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Assignment - 3

(1.) The magnetic field at a point (x, y) on the x-y plane due to an ∞ current carrying conductor along z-dirn. is given by Ampere's Law:

$$\vec{B}_x = -\frac{\mu_0 I}{2\pi r^2}(+y) \quad , \; r = \sqrt{x^2+y^2}$$

$$\& \; B_y = \frac{\mu_0 I x}{2\pi r^2} \quad \& \; B_z = 0$$

$\Rightarrow \vec{E} = 0$; i.e. $E_x = E_y = E_z = 0$

So:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\mu_0 I x}{2\pi r^2} \\ 0 & 0 & 0 & -\frac{\mu_0 I y}{2\pi r^2} \\ 0 & \frac{\mu_0 I x}{2\pi r^2} & \frac{\mu_0 I y}{2\pi r^2} & 0 \end{pmatrix}$$

&

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\mu_0 I y}{2\pi r^2} & \frac{\mu_0 I x}{2\pi r^2} & 0 \\ \frac{\mu_0 I y}{2\pi r^2} & 0 & 0 & 0 \\ -\frac{\mu_0 I x}{2\pi r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(2) At point $(0, y, 0)$ (Let $y>0$) : $\boxed{I = \lambda v}$

The fields are $|\vec{E}_x| = 0$

$$E_y = \frac{\lambda}{2\pi\varepsilon_0 y}$$

$$E_z = 0$$

$$B_x = \frac{-\mu_0 \lambda v}{2\pi}$$

$$B_y = 0$$

$$B_z = 0$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\lambda}{2\pi\varepsilon_0 y c} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-\lambda}{2\pi\varepsilon_0 y c} & 0 & 0 & \frac{-\mu_0\lambda v}{2\pi} \\ 0 & 0 & \frac{\mu_0\lambda v}{2\pi} & 0 \end{pmatrix}$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{-\mu_0\lambda v}{2\pi} & 0 & 0 \\ \frac{\mu_0\lambda v}{2\pi} & 0 & 0 & \frac{\lambda}{2\pi\varepsilon_0 y c} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-\lambda}{2\pi\varepsilon_0 y c} & 0 & 0 \end{pmatrix}$$

$G^{\mu\nu}$ is the dual tensor

(3) $\because \Lambda^{\mu}_{\ \alpha} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Lorentz Transformation Matrix

The Transformed tensor is ;

$$\bar{G}^{\mu\nu} = \Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} G^{\alpha\beta}$$

This in matrix form is $\Lambda G \Lambda^T$

So; Let $M = G\lambda^T$

$$\Rightarrow M = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix} \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} B_x(\gamma\beta) & B_x\gamma & B_y & B_z \\ -B_x\gamma & -B_x\gamma\beta & -E_z/c & E_y/c \\ \left[B_y\gamma + (E_z/c)\gamma\beta\right] & \left[-B_y\gamma\beta + (E_z/c)\gamma\right] & 0 & -\frac{E_x}{c} \\ \left[-B_z\gamma - \left(\frac{E_y}{c}\right)\gamma\beta\right] & \left[-B_z\gamma\beta - \left(\frac{E_y}{c}\right)\gamma\right] & \frac{E_x}{c} & 0 \end{bmatrix}$$

Now; $\bar{G} = \lambda M$

$$= \begin{bmatrix} 0 & B_x & \gamma\left(B_y + \frac{vE_z}{c^2}\right) & \gamma\left(B_z - \frac{vE_y}{c^2}\right) \\ -B_x & 0 & \gamma\left(\frac{vB_y}{c} - \frac{E_z}{c}\right) & \gamma\left(\frac{vB_z}{c} + \frac{E_y}{c}\right) \\ \gamma\left(-B_y - \frac{vE_z}{c^2}\right) & \gamma\left(-\frac{vB_y}{c} + \frac{E_z}{c}\right) & 0 & -\frac{E_x}{c} \\ \gamma\left(-B_z + \frac{vE_y}{c^2}\right) & \gamma\left(-\frac{vB_z}{c} - \frac{E_y}{c}\right) & \frac{E_x}{c} & 0 \end{bmatrix}$$

Now, comparing with $G^{\mu\nu}$:

$\bar{B}_x = B_x$

$\bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$

$\bar{B}_z = \gamma\left(B_z - \frac{vE_y}{c^2}\right)$

$\bar{E}_x = E_x$

$\bar{E}_y = \gamma\left(E_y - vB_z\right)$

$\bar{E}_z = \gamma\left(E_z + vB_y\right)$

(4) $\vec{E}'\cdot\vec{B}' = \bar{E}_x\bar{B}_x + \bar{E}_y\bar{B}_y + \bar{E}_z\bar{B}_z$

$= E_xB_x + \gamma^2(E_y - vB_z)(B_y + \frac{v}{c^2}E_z)$

$+ \gamma^2(E_z + vB_y)(B_z - \frac{v}{c^2}E_y)$

$= E_xB_x + \gamma^2\Big(E_yB_y + \cancel{\frac{v}{c^2}E_yE_z} - \cancel{vB_zB_y} - \frac{v^2}{c^2}B_zE_z$

$+ E_zB_z - \cancel{\frac{v}{c^2}E_zE_y} + \cancel{vB_yB_z} - \frac{v^2}{c^2}B_yE_y\Big)$

$= E_xB_x + E_yB_y\underbrace{(\gamma^2)\left(1-\frac{v^2}{c^2}\right)}_{\to 1} + E_zB_z\underbrace{(\gamma^2)\left(1-\frac{v^2}{c^2}\right)}_{\to 1}$

$= E_xB_x + E_yB_y + E_zB_z$

$= \vec{E}\cdot\vec{B}$

& $\bar{E}^2 - c^2\bar{B}^2 = \vec{\bar{E}}\cdot\vec{\bar{E}} - c^2\,\vec{\bar{B}}\cdot\vec{\bar{B}}$

$= E_x^2 + \gamma^2\left[(E_y - vB_z)^2 + (E_z + vB_y)^2\right]$

$- c^2\left[B_x^2 + \gamma^2(B_y + \frac{v}{c^2}E_z)^2 + (B_z - \frac{v}{c^2}E_y)^2\right]$

$= E_x^2 + \gamma^2\left[E_y^2 + v^2B_z^2 - \cancel{2vE_yB_z} + E_z^2 + v^2B_y^2 + \cancel{2vE_zB_y}\right]$

$- c^2B_x^2 - c^2\gamma^2(B_y)^2 - \cancel{2c^2\gamma^2B_y\left(\frac{v}{c^2}E_z\right)} - \frac{c^2\gamma^2v^2}{c^4}E_z^2$

$- c^2B_z^2 - \frac{c^2v^2}{c^4}E_y^2 + \cancel{2c^2B_z\frac{v}{c^2}E_y}\Big]$

$$= E_x^2 + E_y^2\underbrace{\left(\gamma^2\left(1-\frac{v^2}{c^2}\right)\right)}_{\to 1} + E_z^2\left(\gamma^2\left(1-v^2/c^2\right)\right)$$

$$- c^2\left[B_x^2 + B_y^2(\gamma^2)\left(1-\frac{v^2}{c^2}\right) + B_z^2(\gamma^2)\left(1-\frac{v^2}{c^2}\right)\right]$$

$$= E_x^2 + E_y^2 + E_z^2 - c^2\left[B_x^2 + B_y^2 + B_z^2\right]$$

$$= \vec{E}^2 - c^2\vec{B}^2$$

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(5) For an anti-symmetric 2nd rank tensor $t^{\mu\nu}$,

$t^{00} = (-1)^2 t_{00}$, $t^{0i} = (-1)^1 t_{0i}$ & $t^{ij} = t_{ij}$.

So;

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

So; $F^{00} = (-1)^2 F_{00} = F_{00} = 0$

$F^{0i} = -F_{0i} \Rightarrow F_{0i} = \frac{-E_i}{c}$

$F_{ij} = -\epsilon^{ijk} B_k$ & $G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

$$\Rightarrow F^{\mu\nu}F_{\mu\nu} = 2\sum_i F^{0i}F_{0i} + \sum_{i,j} F^{ij}F_{ij}$$

$$= 2\left(B^2 - \frac{E^2}{c^2}\right)$$

$$\Rightarrow F^{\mu\nu}G_{\mu\nu} = -\frac{4}{c} E\cdot B \quad \& \quad G^{\mu\nu}G_{\mu\nu} = 2\left(\frac{E^2}{c^2} - B^2\right)$$