NCERT 9.5.5

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Question: Solve the homogenous differential equation given below with initial conditions $x = 1$ and $y = 0$.

$$x^2\frac{dy}{dx} = x^2 - 2y^2 + xy \tag{1}$$

Solution:

1) Rewrite the equation: Divide through by x^2 :

$$\frac{dy}{dx} = 1 - \frac{2y^2}{x^2} + \frac{y}{x}, \tag{2}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x} - \frac{2y^2}{x^2}. \tag{3}$$

2) Substitution: Let $v = \frac{y}{x}$, so that $y = vx$. Then:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}. \tag{4}$$

Substituting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ into the equation:

$$x^2\left(v + x\frac{dv}{dx}\right) = x^2 - 2(vx)^2 + x(vx). \tag{5}$$

Simplify:

$$x^2v + x^3\frac{dv}{dx} = x^2 - 2v^2x^2 + vx^2. \tag{6}$$

Divide through by x^2 (assuming$x \neq 0$):

$$v + x\frac{dv}{dx} = 1 - 2v^2 + v. \tag{7}$$

Simplify further:

$$x\frac{dv}{dx} = 1 - 2v^2. \tag{8}$$

3) Separate variables: Rearrange to separate v and x:

$$\frac{dv}{1 - 2v^2} = \frac{dx}{x}. \tag{9}$$

4) Integrate both sides: For the right-hand side:

$$\int \frac{dx}{x} = \ln|x| + C_1. \tag{10}$$

For the left-hand side, use the substitution $u = \sqrt{2}v$, so that $du = \sqrt{2}dv$:

$$\int \frac{dv}{1-2v^2} = \frac{1}{\sqrt{2}} \int \frac{du}{1-u^2}. \tag{11}$$

The integral of $\frac{1}{1-u^2}$ is:

$$\frac{1}{2} \ln \left| \frac{1+u}{1-u} \right|. \tag{12}$$

Substituting back $u = \sqrt{2}v$:

$$\int \frac{dv}{1-2v^2} = \frac{1}{\sqrt{2}} \ln \left| \frac{1+\sqrt{2}v}{1-\sqrt{2}v} \right|. \tag{13}$$

Thus:

$$\frac{1}{\sqrt{2}} \ln \left| \frac{1+\sqrt{2}v}{1-\sqrt{2}v} \right| = \ln|x| + C_1. \tag{14}$$

5) Combine results: Multiply through by $\sqrt{2}$:

$$\ln \left| \frac{1+\sqrt{2}v}{1-\sqrt{2}v} \right| = \sqrt{2} \ln|x| + C_2, \tag{15}$$

where $C_2 = \sqrt{2}C_1$.
Exponentiate both sides:

$$\frac{1+\sqrt{2}v}{1-\sqrt{2}v} = kx^{\sqrt{2}}, \tag{16}$$

where $k = e^{C_2}$.

6) **Solve for v**: Rearrange:

$$1 + \sqrt{2}v = kx^{\sqrt{2}} \left(1 - \sqrt{2}v\right). \tag{17}$$

Simplify:

$$1 + \sqrt{2}v = kx^{\sqrt{2}} - kx^{\sqrt{2}} \sqrt{2}v. \tag{18}$$

Group terms involving v:

$$v \left(\sqrt{2} + kx^{\sqrt{2}} \sqrt{2} \right) = kx^{\sqrt{2}} - 1. \tag{19}$$

Solve for v:

$$v = \frac{kx^{\sqrt{2}} - 1}{\sqrt{2}\left(1 + kx^{\sqrt{2}}\right)}. \tag{20}$$

7) **Back-substitute** $v = \frac{y}{x}$:

$$\frac{y}{x} = \frac{kx^{\sqrt{2}} - 1}{\sqrt{2}\left(1 + kx^{\sqrt{2}}\right)}. \tag{21}$$

Multiply through by x:

$$y = \frac{x\left(kx^{\sqrt{2}} - 1\right)}{\sqrt{2}\left(1 + kx^{\sqrt{2}}\right)}. \tag{22}$$

8) **Apply initial conditions**: The initial conditions are $x = 1$ and $y = 0$. Substituting into the solution:

$$0 = \frac{1\left(k \cdot 1^{\sqrt{2}} - 1\right)}{\sqrt{2}\left(1 + k \cdot 1^{\sqrt{2}}\right)}. \tag{23}$$

Simplify:

$$0 = \frac{k - 1}{\sqrt{2}\,(1 + k)}. \tag{24}$$

The numerator must equal zero:

$$k - 1 = 0 \implies k = 1. \tag{25}$$

9) Final solution: Substitute $k = 1$ into the general solution:

$$y = \frac{x\left(x^{\sqrt{2}} - 1\right)}{\sqrt{2}\left(1 + x^{\sqrt{2}}\right)}. \tag{26}$$

10) CODING LOGIC: The solution for the differential equation can be graphically solved using coding by using below logic :

$$x_0 = 1 \tag{27}$$

$$y_0 = 0 \tag{28}$$

$$h = 0.00001 \tag{29}$$

$$y_{n+1} = y_n + h\left(1 + \frac{y_n}{x_n} - \frac{2y_n^2}{x_n^2}\right) \tag{30}$$

$$x_{n+1} = x_n + h \tag{31}$$

Below is verification 10 :

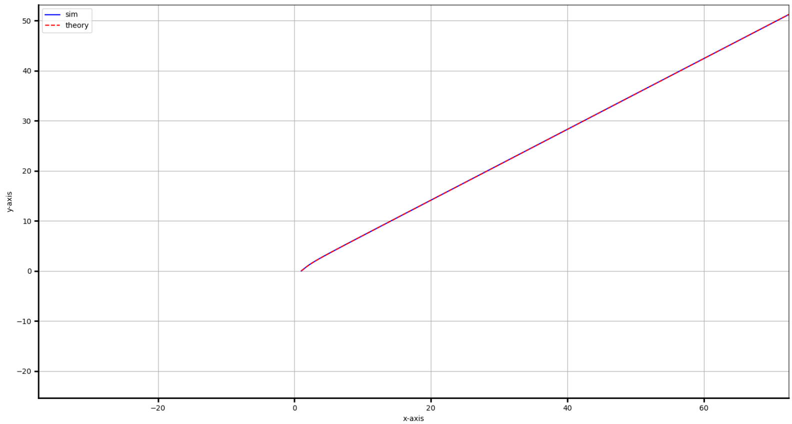

Fig. 10: Verification