Question: Derive the expression for the moment of inertia of a solid cylinder about its central axis.

Answer:

Derivation:

The moment of inertia (I) is the rotational equivalent of mass in linear motion. For a solid cylinder of mass M and radius R:

Consider a thin disk of thickness dx at a distance x from the axis. The mass of this disk dm is given by:

$dm = \frac{M}{L} dx$

where L is the length of the cylinder.

The moment of inertia of this thin disk about the central axis is:

$dI = \frac{1}{2} dm \cdot R^2$

$dI = \frac{1}{2} \left(\frac{M}{L} dx\right) R^2$

$dI = \frac{1}{2} \frac{M}{L} R^2 dx$

Integrating this from 0 to L:

$I = \int_0^L \frac{1}{2} \frac{M}{L} R^2 dx$

$I = \frac{1}{2} \frac{M}{L} R^2 \int_0^L dx$

$I = \frac{1}{2} \frac{M}{L} R^2 \cdot L$

$I = \frac{1}{2} M R^2$

So, the moment of inertia of a solid cylinder about its central axis is:

$I = \frac{1}{2} M R^2$

Question: A torque of 10 Nm is applied to a wheel of moment of inertia 2 kg·m^2. Calculate the angular acceleration produced.

Answer:

Given:

- Torque, $\tau = 10$ Nm
- Moment of inertia, $I = 2$ kg·m^2

Angular Acceleration (α):

The relationship between torque and angular acceleration is given by:

$\tau = I\alpha$

$\alpha = \frac{\tau}{I}$

$\alpha = \frac{10}{2}$

$\alpha = 5 \text{ rad/s}^2$

So, the angular acceleration produced is 5 rad/s^2.

Question: Calculate the work done by a torque of 15 Nm in rotating a wheel through an angle of 2 radians.

Answer:

Given:

- Torque, $\tau = 15$ Nm
- Angle, $\theta = 2$ radians

Work Done (W):

Work done by torque is given by:

$W = \tau\theta$

$W = 15 \times 2$

$W = 30 \text{ J}$

Question: A figure skater with a moment of inertia of 3 kg·m^2 is spinning at 4 rad/s. If she pulls in her arms and reduces her moment of inertia to 2 kg·m^2, what will be her new angular speed?

Answer:

Given:

- Initial moment of inertia, $I_1 = 3$ kg·m^2
- Initial angular speed, $\omega_1 = 4$ rad/s
- Final moment of inertia, $I_2 = 2$ kg·m^2

Using conservation of angular momentum:

$L_1 = L_2$

$I_1\omega_1 = I_2\omega_2$

$3 \times 4 = 2 \times \omega_2$

$12 = 2\omega_2$

$\omega_2 = \frac{12}{2}$

$\omega_2 = 6 \text{ rad/s}$

So, the new angular speed is 6 rad/s.

Question: A solid sphere of radius 0.1 m and mass 0.5 kg rolls down an inclined plane without slipping. Calculate the linear acceleration of the center of mass.

- Radius, $R = 0.1$ m
- Mass, $M = 0.5$ kg
- Acceleration due to gravity, $g = 9.8 \text{ m/s}^2$

Using the equation of motion for rolling objects:

$a = \frac{g\sin(\theta)}{1+\frac{I}{MR^2}}$

For a solid sphere:

$I = \frac{2}{5}MR^2$

Substitute I:

$a = \frac{g\sin(\theta)}{1+\frac{\frac{2}{5}MR^2}{MR^2}}$

$a = \frac{g\sin(\theta)}{1+\frac{2}{5}}$

$a = \frac{g\sin(\theta)}{\frac{7}{5}}$

$a = \frac{5g\sin(\theta)}{7}$

Assuming the incline angle $\theta = 30°$ (common angle):

$\sin(30°) = 0.5$

$a = \frac{5\times9.8\times0.5}{7}$

$a = \frac{24.5}{7}$

$a = 3.5 \text{ m/s}^2$

So, the linear acceleration of the center of mass is 3.5 m/s^2.

Definitions in Rotational Dynamics

1. **Rotational Dynamics:** The study of objects in rotation and the forces and torques that affect them.
2. **Torque (τ):** A force that causes an object to rotate. It is calculated as force times the distance from the pivot point ($\tau = F \times r$).
3. **Moment of Inertia (I):** The resistance of an object to changes in its rotational motion. It depends on the mass and shape of the object.
4. **Angular Velocity (ω):** How fast an object is rotating, measured in radians per second (rad/s).
5. **Angular Acceleration (α):** The rate at which the angular velocity changes with time, measured in radians per second squared (rad/s^2).
6. **Angular Momentum (L):** The quantity of rotation of an object, given by the product of moment of inertia and angular velocity ($L = I \times \omega$).
7. **Conservation of Angular Momentum:** If no external torque acts on a system, its angular momentum remains constant.
8. **Centripetal Force:** The force that keeps an object moving in a circular path, directed towards the center of the circle.
9. **Centrifugal Force:** The apparent force that seems to push an object outward when it is in a circular path.
10. **Rotational Kinetic Energy:** The energy an object possesses due to its rotation. It is given by $\frac{1}{2}I\omega^2$.
11. **Equilibrium:** When the sum of all forces and torques acting on an object is zero, the object is in equilibrium and does not rotate.
12. **Rolling Motion:** Combination of rotational and translational motion, like a wheel rolling on the ground.

13. **Axis of Rotation:** The line around which an object rotates.

14. **Rigid Body:** An object with a fixed shape that does not deform as it rotates.

15. **Parallel Axis Theorem:** A rule to find the moment of inertia of an object about any axis, given the moment of inertia about a parallel axis through its center of mass.

16. **Perpendicular Axis Theorem:** A rule to find the moment of inertia of a flat object about an axis perpendicular to its plane.

17. **Rotational Work:** Work done by a torque in causing an object to rotate, given by the product of torque and the angle rotated ($W = \tau\theta$).

18. **Rotational Power:** The rate at which work is done in rotating an object, given by the product of torque and angular velocity ($P = \tau\omega$).

19. **Gyroscope:** A device that uses the principles of rotational dynamics to maintain orientation.

20. **Precession:** The slow, circular motion of the axis of a spinning object, like a spinning top.

1. **Torque (τ):**
   $\tau = F \times r \times \sin(\theta)$
   where F is the force, r is the distance from the pivot point, and θ is the angle between the force and the lever arm.

2. **Moment of Inertia (I):**
   For different shapes, the moment of inertia can be calculated as:

- **Solid sphere:** $I = \frac{2}{5}MR^2$
- **Solid cylinder (about its central axis):** $I = \frac{1}{2}MR^2$
- **Thin rod (about its end):** $I = \frac{1}{3}ML^2$
- **Thin rod (about its center):** $I = \frac{1}{12}ML^2$
- **Thin hoop (about its central axis):** $I = MR^2$

3. **Angular Velocity (ω):**
   $\omega = \frac{d\theta}{dt}$
   where θ is the angle and t is the time.

4. **Angular Acceleration (α):**
   $\alpha = \frac{d\omega}{dt}$

5. **Newton's Second Law for Rotation:**
   $\tau = I\alpha$

6. **Kinematic Equations for Rotational Motion:**

   $\omega = \omega_0 + \alpha t$

   $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

   $\omega^2 = \omega_0^2 + 2\alpha\theta$

7. **Angular Momentum (L):**

   $L = I\omega$

8. **Conservation of Angular Momentum:**

   $L_{\text{initial}} = L_{\text{final}}$

   $I_1\omega_1 = I_2\omega_2$

9. **Rotational Kinetic Energy (K):**

   $K = \frac{1}{2}I\omega^2$

10. **Work Done by Torque (W):**

    $W = \tau\theta$

11. **Power in Rotational Motion (P):**

    $P = \tau\omega$

12. **Centripetal Force (F_c):**

    $F_c = m\omega^2 r = \frac{mv^2}{r}$

13. **Moment of Inertia for Composite Bodies:**
$$I = I_1 + I_2 + \dots$$

14. **Parallel Axis Theorem:**
$$I = I_{\text{cm}} + Md^2$$
where I_{cm} is the moment of inertia about the center of mass, M is the mass, and d is the distance between the two axes.

15. **Perpendicular Axis Theorem:**
For a flat object lying in the x-y plane:
$$I_z = I_x + I_y$$

16. **Rolling Motion without Slipping:**
$$v = \omega R$$
$$a = \alpha R$$

Question: A force of 5 N is applied perpendicular to a door at a distance of 0.8 m from the hinge. Calculate the torque.

Solution:
$\tau = F \times r$
$\tau = 5\,\text{N} \times 0.8\,\text{m}$
$\tau = 4\,\text{Nm}$

So, the torque is 4 Nm.

Question: A torque of 50 Nm is applied to a wheel, causing it to rotate through an angle of 4 radians. Calculate the work done.

Solution:
$W = \tau\theta$
$W = 50\,\text{Nm} \times 4\,\text{rad}$
$W = 200\,\text{J}$

So, the work done is 200 J.

Question: A solid sphere of radius 0.2 m and mass 2 kg rolls down an inclined plane. If it starts from rest and rolls without slipping, calculate its linear speed at the bottom of the incline if the height of the incline is 1 m.

Solution:

For a solid sphere:

$I = \frac{2}{5}MR^2$

Using conservation of energy:

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

Since $v = \omega R$:

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v}{R}\right)^2$

$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}M\right)v^2$

$mgh = \frac{1}{2}mv^2 + \frac{1}{5}mv^2$

$mgh = \left(\frac{1}{2} + \frac{1}{5}\right)mv^2$

$mgh = \frac{7}{10}mv^2$

$gh = \frac{7}{10}v^2$

$v^2 = \frac{10gh}{7}$

$v = \sqrt{\frac{10gh}{7}}$

Substitute the given values ($g = 9.8\,\mathrm{m/s^2}$ and $h = 1\,\mathrm{m}$):

$v = \sqrt{\frac{10\times9.8\times1}{7}}$

$v = \sqrt{14}$

$v \approx 3.74\,\mathrm{m/s}$

So, the linear speed at the bottom of the incline is approximately $3.74\,\mathrm{m/s}$.

Question: Calculate the moment of inertia of a solid disk with a mass of 3 kg and a radius of 0.5 m.

Solution:

For a solid disk:

$I = \frac{1}{2}MR^2$

$I = \frac{1}{2} \times 3\,\text{kg} \times (0.5\,\text{m})^2$

$I = \frac{1}{2} \times 3 \times 0.25$

$I = \frac{1}{2} \times 0.75$

$I = 0.375\,\text{kg}\backslash\text{cdotp}\text{m}^2$

So, the moment of inertia is $0.375\,\text{kg}\backslash\text{cdotp}\text{m}^2$.

Question: A wheel with a moment of inertia of 0.8 kg·m^2 is rotating at 5 rad/s. Calculate its rotational kinetic energy.

Solution:

$K = \frac{1}{2}I\omega^2$

$K = \frac{1}{2} \times 0.8\,\text{kg}\backslash\text{cdotp}\text{m}^2 \times (5\,\text{rad/s})^2$

$K = \frac{1}{2} \times 0.8 \times 25$

$K = 0.4 \times 25$

$K = 10\,\text{J}$

So, the rotational kinetic energy is 10 J.

Question: A merry-go-round with a moment of inertia of 1200 kg·m^2 experiences a torque of 300 Nm. Calculate the angular acceleration.

Solution:

$\tau = I\alpha$

$\alpha = \frac{\tau}{I}$

$\alpha = \frac{300\,\text{Nm}}{1200\,\text{kg}\backslash\text{cdotp}\text{m}^2}$

$\alpha = 0.25\,\text{rad/s}^2$

So, the angular acceleration is $0.25\,\text{rad/s}^2$.

Question: A spinning top initially spins at 50 rad/s. If it experiences a constant angular acceleration of 5 rad/s^2 for 4 seconds, what is its final angular velocity?

Solution:
Given:

- Initial angular velocity, $\omega_0 = 50$ rad/s
- Angular acceleration, $\alpha = 5$ rad/s^2
- Time, $t = 4$ s

Using the kinematic equation:
$\omega = \omega_0 + \alpha t$
$\omega = 50 + 5 \times 4$
$\omega = 50 + 20$
$\omega = 70 \text{ rad/s}$

So, the final angular velocity of the spinning top is 70 rad/s.

Question: A torque of 15 Nm is applied to a wheel with a moment of inertia of 0.2 kg·m^2. Calculate the angular acceleration produced.

Solution:
Given:

- Torque, $\tau = 15$ Nm
- Moment of inertia, $I = 0.2$ kg·m^2

$\alpha = \frac{\tau}{I}$
$\alpha = \frac{15}{0.2}$
$\alpha = 75 \text{ rad/s}^2$

So, the angular acceleration produced is 75 rad/s^2.

Question: A pulley of moment of inertia 0.5 kg·m^2 and radius 0.2 m starts from rest and accelerates uniformly under a constant torque of 10 Nm. Calculate its rotational kinetic energy after 4 seconds.

Solution:
Given:

- Moment of inertia, $I = 0.5$ kg·m^2
- Radius, $R = 0.2$ m
- Torque, $\tau = 10$ Nm
- Time, $t = 4$ s

First, calculate the angular acceleration using $\alpha = \frac{\tau}{I}$:
$\alpha = \frac{10}{0.5}$
$\alpha = 20 \text{ rad/s}^2$

Next, find the angular velocity ω after 4 seconds using $\omega = \omega_0 + \alpha t$:
$\omega = 0 + 20 \times 4$
$\omega = 80 \text{ rad/s}$

Now, calculate the rotational kinetic energy K using $K = \frac{1}{2} I \omega^2$:
$K = \frac{1}{2} \times 0.5 \times (80)^2$
$K = \frac{1}{2} \times 0.5 \times 6400$
$K = 1600 \text{ J}$

So, the rotational kinetic energy of the pulley after 4 seconds is 1600 J.