9-9.5-12

EE24BTECH11064 - Harshil Rathan

Question:
For the following differential equation, find the particular solution satisfying the given condition:
$x^2dy + (xy + y^2)dx = 0;\ \ y = 1\ \ x = 1$
Solution:
Let us solve the given equation theoretically and then verify the solution computationally

$$x^2dy + \left(xy + y^2\right)dx = 0 \tag{0.1}$$

$$\frac{dy}{dx} = -\frac{xy + y^2}{x^2} \tag{0.2}$$

$$\frac{dy}{dx} = -\frac{y}{x} - (\frac{y}{x})^2 \tag{0.3}$$

Let

$$F\left(x, y\right) = -\frac{(xy + y^2)}{x^2}$$

$$F\left(\lambda x, \lambda y\right) = \frac{[\lambda x \cdot \lambda y + (\lambda y)^2]}{(\lambda x)^2} = \alpha^0 \times F\left(x, y\right) \tag{0.4}$$

Therefore, this is a homogeneous equation. We take the following substitution

$$y = vx \tag{0.5}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{0.6}$$

$$x\frac{dv}{dx} = \frac{v-1}{v+1} - v \tag{0.7}$$

Substituting values of x and $\frac{dy}{dx}$ in 0.2

$$v + x\frac{dv}{dx} = -\frac{[x \cdot vx + (vx)^2]}{x^2} = -v - v^2 \tag{0.8}$$

$$x\frac{dv}{dx} = -v(v+2) \tag{0.9}$$

$$\frac{dv}{v(v+2)} = -\frac{dx}{x} \tag{0.10}$$

$$\frac{1}{2}\frac{(v+2)-v}{v(v+2)}dv = -\frac{dx}{x} \tag{0.11}$$

$$\frac{1}{2}(\frac{1}{v} - \frac{1}{v+2})dv = -\frac{dx}{x} \tag{0.12}$$

Integrating on Both sides

$$\frac{1}{2}\int \frac{1}{v}\cdot dv - \frac{1}{2}\int \frac{1}{v+2}\cdot dv = -\int \frac{dx}{x} \tag{0.13}$$

$$\frac{1}{2}\log v - \frac{1}{2}\log(v+2) = -\log x + \log c \tag{0.14}$$

$$\frac{1}{2}\log(\frac{v}{v+2}) = \log\frac{C}{x} \tag{0.15}$$

$$\frac{v}{v+2} = (\frac{C}{x})^2 \tag{0.16}$$

By substituting $v = \frac{y}{x}$,

$$\frac{x^2 y}{y+2x} = C^2 \tag{0.17}$$

at x=1 and y =1

$$C^2 = \frac{1}{3} \tag{0.18}$$

Substituting $C^2 = \frac{1}{3}$ in 0.17

$$\frac{x^2 y}{y+2x} = \frac{1}{3} \tag{0.19}$$

$$y + 2x = 3x^2 y \tag{0.20}$$

Now lets verify the solution computationally from the definition of $\frac{dy}{dx}$

$$y_{n+1} = y_n + \frac{dy}{dx}\cdot h \tag{0.21}$$

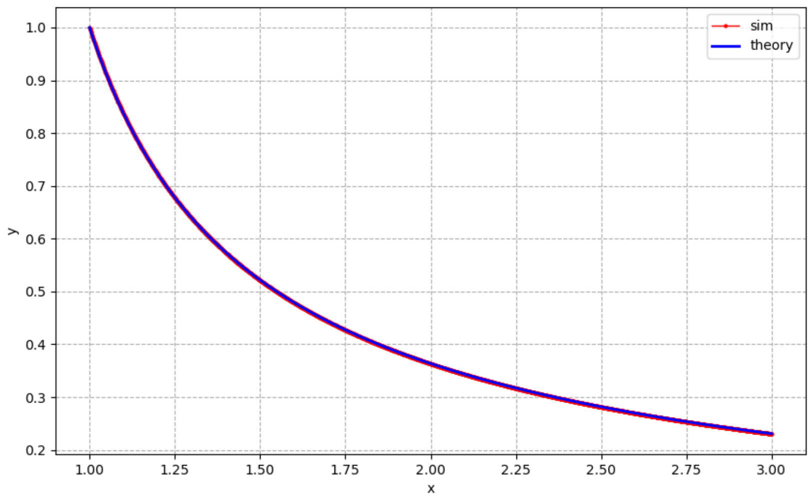
From the differential equation given,

$$\frac{dy}{dx} = -\frac{(xy+y^2)}{x^2} \tag{0.22}$$

Substituting 0.22 in 0.21

$$y_{n+1} = y_n + \left(-\frac{(x_n y_n + y_n^2)}{x_n^2}\right)\cdot h \tag{0.23}$$

The comparison between theoretical and simulation curves is shown in the figure, we can clearly see that both the curves are coincides which verifies our solution



sim
theory
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
y
1.00
1.25
1.50
1.75
2.00
2.25
2.50
2.75
3.00
x