9.3.11

EE24BTECH11024 - G. Abhimanyu Koushik

Question:
Solve the differential equation $\frac{d^2y}{dx^2} + 1 = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 0$
Solution:

Variable	Description
c_1	First Integration constant
c_2	Second Integration constant
n	Order of given differential equation
a_i	Coeefficient of ith derivative of the function in the equation
c	constant in the equation
y^i	ith derivative of given function
$\mathbf{y}(t)$	$\begin{pmatrix} c \\ y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{pmatrix}$
h	stepsize, taken to be 0.001
$u(x)$	Unit step function

TABLE 0: Variables Used

Theoritical Solution:
Laplace Transform definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt \tag{0.1}$$

Properties of Laplace tranform

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0) \tag{0.2}$$

$$\mathcal{L}(1) = \frac{1}{s} \tag{0.3}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = x^2 u(x) \tag{0.4}$$

$$\mathcal{L}(cf(t)) = c\mathcal{L}(f(t)) \tag{0.5}$$

Applying the properties to the given equation

$$y'' + 1 = 0 \tag{0.6}$$

$$\mathcal{L}(y'') + \mathcal{L}(1) = 0 \tag{0.7}$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \frac{1}{s} = 0 \tag{0.8}$$

$$\tag{0.9}$$

Substituting the initial conditions gives

$$s^3\mathcal{L}(y) + 1 = 0 \tag{0.10}$$

$$\mathcal{L}(y) = \frac{-1}{s^3} \tag{0.11}$$

$$y = \mathcal{L}^{-1}\left(\frac{-1}{s^3}\right) \tag{0.12}$$

$$y = \frac{-1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) \tag{0.13}$$

$$y = \frac{-1}{2}x^2u(x) \tag{0.14}$$

The theoritical solution is

$$f(x) = \frac{-x^2}{2}u(x) \tag{0.15}$$

Computational Solution:
Consider the given linear differential equation

$$a_ny^n + a_{n-1}y^{n-1} + \cdots + a_1y' + a_0y + c = 0 \tag{0.16}$$

Then

$$y'(t) = \lim_{h\to 0}\frac{y(t+h) - y(t)}{h} \tag{0.17}$$

$$y(t+h) = y(t) + hy'(t) \tag{0.18}$$

Similarly

$$y^i(t+h) = y^i(t) + hy^{i+1}(t) \tag{0.19}$$

$$y^{n-1}(t+h) = y^{n-1}(t) + hy^n(t) \tag{0.20}$$

$$y^{n-1}(t+h) = y^{n-1}(t) + h\left(-\frac{a_{n-1}}{a_n}y^{n-1} - \frac{a_{n-2}}{a_n}y^{n-2} - \cdots - \frac{a_0}{a_n}y - \frac{c}{a_n}\right) \tag{0.21}$$

Where i ranges from 0 to $n-1$

$$\mathbf{y}(t+h) = \mathbf{y}(t) + \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{a_n} & -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{pmatrix} (h\mathbf{y}(t)) \tag{0.22}$$

$$\mathbf{y}(t+h) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1-\frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}(t)) \tag{0.23}$$

Discretizing the steps gives us

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & h & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & h & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & h \\ -\frac{h}{a_n} & -\frac{a_0 h}{a_n} & -\frac{a_1 h}{a_n} & -\frac{a_2 h}{a_n} & \dots & -\frac{a_{n-2} h}{a_n} & 1-\frac{a_{n-1} h}{a_n} \end{pmatrix} (\mathbf{y}_k) \tag{0.24}$$

where k ranges from 0 to number of data points with y_0^i being the given initial condition and vector $\mathbf{y}_0 = \begin{pmatrix} c \\ y(0) \\ y'(0) \\ \vdots \\ y^{n-1}(0) \end{pmatrix}$

For the given question

$$\mathbf{y}_{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & h \\ -h & 0 & 1 \end{pmatrix} \mathbf{y}_k \tag{0.25}$$

Record the y_k for

$$x_k = lowerbound + kh \tag{0.26}$$

and then plot the graph. The result will be as given below.

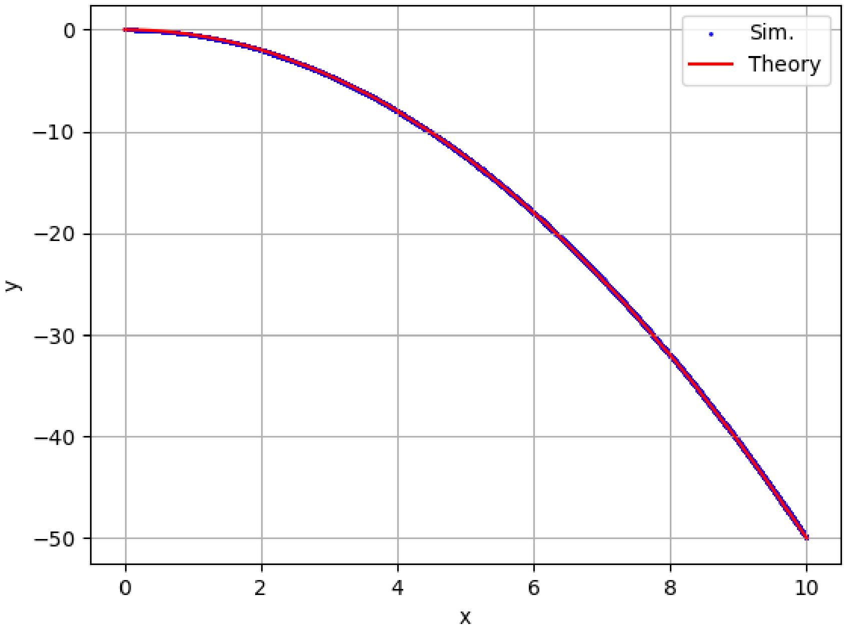

Fig. 0.1: Comparison between the Theoritical solution and Computational solution