Plan

- Directions of Fea(LPP)
- Extreme directions of Fea(LPP)
- Representation Theorem for Fea(LPP)
- Necessary and sufficient conditions for the existence of optimal solutions
- Optimal solutions in atleast one corner point

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- If $\mathbf{d}$ is a direction of a convex set S, then for all $\gamma > 0$, $\mathbf{x} + \alpha\mathbf{d} = \mathbf{x} + (\frac{\alpha}{\gamma})\gamma\mathbf{d} \in S$ for all $\alpha > 0$,

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- Directions $\mathbf{d}_1, \mathbf{d}_2$ of S are said to be distinct if $\mathbf{d}_1 \neq \gamma \mathbf{d}_2$ for any $\gamma > 0$
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  are all equal as directions of $Fea(LPP)$.
- Whereas $\mathbf{d}_1 = [1, 1]^T$, $\mathbf{d}_2 = [1, 0]^T$ give two distinct directions.

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- The set of all directions of $S = Fea(LPP)$ is a convex set.

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  $D' = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \geq \mathbf{0}, A\mathbf{d} \leq \mathbf{0}, \sum_i d_i = 1\}$ is a set of all distinct directions of S.

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- Also each $\mathbf{d} \in D$ is of the form $\mathbf{d} = \alpha\mathbf{d}'$ for some $\mathbf{d}' \in D'$ where $\alpha = \sum_i d_i (> 0)$.
- $D' =$

  $$\left\{ \mathbf{d} \in \mathbb{R}^n : \mathbf{d} \geq \mathbf{0}, \begin{bmatrix} & A & \\ 1 & 1, .., & 1 \\ -1 & -1, .., & -1 \end{bmatrix} \mathbf{d} \leq \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix} \right\}.$$

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  The set D' now looks exactly like the feasible region of a LPP.

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- If $Fea(LPP) = S \neq \phi$ is unbounded then $D \neq \phi$ and S must have atleast one extreme direction.
- The number of distinct extreme directions of S is finite (why?).

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- Since $\mathbf{d} \in \mathbb{R}^n$, $\mathbf{d} \neq \mathbf{0}$ cannot be orthogonal to n **LI** vectors, so $\mathbf{d}$ cannot lie on n LI hyperplanes of the $(m+n)$ hyperplanes given by,

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  $\{\mathbf{d} \in \mathbb{R}^n : -\mathbf{e}_j^T\mathbf{d} = \mathbf{0}\}$ for $j = 1, 2, \ldots, n$.
- If $\mathbf{d} \in D'$, is an extreme direction of S then it should lie on $(n-1)$ LI hyperplanes of the above mentioned $(m+n)$ hyperplanes, and the hyperplane

  $\{\mathbf{d} \in \mathbb{R}^n : [1, 1, \ldots, 1]\mathbf{d} = 1\}$,

  which gives a collection of n LI hyperplanes.

- **Exercise:** Check that if a $\mathbf{d} \in D$ lies on $(n-1)$ LI hyperplanes (out of the $(m+n)$ defining hyperplanes of D) given by $\{H_1, \ldots, H_{n-1}\}$, then $\{H, H_1, \ldots, H_{n-1}\}$ is LI where $H = \{\mathbf{d} \in \mathbb{R}^n : [1, 1, \ldots, 1]\mathbf{d} = 1\}$.

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- These are the only extreme directions of $S = Fea(LPP)$.

- **Theorem:**
  If $S = Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is nonempty, then S has atleast one extreme point.

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- **Remark:** Note that the above result is not necessarily true for all polyhedral sets.
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- The theorem works for $Fea(LPP)$ because of the non negativity constraints, that is because $Fea(LPP)$ is given a supply of n LI hyperplanes, defining hyperplanes.

- **Exercise:** Can you find a nonempty polyhedral set S, $S \subset \mathbb{R}^2$ which has two defining hyperplanes but does not have any extreme point?

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- **Definition:** Given S, a nonempty subset of $\mathbb{R}^n$, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in S$, $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i$, is called a convex combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, where $0 \leq \lambda_i \leq 1$ for all $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$.

- All possible convex combinations of two distinct points gives a straight line segment with those two points as boundary points.

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- **Result:** Given $\phi \neq S \subset \mathbb{R}^n$, S is a convex set if and only if for all $k \in \mathbb{N}$,
  the convex combination of any k elements of S is again an element of S.

- **Theorem: (Representation Theorem)**
  If $S = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is nonempty and if $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ are the extreme points of S and $\mathbf{d}_1, \mathbf{d}_2, .., \mathbf{d}_r$ are the distinct extreme directions of S (the set of directions is empty if S is bounded) then $\mathbf{x} \in S$ if and only if

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  and $\mu_j \geq 0$, for all $j = 1, 2, \ldots, r$.
- That is, $\mathbf{x} \in S \Leftrightarrow \mathbf{x}$ can be written as a convex combination of the extreme points of S plus a non negative linear combination of the extreme directions of S.

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- So any direction $\mathbf{d} \in D$ can be written as a nonnegative linear combination of the extreme directions of S.

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- **Exercise:** If $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all extreme directions $\mathbf{d}_j$ of the nonempty and unbounded feasible region S of a LPP, then does it imply that $\mathbf{c}^T\mathbf{d} \geq 0$ for all directions $\mathbf{d} \in D$, of the feasible region S?

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  Ans is **yes.**

- **Observation 9:** From the representation theorem of S we can see that if $S \neq \phi$ and $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all $j = 1, 2, .., r$, then LPP(*) has an optimal solution, and atleast one optimal solution is attained at an extreme point of S.

- **Observation 10:** From the representation theorem of S we can also see that if $S = \mathit{Fea}(\mathit{LPP})$ is nonempty and bounded then the LPP(*) has an optimal solution and the optimal value is attained in atleast one extreme point.

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  From the above observations we can conclude the following:

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- **Conclusion 1:** If $S = Fea(LPP) \neq \phi$, then the LPP (*) has an optimal solution if and only if one of the following is true:

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  From the above observations we can conclude the following:

- **Conclusion 1:** If $S = Fea(LPP) \neq \phi$, then the LPP (*) has an optimal solution if and only if one of the following is true:

- **(i)** $S = Fea(LPP)$ is bounded (also seen before by using extreme value theorem)
  (ii) $S = Fea(LPP)$ is unbounded and $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all extreme directions $\mathbf{d}_j$ of the feasible region S.

- **Observation 10:** From the representation theorem of S we can also see that if $S = Fea(LPP)$ is nonempty and bounded then the LPP(*) has an optimal solution and the optimal value is attained in atleast one extreme point.

  From the above observations we can conclude the following:

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- **Exercise:** Give an example of a **nonlinear** function $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}$ is a closed and bounded polyhedral subset of $\mathbb{R}$, (what are these sets?) such that f has a minimum value in S but the minimum value is not attained at any extreme point of S.

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- **Conclusion 3:** If $S = Fea(LPP)$ is nonempty, and there exists an $M \in \mathbb{R}$ such that for all $\mathbf{x} \in S$, $\mathbf{c}^T\mathbf{x} \geq M$,

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