Practice problems 3

1. Write the dual of the following problems:

   (a)
$$\begin{aligned} &\text{Maximize } x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \geq 1 \\ & x_1 + 2x_2 + 3x_3 = 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

   (b)
$$\begin{aligned} &\text{Maximize } x_1 + x_2 + x_3 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \leq 2 \\ & x_1 + 2x_2 + 3x_3 \geq 2 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

   (c)
$$\begin{aligned} &\text{Maximize } x_1 - x_2 + x_3 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \leq 2 \\ & x_1 + 2x_2 + 3x_3 = 2. \end{aligned}$$

   Hint: Write x_i as $x_i = z_i - w_i$, where $z_i, w_i \geq 0$.

   Check whether the above problems have optimal solutions by using the dual(you need not find the optimal solutions).

2. Find an infeasible (that has no feasible solution) primal problem which also has an infeasible dual.

3. (a) Using similar results done in class show that exactly one of the following two systems has a solution.
   $\mathbf{x} \geq \mathbf{0}$, $A\mathbf{x} > \mathbf{0}$ (all components of $A\mathbf{x}$ is positive). (1)
   $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$, $A^T\mathbf{y} \leq \mathbf{0}$. (2)
   Hence can the feasible region of both a primal problem (in standard form) and its dual be bounded?
   Hint: For the first part use Farka's lemma and convert the system (2) to
   $\begin{bmatrix} A^T & I \\ \mathbf{e}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$, $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$ (2)'
   Then by Farka's lemma exactly one of the following two systems (1)' and (2)' has a solution,
   where (1)' is given by:
   $\begin{bmatrix} A & \mathbf{e} \\ I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ a \end{bmatrix} \geq \mathbf{0}$, $[\mathbf{0} \;\; 1] \begin{bmatrix} \mathbf{x} \\ a \end{bmatrix} < 0$
   Show that (1)' is equivalent to the system (1) above.
   For the second part, answer is **No**.

(b) Can both the feasible regions be unbounded?
Hint: Answer is Yes.

4. Consider a LPP of the type:

$$\text{Maximize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } A_{m\times n}\mathbf{x} \leq \mathbf{b}.$$

Write the dual of the above problem such that it (the dual) has only n constraints (leaving out the non negativity constraints).

5. Given a LPP of the type:

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}.$$

(a) Write the dual of the above problem.

(b) Obtain the complementary slackness conditions of the above problem.

6. Using the complementary slackness conditions obtain the optimal solution of the following problem and show that it is unique.

$$\text{Minimize } x_1 + 3x_2 + x_3$$

$$\begin{aligned} \text{subject to} \quad 3x_1 + x_2 &\geq 6 \\ x_1 + x_2 - x_3 &\geq 2 \\ x_1 + x_3 &\geq 2 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\geq 0. \end{aligned}$$

Does the dual have a unique solution? Obtain feasible solutions $\mathbf{x}$ and $\mathbf{y}$ of the primal(P) and the dual(D) respectively, which does **not** mutually satisfy the complementary slackness condition.

7. Consider the problem

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$

$$\text{subject to } A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$$

where A has m rows and n columns. Suppose an optimal solution exists such that only $x_1, x_2, ..., x_k$ are positive and $x_j = 0$ for $j = k+1, k+2, ..., n$. Now change the vector $\mathbf{b}$ to $\mathbf{b}'$. Prove that if there exists a $\mathbf{z} \geq \mathbf{0}$ such that $A\mathbf{z} = \mathbf{b}'$ and $z_j = 0$ for $j = k+1, k+2, ..., n$, then this $\mathbf{z}$ is optimal for the changed problem.