Lecture Notes For Probability

Probability is one of the most important mathematical concepts that we use in our daily life. *Probability means possibility of something.* It is a mathematical tool which deals with the occurrence of random events. Value of the probability lies in between 0 and 1.

Probability of an event is defined by the number of ways in which the event occurs divided by the number of outcomes in the sample space.

P(event) = n(E)/n(S)

Sample space: *Sample space of an event is the set of all possible outcomes of that event.*

For example:

1. You tossed a coin. Your sample space is head or tail.
   P(H) = 1/2.
2. You throw a dice. Your sample space {1,2,3,4,5,6}
   P(6) = 1/6.
3. England and India play a one day match.
   In this case 3 events will happen. 1. England wins 2. India wins 3. Match tie.
   P(tie) $\neq$ 1/3 because the possible outcomes of the India Vs England match is not the sample space in this situation.

Two things happen to form a sample space;
1. Exhaustive or complete list of all possible outcomes.
2. A list to become a sample space is that the outcome should be equally likely.

So, in India Vs England match, tie is not an equally like outcome. Hence it is not in the sample space.

1st kind of questions based on coins:

Problem 1:
A coin tossed three times. What is the probability of a) All heads. b) Exactly two heads. c) Minimum two heads. d) At Least one head.

Solution:

List of the possible outcomes {HHH,HHT,HTH,THH,TTH,THT,HTT,TTT}

Total number of outcomes = 8. i.e. n(S) = 8.

a) All heads n(E) = 1.

P(All heads) = n(E)/n(S) = 1/8.

b) exactly two heads n(E) = 3

P(Exactly two heads) = 3/8.

c) minimum two heads n(E) = 4.

P(Minimum two heads) = 4/8.

d) P(At least 1 head) = 1 - P(not heads)

= 1 - P(all tails) = 1 - 1/8 = 7/8.

NOTE: 1. None event in probability is denoted by $\bar{E}$ or E' and P(E) + P($\bar{E}$) = 1.

2. The probability of all events in a sample space is 1.

2nd method: Without forming sample space

a) All heads.

If you do not want to form a sample space, you can define this in 3 events.

Event definition: All heads.

H&H&H i.e. $1/2 \times 1/2 \times 1/2 = 1/8$.

b) Exactly two heads.

Event definition: Exactly two heads.

H&H&T or H&T&H or T&H&H i.e. $1/2 \times 1/2 \times 1/2 + 1/2 \times 1/2 \times 1/2 + 1/2 \times 1/2 \times 1/2 =$ 1/8 + 1/8 + 1/8 = 3/8.

Biased coin question

Problem 1:

A coin toss three times, what is the probability of getting 2 Heads and 1 Tail if the probability of head is 0.6 and tail is 0.4?

Solution:
2 Heads and 1Tail.
Event definition: 2 Heads and 1 Tail
H&H&T or H&T&H or T&H&H i.e. $0.6 \times 0.6 \times 0.4 + 0.6 \times 0.6 \times 0.4 + 0.6 \times 0.6 \times 0.4 = 3(0.6 \times 0.6 \times 0.4) = 3 \times 0.144 = 0.432$.

Probability based on dice

Single dice situation is very simple. Normally we get in dice question type is the 2 dice situation. In such cases normally questions are asked on the sum of the dice.

In a 2 dice situation you need to understand that there is a certain pattern for different numbers.
For example:

Sum 2 can happen in only 1 way.(i.e. 1,1)	$\leftarrow$ Sum 12 can happen in 1 way.
Sum 3 can happen in 2 ways.	$\leftarrow$ Sum 11 can happen in 2 ways.
Sum 4 can happen in 3 ways.	$\leftarrow$ Sum 10 can happen in 3 ways.
Sum 5 can happen in 4 ways.	$\leftarrow$ Sum 9 can happen in 4 ways.
Sum 6 can happen in 5 ways.	$\leftarrow$ Sum 8 can happen in 5 ways.
Sum 7 can happen in 6 ways.	

The pair which have same number of ways;

$2 \Leftrightarrow 12$ **sum of each pair = 14.(i.e. 2+12=14).**
$3 \Leftrightarrow 11$
$4 \Leftrightarrow 10$
$5 \Leftrightarrow 9$
$6 \Leftrightarrow 8$

Problem 1:
Two dice are thrown together. Find the probability of :
1. Getting a number greater than 10.
2. Getting a sum of 5.
3. Getting a sum is prime.
4. Getting a multiple of 3 or 4.

Solution:
1. Total number of possible outcome = 36
   Getting a number greater than 10 means we want 11 or 12.

Sum 11 can happen in 2 ways or Sum 12 can happen in 1 way.
Number of events of getting number greater than 10 = 2+1=3
There for probability of Getting a number greater than 10
P(E) = 3/36 = 1/12.

2. Total number of possible outcome = 36
   Getting a sum of 5:

Sum 5 can happen in 4 ways.
Number of events of getting sum of 5 = 4.
There for probability of getting a sum of 5
P(E) = 4/36 = 1/9.

3. Total number of possible outcome = 36

Getting a sum is prime. In this case we will go and search situations for sum2,sum3,sum5,sum7 and sum 11.
Sum 2 can happen in only 1 way.
Sum 3 can happen in 2 ways.
Sum 5 can happen in 4 ways.
Sum 7 can happen in 6 ways.
Sum 11 can happen in 2 ways.
Number of events of getting sum is prime = 1+2+4+6+2 =15
There for probability of getting a sum is prime
P(E) = 15/36 = 5/12.

4. Total number of possible outcome = 36

Getting a sum is multiple of 3or4. Multiple of 3 or 4 is 3,4,6,8,9,12
Sum 3 can happen in 2 ways.
Sum 4 can happen in 3 ways.
Sum 6 can happen in 5 ways.
Sum 8 can happen in 5 ways.
Sum 9 can happen in 4 ways.
Sum 12 can happen in 1 way.
Number of events of getting sum is multiple of 3or4 = 2+3+5+5+4+1 = 20
There for probability of getting a sum is multiple of 3or4
P(E) = 20/36 = 5/9.

Probability based on cards

Some basic information about cards:

1. Pack of cards = 52
2. There are 4 suits in a pack of 52 cards.(**clubs,spades,diamonds,hearts)**
3. 13 cards in each of the 4 suits.
4. Each of 4 suits has an ace,2,3,4………,10,jack.queen,king.
5. Clubs and spades are in black color.
6. Diamonds and hearts are in red color.
7. Jack is at the same time in problems also referred to as Knave.
8. Jack,Queen and King are face cards.

Problem 1:
A card is drawn from a pack of 52 cards. Find the probability:

1. A spade.
2. A king.
3. A Black card.
4. A king or a queen.
5. A face card.
6. A king or a spade.

Solution:

1. Total number of possible outcomes = 52.

Number of events of drawing a spade = 13.
Therefore the probability of a spade
P(E) = 13/52.

2. Total number of possible outcomes = 52.

Number of events of drawing a king = 4.
Therefore the probability of a king
P(E) = 4/52.

3. Total number of possible outcomes = 52.

Number of events of drawing a black card = 26.
Therefore the probability of a black card
P(E) = 26/52.

4. Total number of possible outcomes = 52.

Number of events of drawing a king or queen = 4+4=8.
Therefore the probability of a spade
P(E) = 8/52.

5. Total number of possible outcomes = 52.

Number of events of drawing a face card = 12.
Therefore the probability of a face card
P(E) = 12/52.

6. Total number of possible outcomes = 52.

A king or a spade: there are 4 kings(among 4 king one king of spades) out of 52 cards and 13 cards of spades.

Number of events of drawing a king or a spade = 4+12 = 16
Therefore the probability of a king or a spade
P(E) = 16/52.

Problem 2:
Two cards are drawn at random **without replacement** from a pack of 52 cards. Find the probability of:

1. 1 queen and 1 king.
2. 1 red and 1 black.

Solution:

1. **1 queen and 1 king :**

Total number of possible outcomes = 52.
From a pack of 52 cards probability of queen = 4/52.
From a pack of 52 cards probability of king = 4/52.

1 queen and 1 king :
In this case 1st is queen & 2nd is king or 1st is king and 2nd is queen
Q&K or K&Q i.e. $4/52 \times 4/51 + 4/52 \times 4/51 = 8/(52 \times 51)$.
Therefore P(1Q & 1K) = $8/(52 \times 51)$.

2. **1 red and 1 black :**

Total number of possible outcomes = 52.
From a pack of 52 cards probability of red = 26/52.
From a pack of 52 cards probability of black = 26/52.

1 red and 1black :
In this case 1st is red & 2nd is black or 1st is black and 2nd is red
R&B or B&R i.e. $26/52 \times 26/51 + 26/52 \times 26/51 = 52/(52 \times 51)$.
Therefore P(1Q & 1K) = 1/51.

Probability based on balls from boxes

Problem 1:
A box contains 10 red, 5 blue and 1 black. All the balls are identical and 1 ball drawn at random. What is the probability that :

1. Ball is red.
2. Ball is blue.
3. Ball is black.

Solution:
Total number of balls = 10+5+1=16. i.e. n(S) = 16.

1. **Ball is red:**

n(E) = number of ways of drawing red balls = 10.
Therefore probability of drawing red balls
p(E) = n(E)/n(S) = 10/16.

2. **Ball is blue:**

n(E) = number of ways of drawing blue balls = 4.
Therefore probability of drawing blue balls
p(E) = n(E)/n(S) = 5/16.

3. **Ball is black:**

n(E) = number of ways of drawing black balls = 1.
Therefore probability of drawing black balls
p(E) = n(E)/n(S) = 1/16.

One ball question is very simple, but the main question here draws two balls. In such cases there are two kinds of questions.

1. Ball drawn with replacement.
2. Ball drawn without replacement.

Problem 2:

A box contains 10 red, 5 blue and 1 black. All the balls are identical and 3 balls drawn at random one after the other with replacement. What is the probability that all 3 balls are red?

Solution:

Total number of balls = 10+5+1=16. i.e. n(S) = 16.
n(E) = number of ways of drawing red balls = 10.
Probability of a red ball = 10/16
Therefore probability of drawing 3 red balls with replacement
1st red & 2nd red & 3rd red
$10/16 \times 10/16 \times 10/16$.

Problem 3:

A box contains 10 red, 5 blue and 1 black. All the balls are identical and 3 balls drawn at random one after the other with replacement or without replacement. What is the probability that all 3 balls are of the same color?

Solution:

Total number of balls = 10+5+1=16. i.e. n(S) = 16.
Probability of a red ball = 10/16.
Probability of a blue ball = 5/16.
Probability of a black ball = 1/16.

With replacement:

R - Red ball, b - Blue ball , B - Black ball
P(all 3 balls of same color) = R&R&R or b&b&b or B&B&B
$= 10/16 \times 10/16 \times 10/16 + 5/16 \times 5/16 \times 5/16 + 1/16 \times 1/16 \times 1/16$

Without replacement:

R - Red ball, b - Blue ball , B - Black ball
In this case we have only 1 black ball.
P(all 3 balls of same color) = R&R&R or b&b&b
$= 10/16 \times 9/15 \times 8/14 + 5/16 \times 4/15 \times 3/14$

Problem 4:

A box contains 10 red, 5 blue and 1 black. All the balls are identical and 3 balls drawn at random one after the other without replacement. What is the probability that all 3 balls are of the different color?

Solution:
Total number of balls = 10+5+1=16. i.e. n(S) = 16.
Probability of a red ball = 10/16.
Probability of a blue ball = 5/16.
Probability of a black ball = 1/16.

R - Red ball, b - Blue ball , B - Black ball
Arrangement of three different balls = 3!

P(all 3 balls of different color) = R&b&B × 3!
= (10/16 × 5/15 × 1/14) × 3!

Draw 1 ball from 2 boxes or 3 boxes

Problem 1:
A box contains 10 red, 5 blue and 2 black and another box contains 5 red, 7 blue and 8 black. 1 ball drawn at random from any of the 2 boxes. Find the probability that the ball is black?

Solution:
Probability of black ball form 1st box = 2/16
Probability of black ball from 2nd box = 8/20.
Selection of 1st box = 1/2
Selection of 2nd box = 1/2
P(Ball is black) = 1st box & Black ball or 2nd box & Black ball
= 1/2 × 2/16 + 1/2 × 8/20
= 1/16 + 8/40.

Word based question on probability

Problem 1:
What is the probability that there are 53 Sundays in a normal non leap year?

Solution:
In a non leap year = 365 days.
365 days has 52 complete weeks and 1 day.
The 365 days calendar will start on 1st january and 1st week will end on 7th january ……. And so on …. the 52nd week will end on 30th december.

For 53 sundays in a non leap year, the last day of the year 31th december has to be a sunday and probability of 31th dec being a sunday = 1/7.
Hence the answer = 1/7.

Problem 2:
What is the probability that there are 53 Sundays in a leap year?

Solution:
In a leap year = 366 days.
366 days has 52 complete weeks and 2 days.

The 52nd week would end on the 364th day of the year and that day would be 29th december.
For 53 sundays in a non leap year, the last 2 days of the year would be

1. Sunday or Monday
2. Saturday or sunday
3. Monday or Tuesday
4. Tuesday or Wednesday
5. Wednesday or Thursday
6. Thursday or Friday and
7. Friday or Saturday

Last 2 days of the year out of 7 cases. Out of 7 cases only 2 cases have Sundays in them.
Hence probability of 53 sundays in a leap year = 1/7.

Problem 3:
N1,N2,N3,N4 and N5 are the natural numbers. What is the probability that the product of these numbers ends in an odd number that is not a multiple of 5.

Solution:
Product of 5 numbers to be odd, 1st of all the last digit should not be even and last digit should not be 5.
Probability of any unit place = 1/10
All the numbers should end with 1,3,7 and 9 i.e. 4 numbers.

Probability of number P(N1) = 4/10.
Probability of number P(N2) = 4/10.
Probability of number P(N3) = 4/10.

Probability of number $P(N4) = 4/10$.

$$P(N1,N2,N3,N4 \text{ and } N5) = P(N1) \times P(N2) \times P(N3) \times P(N4) \times P(N5)$$
$$= 4/10 \times 4/10 \times 4/10 \times 4/10 \times 4/10 = 0.4^5$$

Problem 4:

Amit orders a gift from 4 different websites for his friend's birthday. The probability of the sites delivering on time are 0.9,0.8,0.7 and 0.6 respectively. What is the probability that the friend would get the gift on time?

Solution:

Lets say $\overline{E}$ be the event that a friend doesn't get the gift on time.

$P(E) = 1 - P(\overline{E})$.

1st site should fail deliver on time & 2nd site should fail deliver on time & 3rd site should fail deliver on time & 4th site should fail deliver on time.

P(1st site fail) = 1 - 0.9 = 0.1
P(2nd site fail) = 1 - 0.8 = 0.2
P(3rd site fail) = 1 - 0.7 = 0.3
P(4th site fail) = 1 - 0.6 = 0.4

$P(\overline{E}) = 0.1 \times 0.2 \times 0.3 \times 0.4 = 0.0024$
$P(E) = 1 - P(\overline{E}) = 1 - 0.0024 = 0.9976$.

Problem 5:

Probability of a man living for 50 years from today is 0.6 and the probability for his wife to live for 50 year from today is 0.5. Find the probability that both are alive after 50 years and one of them is dead ?

Solution:

P(man alive) = 0.6 and P(wife alive) = 0.5
P(both are alive) = P(man alive) & P(wife alive)
$= 0.6 \times 0.5 = 0.3$.
P (man not alive) = 1 - 0.6 = 0.4 and P (wife not alive) = 1 - 0.5 = 0.5.
P(one of them is dead) = (man alive & wife dead) or (man dead & wife alive)
$= 0.6 \times 0.5 + 0.4 \times 0.5 = 0.5$.

Problem 6:
Probability that India wins the match is 0.6 and probability that England wins the match is 0.4. India and England play 3 one-day matches. What is the probability that India wins the series?

Solution:
Events:
India can win the series by 3-0 or 2-1
3-0 means India wins 1st match & 2nd match & 3rd match
2-1 means India wins 2 matches and England wins 1 match and 3 arrangements.
P(India wins series) = $0.6 \times 0.6 \times 0.6 + (0.6 \times 0.6 \times 0.4) \times 3 = 0.648$.

Some questions for practice

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1. What is the probability of getting a number greater than 9, in a throw of two normal unbiased dice having 6 faces?
   Ans : 1/6.

2. In a throw of two dice, find the probability of getting one prime and one composite number.
   Ans : 1/3.

3. There are two bags containing white and black balls. In the first bag, there are 8 white and 6 black balls and in the second bag, there are 4 white and 7 black balls. One ball is drawn at random from any of these two bags. Find the probability of this ball being black.
   Ans : 41/77.

4. The letters of the word LUCKNOW are arranged among themselves. Find the probability of always having NOW in the word.
   Ans : 1/42.

5. Out of 13 applicants for a job, there are 5 women and 8 men. Two persons are to be selected for the job. The probability that at least one of the selected persons will be a woman is:
   Ans : 25/39.

6. The probability that A can solve the problem is 2/3 and B can solve it is 3/4. If both of them attempt the problem, then what is the probability that the problem gets solved.
   Ans : 11/12.