Question-9.6.7

EE24BTECH11038 - MALAKALA BALA SUBRAHMANYA ARAVIND

Question: Find the general solution of x$\frac{dy}{dx}$+y=$\frac{2}{x}\log x$

Solution:
We can rewrite the equation in the form of a linear first-order differential equation

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2\log^2 x} \tag{0.1}$$

This matches the general form of a first-order linear differential equation

$$\frac{dy}{dx} + P(x)\,y = Q(x) \tag{0.2}$$

where

$$P(x) = \frac{1}{x\log x} \tag{0.3}$$

$$Q(x) = \frac{2}{x^2\log^2 x} \tag{0.4}$$

The integrating factor $\mu(x)$ is calculated as

$$\mu(x) = e^{\int P(x)\,dx} \tag{0.5}$$

$$\mu(x) = e^{\int \frac{1}{x\log x}\,dx} \tag{0.6}$$

$$\log x = t \tag{0.7}$$

$$\frac{1}{x}\,dx = dt \tag{0.8}$$

$$\int \frac{1}{xlogx}\,dx = \int \frac{1}{t}\,dt \tag{0.9}$$

$$\int \frac{1}{x\log x} = \log(\log x) \tag{0.10}$$

$$\mu(x) = e^{\log(\log x)} \tag{0.11}$$

$$\mu(x) = \log x \tag{0.12}$$

Now, multiply the entire differential equation by $\mu(x)$=$\log x$

$$\log x \frac{dy}{dx} + \frac{y \log x}{x \log x} = \frac{2 \log x}{x^2 \log^2 x} \tag{0.13}$$

$$\frac{dy}{dx} \log x + \frac{y}{x} = \frac{2}{x^2 \log x} \tag{0.14}$$

$$\int \frac{d}{dx} (y \log x) \ dx = \int \frac{2}{x^2 \log x} \ dx \tag{0.15}$$

$$\log x = t \tag{0.16}$$

$$\frac{1}{x} dx = dt \tag{0.17}$$

$$\int \frac{2}{x^2 \log x} \ dx = \int \frac{2}{te^t} dt \tag{0.18}$$

$$\int \frac{2}{x^2 \log x} \ dx = \frac{-2}{te^t} \tag{0.19}$$

$$\int \frac{2}{x^2 \log x} dx = \frac{-2}{x \log x} \tag{0.20}$$

$$y \log x = \frac{-2}{x \log x} + c \tag{0.21}$$

Final solution

$$y = \frac{-2}{x \log^2 x} + \frac{c}{\log x} \tag{0.22}$$

Let the initial conditions be $y_0 = \frac{-2}{e}$,$x_0 = e$

$$\frac{-2}{e} = \frac{-2}{e} + c \tag{0.23}$$

$$c = 0 \tag{0.24}$$

Numerical Approach:
1. I used a for loop for finding the y values as the loop proceeds with iterative formula given below. I took some initial value of x and as loop proceeds I assigned it the value as $x + h$. where h is the step size, representing the rate of change.
2. Assigned the values of y for different x-values using a for loop.

Using the Method of Finite Differences
The Method of Finite Differences is a numerical technique used to approximate solutions to differential equations.

We know that:

$$\lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = \frac{dy}{dx} \tag{0.25}$$

For the given differential equation,

$$\frac{dy}{dx} = \frac{2}{x^2 \log^2 x} - \frac{y}{x \log x} \tag{0.26}$$

$$\frac{y_{n+1} - y_n}{h} \approx \frac{2}{x_n^2 \log^2 x_n} - \frac{y_n}{x_n \log x_n} \tag{0.27}$$

$$y_{n+1} = y_n + h \cdot \left(\frac{2}{x_n^2 \log^2 x_n} - \frac{y_n}{x_n \log x_n} \right) \tag{0.28}$$

The iterative formula for updating x-values is:

$$x_n = x_{n-1} + h \tag{0.29}$$

Using Matplotlib, I plotted the computed points and the graph of the exact solution to verify that they approximately match