Practice sheet 4

Notations: $\mathbf{a}_k^T$ denotes the k-th row of A and $\tilde{\mathbf{a}}_k$ denotes the k-th column of A.
$\mathbf{a'}_k^T$ denotes the i th row of the matrix $[A^T : -I]^T$, or $\mathbf{a'}_k^T$ is either the k th row of A or the kth row of $-I$
b'_i is either the i th component of $\mathbf{b}$ or is equal to 0.

1. Consider the problem (P) given below:

$$\begin{aligned} &\text{Maximize } x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 + x_3 \leq 1 \\ & x_1 + 2x_2 + 3x_3 = 2 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

   Consider the extreme point $\mathbf{x}^0$ which lies in the hyperplanes corresponding to the first constraint, second constraint and the constraint $x_1 \geq 0$. Write the corresponding feasible solution $[\mathbf{x}^0, s_1]^T$ of the system,

$$-x_1 + x_2 + x_3 + s_1 = 1$$

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0.$$

   Verify that $[\mathbf{x}^0, s_1]^T$ is a basic feasible solution of the above system.
   Verify that $[0, 0, \frac{2}{3}, \frac{1}{3}]^T$ is a basic feasible solution of the above system. Verify that $[0, 0, \frac{2}{3}]^T$ is an extreme point of $Fea(P)$.

2. Use simplex algorithm to solve the following problem:
   Min $4x_1 + 4x_2 - 2x_3$
   Subject to
   $2x_1 + 3x_2 - 2x_3 \leq 10$
   $2x_1 - x_2 + 3x_3 \leq 4$
   $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

3. For a LPP (P) of the form,

$$\begin{aligned} &\text{Minimize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } A_{m\times n}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}, rank(A) = m \end{aligned}$$

   check the correctness of the following statements with proper justification.

   (a) If $c_1 - z_1 > 0$ in some iteration (table) and if $\tilde{\mathbf{a}}_1$ is made to enter the basis, then the new solution obtained will not be feasible for (P).

   (b) If for some basic feasible solution $\overline{\mathbf{x}}$, $\mathbf{c}^T\overline{\mathbf{x}} = 10$ and for no feasible $\mathbf{x}$, $\mathbf{c}^T\mathbf{x} = 5$, then (P) has an optimal solution.

(c) For $m = 2$ and $n = 4$ in (P), if $[1, 2, 3, 0]^T$ is an optimal solution, then (P) has infinitely many solutions.

(d) For $m = 2$ and $n = 4$ in (P), if in some iteration (table) of the simplex method, $\tilde{\mathbf{a}_2}$ enters the basis, then $\tilde{\mathbf{a}_2}$ can leave the basis in the next iteration (table).

(e) For $m = 2$ and $n = 4$ in (P), if in some iteration (table) of the simplex method, $\tilde{\mathbf{a}_2}$ leaves the basis, then $\tilde{\mathbf{a}_2}$ can again enter the basis in the next iteration (table).

(f) If $\mathbf{B}$ is the basis matrix corresponding to the optimal table (of the simplex method) where $\mathbf{c}_1 - \mathbf{z}_1 = 0$ and $\mathbf{B}^{-1}\tilde{\mathbf{a}_1} \leq \mathbf{0}$, then the set of optimal solutions of (P) is an unbounded set.

(g) If $\mathbf{c}^T\mathbf{x}^0 = \mathbf{b}^T\mathbf{y}^0$ for some $\mathbf{x}^0$ feasible but not optimal for (P), then $\mathbf{y}^0$ is not feasible for the Dual of (P).

(h) For $m = 2$ and $n = 4$ in (P) if $x_1 = 1$ and $x_2 = 3$ are the basic variables of a BFS of (P), then there exists a feasible solution of (P) with $x_1 > 0, x_2 > 0, x_3 > 0$.

(i) If for some BFS $\mathbf{x}$ and in a nonbasic column $B^{-1}\tilde{\mathbf{a}_s}$ corresponding to $\mathbf{x}$, $min\{\frac{x_i}{u_{is}} : u_{is} > 0\} = \frac{x_t}{u_{ts}} = \frac{x_r}{u_{rs}} > 0$ for some $t \neq r$, then there exists a BFS of (P) which corresponds to two different basis.

4. Given a LPP with feasible region F as follows
$S = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.
It is given that the feasible region S has only the three extreme points $[0, 0]^T, [0, 3]^T$ and $[2, 5]^T$.
Convert the feasible region S (by adding variables) to the form $S' = \{[\mathbf{x}, \mathbf{s}]^\mathbf{T} \in \mathbb{R}^4 : A'[\mathbf{x}, \mathbf{s}]^\mathbf{T} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}\}$, where $A' = [A : I]$.
The following is the simplex table for the basic feasible solution corresponding to the basis $B = [\mathbf{a}_1, \mathbf{a}_2]$.

	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}_1}$			-2		
$\tilde{\mathbf{a}_2}$			-1		

(a) How many constraints (other than the non negativity constraints) does S have?

(b) Obtain a feasible region S of the above problem.

(c) If possible give all the missing entries of the above table.

(d) If possible give a direction $[\mathbf{d}, \mathbf{s}]^T = [d_1, d_2, s_1, s_2]^T$ of S' such that the corresponding direction $\mathbf{d} = [d_1, d_2]^T$ of the feasible region S satisfies $\mathbf{a}_1^T\mathbf{d} < 0$ while $\mathbf{a}_2^T\mathbf{d} = 0$.

(e) If possible give a direction $[\mathbf{d}', \mathbf{s}]^T = [d_1', d_2', s_1, s_2]^T$ of S' such that the corresponding direction $\mathbf{d}' = [d_1', d_2']^T$ of the feasible region S satisfies $\mathbf{a}_1^T\mathbf{d}' < 0$ and $\mathbf{a}_2^T\mathbf{d}' < 0$.

(f) Let $\mathbf{u}$ and $\mathbf{v}$ be the extreme points of S' corresponding to the extreme points $[0, 0]^T$ and $[0, 3]^T$, respectively. Give a $[\mathbf{d}, \mathbf{s}]^T$ such that $\mathbf{u} + \alpha[\mathbf{d}, \mathbf{s}]^T = \mathbf{v}$. Give the value of α.

(g) Similarly let $\mathbf{v}$ and $\mathbf{w}$ be the extreme points of S' corresponding to the extreme points $[0,3]^T$ and $[2,5]^T$, respectively. If $[\mathbf{d},\mathbf{s}]^T$ is such that $\mathbf{v}+\alpha[\mathbf{d},\mathbf{s}]^T=\mathbf{w}$, $\alpha>0$ then can $\mathbf{d}$ be a nonnegative vector?

(h) Which of the following can possibly constitute a basis matrix for the above problem:
$B=[\tilde{\mathbf{a}_2},\mathbf{e}_1], B=[\tilde{\mathbf{a}_1},\mathbf{e}_1], B=[\mathbf{e}_1,\mathbf{e}_2]$ and $B=[\tilde{\mathbf{a}_2},\mathbf{e}_2]$, where $\mathbf{e}_i$ denotes the i th column of the identity matrix.

(i) Find a $\mathbf{c}$ such that if the objective function is of the form:
Minimize $\mathbf{c}^T\mathbf{x}$
then the problem does not have an optimal solution.

5. Consider a LPP for which we get the following optimal table.

$c_j - z_j$	0	0	1	0	0	
	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\tilde{\mathbf{a}_3}$	$B^{-1}\tilde{\mathbf{a}_4}$	$B^{-1}\tilde{\mathbf{a}_5}$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}_1}$		-1				4
$\tilde{\mathbf{a}_4}$		-2				4
		-3				3

(a) Can you suggest two different optimal solutions for this problem?

(b) If you change the entry -1 in the above table to +1, can you find two optimal BFS, by entering a new variable in the basis?

(c) If there are two optimal BFS's then is it true that there are infinitely many optimal solutions of a LPP? What about the converse?

6. Consider a LPP (P) of the form,

$$\text{Minimize } \mathbf{c}^T\mathbf{x}$$
$$\text{subject to } A_{4\times 2}\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}.$$

Let the simplex table of a BFS $[\mathbf{x}_0^T,\mathbf{s}_0^T]^T$ of the corresponding problem with equality constraints (P') be given by:

$c_j - z_j$			$\frac{5}{2}$		$\frac{3}{2}$		
	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{s}_3$	$B^{-1}\mathbf{s}_4$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}_1}$			-1				5
$\tilde{\mathbf{a}_2}$			2				3
$\mathbf{s}_2$			-3				0
$\mathbf{s}_4$			0				1

(a) Let $[\mathbf{d}^T,\mathbf{u}^T]^T = \begin{bmatrix} -B^{-1}\mathbf{s}_1 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$, where $\mathbf{d}\in\mathbb{R}^2$ (corresponds to the variables x_1,x_2) and $\mathbf{u}\in\mathbb{R}^4$.

Give a hyperplane H_{i_1} (just indicate which one) with normal $\mathbf{a}'_{i_1}$ such that, ${\mathbf{a}'_{i_1}}^T(\mathbf{x}_0 + \alpha\mathbf{d}) < b'_{i_1}$, ${\mathbf{a}'_{i_1}}^T(\mathbf{x}_0 - \alpha\mathbf{d}) < b'_{i_1}$ but ${\mathbf{a}'_{i_1}}^T(\mathbf{x}_0 + \beta\mathbf{d}) > b'_{i_1}$ for some $\alpha, \beta > 0$.

Given the above information is it possible to give another hyperplane satisfying similar conditions?

(b) If $\mathbf{d}$ is as defined in part(a) then if possible give a hyperplane H_{i_2} (just indicate which one) with normal $\mathbf{a}'_{i_2}$ such that,
${\mathbf{a}'_{i_2}}^T(\mathbf{x}_0 + \alpha\mathbf{d}) < b'_{i_2}$, ${\mathbf{a}'_{i_2}}^T(\mathbf{x}_0 - \alpha\mathbf{d}) < b'_{i_2}$ for all $\alpha > 0$.

(c) If $\mathbf{d}$ is as defined in part(a) then if possible give a hyperplane H_{i_3} (just indicate which one) with normal $\mathbf{a}'_{i_3}$ such that,
${\mathbf{a}'_{i_3}}^T(\mathbf{x}_0 + \alpha\mathbf{d}) < b'_{i_3}$, for all $\alpha > 0$ and ${\mathbf{a}'_{i_3}}^T(\mathbf{x}_0 - \beta\mathbf{d}) < b'_{i_3}$ for some $\beta > 0$.

(d) The normals to how many hyperplanes (defining $Fea(P')$) is $[\mathbf{d}^T, \mathbf{u}^T]^T$ (defined in part(a)) orthogonal to ? In that case indicate also the corresponding hyperplanes. $\mathbf{d}$ is orthogonal to the normals of how many hyperplanes defining Fea(P)?

(e) How many optimal solutions does the problem (P) have? Justify.

(f) How many optimal solution/s does the dual of (P) have? Justify.

(g) Give the basic and nonbasic variables for the optimal solution/s of the dual of this problem.

(h) Can you guess the sign of the entries in the row corresponding to the basic variable y_1 in the corresponding dual table? Justify.

(i) Can you guess the exact entries in the row corresponding to the basic variable y_1 in the corresponding dual table? Justify.

(j) Given the above information is it possible to give a feasible solution of the dual of (P) with $y_1 > 0, y_2 > 0, y_3 > 0$?