

NCERT-9.5.13

EE24BTECH11065 - Spoorthi yellamanchali

Question:

Find the solution of the following differential equation:

$$\left[x \sin^2 \left(\frac{y}{x} \right) - y \right] dx + x dy = 0; \quad y = \frac{\pi}{4} \text{ when } x = 1$$

Theoretical Solution:

From the question,

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x}; \quad (0.1)$$

Let $t = \frac{y}{x}$, then,

$$\frac{dy}{dx} = x \frac{dt}{dx} + t \quad (0.2)$$

On substituting the value of $\frac{dy}{dx}$ in the equation (0.1), we get,

$$x \frac{dt}{dx} = -\sin^2 t; -\operatorname{cosec}^2 t \, dt = \frac{dx}{x}; \quad (0.3)$$

On integrating on both sides,

$$\int -\operatorname{cosec}^2 t \, dt = \int \frac{dx}{x}; \quad (0.4)$$

$$\cot t = \ln x + c; \quad (0.5)$$

$$\cot \frac{y}{x} = \ln x + c \quad (0.6)$$

On Substituting given initial conditions in the equation (0.6) we get $c = 1$.

$$\therefore y = x \cot^{-1} (\ln x + 1) \quad (0.7)$$

Solution by the method of finite differences:

The finite difference method is a numerical technique for solving differential equations by approximating derivatives with differences.

The first forward difference approximation of the derivative of $f(x)$ at x is given by:

$$\frac{dy}{dx} = \frac{f(x+h) - f(x)}{h} \quad (0.8)$$

On taking the given point on the curve as the initial conditions (x_0, y_0) , we can get ,

$$x_1 = x_0 + h; \quad (0.9)$$

And from the above equation, we can get

$$y_1 = y_0 + h \left(\frac{dy}{dx} \Big|_{x=x_0} \right) \quad (0.10)$$

we know that our derivative is given by:

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2 \frac{y}{x}; \quad (0.11)$$

On substituting the expression of the derivative in equation (0.11), we get

$$y_1 = y_0 + h \left(\frac{y_0}{x_0} - \sin^2 \frac{y_0}{x_0} \right) \quad (0.12)$$

On assuming a value for h which is close to zero and by substituting the values of x_0 and y_0 in the above equations we get the point (x_1, y_1) .

what we have essentially done above is, obtaining a point which is very close to the initial point along the direction of derivative at that point.

similarly we get,

$$x_n = x_{n-1} + h; \quad (0.13)$$

$$y_n = y_{n-1} + h \left(\frac{y_{n-1}}{x_{n-1}} - \sin^2 \frac{y_{n-1}}{x_{n-1}} \right) \quad (0.14)$$

we can obtain points on the curve by using the above expressions for y_n and x_n

∴ we can plot the curve by the points obtained.

