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1 Dynamic Programming

(a) For the given dynamical system with

$$x_n \in -2, -1, 0, 1, 2 \text{ and } u_n \in -1, 0, 1$$

and cost function:

$$J = \left(\sum_{k=0}^{2} 2|x_k| + |u_n| \right) + (x_3)^2$$

we can apply dynamic programming starting from the last stage and calculating cost thereon as follows:
Stage 3:

$$J_3(x_3) = (x_3)^2$$
$$x_3 = -2 \rightarrow J_3(-2) = 4$$
$$x_3 = -1 \rightarrow J_3(-1) = 1$$
$$x_3 = 0 \rightarrow J_3(0) = 0$$
$$x_3 = 1 \rightarrow J_3(1) = 1$$
$$x_3 = 2 \rightarrow J_3(2) = 4$$

Stage 2:

$$J_2(x_2) = min\Big(2|x_2| + |u_2|\Big) + J_3(-x_2 + 1 + u_2)$$

for $\boldsymbol{x_2 = -2}$ we have,

$$J_2(-2) = (4 + |u_2|) + J_3(3 + u_2)$$

further for each value of u_2 :

$$u_2 = -1 : J_2(-2) = (4 + |-1|) + J_3\big(3 + (-1)\big) = 5 + J_3(2) = 9$$
$$u_2 = 0 : J_2(-2) = (4 + |0|) + J_3\big(3 + (0)\big) = 4 + J_3(2) = 8$$
$$u_2 = 1 : J_2(-2) = (4 + |1|) + J_3\big(3 + (1)\big) = 5 + J_3(2) = 9$$

Hence, by selecting minimizing u_2,

$$\boldsymbol{J_2(-2) = 8,} \text{ and } \boldsymbol{\mu_2^*(-2) = 0}$$

Similarly for $\boldsymbol{x_2 = -1}$ we have,

$$J_2(-1) = (2|-1| + |u_2|) + J_3(2 + u_2)$$

further for each value of u_2 :

$$u_2 = -1 : J_2(-1) = (2 + |-1|) + J_3\big(2 + (-1)\big) = 3 + J_3(1) = 4$$

$$u_2 = 0 : J_2(-1) = (2 + |0|) + J_3\big(2 + (0)\big) = 2 + J_3(2) = 6$$
$$u_2 = 1 : J_2(-1) = (2 + |1|) + J_3\big(2 + (1)\big) = 3 + J_3(2) = 7$$

Hence, by selecting minimizing u_2,

$$\boldsymbol{J_2(-1) = 4,} \text{ and } \boldsymbol{\mu_2^*(-1) = -1}$$

Similarly for $\boldsymbol{x_2 = 0}$ we have,

$$J_2(0) = (2|0| + |u_2|) + J_3(1 + u_2)$$

further for each value of u_2 :

$$u_2 = -1 : J_2(0) = (|-1|) + J_3\big(1 + (-1)\big) = 1 + J_3(0) = 1$$
$$u_2 = 0 : J_2(0) = (|0|) + J_3\big(1 + (0)\big) = 0 + J_3(1) = 1$$
$$u_2 = 1 : J_2(0) = (|1|) + J_3\big(1 + (1)\big) = 1 + J_3(2) = 5$$

Hence, by selecting minimizing u_2,

$$\boldsymbol{J_2(0) = 1,} \text{ and } \boldsymbol{\mu_2^*(0) = -1, 0}$$

Similarly for $\boldsymbol{x_2 = 1}$ we have,

$$J_2(1) = (2|1| + |u_2|) + J_3(0 + u_2)$$

further for each value of u_2 :

$$u_2 = -1 : J_2(1) = (2 + |-1|) + J_3\big(0 + (-1)\big) = 3 + J_3(-1) = 4$$
$$u_2 = 0 : J_2(1) = (2 + |0|) + J_3\big(0 + (0)\big) = 2 + J_3(0) = 2$$
$$u_2 = 1 : J_2(1) = (2 + |1|) + J_3\big(0 + (1)\big) = 3 + J_3(1) = 4$$

Hence, by selecting minimizing u_2,

$$\boldsymbol{J_2(1) = 2,} \text{ and } \boldsymbol{\mu_2^*(1) = 0}$$

finally for $\boldsymbol{x_2 = 2}$ we have,

$$J_2(2) = (4 + |u_2|) + J_3(-1 + u_2)$$

further for each value of u_2 :

$$u_2 = -1 : J_2(-2) = (4 + |-1|) + J_3\big(-1 + (-1)\big) = 5 + J_3(-2) = 9$$
$$u_2 = 0 : J_2(-2) = (4 + |0|) + J_3\big(-1 + (0)\big) = 4 + J_3(-1) = 5$$
$$u_2 = 1 : J_2(-2) = (4 + |1|) + J_3\big(-1 + (1)\big) = 5 + J_3(0) = 5$$

Hence, by selecting minimizing u_2,

$$\boldsymbol{J_2(2) = 5,} \text{ and } \boldsymbol{\mu_2^*(-2) = 0, 1}$$

Stage 1:

$$J_1(x_1) = min\Big(2|x_1| + |u_1|\Big) + J_2(-x_1 + 1 + u_1)$$

for $\boldsymbol{x_1 = -2}$ and for each value of u_1 :

$$u_1 = -1 : J_1(-2) = (4 + |-1|) + J_2\big(3 + (-1)\big) = 5 + J_2(2) = 10$$

$$u_1 = 0 : J_1(-2) = (4 + |0|) + J_2\big(3 + (0)\big) = 4 + J_2(2) = 9$$

$$u_1 = 1 : J_1(-2) = (4 + |1|) + J_2\big(3 + (1)\big) = 5 + J_2(2) = 10$$

Hence, by selecting minimizing u_1,

$$\boldsymbol{J_1(-2) = 9,} \text{ and } \boldsymbol{\mu_1^*(-2) = 0}$$

Similarly for $\boldsymbol{x_1 = -1}$ we have,

$$J_1(-1) = (2|-1| + |u_1|) + J_2(2 + u_1)$$

further for each value of u_1 :

$$u_1 = -1 : J_1(-1) = (2 + |-1|) + J_2\big(2 + (-1)\big) = 3 + J_2(1) = 5$$

$$u_1 = 0 : J_1(-1) = (2 + |0|) + J_2\big(2 + (0)\big) = 2 + J_2(2) = 7$$

$$u_1 = 1 : J_1(-1) = (2 + |1|) + J_2\big(2 + (1)\big) = 3 + J_2(2) = 8$$

Hence, by selecting minimizing u_1,

$$\boldsymbol{J_1(-1) = 5,} \text{ and } \boldsymbol{\mu_1^*(-1) = -1}$$

Similarly for $\boldsymbol{x_1 = 0}$ we have,

$$J_1(0) = (2|0| + |u_1|) + J_2(1 + u_1)$$

further for each value of u_1 :

$$u_1 = -1 : J_1(0) = (|-1|) + J_2\big(1 + (-1)\big) = 1 + J_2(0) = 2$$

$$u_1 = 0 : J_1(0) = (|0|) + J_2\big(1 + (0)\big) = 0 + J_2(1) = 2$$

$$u_1 = 1 : J_1(0) = (|1|) + J_2\big(1 + (1)\big) = 1 + J_2(2) = 5$$

Hence, by selecting minimizing u_1,

$$\boldsymbol{J_1(0) = 2,} \text{ and } \boldsymbol{\mu_1^*(0) = -1, 0}$$

Similarly for $\boldsymbol{x_1 = 1}$ we have,

$$J_1(1) = (2|1| + |u_1|) + J_2(0 + u_1)$$

further for each value of u_1 :

$$u_1 = -1 : J_1(1) = (2 + |-1|) + J_2\big(0 + (-1)\big) = 3 + J_2(-1) = 7$$

$$u_1 = 0 : J_1(1) = (2 + |0|) + J_2\big(0 + (0)\big) = 2 + J_2(0) = 3$$

$$u_1 = 1 : J_1(1) = (2 + |1|) + J_2\big(0 + (1)\big) = 3 + J_2(1) = 5$$

Hence, by selecting minimizing u_1,

$$\boldsymbol{J_1(1) = 3,} \text{ and } \boldsymbol{\mu_1^*(1) = 0}$$

finally for $\boldsymbol{x_1 = 2}$ we have,

$$J_1(2) = (4 + |u_1|) + J_2(-1 + u_1)$$

further for each value of u_1 :

$$u_1 = -1 : J_1(-2) = (4 + |-1|) + J_2\big(-1 + (-1)\big) = 5 + J_2(-2) = 13$$

$$u_1 = 0 : J_1(-2) = (4 + |0|) + J_2\big(-1 + (0)\big) = 4 + J_2(-1) = 8$$

$$u_1 = 1 : J_1(-2) = (4 + |1|) + J_2\big(-1 + (1)\big) = 5 + J_2(0) = 6$$

Hence, by selecting minimizing u_1,

$$\boldsymbol{J_1(2) = 6,} \text{ and } \boldsymbol{\mu_1^*(-2) = 1}$$

Stage 0:

$$J_0(x_0) = min\Big(2|x_0| + |u_0|\Big) + J_1(-x_0 + 1 + u_0)$$

for $\boldsymbol{x_0 = -2}$ and for each value of u_0 :

$$u_0 = -1 : J_0(-2) = (4 + |-1|) + J_1\big(3 + (-1)\big) = 5 + J_1(2) = 11$$

$$u_0 = 0 : J_0(-2) = (4 + |0|) + J_1\big(3 + (0)\big) = 4 + J_1(2) = 10$$

$$u_0 = 1 : J_0(-2) = (4 + |1|) + J_1\big(3 + (1)\big) = 5 + J_1(2) = 11$$

Hence, by selecting minimizing u_0,

$$\boldsymbol{J_0(-2) = 10,} \text{ and } \boldsymbol{\mu_0^*(-2) = 0}$$

Similarly for $\boldsymbol{x_0 = -1}$ we have,

$$J_0(-1) = (2|-1| + |u_0|) + J_1(2 + u_0)$$

further for each value of u_0 :

$$u_0 = -1 : J_0(-1) = (2 + |-1|) + J_1\big(2 + (-1)\big) = 3 + J_1(1) = 6$$

$$u_0 = 0 : J_0(-1) = (2 + |0|) + J_1\big(2 + (0)\big) = 2 + J_1(2) = 8$$

$$u_0 = 1 : J_0(-1) = (2 + |1|) + J_1\big(2 + (1)\big) = 3 + J_1(2) = 9$$

Hence, by selecting minimizing u_0,

$$\boldsymbol{J_0(-1) = 6,} \text{ and } \boldsymbol{\mu_0^*(-1) = -1}$$

Similarly for $\boldsymbol{x_0 = 0}$ we have,

$$J_0(0) = (2|0| + |u_0|) + J_1(1 + u_0)$$

further for each value of u_0 :

$$u_0 = -1 : J_0(0) = (|-1|) + J_1\big(1 + (-1)\big) = 1 + J_1(0) = 3$$

$$u_0 = 0 : J_0(0) = (|0|) + J_1\big(1 + (0)\big) = 0 + J_1(1) = 3$$

$$u_0 = 1 : J_0(0) = (|1|) + J_1\big(1 + (1)\big) = 1 + J_1(2) = 7$$

Hence, by selecting minimizing u_0,

$$\boldsymbol{J_0(0) = 3,}\ \text{and}\ \boldsymbol{\mu_0^*(0) = -1, 0}$$

Similarly for $\boldsymbol{x_0 = 1}$ we have,

$$J_0(1) = (2|1| + |u_0|) + J_1(0 + u_0)$$

further for each value of u_0 :

$$u_0 = -1 : J_0(1) = (2 + |-1|) + J_1\big(0 + (-1)\big) = 3 + J_1(-1) = 8$$

$$u_0 = 0 : J_0(1) = (2 + |0|) + J_1\big(0 + (0)\big) = 2 + J_1(0) = 4$$

$$u_0 = 1 : J_0(1) = (2 + |1|) + J_1\big(0 + (1)\big) = 3 + J_1(1) = 6$$

Hence, by selecting minimizing u_0,

$$\boldsymbol{J_0(1) = 4,}\ \text{and}\ \boldsymbol{\mu_0^*(1) = 0}$$

finally for $\boldsymbol{x_0 = 2}$ we have,

$$J_0(2) = (4 + |u_0|) + J_1(-1 + u_0)$$

further for each value of u_0 :

$$u_0 = -1 : J_0(-2) = (4 + |-1|) + J_1\big(-1 + (-1)\big) = 5 + J_1(-2) = 14$$

$$u_0 = 0 : J_0(-2) = (4 + |0|) + J_1\big(-1 + (0)\big) = 4 + J_1(-1) = 9$$

$$u_0 = 1 : J_0(-2) = (4 + |1|) + J_1\big(-1 + (1)\big) = 5 + J_1(0) = 7$$

Hence, by selecting minimizing u_0,

$$\boldsymbol{J_0(2) = 7,}\ \text{and}\ \boldsymbol{\mu_0^*(2) = 1}$$

Table:

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Stage 0			Stage 1			Stage 2			Stage 3	
x_0	cost(J_0)	control(u_0)	x_1	cost(J_1)	control(u_1)	x_2	cost(J_2)	control(u_2)	x_3	cost(J_3)
-2	10	0	-2	9	0	-2	8	0	-2	4
-1	6	-1	-1	5	-1	-1	4	-1	-1	1
0	3	0,-1	0	2	0,-1	0	1	0,-1	0	0
1	4	0	1	3	0	1	2	0	1	1
2	7	1	2	6	1	2	5	0,1	2	4

(b) If $\boldsymbol{x_0 = 0}$ then control sequence can be chosen as:

$$x_0 = 0 \rightarrow J_0 = 3 \text{ and } u_0 = -1 \rightarrow x_1 = 0 \text{ (as } J_1(0) < J_1(1))$$

$$x_1 = 0 \rightarrow J_1 = 2 \text{ and } u_1 = -1 \rightarrow x_2 = 0 \text{ (as } J_2(0) < J_2(1))$$
$$x_2 = 0 \rightarrow J_2 = 1 \text{ and } u_2 = -1 \rightarrow x_3 = 0 \text{ (as } J_3(0) < J_3(1))$$
$$x_3 = 0 \rightarrow J_3 = 0$$

Hence, optimal cost in this case = 3 with states:

$$x_0 = 0 \rightarrow x_1 = 0 \rightarrow x_2 = 0 \rightarrow x_3 = 0$$

and control actions:

$$u_0 = -1 \rightarrow u_1 = -1 \rightarrow u_2 = -1$$

If $\boldsymbol{x_0 = -2}$ then control sequence can be chosen as:

$$x_0 = -2 \rightarrow J_0 = 10 \text{ and } u_0 = 0 \rightarrow x_1 = 2$$
$$x_1 = 2 \rightarrow J_1 = 6 \text{ and } u_1 = 1 \rightarrow x_2 = 0$$
$$x_2 = 0 \rightarrow J_2 = 1 \text{ and } u_2 = -1 \rightarrow x_3 = 0$$
$$x_3 = 0 \rightarrow J_3 = 0$$

Hence, optimal cost in this case = 10 with states:

$$x_0 = -2 \rightarrow x_1 = 2 \rightarrow x_2 = 0 \rightarrow x_3 = 0$$

and control actions:

$$u_0 = 0 \rightarrow u_1 = 1 \rightarrow u_2 = -1$$

If $\boldsymbol{x_0 = 2}$ then control sequence can be chosen as:

$$x_0 = 2 \rightarrow J_0 = 7 \text{ and } u_0 = 1 \rightarrow x_1 = 0$$
$$x_1 = 0 \rightarrow J_1 = 2 \text{ and } u_1 = -1 \rightarrow x_2 = 0$$
$$x_2 = 0 \rightarrow J_2 = 0 \text{ and } u_2 = -1 \rightarrow x_3 = 0$$
$$x_3 = 0 \rightarrow J_3 = 0$$

Hence, optimal cost in this case = 11 with states:

$$x_0 = 2 \rightarrow x_1 = 0 \rightarrow x_2 = 0 \rightarrow x_3 = 0$$

and control actions:

$$u_0 = 1 \rightarrow u_1 = -1 \rightarrow u_2 = -1$$

(c) Similar to the problem (a) and with added consideration for Expected value of the cost, we can solve this problem by constructing cost function as:

$$J_k(x_k) = \min\ E\Big\{(2|x_k| + |u_k|) + J_{k+1}(-x_k + w_k + u_k)\Big\}$$

$$\Rightarrow J_k(x_k) = \min\ \Big\{(2|x_k| + |u_k|) + 0.3 \times J_{k+1}(-x_k + u_k) + 0.7 \times J_{k+1}(-x_k + 1 + u_k)\Big\}$$

Stage 3:

$$J_3(-2) = 4$$
$$J_3(-1) = 1$$
$$J_3(0) = 0$$
$$J_3(1) = 1$$
$$J_3(2) = 4$$

Stage 2:
for $\boldsymbol{x_2 = -2}$:,

$$u_2 = -1 : J_2(-2) = 8.1$$
$$u_2 = 0 : J_2(-2) = 8$$
$$u_2 = 1 : J_2(-2) = 9$$
$$\boldsymbol{J_2(-2) = 8} \text{ and } \boldsymbol{u_2 = 0}$$

for $\boldsymbol{x_2 = -1}$:,

$$u_2 = -1 : J_2(-2) = 3.7$$
$$u_2 = 0 : J_2(-2) = 5.1$$
$$u_2 = 1 : J_2(-2) = 6$$
$$\boldsymbol{J_2(-1) = 3.7} \text{ and } \boldsymbol{u_2 = -1}$$

for $\boldsymbol{x_2 = 0}$:,

$$u_2 = -1 : J_2(-2) = 1.3$$
$$u_2 = 0 : J_2(-2) = 0.7$$
$$u_2 = 1 : J_2(-2) = 4.1$$
$$\boldsymbol{J_2(0) = 8} \text{ and } \boldsymbol{u_2 = 0}$$

for $\boldsymbol{x_2 = 1}$:,

$$u_2 = -1 : J_2(-2) = 4.9$$
$$u_2 = 0 : J_2(-2) = 2.3$$
$$u_2 = 1 : J_2(-2) = 3.7$$
$$\boldsymbol{J_2(1) = 2.3} \text{ and } \boldsymbol{u_2 = 0}$$

for $\boldsymbol{x_2 = 2}$:,

$$u_2 = -1 : J_2(-2) = 9$$
$$u_2 = 0 : J_2(-2) = 5.9$$
$$u_2 = 1 : J_2(-2) = 5.3$$
$$\boldsymbol{J_2(2) = 5.3} \text{ and } \boldsymbol{u_2 = 1}$$

Stage 1:
for $\boldsymbol{x_1 = -2}$:,

$$u_1 = -1 : J_1(-2) = 9.4$$

$$u_1 = 0 : J_1(-2) = 9.3$$

$$u_1 = 1 : J_1(-2) = 10.3$$

$$\boldsymbol{J_1(-2) = 9.3} \text{ and } \boldsymbol{u_1 = 0}$$

for $\boldsymbol{x_1 = -1}$:,

$$u_1 = -1 : J_1(-2) = 4.82$$

$$u_1 = 0 : J_1(-2) = 6.4$$

$$u_1 = 1 : J_1(-2) = 8.3$$

$$\boldsymbol{J_1(-1) = 4.82} \text{ and } \boldsymbol{u_1 = -1}$$

for $\boldsymbol{x_1 = 0}$:,

$$u_1 = -1 : J_1(-2) = 2.6$$

$$u_1 = 0 : J_1(-2) = 1.82$$

$$u_1 = 1 : J_1(-2) = 5.4$$

$$\boldsymbol{J_1(0) = 1.82} \text{ and } \boldsymbol{u_1 = 0}$$

for $\boldsymbol{x_1 = 1}$:,

$$u_1 = -1 : J_1(-2) = 7.99$$

$$u_1 = 0 : J_1(-2) = 3.6$$

$$u_1 = 1 : J_1(-2) = 4.82$$

$$\boldsymbol{J_1(1) = 3.6} \text{ and } \boldsymbol{u_1 = 0}$$

for $\boldsymbol{x_1 = 2}$:,

$$u_1 = -1 : J_1(-2) = 13$$

$$u_1 = 0 : J_1(-2) = 8.99$$

$$u_1 = 1 : J_1(-2) = 6.6$$

$$\boldsymbol{J_1(2) = 6.6} \text{ and } \boldsymbol{u_1 = 1}$$

Stage 0:
for $\boldsymbol{x_0 = -2}$:,

$$u_0 = -1 : J_0(-2) = 10.7$$

$$u_0 = 0 : J_0(-2) = 10.6$$

$$u_0 = 1 : J_0(-2) = 11.6$$

$$\boldsymbol{J_0(-2) = 10.6} \text{ and } \boldsymbol{u_0 = 0}$$

for $\boldsymbol{x_0 = -1}$:,

$$u_0 = -1 : J_0(-2) = 6.066$$
$$u_0 = 0 : J_0(-2) = 7.7$$
$$u_0 = 1 : J_0(-2) = 9.6$$
$$\boldsymbol{J_0(-1) = 6.066} \text{ and } \boldsymbol{u_0 = -1}$$

for $\boldsymbol{x_0 = 0}$:,

$$u_0 = -1 : J_0(-2) = 3.72$$
$$u_0 = 0 : J_0(-2) = 3.066$$
$$u_0 = 1 : J_0(-2) = 6.7$$
$$\boldsymbol{J_0(0) = 3.066} \text{ and } \boldsymbol{u_0 = 0}$$

for $\boldsymbol{x_0 = 1}$:,

$$u_0 = -1 : J_0(-2) = 9.164$$
$$u_0 = 0 : J_0(-2) = 4.72$$
$$u_0 = 1 : J_0(-2) = 6.066$$
$$\boldsymbol{J_0(1) = 4.72} \text{ and } \boldsymbol{u_0 = 0}$$

for $\boldsymbol{x_0 = 2}$:,

$$u_0 = -1 : J_0(-2) = 14.3$$
$$u_0 = 0 : J_0(-2) = 10.164$$
$$u_0 = 1 : J_0(-2) = 7.72$$
$$\boldsymbol{J_0(2) = 7.72} \text{ and } \boldsymbol{u_0 = 1}$$

Table:

Stage 0			Stage 1			Stage 2			Stage 3	
x_0	cost(J_0)	control(u_0)	x_1	cost(J_1)	control(u_1)	x_2	cost(J_2)	control(u_2)	x_3	cost(J_3)
-2	10.6	0	-2	9.3	0	-2	8	0	-2	4
-1	6.066	-1	-1	4.82	-1	-1	3.7	-1	-1	1
0	3.066	0	0	1.82	0	0	0.7	0	0	0
1	4.72	0	1	3.6	0	1	2.3	0	1	1
2	7.72	1	2	6.6	1	2	5.3	1	2	4

(d) Cost-to-go in the case of probabilistic model is different when compared with deterministic model. These variations arise due to incorporation of Expected value of the cost at each stage and stage. Moreover, the optimal control sequence in case of probabilistic model is relying on the uncertainty term as well as state sequence which is different from the deterministic case where the control sequence was dependent only on states of the dynamical system. In the probabilistic case we can not make decision unless that stage is reached but in deterministic case we can plan beforehand.

2 DP Python

All the questions are answered in the attached jupyter notebook (Exercise 2-DP Sol.ipynb)

3 Shortest Path

Jupyter Notebook (exercise 3 - shortest paths.ipynb) is attached with this file.
(d) Pros and Cons of the algorithms

Depth First Search
Pros: In this algorithm we can quickly find a path in cases when we need to find 'a path' as compared to finding optimal/shortest path. **Cons:** This is not a very good algorithm in terms of efficiency. As we saw in the given cases it checked the same nodes multiple times and hence performing poor in terms of time when need to find an optimal/shortest path solution. Also it will get stuck in a traversing a distant node even if that node connects with the target node or not.

Breath First Search
Pros: In this algorithm we can cover a broad range while finding the path, It is more efficient than the DFS and at the same time quicker in finding the optimal path even in the cases where some distant nodes do not connect with the target node.
Cons: This algorithm is in general slower in finding 'a path' as it keeps exploring all the branches available.

A * Algorithm
Pros: This algorithm is very efficient over the above discussed two cases. Usage of proper heuristic can make it quickly reach optimal path, and optimistic cost-to-go ensures optimality. **Cons:** In this algorithm if we overestimate the cost-to-go we will arrive at an sub-optimal path which is not good. Moreover bad heuristic can slow down the search.

(a) Depth First Search

Maze	Nodes Tested	length of shortest path
World Map	17	8
1	223241	98
2	176721	130
3	332734	140

(b) Breath First Search

Maze	Nodes Tested	length of shortest path
World Map	16	8
1	2024	98
2	2172	130
3	2048	140

(c) A Star $H_{i,j} = |i - \text{goal}_i| + |j - \text{goal}_j|$
Yes this heuristic is an under-estimator as it is significantly reducing the number of nodes evaluated.

Maze	Nodes Tested	length of shortest path
World Map	15	8
1	662	98
2	1842	130
3	1705	140

A Star $h_{ij} = 0$

Maze	Nodes Tested	length of shortest path
World Map	16	8
1	2024	98
2	2172	130
3	2048	140