Question: What is Simple Harmonic Motion? Derive the equation of motion for a simple harmonic oscillator.

Answer:
Simple Harmonic Motion (SHM) is a type of oscillatory motion in which the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

Derivation:
Consider a mass m attached to a spring with a spring constant k. The mass is displaced by x from its equilibrium position.

According to Hooke's law, the restoring force F is given by:
$F = -kx$

Using Newton's second law of motion:
$F = ma$

Equating the two expressions for force:
$ma = -kx$

The acceleration a is the second derivative of displacement x with respect to time t:
$m\frac{d^2x}{dt^2} = -kx$

Rearranging the terms:
$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

Let $\omega^2 = \frac{k}{m}$, where ω is the angular frequency of the oscillation:
$\frac{d^2x}{dt^2} + \omega^2 x = 0$

This is the differential equation of SHM.

Question: Derive the expression for the total energy in a simple harmonic oscillator.

Answer:

In SHM, the total energy E is the sum of kinetic energy K and potential energy U.

Kinetic Energy (K):

$K = \frac{1}{2}mv^2$

The velocity v in SHM is given by:

$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$

Substituting v in the kinetic energy expression:

$K = \frac{1}{2}m(A\omega\sin(\omega t + \phi))^2$

$K = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$

Potential Energy (U):

$U = \frac{1}{2}kx^2$

The displacement x in SHM is:

$x = A\cos(\omega t + \phi)$

Substituting x in the potential energy expression:

$U = \frac{1}{2}k(A\cos(\omega t + \phi))^2$

$U = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$

Since $k = m\omega^2$:

$U = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$

Total Energy (E):

$$E = K + U$$

$$E = \frac{1}{2} m A^2 \omega^2 \sin^2(\omega t + \phi) + \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t + \phi)$$

$$E = \frac{1}{2} m A^2 \omega^2 (\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi))$$

$$E = \frac{1}{2} m A^2 \omega^2$$

Since $\sin^2(\theta) + \cos^2(\theta) = 1$:

$$E = \frac{1}{2} m A^2 \omega^2$$

This is the expression for the total energy of a simple harmonic oscillator, and it remains constant throughout the motion.

Question: Explain damped oscillations and derive the equation for a damped harmonic oscillator.

Answer:

Damped oscillations occur when a resistive force, such as friction or air resistance, acts on the oscillating system, causing the amplitude of oscillations to decrease over time.

Derivation:

Consider a mass m attached to a spring with a damping force proportional to the velocity. The damping force F_d is given by:

$$F_d = -b\frac{dx}{dt}$$

Where b is the damping constant.

The total restoring force is:

$$F = -kx - b\frac{dx}{dt}$$

Using Newton's second law:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

Rearranging the terms:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Dividing through by m:

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Let $\gamma = \frac{b}{2m}$ and $\omega_0^2 = \frac{k}{m}$:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

This is the differential equation for a damped harmonic oscillator.

Question: Derive the expression for the time period of a simple pendulum.

Answer:
A simple pendulum consists of a mass m (bob) attached to a string of length L and is displaced by a small angle θ from its equilibrium position.

Derivation:
For small angles (θ in radians), the restoring force is:
$$F = -mg\sin(\theta) \approx -mg\theta$$

Since $\theta = \frac{x}{L}$ where x is the arc length:
$$F = -mg\frac{x}{L}$$

Using Newton's second law:
$$ma = -mg\frac{x}{L}$$
$$m\frac{d^2x}{dt^2} = -mg\frac{x}{L}$$

Rearranging the terms:
$$\frac{d^2x}{dt^2} + \frac{g}{L}x = 0$$

This is the differential equation for SHM with angular frequency ω:
$$\omega^2 = \frac{g}{L}$$
$$\omega = \sqrt{\frac{g}{L}}$$

The time period T is given by:
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

This is the time period of a simple pendulum.

Question: Derive the expression for the angular frequency of a torsional pendulum.

Answer:
A torsional pendulum consists of a disk or wheel suspended from a wire or rod that twists when the disk is rotated.

Derivation:

The restoring torque τ is proportional to the angle of twist θ:

$$\tau = -k\theta$$

where k is the torsional constant.

Using Newton's second law for rotational motion:

$$I\frac{d^2\theta}{dt^2} = -k\theta$$

where I is the moment of inertia.

Rearranging the terms:

$$\frac{d^2\theta}{dt^2} + \frac{k}{I}\theta = 0$$

This is the differential equation for angular SHM with angular frequency ω:

$$\omega^2 = \frac{k}{I}$$

$$\omega = \sqrt{\frac{k}{I}}$$

The angular frequency ω of the torsional pendulum is:

$$\omega = \sqrt{\frac{k}{I}}$$

Key Definitions in Oscillations

1. **Oscillation:** Repeated back-and-forth movement around a central point.
2. **Simple Harmonic Motion (SHM):** Type of oscillation where the force pulling back is directly proportional to how far you are from the middle.
3. **Amplitude:** Maximum distance from the middle point in an oscillation.
4. **Period (T):** Time taken to complete one full oscillation.
5. **Frequency (f):** Number of oscillations per second.
6. **Angular Frequency (ω):** How quickly an object oscillates, measured in radians per second. Calculated as $\omega = 2\pi f$.
7. **Phase:** Position of the point in the oscillation cycle at a specific time.
8. **Phase Constant (ϕ):** The initial angle or phase at time $t = 0$.
9. **Restoring Force:** Force that brings the object back to its middle position.
10. **Damped Oscillation:** Oscillation where the amplitude decreases over time due to resistive forces like friction.
11. **Damping Force:** The force that reduces the amplitude of oscillations.
12. **Underdamped:** A damped system where oscillations slowly die out.

13. **Critically Damped:** A damped system where oscillations return to the middle point as quickly as possible without oscillating.
14. **Overdamped:** A damped system where oscillations return to the middle point slowly without oscillating.
15. **Forced Oscillation:** Oscillation where an external force is regularly applied.
16. **Resonance:** Condition where the frequency of the applied force matches the natural frequency, causing maximum amplitude.
17. **Natural Frequency:** Frequency at which a system naturally oscillates when not disturbed by an external force.
18. **Potential Energy (U):** Stored energy due to position, such as height or stretch in SHM.
19. **Kinetic Energy (K):** Energy due to motion.
20. **Total Energy:** Sum of kinetic and potential energy in the system.
21. **Torsional Pendulum:** A type of pendulum that twists around its axis instead of swinging back and forth.
22. **Torsional Constant (k):** Measure of how resistant a material is to twisting.
23. **Moment of Inertia (I):** Measure of an object's resistance to changes in its rotation.
24. **Simple Pendulum:** A mass (bob) attached to a string or rod that swings back and forth under gravity.
25. **Equilibrium Position:** The central point around which an object oscillates.

Forced oscillations occur when an external periodic force is applied to a system. Resonance occurs when the frequency of the external force matches the natural frequency of the system, resulting in maximum amplitude.

Numericals for Oscillations

Problem:

A mass of 0.5 kg is attached to a spring with a spring constant of 200 N/m. Calculate the angular frequency and period of the oscillation.

Solution:

Given:

- Mass, $m = 0.5$ kg
- Spring constant, $k = 200$ N/m

Angular Frequency (ω):

$$\omega = \sqrt{\frac{k}{m}}$$
$$\omega = \sqrt{\frac{200}{0.5}}$$
$$\omega = \sqrt{400}$$
$$\omega = 20 \text{ rad/s}$$

Period (T):

$$T = \frac{2\pi}{\omega}$$
$$T = \frac{2\pi}{20}$$
$$T = \frac{\pi}{10}$$
$$T = 0.314 \text{ s}$$

Problem:

Calculate the total energy of a simple harmonic oscillator with a mass of 1 kg, amplitude of 0.1 m, and angular frequency of 5 rad/s.

Solution:

Given:

- Mass, $m = 1$ kg
- Amplitude, $A = 0.1$ m
- Angular frequency, $\omega = 5$ rad/s

Total Energy (E):

$E = \frac{1}{2}mA^2\omega^2$

$E = \frac{1}{2} \times 1 \times (0.1)^2 \times 5^2$

$E = \frac{1}{2} \times 0.01 \times 25$

$E = \frac{1}{2} \times 0.25$

$E = 0.125\ \text{J}$

Problem:

A damped harmonic oscillator has a mass of 2 kg, damping constant of 4 kg/s, and spring constant of 100 N/m. Calculate the damping ratio and determine if the system is underdamped, critically damped, or overdamped.

Solution:

Given:

- Mass, $m = 2$ kg
- Damping constant, $b = 4$ kg/s
- Spring constant, $k = 100$ N/m

Natural Frequency (ω_0):

$\omega_0 = \sqrt{\frac{k}{m}}$

$\omega_0 = \sqrt{\frac{100}{2}}$

$\omega_0 = \sqrt{50}$

$\omega_0 = 7.07\ \mathrm{rad/s}$

Damping Ratio (ζ):

$\zeta = \frac{b}{2\sqrt{mk}}$

$\zeta = \frac{4}{2\sqrt{2\times100}}$

$\zeta = \frac{4}{2\sqrt{200}}$

$\zeta = \frac{4}{2\times14.14}$

$\zeta = \frac{4}{28.28}$

$\zeta = 0.141$

Since $\zeta < 1$, the system is **underdamped**.

Problem:

A mass-spring system with a natural frequency of 3 Hz is subjected to an external force with a frequency of 3 Hz. If the amplitude of the external force is 10 N and the damping constant is 0.1 kg/s, calculate the amplitude of the forced oscillation. The mass of the system is 1 kg.

Solution:

Given:

- Natural frequency, $f_0 = 3$ Hz
- Driving frequency, $f = 3$ Hz
- Amplitude of external force, $F_0 = 10$ N
- Damping constant, $b = 0.1$ kg/s
- Mass, $m = 1$ kg

Angular Frequency (ω):

$\omega = 2\pi f$

$\omega = 2\pi \times 3$

$\omega = 6\pi \text{ rad/s}$

Amplitude (A):

$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}$

$\omega_0 = 2\pi f_0 = 6\pi \text{ rad/s}$

$\gamma = \frac{b}{2m} = \frac{0.1}{2\times1} = 0.05 \text{ s}^{-1}$

Since $\omega = \omega_0$ (resonance):

$A = \frac{F_0/m}{2\gamma\omega_0}$

$A = \frac{10/1}{2\times0.05\times6\pi}$

$A = \frac{10}{0.6\pi}$

$A = \frac{10}{1.884}$

$A \approx 5.31 \text{ m}$

Problem:

Calculate the time period of a simple pendulum with a length of 2 meters. Take $g = 9.8 \ \mathrm{m/s^2}$.

Solution:

Given:

- Length, $L = 2$ m
- Acceleration due to gravity, $g = 9.8 \ \mathrm{m/s^2}$

Time Period (T):

$T = 2\pi\sqrt{\frac{L}{g}}$

$T = 2\pi\sqrt{\frac{2}{9.8}}$

$T = 2\pi\sqrt{0.204}$

$T = 2\pi \times 0.451$

$T \approx 2.83 \ \mathrm{s}$

Problem:

A torsional pendulum has a moment of inertia of 0.05 kg·m^2 and a torsional constant of 0.2 N·m/rad. Calculate its angular frequency.

Solution:

Given:

- Moment of inertia, $I = 0.05$ kg·m^2
- Torsional constant, $k = 0.2$ N·m/rad

Angular Frequency (ω):

$\omega = \sqrt{\frac{k}{I}}$

$\omega = \sqrt{\frac{0.2}{0.05}}$

$\omega = \sqrt{4}$

$\omega = 2 \ \mathrm{rad/s}$