Plan

- Dual of a Primal
- Fundamental theorem of Duality
- The complementary slackness Theorem

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- The LPP(*) is called a **Primal** problem.

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- **Theorem 3:** The Dual of the Dual (D) (of the Primal (P)) is the Primal (P).

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- Hence $\mathbf{x} = [2, 0]^T,$ is optimal for the **Primal (P)** and $\mathbf{y} = [\frac{5}{3}, 0]^T$ is optimal for the **Dual (D)**.

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- Clearly (D) **does not** have any feasible solution, since the first constraint cannot be satisfied by any non negative u and v.

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- Corollary 4 implies if the Primal does **not** have a feasible solution then the dual does not have an optimal solution.

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- If $\mathbf{x} \in Fea(P)$ and $\mathbf{y} \in Fea(D)$ then $\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$, hence system (1) has a solution $\Leftrightarrow$ system $(1)''$ has a solution:
-

$$A_{m \times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0} \qquad (1)''$$

$$A^T\mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0}$$

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{b}^T\mathbf{y}$$

- System $(1)''$ can be written as:

$$\begin{bmatrix} A_{m\times n} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n\times n} & -A^T_{n\times m} \\ -\mathbf{c}^T_{1\times n} & \mathbf{b}^T_{1\times m} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{c} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}_{n\times 1} \\ \mathbf{0}_{m\times 1} \end{bmatrix}$$

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- By **corollary 4**, exactly one of the two systems, $(1'')$ (given above) and $(2)''$ (given below) has a solution.

$$\begin{bmatrix} \mathbf{z}^T_{1\times m} & \mathbf{w}^T_{1\times n} & a_{1\times 1} \end{bmatrix} \begin{bmatrix} A_{m\times n} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{n\times n} & -A^T_{n\times m} \\ -\mathbf{c}^T_{1\times n} & \mathbf{b}^T_{1\times m} \end{bmatrix} \geq \begin{bmatrix} \mathbf{0}_{1\times n} & \mathbf{0}_{1\times m} \end{bmatrix},$$

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  $x_j = 0$ whenever $(A^T\mathbf{y})_j > c_j$, $j = 1, 2, .., n$ **(1)**
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- Check that $[\frac{1}{2}, 3, 0]^T \in Opt(P)$ and $[1, 1]^T \in Opt(D)$.