NCERT - 9.5.10

EE1003

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Question: Find a general solution of the differential equation $(x+y)\frac{dy}{dx}=1$
Theoretical Solution:
rearranging, we get $\frac{dx}{dy}-y=x$,
solving this as linear differential equation of first order by taking Integrating factor as

$$e^{\int -1\,dy} \tag{1}$$
$$e^{-y} \tag{2}$$

We get the equation as

$$xe^{-y}=\int ye^{-y}dy+c \tag{3}$$

Solving this, we get

$$xe^{-y}=-(y+1)e^{-y}+c \tag{4}$$
$$x+y=ce^{y}-1 \tag{5}$$

Theoretical solution using Laplace transformation:
The Laplace transformation of a function $f(t)$ is

$$\mathcal{L}(f)=\int_0^\infty e^{-st}f(t)\,dt=F(s) \tag{6}$$

for the given differential equation

$$(x+y)\frac{dy}{dx}=1 \tag{7}$$

let us take

$$x+y=t \tag{8}$$

differentiating on both sides with respect to y,

$$\frac{dx}{dy}+1=\frac{dt}{dy} \tag{9}$$

Substituting equation (9) in (7),

$$\frac{dt}{dy}=t+1 \tag{10}$$

Applying Laplace on both sides,

$$\mathcal{L}(t') = \mathcal{L}(t+1) \tag{11}$$

Using the formula

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) \tag{12}$$

$$s(T(s)) - t(0) = \frac{1}{s} + T(s) \tag{13}$$

$$T(s) = \frac{\frac{1}{s} + t(0)}{s-1} \tag{14}$$

Applying inverse Laplace transformation,

$$\mathcal{L}^{-1}(T(s)) = \mathcal{L}^{-1}\left(\frac{1}{s(s-1)} + \frac{t(0)}{s-1}\right) \tag{15}$$

$$t = (t(0)+1)e^{y} - 1 \tag{16}$$

$$t = (c)e^{y} - 1 \tag{17}$$

substituting $x + y = t$ we get final equation as

$$x + y = ce^{y} - 1 \tag{18}$$

This is the same equation we obtained earlier.
Theory: The derivative $\frac{dy}{dx}$ of a function $y = f(x)$ is given by

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{19}$$

Let us say (x_0, y_0) are points that satisfy $f(x)$, then

$$\frac{dy}{dx} = \frac{y_1 - y_0}{h} \tag{20}$$

$$y_1 = y_0 + \frac{dy}{dx}h \tag{21}$$

Similarly,

$$y_2 = y_1 + \frac{dy}{dx}h \tag{22}$$

$$y_{n+1} = y_n + h\frac{dy}{dx}, \tag{23}$$

$$x_{n+1} = x_n + h \tag{24}$$

The difference equation will be of the form

$$y_{n+1} = y_n + h\left(\frac{1}{x_n + y_n}\right), \tag{25}$$

$$x_{n+1} = x_n + h \tag{26}$$

Difference equation for $\frac{dx}{dy}$: This is similar to that of $\frac{dy}{dx}$ but the roles of x and y are reversed.

$$\frac{dx}{dy} = \lim_{h\to 0} \frac{x(y+h) - x(y)}{h} \tag{27}$$

$$\frac{dx}{dy} = \frac{x_1 - x_0}{h} \tag{28}$$

$$x_1 = x_0 + \frac{dx}{dy}h, \tag{29}$$

$$y_1 = y_0 + h \tag{30}$$

Thus the difference equation will become

$$x_{n+1} = x_n + h(x_n + y_n), \tag{31}$$

$$y_{n+1} = y_n + h \tag{32}$$

Solving difference equation using one-sided z transformation:

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} \tag{33}$$

From difference equations,

$$x_{n+1} - x_n = h\left(\frac{dx}{dy}\right) \tag{34}$$

$$y_{n+1} - y_n = h \tag{35}$$

Applying one sided z transform on both sides,

$$\mathcal{Z}(x_{n+1} - x_n) = \mathcal{Z}\left(h\left(\frac{dx}{dy}\right)\right) \tag{36}$$

$$\mathcal{Z}(y_{n+1} - y_n) = \mathcal{Z}(h) \tag{37}$$

Applying the formula

$$X(z+1) = zX(z) - x_0, \tag{38}$$

$$zX(z) - x_0 - X(z) = \mathcal{Z}(h(x_n + y_n)) \tag{39}$$

$$zY(z) - y_0 - Y(z) = h\left(\frac{1}{1 - z^{-1}}\right) \tag{40}$$

$$Y(z) = \frac{y_0 z^{-1}}{1 - z^{-1}} + \frac{h z^{-1}}{1 - z^{-1}} \tag{41}$$

$$X(z) = \frac{x_0 + hY(z)}{z - 1 - h} \tag{42}$$

$$X(z) = \frac{\left(z^{-2}\right)\left(h^2 + hy_0\left(1 - z^{-1}\right)\right)}{\left(1 - z^{-1} - hz^{-1}\right)\left(1 - z^{-1}\right)^2} + \frac{x_0 z^{-1}}{1 - z^{-1} - hz^{-1}} \tag{43}$$

$$\tag{44}$$

Splitting into partial fractions,

$$X(z) = \frac{z^{-1}(-1-2h-y_0)}{1-z^{-1}} + \frac{-hz^{-2}}{(1-z^{-1})^2} + \frac{z^{-1}\left(h^2+2h+y_0+x_0\right)}{1-z^{-1}-hz^{-1}} \tag{45}$$

ROC: Casual ROC: $|z| > (1+h)$
Anti-casual ROC: $|z| < 1$
two sided ROC: $1 < |z| < (1+h)$
Since we are considering $n \geq 0$, casual ROC can be taken.
Applying inverse z transform,

$$x_n = \left(h^2+2h+1+y_0+x_0\right)(1+h)^n - (1+2h+y_0+hn) \tag{46}$$

$$y_n = y_0 + nh \tag{47}$$

Plotting: For this particular question, when we consider $c = 0$, we get the equation as

$$y = -x - 1 \tag{48}$$

and a point

$$(x_0, y_0) = (-1, 0) \tag{49}$$

Using these, we can get points on the curve and plot it.
sim-1 is done using difference equation of $\frac{dy}{dx}$
sim-2 is done using difference equation of $\frac{dx}{dy}$
sim-3 is done using solving the difference equation by using one-sided z transformation

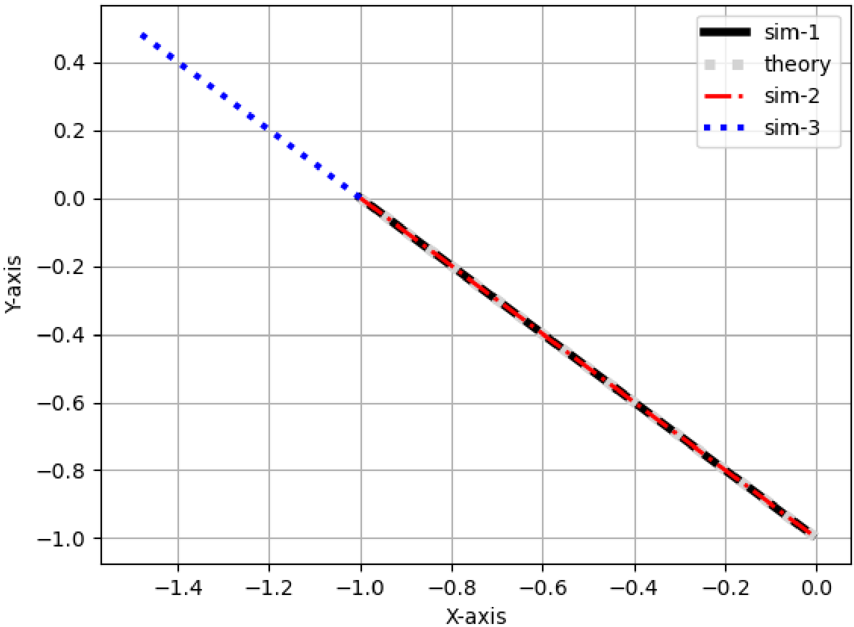
sim-1
theory
sim-2
sim-3
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
Y-axis
−1.4
−1.2
−1.0
−0.8
−0.6
−0.4
−0.2
0.0
X-axis

Fig. 0: Plot