NCERT - 9.4.11

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Question:
For the differential equation given below, find a particular solution that satisfies y=1 when x=0:

$$\left(x^3 + x^2 + x + 1\right) \frac{dy}{dx} = 2x^2 + x \tag{0.1}$$

Solution (using the method of finite differences):
The required particular solution can be found using the method of finite differences.

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} \tag{0.2}$$

$$\implies y(x+h) = y(x) + h \cdot \frac{dy}{dx} \tag{0.3}$$

As can be seen from the question above,

$$\frac{dy}{dx} = \frac{2x^2 + x}{(x^3 + x^2 + x + 1)} \tag{0.4}$$

$$\implies y(x+h) = y(x) + h \cdot \frac{2x^2 + x}{(x^3 + x^2 + x + 1)} \tag{0.5}$$

Let $x_0 = 0$ and $y_0 = 1$ (as per the given condition).
Let some $x_1 = x_0 + h$. Then

$$y_1 = y_0 + h \cdot \frac{2x_0^2 + x_0}{\left(x_0^3 + x_0^2 + x_0 + 1\right)} \tag{0.6}$$

Iterating through the above-mentioned process to generate y_2, y_3, y_4 and so on generalises equation (0.6) to

$$y_{n+1} = y_n + h \cdot \frac{2x_n^2 + x_n}{\left(x_n^3 + x_n^2 + x_n + 1\right)} \tag{0.7}$$

The smaller the value of h, the more accurate the curve is.

Solution (using manual methods):

$$\frac{dy}{dx} = \frac{2x^2 + x}{(x^3 + x^2 + x + 1)} \tag{0.8}$$

$$\implies \frac{dy}{dx} = \frac{2x^2 + x}{(x^2 + 1)(x + 1)} \tag{0.9}$$

To split equation (0.9) into partial fractions, assume that it can be rewritten as given below:

$$\frac{dy}{dx} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} \tag{0.10}$$

Here A, B, and C are real numbers.

From equation (0.10),

$$\frac{dy}{dx} = \frac{A(x^2 + 1) + (Bx + C)(x + 1)}{(x + 1)(x^2 + 1)} \tag{0.11}$$

$$\implies \frac{dy}{dx} = \frac{(A + B)x^2 + (B + C)x + (A + C)}{(x + 1)(x^2 + 1)} \tag{0.12}$$

Equating equations (0.9) and (0.12), we get

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2} \text{ and } C = \frac{-1}{2} \tag{0.13}$$

Substituting equation (0.13) in equation (0.10) and integrating, we get

$$\int dy = \int \left(\frac{1}{2(x + 1)} + \frac{3x - 1}{2(x^2 + 1)} \right) dx \tag{0.14}$$

$$\int dy = \frac{1}{2} \int \frac{1}{(x + 1)} dx + \frac{3}{4} \int \frac{2x}{x^2 + 1} dx - \frac{1}{2} \int \frac{1}{x^2 + 1} dx \tag{0.15}$$

$$\implies y = \frac{1}{2} \ln(x + 1) + \frac{3}{4} \ln(x^2 + 1) - \frac{1}{2} \tan^{-1} x + c \tag{0.16}$$

Here, c is the constant of integration.
Substituting the initial conditions of $x = 0$ and $y = 1$ in equation (0.16), we get

$$c = 1 \tag{0.17}$$

Therefore, the equation of the curve found by manual methods is

$$y = \frac{1}{4} \ln\left((x + 1)^2 \left(x^2 + 1\right)^3\right) - \frac{1}{2} \tan^{-1} x + 1 \tag{0.18}$$

The curve generated using both described methods for the given question, taking $h = 0.1$ and running iterations 100 times is given below.

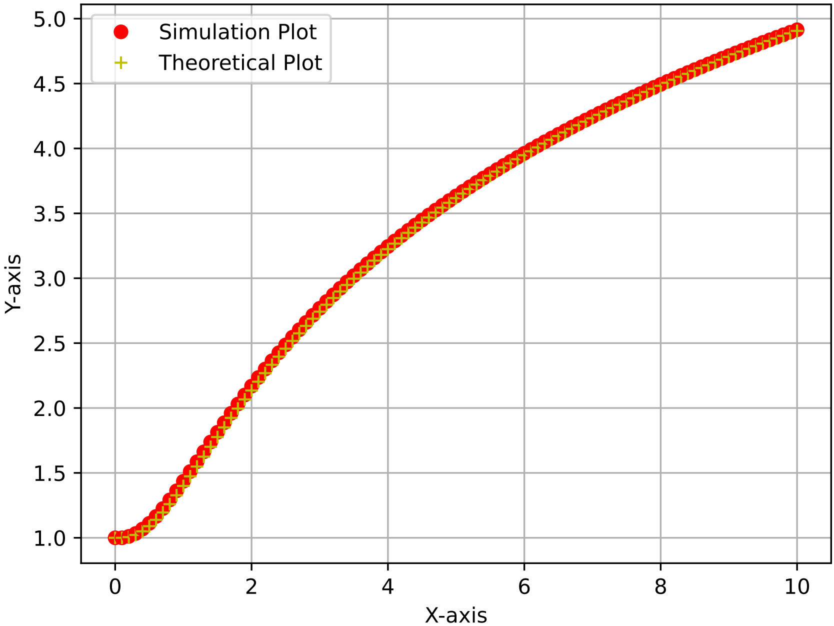

Fig. 0.1: Solution of given DE