NCERT-9.5.2

S. Sai Akshita - EE24BTECH11054

Question: Find the general solution of the Differential Equation

$$\frac{dy}{dx} + 3y = e^{-2x} \tag{1}$$

Obtaining Solution using Integrating Factor: The given Differential Equation is a first-order linear equation of the form:

$$\frac{dy}{dx} + p(x) \times y = q(x) \tag{2}$$

The Integrating Factor for such equations is given by:

$$\text{IF} = e^{\int p(x)dx} \tag{3}$$

For this question, it is:

$$\text{IF} = e^{\int 3dx} = e^{3x} \tag{4}$$

Multiplying the entire equation (1) by the Integrating Factor , we get:

$$\frac{dy}{dx}e^{3x} + 3ye^{3x} = e^{3x}e^{-2x} \tag{5}$$

$$\frac{d}{dx}\left(ye^{3x}\right) = e^{x} \tag{6}$$

Integrating both sides:

$$\int \frac{d}{dx}\left(ye^{3x}\right)dx = \int e^{x}dx \tag{7}$$

$$ye^{3x} = e^{x} + C \tag{8}$$

Dividing through by :

$$y = e^{-2x} + Ce^{-3x} \tag{9}$$

Thus, (9) is the general solution of the given Differential Equation.
Assume $c = 1$. The solution becomes:

$$y = e^{-2x} + e^{-3x} \tag{10}$$

From here, initial conditions=$(x_0, y_0) = (0, 2)$
Obtaining solution by Laplace Transform:
The fundamental formula for Laplace Transform is:

$$\mathcal{L}(f(t)) = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt \tag{11}$$

Taking the Laplace transform on both sides of 1

$$\mathcal{L}\left(\frac{dy}{dx}\right) + 3\mathcal{L}(y) = \mathcal{L}\left(e^{-2x}\right) \tag{12}$$

Using the Laplace transform properties:

$$\mathcal{L}\left(\frac{dy}{dx}\right) = sY(s) - y(0), \quad \mathcal{L}(y) = Y(s), \quad \mathcal{L}\left(e^{-2x}\right) = \frac{1}{s+2}. \tag{13}$$

Substituting these into the equation:

$$sY(s) - 2 + 3Y(s) = \frac{1}{s+2}. \tag{14}$$

After simplification,

$$Y(s) = \frac{1}{(s+3)(s+2)} + \frac{2}{s+3}. \tag{15}$$

Applying partial fractions to the first term of above equation,

$$\frac{1}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}. \tag{16}$$

$$1 = A(s+2) + B(s+3). \tag{17}$$

$$1 = A(s) + 2A + B(s) + 3B. \tag{18}$$

$$1 = (A+B)s + (2A+3B). \tag{19}$$

By comparing the coefficients, we get

$$A + B = 0 \tag{20}$$

$$2A + 3B = 1 \tag{21}$$

Solving this system of equations:

$$B = -A, \tag{22}$$

$$2A + 3(-A) = 1 \tag{23}$$

$$B = -A = 1. \tag{24}$$

Thus:

$$\frac{1}{(s+3)(s+2)} = \frac{-1}{s+3} + \frac{1}{s+2}. \tag{25}$$

Substitute back into $Y(s)$:

$$Y(s) = \frac{-1}{s+3} + \frac{1}{s+2} + \frac{2}{s+3} \tag{26}$$

$$Y(s) = \frac{1}{s+3} + \frac{1}{s+2}. \tag{27}$$

Finding the Region of Convergence (ROC):
Each term in $Y(s)$ contributes a pole: The term $\frac{1}{s+3}$ has a pole at $s = -3$, and the term $\frac{1}{s+2}$ has a pole at $s = -2$.Thus, the poles of $Y(s)$ are located at $s = -3$ and $s = -2$.

The ROC of a Laplace transform depends on whether the signal is:

1) **Right-sided (causal):** The ROC is to the right of the rightmost pole, i.e., $\mathrm{Re}(s) > -2$.
2) **Left-sided (anti-causal):** The ROC is to the left of the leftmost pole, i.e., $\mathrm{Re}(s) < -3$.
3) **Two-sided:** The ROC lies between the poles, i.e., $-3 < \mathrm{Re}(s) < -2$.

To determine the exact ROC, additional information about the time-domain behavior of the signal is needed:

1) For a **causal signal**, the ROC is: $\mathrm{Re}(s) > -2$.

2) For an **anti-causal signal**, the ROC is: $\text{Re}(s) < -3$.
3) For a **two-sided signal**, the ROC is: $-3 < \text{Re}(s) < -2$.

The ROC depends on the nature of the signal (causal, anti-causal, or two-sided).
In causal systems, The function $y(x)$ depends only on present and past values ($x \geq 0$).

Using the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) = e^{-ax}. \tag{28}$$

Substituting in the above equation :

$$y(x) = \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) + \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \tag{29}$$

$$y(x) = e^{-3x} + e^{-2x} \tag{30}$$

The Laplace Transform converts linear differential equations with constant coefficients into simpler algebraic equations in the Laplace domain.This simplification makes complex systems easier to analyze and solve.

Plotting the Curve: To trace the curve, we calculate the slope of the tangent, at a point on the curve. From (1), the slope is given by:

$$\frac{dy}{dx} = e^{-2x} - 3y \tag{31}$$

Using this slope, we calculate successive points along the curve using a small step size :

$$x_1 = x_0 + h, \tag{32}$$

$$y_1 = y_0 + h\left.\frac{dy}{dx}\right|_{(x_0,y_0)} \tag{33}$$

Similarly, for subsequent points:

$$x_2 = x_1 + h, \tag{34}$$

$$y_2 = y_1 + h\left.\frac{dy}{dx}\right|_{(x_1,y_1)} \tag{35}$$

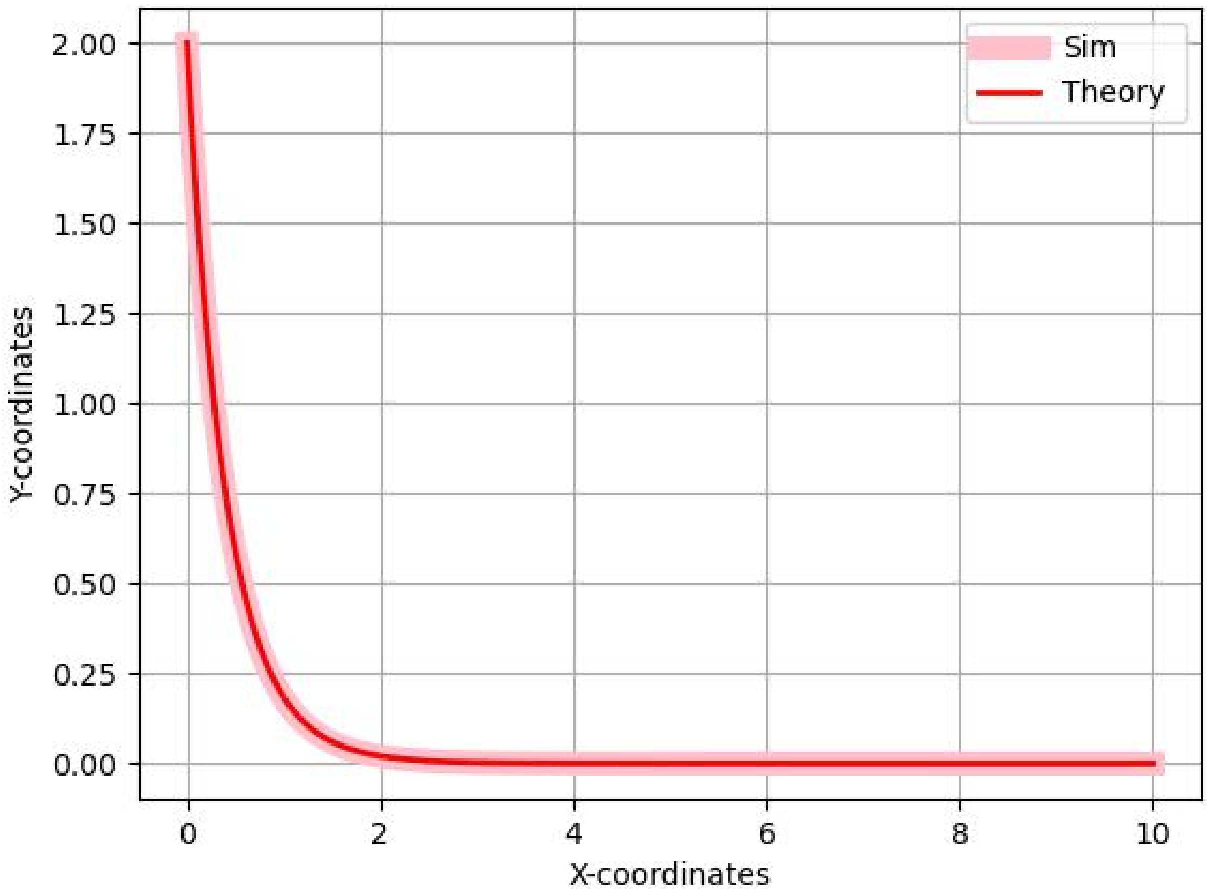
For generating $(n+1)^{th}$ point,

$$x_{n+1} = x_n + h \tag{36}$$

$$y_{n+1} = y_n + h \times \left(e^{-2x_n} - 3y_n\right) \tag{37}$$

$$y_{n+1} = y_n(1-3h) + he^{-2x_n} \tag{38}$$

Repeating this process for a large number of points, we can trace the curve that represents one of the solutions of the Differential Equation. To generate the plot, start from an initial point satisfying (10), choose a small value of h, and calculate successive points,using the equations mentioned above.



Sim
Theory
Y-coordinates
X-coordinates
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
2
4
6
8
10