quarter	Portfolio returns	Benchmark returns	Difference
1	2.30%	2.70%	-0.40%
2	-3.60%	-4.60%	1.00%
3	11.20%	10.10%	1.10%
4	1.20%	2.20%	-1.00%
5	1.50%	0.40%	1.10%
6	3.20%	2.80%	0.40%
7	8.90%	8.10%	0.80%
8	-0.80%	0.60%	-1.40%
Average	2.99%	2.79%	0.20%
SD			1.00%

Information ratio measure (IR)

- IR= 0.2%/1%=0.20; this represents the manager's incremental performance (alpha, relative to the index) per unit of risk incurred in the pursuit of those active returns
- IR will be only positive when the manager outperforms his benchmark

Quarter	Portfolio returns	Benchmark returns	Difference
1	2.30%	2.70%	-0.40%
2	-3.60%	-4.60%	1.00%
3	11.20%	10.10%	1.10%
4	1.20%	2.20%	-1.00%
5	1.50%	0.40%	1.10%
6	3.20%	2.80%	0.40%
7	8.90%	8.10%	0.80%
8	-0.80%	0.60%	-1.40%
Average	2.99%	2.79%	0.20%
SD			1.00%

Sortino’s ratio: Performance measurement with downside risk

- **Sortino’s ratio :** $\frac{\overline{R_p} - T}{D_R}$
- Here, D_R is the downside risk; Total risk, i.e., SD includes upside and downside both risks
- T is the target rate of return
- In most of the computations T= Rf (risk-free rate) or some target return as set by fund management
- $D_R = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(Min(0, R_i - MAR)\right)^2}$

Sortino's ratio: Performance measurement with downside risk

- $D_R = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(Min(0, R_i - MAR)\right)^2}$
- Also in most of the computations MAR= Risk free returns (Rf)
- Sortino's measure measures returns in excess of pre-defined target rate
- This excess return is adjusted by not the total risk (SD) but only the downside risk
- The downside risk is computed against some minimum acceptable returns

Sortino's ratio: Performance measurement with downside risk

- $D_R = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(Min(0, R_i - MAR)\right)^2}$
- This kind of downside risk is often considered more appropriate, because the downside volatility is often associated with shortfall
- Thus, this downside risk can be considered to capture the fear of investors more efficiently

Sortino's ratio: Performance measurement with downside risk

- Consider the example below, where we compare the Sharpe and Sortino measures
- Considering a riskfree rate of 2%

Year (Rf=2%)	Portfolio A return (%)	Portfolio B return (%)
1	-5	-1
2	-3	-1
3	-2	-1
4	3	-1
5	3	0
6	6	4
7	7	4
8	8	7
9	10	13
10	13	16
Average	?	?
SD	?	?

Sortino's ratio: Performance measurement with downside risk

- Consider the example below, where we compare the Sharpe and Sortino measures
- Considering a riskfree rate of 2%

Year (Rf=2%)	Portfolio A return (%)	Portfolio B return (%)
1	-5	-1
2	-3	-1
3	-2	-1
4	3	-1
5	3	0
6	6	4
7	7	4
8	8	7
9	10	13
10	13	16
Average	4	4
SD	5.60	5.92

Sortino's ratio: Performance measurement with downside risk

- The Sharpe ratios of portfolio A and B are computed as follows
- $S_A = \frac{4-2}{5.60} = 0.357$ and $S_B = \frac{4-2}{5.92} = 0.338$
- Based on these numbers it appears that A outperformed B
- Let us see what happens when we only consider the downside risk
- Let us use the average return of 4% as MAR to compute the downside return, and target return T as riskfree return
- All the positive returns are considered as Zero
- **Sortino's ratio :** $\frac{\overline{R_p}-T}{D_R}$
- $D_R = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(Min(0, R_i - MAR)\right)^2}$

Sortino's ratio: Performance measurement with downside risk

- Assume target rate of 4% (average return)
- MAR= riskfree= 2%
- $DR_A = \sqrt{\frac{[(-5-4)^2 + \cdots .. + \cdots ... + \cdots .. + \cdots ....]}{10}}$=?
- $DR_B = \sqrt{\frac{[(-1-4)^2 + \cdots .. + \cdots ... + \cdots ... + ...^2]}{10}}$= ?
- The Sortino's ratio for both of these funds are computed as
- $ST_A = : \frac{\overline{R_A} - T}{DR_A}$ =? and $ST_B = \frac{\overline{R_B} - T}{DR_B}$ =?

Year (Rf=2%)	Portfolio A return (%)	Portfolio B return (%)
1	-5	-1
2	-3	-1
3	-2	-1
4	3	-1
5	3	0
6	6	4
7	7	4
8	8	7
9	10	13
10	13	16
Average	4	4
SD	5.60	5.92

Sortino's ratio: Performance measurement with downside risk

- Let us see what happens when we only consider the downside risk
- Let us use the average return of 4% as MAR to compute the downside return
- All the positive returns are considered as Zero
- $DR_A = \sqrt{\frac{[(-5-4)^2+(-3-4)^2+(-2-4)^2+(3-4)^2+(-3-4)^2]}{10}} = 4.10$
- $DR_B = \sqrt{\frac{[(-1-4)^2+(-1-4)^2+(-1-4)^2+(-1-4)^2+(0-4)^2]}{10}} = 3.41$
- The Sortino's ratio for both of these funds are computed as
- $ST_A = \frac{4-2}{4.10} = 0.488$ and $S_B = \frac{4-2}{3.41} = 0.587$

Sortino's ratio: Performance measurement with downside risk

- With Sortino's ratio, portfolio B appears to perform better
- This happens because portfolio A appears to have more extreme negative returns
- Various risk averse investors would be uncomfortable with this aspect of portfolio A.

Year (Rf=2%)	Portfolio A return (%)	Portfolio B return (%)
1	-5	-1
2	-3	-1
3	-2	-1
4	3	-1
5	3	0
6	6	4
7	7	4
8	8	7
9	10	13
10	13	16
Average	4	4
SD	5.60	5.92

Last words on one-parameter measures

- If portfolios are well diversified, then Treynor's measure and Sharpe give same results
- However, for poorly diversified portfolios, one can get a high rank on Treynor's measure (as it ignores the diversifiable risk), despite performing poorly on Sharpe's measure
- Also to be noted, that these measures provide comparisons, and produce relative rankings not absolute performance rankings

Last words on one-parameter measures

- In this regard, the advantage with Jensen's alpha is that it produces an absolute measure
- For example, an alpha value of 2% would indicate that manager generated excess return of 2% per period more than the expected returns
- Also, the result from Jensen's alpha has certain statistical significance
- Moreover, Jensen's alpha has the flexibly to compute the alpha with respect to any given model.

Thanks

Portfolio Performance Evaluation: Timing and selection

Course: Portfolio Management

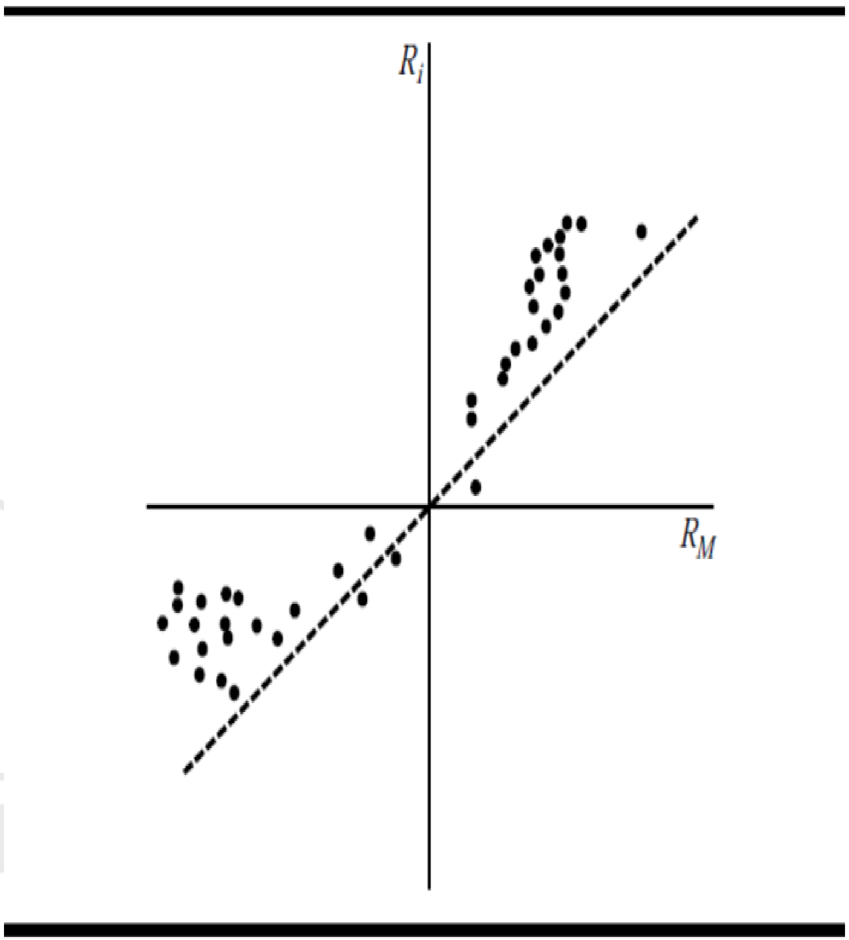
Instructor: Abhinava Tripathi

Portfolio Performance Evaluation: Timing

- Timing involves changing the sensitivity of the portfolio to one or more systematic influences in the anticipation of future movements
- For example, in anticipation of market movements, the manager would want to adjust the portfolio
- If you believe that market will go up and want to exploit this, you can buy high beta stocks and sell low beta stocks
- Alternatively you can buy equity and sell debt
- Vice-versa for opposite scenario
- Another less costly method is to buy sell stock index futures

Portfolio Performance Evaluation: Timing

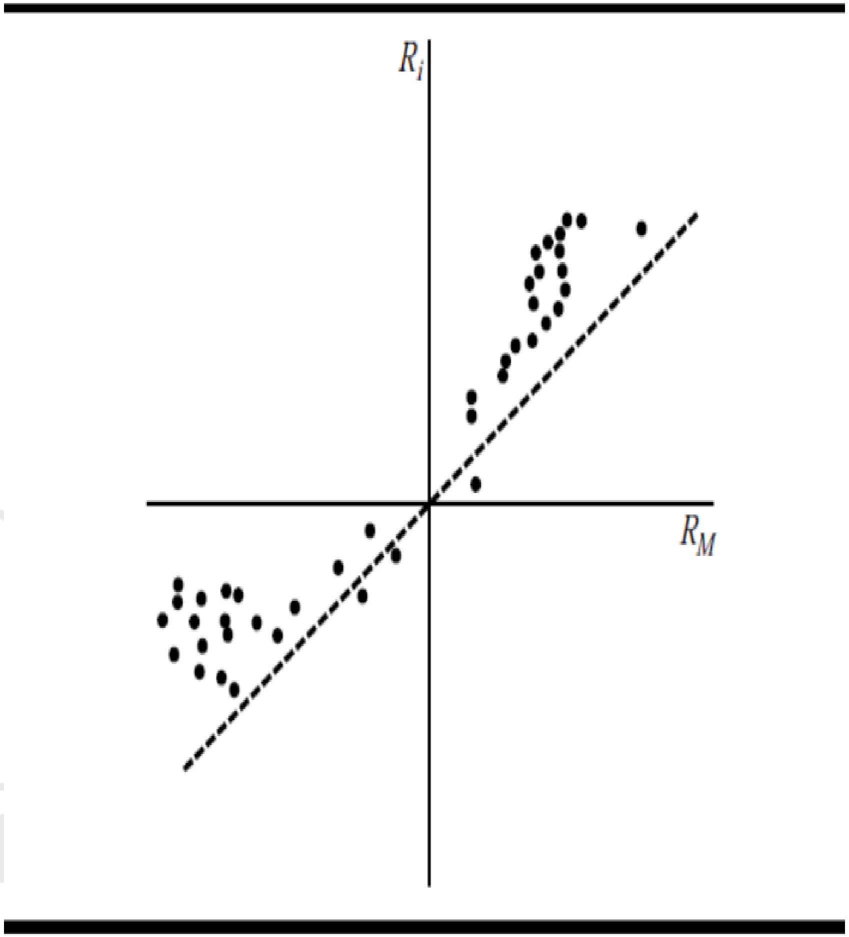
- To test the ability of a manager to time the market can be done as follows
- One can plot the returns with market returns
- When the market increases substantially, the fund would have higher beta than the normal
- Also the returns would be above the normal returns



Returns for manager with timing.

Portfolio Performance Evaluation: Timing

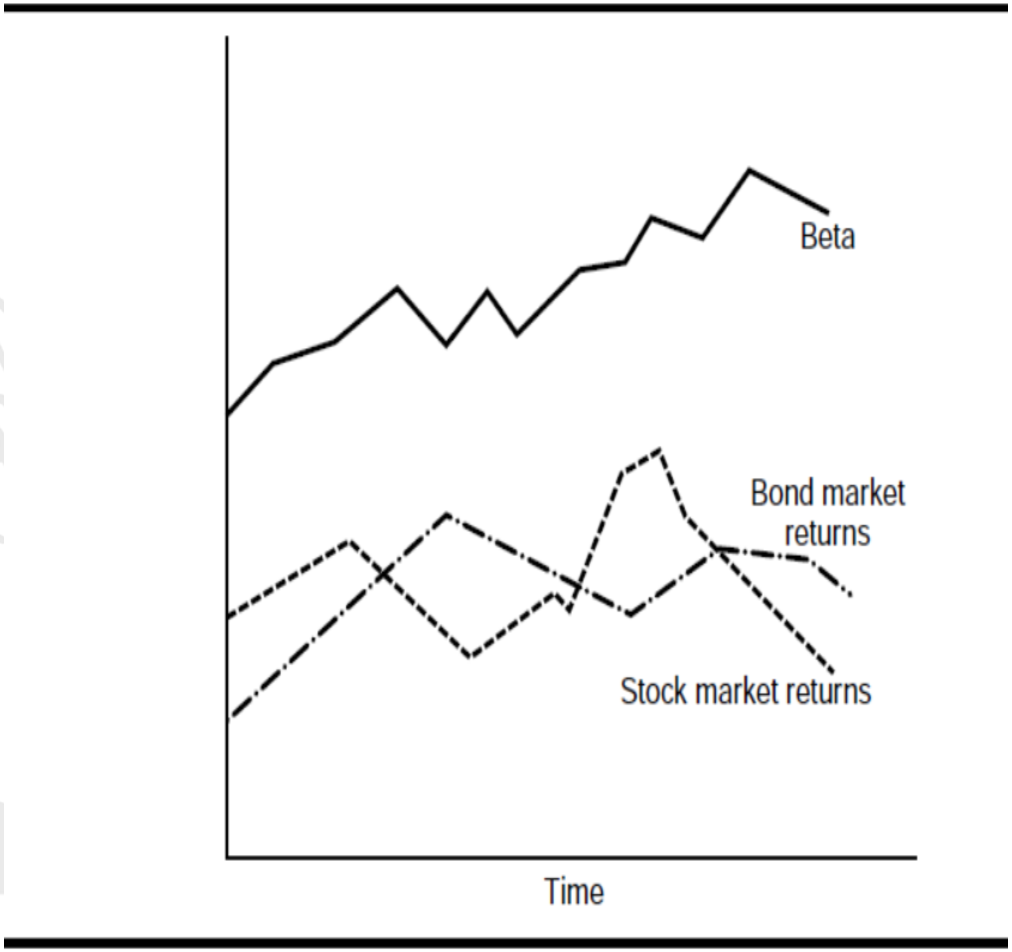
- Likewise, in the cases of market declines, low beta would be observed
- Also, the decline in portfolio would be less than the normal decline in the market downturn



Returns for manager with timing.

Evaluation of market timing

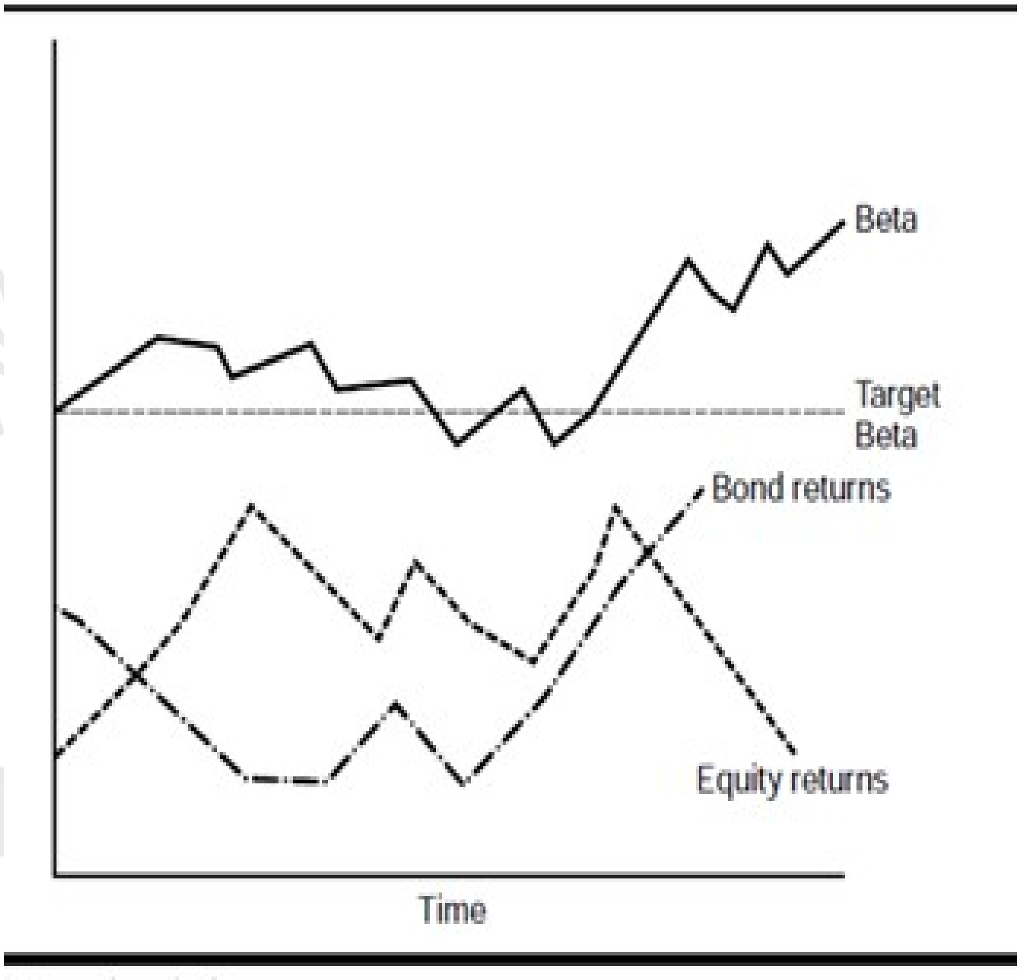
- One way to observe whether the manager is trying to time the market is to visualize (i.e., graphically examine) the movements of the market versus the beta of the fund
- Or bond-stock mix (capital allocation in the fund), what is this?
- If a fund is successfully able to time the market, then the beta of the firm should mimic the market movements in advance



Evaluation of market timing

- In order to perform the timing analysis, we examine the beta policy of the fund

- Then we examine the deviations from the policy and their relation to market movements.

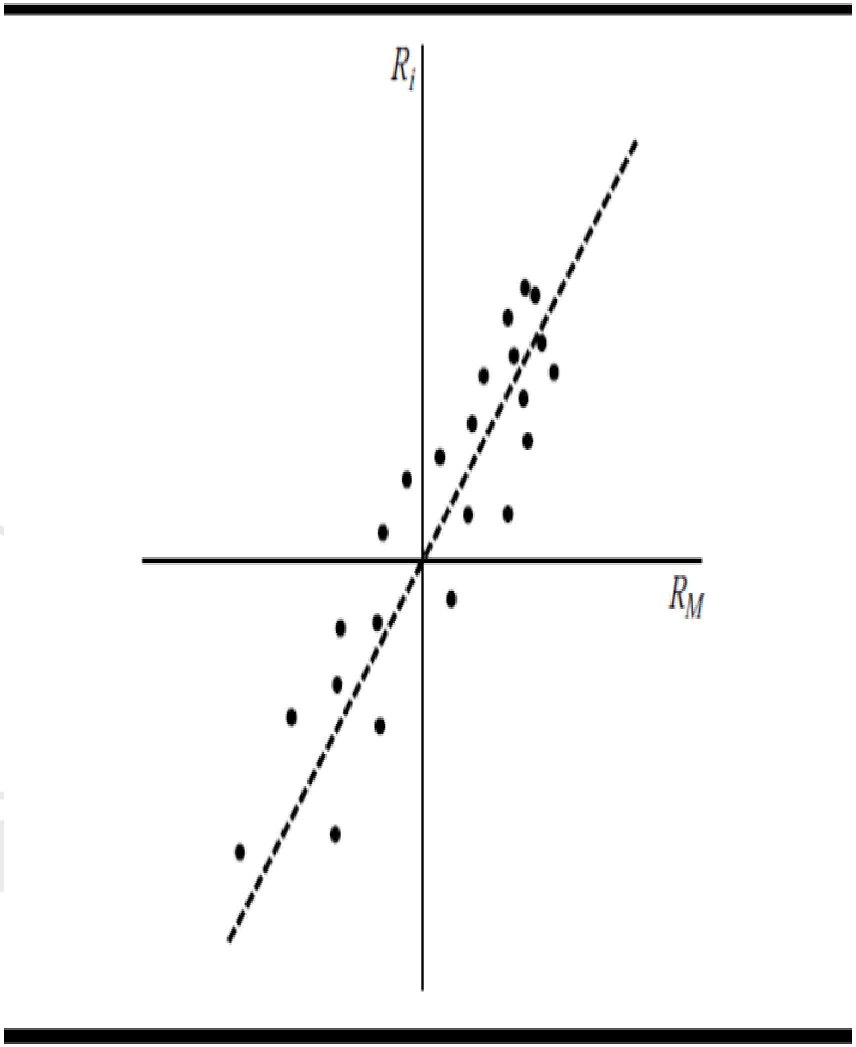


Evaluation of market timing

- If there some relation in portfolio beta (or bond stock mix) and market return, then this should be apparent from the plot
- If the portfolio is fairly diversified with no stock specific variations, and only risk that is present in the stock is represented by beta
- This means that the relation in portfolio return and market return would essentially represent this beta

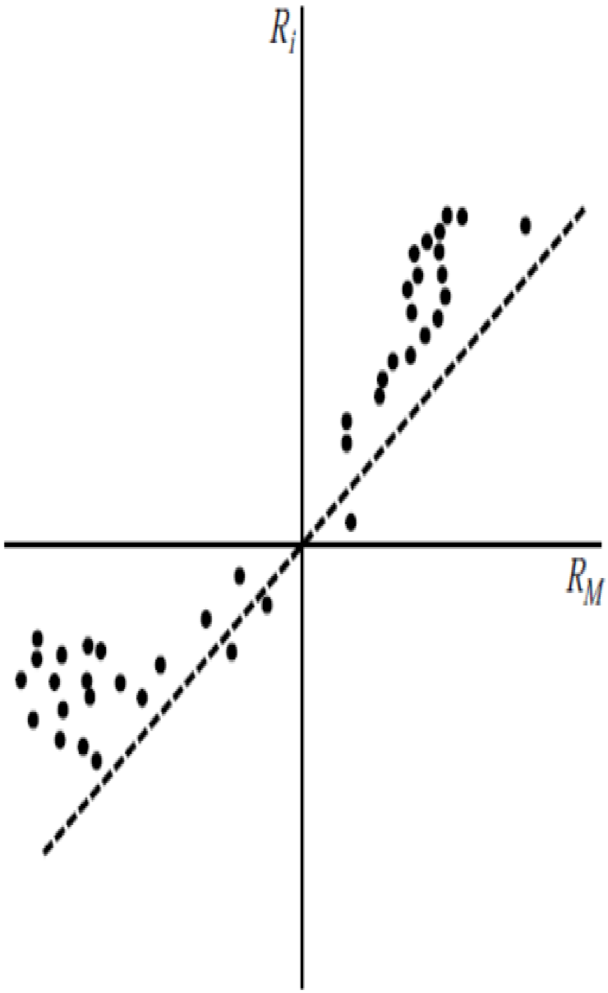
Evaluation of market timing

- In the Figure, we can see that on average the stock return and fund (portfolio) return are in the form of a straight line
- This means that no timing strategy
- The scattered nature of points around the line indicate the presence of diversifiable risk in small quantities.



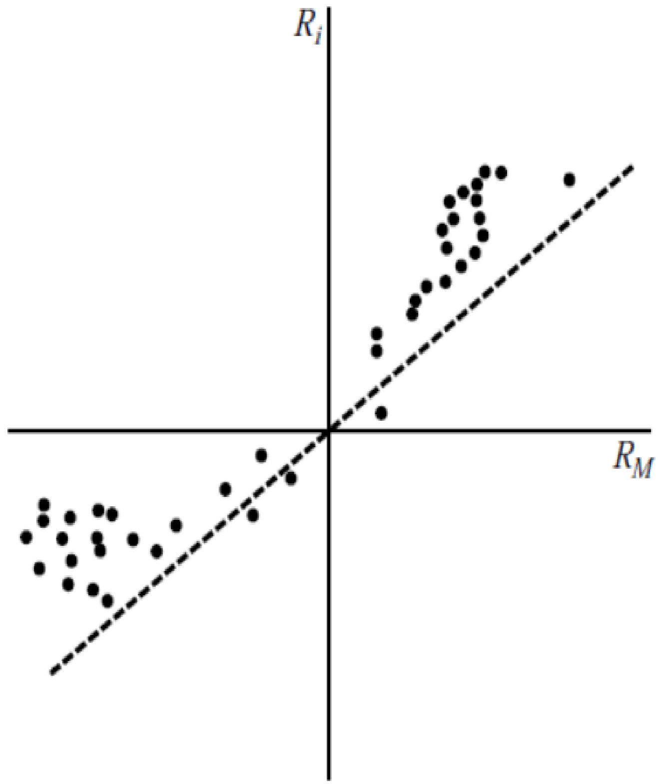
Evaluation of market timing

- If the fund is successfully following the timing strategy by changing the beta of the fund, then if it anticipates rising market, the fund would exhibit a higher beta in advance and tend to do well as compared to normal conditions
- This would cause the return points to be above the line of showing the average relationship between the fund and market during normal times



Evaluation of market timing: Manager with timing

- Similarly, in the cases of market declines, the fund would decrease beta
- Therefore, the fall in prices would be less and the return points would still be above the line showing the average relationships during normal times
- In both cases the points will be above the normal relationship line and exhibit a curvature



Simple statistical measurement of performance

- How to statistically measure this performance
- $R_{it} - R_{Ft} = a_i + b_i(R_{mt} - R_{Ft}) + c_i(R_{mt} - R_{Ft})^2 + e_{it}$
- Here, R_{it} is the return on fund 'i' in period t
- R_{mt} is the return on the market index in period t
- R_{Ft} is the riskless asset return, and e_{it} is the residual return
- Here, if there is no strategy then the coefficient c_i that captures the relationship of excess returns ($R_{it} - R_{Ft}$) with the curvature term '$(R_{mt} - R_{Ft})^2$' should be insignificant and zero

Simple statistical measurement of performance

- Here, if there is no strategy then the coefficient c_i that captures the relationship of excess returns ($R_{it} - R_{Ft}$) with the curvature term '$(R_{mt} - R_{Ft})^2$' should be insignificant and zero
- Suppose we find that the coefficient c_i is significantly positive This would indicate the ability of the fund to time the market
- Therefore, c_i here becomes a measure of fund's timing ability
- What if c_i is negative
- Also please note, here we are believing in the CAPM/single factor APT
- The model can be adjusted to reflect multi-factor APT as well

Holding measure of timings

- In contrast to examining the relationship between the fund returns and market returns, holding measures rely on the portfolio holdings

- Beta is estimated as the weighted average beta of securities that comprise the portfolio

- This requires beta estimation of each security in the portfolio

- Then, using holdings data, one can compute the security proportions in the fund, and thereby, estimate portfolio beta

Holding measure of timings

- One measure of performance evaluation is discussed as follows [Elton, Gruber, Blage: EGB measure]

- $\boldsymbol{Timing} = \sum_{t=1}^{T} \frac{(\boldsymbol{\beta}_{At} - \boldsymbol{\beta}_t^*) \boldsymbol{R}_{pt}}{\boldsymbol{T}}$

- Here β_t^* is the target beta and β_{At} is the actual beta for the beginning of the period

- T is the number of time-periods and R_{pt} is the return in the period

- The measure captures whether the fund deviated from the target beta in the same direction as the return on the index deviated from its normal pattern

Holding measure of timings

- $\boldsymbol{Timing = \sum_{t=1}^{T} \frac{(\beta_t^* - \beta_{At}) R_{pt}}{T}}$

- Does the fund increase its beta when index returns are high, and decrease when the index returns are low

- Target beta is determined by the firm policy and an agreed upon normal beta

- Often an average beta overtime can also be used as a proxy of target beta

Holding measure of timings

- $\boldsymbol{Timing} = \sum_{t=1}^{T} \frac{(\beta_t^* - \beta_{At}) R_{pt}}{T}$
- Some researcher believe that certain aspects of the market can be forecasted with reasonable accuracy
- For example, dividend price ratio can be employed to forecast prices
- Therefore, fund manager should not get credit for price changes that can be easily forecasted using metrics such as dividend price ratio
- Then the beta (price) forecasted using the metrics (e.g., dividend price ratio) may also be used as target beta.

Holding measure of security selection

- By looking at the portfolio holdings, the investor can find which securities the manager buys or sells in the portfolio
- Then one can establish which stock or bond positions led to these performances
- For example, consider the Grinblatt-Titman (GT) Performance Measure below
- $GT_t = \sum_{j=1}^{N}\left(w_{jt} - w_{jt-1}\right)R_{jt}$
- The manager's security selection ability can be established by understanding how he adjusted these weights

Holding measure of security selection

- $GT_t = \sum_{j=1}^{N}\left(w_{jt} - w_{jt-1}\right)R_{jt}$
- $w_{it} - w_{jt-1}$= change in the weights for the jth security between the periods 't' and 't-1'
- R_{jt}= Return on the security 'j' during period 't'.
- The manager's security selection ability can be established by understanding how he adjusted these weights
- A series of GTs can be averaged over several periods to get an average measure ($\frac{\sum_{i=1}^{T} GT_t}{T}$)
- This average GT is an indicator of the quality of his decision making

Holding measure of security selection-Example

- $GT_t = \sum_{j=1}^{N}(w_{jt} - w_{jt-1})R_{jt}$
- Different portfolio performances are shown
- (1) Passive portfolio that carries value-weighted index of all the stocks in the market
- (2) Active portfolio manager
- Panel A shows the share prices of all the five (5) stocks available for investment
- These are shown for six different dates relative to the current date 0
- For these stocks the returns are computed and are shown
- Panel B shows the shares outstanding at the beginning dates for each of the periods

Holding measure of security selection-Example

- Different portfolio performances are shown
- (1) Passive portfolio that carries value-weighted index of all the stocks in the market; (2) Active portfolio manager
- Panel A shows the share prices of all the five (5) stocks available for investment. These are shown for six different dates relative to the current date 0. For these stocks the returns are computed and are shown. Panel B shows the shares outstanding at the beginning dates for each of the periods. The index weights are also shown at the beginning of the periods. The index weights (28% for A at the beginning of 2) are computed by multiplying the stock price (14 for A Date 1) with the number of stocks (200 for A Date 1) for numerator.

Holding measure of security selection-Example

- Panel A shows the share prices of all the five (5) stocks available for investment. These are shown for six different dates relative to the current date 0. For these stocks the returns are required.

A. Stock Market Data

	SHARE PRICE ($):					
Stock	Date -1	Date 0	Date 1	Date 2	Date 3	Date 4
A	10	10	14	13	13	14
B	10	10	8	8	8	6
C	10	10	8	8	7	6
D	10	10	10	11	12	12
E	10	10	10	10	10	10

A. Stock Market Data

	RETURN (%):			
Stock	Period 1	Period 2	Period 3	Period 4
A	14/10-1=40%			
B				
C				
D				
E				

Holding measure of security selection-Example

- Panel A shows the share prices of all the five (5) stocks available for investment. These are shown for six different dates relative to the current date 0. For these stocks the returns are required.

A. Stock Market Data

	SHARE PRICE ($):					
Stock	Date -1	Date 0	Date 1	Date 2	Date 3	Date 4
A	10	10	14	13	13	14
B	10	10	8	8	8	6
C	10	10	8	8	7	6
D	10	10	10	11	12	12
E	10	10	10	10	10	10

A. Stock Market Data

	RETURN (%):			
Stock	Period 1	Period 2	Period 3	Period 4
A	40	-7.14	0	7.69
B	-20	0	0	-25
C	-20	0	-12.5	-14.29
D	0	10	9.09	0
E	0	0	0	0

Holding measure of security selection-Example

- Panel B shows the shares outstanding at the beginning dates for each of the periods
- The index weights (28% for A at the beginning of 2) are computed by multiplying the stock price (14 for A Date 1) with the number of stocks (200 for A Date 1) for numerator
- Denominator=200*(14+8+8+10+10)=10000
- This is a value-weighted passive portfolio

B. Value-Weighted Index Holding Data					
	SHARES OUTSTANDING ON:				
Stock	Date -1	Date 0	Date 1	Date 2	Date 3
A	200	200	200	200	200
B	200	200	200	200	200
C	200	200	200	200	200
D	200	200	200	200	200
E	200	200	200	200	200

B. Value-Weighted Index Holding Data					
		INDEX WEIGHT (wjt) AT BEGINNING OF:			
Stock	Period 0	Period 1	Period 2	Period 3	Period 4
A			14*200 /10000= 0.28		
B					
C					
D					
E					

Holding measure of security selection-Example

- Panel B shows the shares outstanding at the beginning dates for each of the periods
- The index weights (28% for A at the beginning of 2) are computed by multiplying the stock price (14 for A Date 1) with the number of stocks (200 for A Date 1) for numerator
- Denominator=200*(14+8+8+10+10)=10000
- This is a value-weighted passive portfolio

B. Value-Weighted Index Holding Data

	SHARES OUTSTANDING ON:				
Stock	Date -1	Date 0	Date 1	Date 2	Date 3
A	200	200	200	200	200
B	200	200	200	200	200
C	200	200	200	200	200
D	200	200	200	200	200
E	200	200	200	200	200

B. Value-Weighted Index Holding Data

		INDEX WEIGHT (wjt) AT BEGINNING OF:			
Stock	Period 0	Period 1	Period 2	Period 3	Period 4
A	0.2	0.2	0.28	0.26	0.26
B	0.2	0.2	0.16	0.16	0.16
C	0.2	0.2	0.16	0.16	0.14
D	0.2	0.2	0.2	0.22	0.24
E	0.2	0.2	0.2	0.2	0.2

Holding measure of security selection-Example

- Panel C shows the holdings of active manager at the beginning dates for each of the periods
- The portfolio weights (33.3% for A at the beginning of 2) are computed by multiplying the stock price (14 for A Date 1) with the portfolio holdings (10 for A Date 1) for numerator
- Denominator is =14*10+8*5+8*5+10*10 +10*10=420

C. Active Manager Holding Data

	SHARES HELD ON:				
Stock	Date -1	Date 0	Date 1	Date 2	Date 3
A	0	10	10	10	10
B	10	5	5	0	0
C	10	5	5	10	10
D	10	10	10	10	10
E	10	10	10	10	10

C. Active Manager Holding Data

	PORTFOLIO WEIGHT (wjt) AT BEGINNING OF:				
Stock	Period 0	Period 1	Period 2	Period 3	Period 4
A			140/420 =0.333		
B					
C					
D					
E					

Holding measure of security selection-Example

- Panel C shows the holdings of active manager at the beginning dates for each of the periods
- The portfolio weights (33.3% for A at the beginning of 2) are computed by multiplying the stock price (14 for A Date 1) with the portfolio holdings (10 for A Date 1) for numerator
- Denominator is =14*10+8*5+8*5+10*10 +10*10=420

C. Active Manager Holding Data

	SHARES HELD ON:				
Stock	Date -1	Date 0	Date 1	Date 2	Date 3
A	0	10	10	10	10
B	10	5	5	0	0
C	10	5	5	10	10
D	10	10	10	10	10
E	10	10	10	10	10

C. Active Manager Holding Data

	PORTFOLIO WEIGHT (wjt) AT BEGINNING OF:				
Stock	Period 0	Period 1	Period 2	Period 3	Period 4
A	0	0.25	0.333	0.31	0.31
B	0.25	0.125	0.095	0	0
C	0.25	0.125	0.095	0.19	0.167
D	0.25	0.25	0.238	0.262	0.286
E	0.25	0.25	0.238	0.238	0.238

Holding measure of security selection-Example

- Now let us compute GT measure

Value-Weighted Index				
Stock	(w1 - w0)× R1	(w2 - w1)× R2	(w3 - w2)× R3	(w4 - w3)× R4
A				
B				
C				
D				
E				
GT				

Active Manager				
Stock	(w1 - w0)× R1	(w2 - w1)× R2	(w3 - w2)× R3	(w4 - w3)× R4
A				
B				
C				
D				
E				
GT				

Holding measure of security selection-Example

- Now let us compute GT measure

Value-Weighted Index				
Stock	(w1 - w0)× R1	(w2 - w1)× R2	(w3 - w2)× R3	(w4 - w3)× R4
A	0	-0.57	0	0
B	0	0	0	0
C	0	0	0	0.29
D	0	0	0.18	0
E	0	0	0	0
GT	0.00%	-0.57%	0.18%	0.29%

Active Manager				
Stock	(w1 - w0)× R1	(w2 - w1)× R2	(w3 - w2)× R3	(w4 - w3)× R4
A	10	-0.59	0	0
B	2.5	0	0	0
C	2.5	0	-1.19	0.34
D	0	-0.12	0.22	0
E	0	0	0	0
GT	15.00%	-0.71%	-0.97%	0.34%

Holding measure of security selection-Example

- Average GT for VWA index= (0.00-0.57+0.18+0.29)/5= 0.02%
- Average GT for Active manager= (15.00-0.71+0.97+0.34)/5= 3.12%
- For the index, average GT across the investments is close to zero (-0.02%)
- This is expected for the passive buy-and-hold portfolio
- In contrast, the average GT for an active portfolio should be positive (3.41% in this case) or negative when he has not done well

Holding measure of security selection-Example

- This positive GT indicates that the manager added substantial value through his stock selection skills
- In period 1, the decision to buy stock A at date 0, contributed 10%. Whereas, the decisions to sell stocks B and C contributed 2.5% each
- In contrast, the decision to repurchase stock C on date 2 subtracted 1.2% if the value

Characteristic Selectivity (CS) Performance Measure

- This is an improvement over GT
- One shortcoming of the GT measure is that it does not control for the market trends (that are public) causing increasing returns
- For example, a stock return may simply perform well because the market (or some benchmark index of which this stock was part of) did well, and it was public knowledge
- Thus, an improvement over GT measure is suggested by comparing the returns of the actively managed fund to those of a benchmark fund that has the same aggregate investment characteristics

Characteristic Selectivity (CS) Performance Measure

- This measure is described as follows
- $CS_t = \sum_{j=1}^{N} w_{jt}(R_{jt} - R_{Bjt})$
- Here, R_{Bjt} is the return to a passive portfolio whose investment characteristics are matched at the beginning of period ‘t’ with those of stock j
- With this, the values of CS_t can be averaged over a period to indicate the manager’s ability to pick specific stocks
- Average CS= $\frac{\sum_{t=1}^{T} CS}{T}$

Characteristic Selectivity (CS) Performance Measure

- This measure credits the manager for selecting the stock that outperforms a style-matched index investment and penalizes when the opposite is true
- The argument is that why should an investor pay the management fee of actively managed stock, when he can simply buy these indexes that suit certain investment styles
- Thus, the manager is rewarded only when his portfolio outperforms the passive portfolio matched in terms of investment style indices
- One challenge to this measure is identification of risk and style characteristics of stocks that the active manager plans to hold.

Thanks