9.6.8

EE24BTECH11005 - Arjun Pavanje

Question: Solve the differential equation $\left(1+x^2\right)dy+(2xy)\,dx=\cot(x)\,dx$, with initial conditions $y\left(\frac{\pi}{2}\right)=0$

Solution:

Theoretical Solution:

This is a linear differential equation of the first order.

$$\frac{dy}{dx}=\frac{\cot(x)-2xy}{1+x^2} \tag{1}$$

$$\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\cot(x)}{1+x^2} \tag{2}$$

Finding integrating factor

$$e^{\int\frac{2x}{1+x^2}dx} \tag{3}$$

Taking $1+x^2=t$, then the integrating factor becomes

$$e^{\int\frac{dt}{t}} \tag{4}$$

$$=e^{\log t} \tag{5}$$

$$=t=1+x^2 \tag{6}$$

Multiplying both sides of (2) with integrating factor,

$$\frac{dy}{dx}\left(1+x^2\right)+2xy=\cot(x) \tag{7}$$

$$\frac{d\left(\left(1+x^2\right)y\right)}{dx}=\cot(x) \tag{8}$$

$$y\left(1+x^2\right)=\int\cot(x)\,dx+c \tag{9}$$

$$y\left(1+x^2\right)=\log|\sin(x)|+c \tag{10}$$

On substituting initial conditions we get,

$$y=\frac{\log|\sin(x)|}{1+x^2} \tag{11}$$

Computational Solution:

By first principle of derivatives,

$$y'(t)=\lim_{h\to0}\frac{y(t+h)-y(t)}{h} \tag{12}$$

$$y(t+h)=y(t)+hy'(t) \tag{13}$$

If we repeat the above proccess iteratively, we obtain the points to plot. Taking smaller step-size h will give more accurate plots. On discretizing the proccess we get,

$$y(x_{n+1}) = y(x_n) + hy'(x_n) \tag{14}$$

$$x_{n+1} = x_n + h \tag{15}$$

If we denote $y(x_n)$ as y_n, the equation (14) becomes,

$$y_{n+1} = y_n + hy'_n \tag{16}$$

The above equation is the general difference equation.

In the given question,

$$y' = \frac{\cot(x) - 2xy}{1 + x^2} \tag{17}$$

Difference Equation can be written as,

$$y_{n+1} = y_n + h\left(\frac{\cot(x_n) - 2x_n y_n}{1 + x_n^2}\right) \tag{18}$$

Below is a comparission between Simulated Plot and Theoretical Plot.

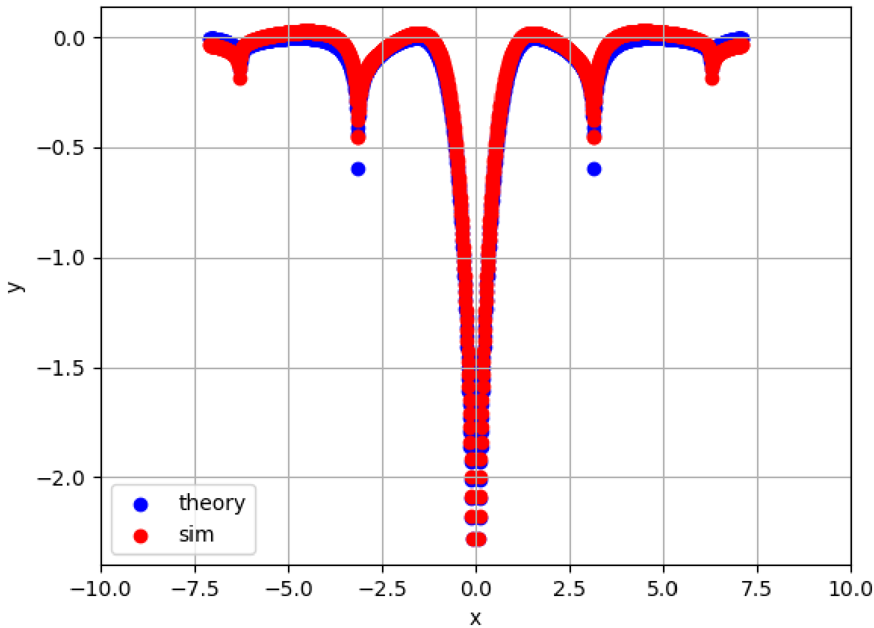

Fig. 1: Computational vs Theoretical solution of $\frac{dy}{dx} = \frac{\cot(x) - 2xy}{1+x^2}$