

**Notations:**

$\mathbf{x}, \mathbf{d}, \mathbf{b}$ , etc, that is characters in boldface represent (column) vectors.  
 $\tilde{\mathbf{a}}_i$ , the  $i$ th column of  $A$ .

**Simplex Algorithm**

From now on we will consider linear programming problems of the type (P\*):

Max or Min  $\mathbf{c}^T \mathbf{x}$

subject to  $A_{m \times n} \mathbf{x} = \mathbf{b}_{m \times 1}, \mathbf{x} \geq \mathbf{0}$ , (\*)

where  $\text{rank}(A) = m$ .

Let us now denote the set  $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$  by  $\text{Fea}(LPP)$ .

Note that if we are given a problem of the type:

Max or Min  $\mathbf{c}^T \mathbf{x}$

subject to  $A_{m \times n} \mathbf{x} \leq \mathbf{b}_{m \times 1}, \mathbf{x} \geq \mathbf{0}$ ,

we add some variables and we can convert the system of constraints to

$$A_{m \times n} \mathbf{x} + \mathbf{s}_{m \times 1} = [A : I] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \mathbf{b}_{m \times 1}, \mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}.$$

So it is now of the form of problem (P\*).

If suppose we are given a problem of the type (P\*\*):

Max or Min  $\mathbf{c}^T \mathbf{x}$

subject to  $A_{k \times n} \mathbf{x} = \mathbf{b}_{k \times 1}, \mathbf{x} \geq \mathbf{0}$ ,

where  $k > \text{rank}(A) = m$ .

Let (P\*\*) have atleast one feasible solution and WLOG let  $\mathbf{a}_1^T, \dots, \mathbf{a}_m^T$  be a set of  $m$  LI rows of  $A$  ( $\text{rank}(A) = m$ ).

Let  $\mathbf{x}_0 \in \{\mathbf{x} \in \mathbb{R}^n : A_{k \times n} \mathbf{x} = \mathbf{b}_{k \times 1}, \mathbf{x} \geq \mathbf{0}\}$ .

Then for all  $i = 1, \dots, k$ ,

$$\mathbf{a}_i^T = \sum_{j=1}^m u_{ji} \mathbf{a}_j^T \text{ for some real } u_{ji}\text{'s, } i = 1, \dots, k, j = 1, \dots, m, \quad (**)$$

$$\text{hence } \mathbf{a}_i^T \mathbf{x}_0 = b_i = \sum_{j=1}^m u_{ji} \mathbf{a}_j^T \mathbf{x}_0 = \sum_{j=1}^m u_{ji} b_j, \text{ which implies } b_i = \sum_{j=1}^m u_{ji} b_j, \text{ for all } i = 1, \dots, k. \quad (***)$$

Hence from (\*\*) and (\*\*\*) it follows that for any  $\mathbf{x}$

$$\mathbf{a}_i^T \mathbf{x} = b_i, \text{ for } i = 1, \dots, m \iff \mathbf{a}_i^T \mathbf{x} = \sum_{j=1}^m u_{ji} \mathbf{a}_j^T \mathbf{x} = \sum_{j=1}^m u_{ji} b_j = b_i, \text{ for all } i = 1, \dots, k.$$

Hence if the system  $A_{k \times n} \mathbf{x} = \mathbf{b}_{k \times 1}, \mathbf{x} \geq \mathbf{0}$  is consistent (that is it has at least one solution), then we can throw away  $k - m$  rows of  $A$  and the corresponding components of  $\mathbf{b}$  in (P\*\*) to get an equivalent (having the same set of solutions) system of equations of the form (\*).

Hence an LPP of the form (P\*\*) can again be converted to a problem of the type (P\*), by throwing away some of the constraints, such that the feasible region of the changed LPP remains same.

**Note:** Here it is important to point out that if we have a problem (P) of the form:

Max or Min  $\mathbf{c}^T \mathbf{x}$

subject to  $A_{k \times n} \mathbf{x} \leq \mathbf{b}_{k \times 1}, \mathbf{x} \geq \mathbf{0}$ ,

where  $k > \text{rank}(A) = m$ .

Then if we throw away constraints (corresponding to LD rows of  $A$ ) from the above set of constraints, then we may get a different feasible region.

An  $\mathbf{x} \in \text{Fea}(LPP)$  is called a **basic feasible solution (BFS)** of the LPP if the columns of the matrix  $A$  corresponding to the nonzero components of  $\mathbf{x}$  are LI.

An  $\mathbf{x}$  satisfying the system  $A\mathbf{x} = \mathbf{b}$ , and the condition that the columns corresponding to the nonzero components are LI, is called a **basic solution** of the LPP. So a basic solution may not be a nonnegative vector, hence need not be a feasible solution of the LPP.

So a basic feasible solution of a LPP of the form (1), can have at most  $m$  strictly positive components (since  $\text{rank}(A) = m$ ).

**Example 1:** Consider the LPP (P)

Max or Min  $\mathbf{c}^T \mathbf{x}$

subject to  $A_{2 \times 3} \mathbf{x} = \mathbf{b}_{2 \times 1}, \mathbf{x} \geq \mathbf{0}$

where  $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Clearly  $\text{rank}(A) = 2$ . Then  $\mathbf{x} = [1, 1, 0]^T$  is a **BFS** of (P) since it satisfies the conditions  $A_{m \times n} \mathbf{x} = \mathbf{b}_{m \times 1}$ ,  $\mathbf{x} \geq \mathbf{0}$  and the columns of  $A$  corresponding to the positive components of  $\mathbf{x}$  are  $[1, 1]^T$ ,  $[2, 1]^T$ , is LI.

The basis matrix has columns  $[1, 1]^T, [2, 1]^T$ , with basic variables  $x_1, x_2$ .

$\mathbf{x} = [2, 1, 1]^T$  is a feasible solution of (P) since it satisfies the conditions  $A_{m \times n} \mathbf{x} = \mathbf{b}_{m \times 1}$ ,  $\mathbf{x} \geq \mathbf{0}$  but the columns of  $A$  corresponding to the positive components of  $\mathbf{x}$  are  $[1, 1]^T$ ,  $[2, 1]^T, [-1, -1]^T$  is LD, hence is not a BFS.

$\mathbf{x} = [0, 1, -1]^T$  is a basic solution of (P) since it satisfies the conditions  $A_{m \times n} \mathbf{x} = \mathbf{b}_{m \times 1}$  and the columns of  $A$  corresponding to the nonzero components of  $\mathbf{x}$  are  $[2, 1]^T, [-1, -1]^T$ , is LI.

But is not a feasible solution of (P).

A BFS is called a **non degenerate** BFS if it has exactly  $m$  positive components, otherwise it is said to be a **degenerate** BFS.

If  $\mathbf{x}$  is a **non degenerate** BFS, then the columns of  $A$  corresponding the nonzero (positive) components of  $\mathbf{x}$  form a basis of  $\mathbb{R}^m$ . WLOG, let the positive components of  $\mathbf{x}$  be  $x_1, \dots, x_m$ , then  $\tilde{\mathbf{a}}_i, i = 1, \dots, m$ , forms a basis of  $\mathbb{R}^m$ , where  $\tilde{\mathbf{a}}_i$  is the  $i$ th column of  $A$ .

The  $m \times m$  matrix,  $B_{m \times m} = [\tilde{\mathbf{a}}_1 \dots \tilde{\mathbf{a}}_m]$  formed with columns  $\tilde{\mathbf{a}}_i, i = 1, \dots, m$ , of  $A$  is called the **basis matrix** corresponding to  $\mathbf{x}$ .

The variables  $x_1, \dots, x_m$  are called the **basic variables**, and  $x_{m+1} = x_{m+2} = \dots = x_n = 0$ , are called the **non basic variables** of the BFS  $\mathbf{x}$ .

If  $\mathbf{x}$  is a **degenerate** BFS, then consider the columns of  $A$  corresponding to the nonzero components of  $\mathbf{x}$ . WLOG let them be  $\tilde{\mathbf{a}}_1, \dots, \tilde{\mathbf{a}}_k, k < m$ .

Then add  $(m - k)$  LI columns of  $A$ , such that these  $(m - k)$  columns of  $A$  together with  $\tilde{\mathbf{a}}_1, \dots, \tilde{\mathbf{a}}_k$ , form a set of  $m$  LI vectors from the columns of  $A$  (you can always do that), hence a basis of  $\mathbb{R}^m$ .

Let as before the matrix  $B_{m \times m}$  formed with these  $m$  columns, (WLOG let it be  $\tilde{\mathbf{a}}_i, i = 1, \dots, m$ ) be called a basis matrix corresponding to  $\mathbf{x}$ .

The components  $x_1, \dots, x_m$  of  $\mathbf{x}$  are called as before, **basic variables** corresponding to  $\mathbf{x}$  and the basis matrix  $B$ . The components  $x_{m+1} = \dots = x_n = 0$  are the **nonbasic variables**.

Note that depending on the choice of the rest of the  $(m - k)$  columns which are added, the same  $\mathbf{x}$  will correspond to different basis matrices  $B$ , and hence will have different basic and nonbasic variables.

**Example 2:** Consider the LPP (P)

Max or Min  $\mathbf{c}^T \mathbf{x}$

subject to  $A_{2 \times 3} \mathbf{x} = \mathbf{b}_{2 \times 1}, \mathbf{x} \geq \mathbf{0}$

where  $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

Clearly  $\text{rank}(A) = 2$ .

$\mathbf{x} = [0, 1, 1]^T$  is a **non degenerate BFS** of (P) since it satisfies the conditions  $A_{m \times n} \mathbf{x} = \mathbf{b}_{m \times 1}$ ,  $\mathbf{x} \geq \mathbf{0}$  and the columns of  $A$  corresponding to the positive components of  $\mathbf{x}$  are  $[2, 1]^T, [-1, 0]^T$ , is LI.

Also the number of positive components is equal to  $2 = \text{rank}(A)$ .

The basis matrix is unique upto a permutation of the columns.

The basic variables are given by  $x_2, x_3$ .

$\mathbf{x} = [1, 0, 0]^T$  is also a BFS of (P) since it satisfies the conditions

$A_{m \times n} \mathbf{x} = \mathbf{b}_{m \times 1}$ ,  $\mathbf{x} \geq \mathbf{0}$  but the columns of  $A$  corresponding to the positive components of  $\mathbf{x}$  is  $[1, 1]^T$ , is LI. But  $\mathbf{x}$  is a **degenerate** BFS since the number of positive components is strictly less 2.

There are two different basis matrices  $B$  for the same BFS  $\mathbf{x}$ .

- (i) One with columns  $[1, 1]^T, [2, 1]^T$ , with basic variables  $x_1, x_2$ .
- (ii) The other with columns  $[1, 1]^T, [-1, 0]^T$ , with basic variables  $x_1, x_3$ .

Suppose if  $\mathbf{x}$  is a BFS such that  $x_{m+1} = \dots = x_n = 0$ , then since  $\mathbf{x}$  is a feasible solution of the LPP,  $A\mathbf{x} = \mathbf{b}$ ,

hence we get  $\sum_{i=1}^m \tilde{\mathbf{a}}_i x_i = \mathbf{b}$ , which implies  $B\mathbf{x}_B = \mathbf{b}$  or  $\mathbf{x}_B = B^{-1}\mathbf{b}$ , where  $\mathbf{x}_B$  are the components of  $\mathbf{x}$  corresponding to the basic variables.

Hence a BFS  $\mathbf{x}$  is of the form,  $\mathbf{x} = \begin{bmatrix} B^{-1}\mathbf{b}_{m \times 1} \\ \mathbf{0}_{(n-m) \times 1} \end{bmatrix}$ .

Hence in the context of the diet problem, that is if we consider the above problem to be the diet problem (with  $\geq$  inequalities changed to equalities in the original problem), then  $\mathbf{x}$  gives the quantities of the various food products  $F_j$ ,  $j = 1, \dots, n$  which are to be included in the diet. The food products  $F_1, \dots, F_m$ , which correspond to the basic variables  $x_1, x_2, \dots, x_m$  of  $\mathbf{x}$ , are the ones which are included in the diet in the quantities  $x_j$ , the other food products corresponding to the nonbasic variables (they have zero values) are not to be consumed, hence not included in the diet.

Let us assume that (P\*) has atleast one feasible solution and let  $\mathbf{x}$  be a BFS (degenerate or non degenerate) of the LPP (P\*) (the existence of a BFS will be justified later).

WLOG let  $B = [\tilde{\mathbf{a}}_1 \dots \tilde{\mathbf{a}}_m]$  be a (or the) basis matrix corresponding to  $\mathbf{x}$ .

Then note that for all  $k = 1, 2, \dots, n$ ,

$$\tilde{\mathbf{a}}_k = \sum_{i=1}^m u_{ik} \tilde{\mathbf{a}}_i = [\tilde{\mathbf{a}}_1 \dots \tilde{\mathbf{a}}_m] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} \quad (*)$$

for some real  $u_{ik}$ 's,  $i = 1, \dots, m$ ,  $k = 1, \dots, n$ ,

which implies  $\begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = B^{-1} \tilde{\mathbf{a}}_k$  for all  $k = 1, 2, \dots, n$ .

In the context of the diet problem the above equations (\*), mean that in order to obtain the same amount of nutrient as unit amount of  $F_k$ ,  $k = 1, \dots, n$ , one needs to consume  $(u_{1k})$  amount of  $F_1 + (u_{2k})$  amount of  $F_2 + \dots + (u_{mk})$  amount of  $F_m$ .

Hence the value of unit amount of  $F_k$ , if we include only  $F_i, i = 1, \dots, m$  in the diet (which has to be bought from the market) comes out to be

$u_{1k}c_1 + u_{2k}c_2 + \dots + u_{mk}c_m$ , which we denote by  $z_k$ .

$$\text{So } z_k = \sum_{i=1}^m u_{ik}c_i = [c_1, \dots, c_m] \begin{bmatrix} u_{1k} \\ \vdots \\ u_{mk} \end{bmatrix} = \mathbf{c}_B^T B^{-1} \tilde{\mathbf{a}}_k.$$

Note that the cost of the objective function corresponding to this BFS is given by

$$\mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T B^{-1} \mathbf{b},$$

where,  $\mathbf{c}_B^T$  are the components of the vector  $\mathbf{c}^T$  which correspond to the basic variables.

Also for all  $k = 1, \dots, m$ , note that  $z_k = c_k$ .

Now the simplex table corresponding to the BFS  $\mathbf{x}$  will be given by

	$c_1 - z_1 = 0$	$c_2 - z_2 = 0$	...	$c_m - z_m = 0$	...	$c_s - z_s$	...	$c_k - z_k$	...	$c_n - z_n$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	...	$B^{-1}\tilde{\mathbf{a}}_m$	...	$B^{-1}\tilde{\mathbf{a}}_s$	...	$B^{-1}\tilde{\mathbf{a}}_k$	...	$B^{-1}\tilde{\mathbf{a}}_n$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$	1	0	...	0	...	$u_{1s}$	...	$u_{1k}$	...	$u_{1n}$	$x_1$
$\tilde{\mathbf{a}}_2$	0	1	...	0	...	$u_{2s}$	...	$u_{2k}$	...	$u_{2n}$	$x_2$
$\vdots$	0	0	...	0	...	$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_r$	$\vdots$	$\vdots$	...	$\vdots$	...	$u_{rs}$	...	$u_{rk}$	...	$u_{rn}$	$x_r$
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	...	$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_m$	0	0	...	1	...	$u_{ms}$	...	$u_{mk}$	...	$u_{mn}$	$x_m$

**Case 1:**  $c_k - z_k < 0$  for atleast one  $k$ ,  $k = m + 1, \dots, n$ .

So if  $c_k - z_k < 0$  for some  $k = m + 1, \dots, n$ , then the market price of unit amount of  $F_k$  given by  $c_k$  is less than the value of unit amount of  $F_k$  (as obtained by consuming only  $F_1, \dots, F_m$ ) given by  $z_k$ , hence it might be sensible (profitable) to include this  $F_k$  in the diet.

Let  $c_s - z_s = \min\{c_k - z_k : c_k - z_k < 0, k = m + 1, \dots, n\}$ .

**Simplex algorithm** says that, then include  $x_s$  in the diet.

If there exists,  $s, l$ , such that for both  $s, l$ ,

$c_s - z_s = c_l - z_l = \min\{c_k - z_k : c_k - z_k < 0, k = m + 1, \dots, n\}$ ,

then include any **one** of these food products in the diet.

Let  $\mathbf{x}'$  be the new solution to be obtained.

Note that (refer to (\*))  $u_{is}$  amount of  $F_i$ ,  $i = 1, \dots, m$ , is used in the constitution of unit amount of  $F_s$ , which is now included in the diet in an amount say  $x'_s$ . Hence the same corresponding amount of the  $F_i$ 's,  $i = 1, \dots, m$ , given by  $u_{is}x'_s$  need not be consumed any more.

Hence the components of  $\mathbf{x}'$  are given by

$x'_i = x_i - u_{is}x'_s$  for  $i = 1, \dots, m$ ,

$x'_s \geq 0$  and

$x'_i = 0$  for  $i = m + 1, \dots, n$ ,  $i \neq s$ .

Then note that if we want  $\mathbf{x}'$  to be a feasible solution of the LPP, then we have to choose  $x'_s \geq 0$  such that  $x'_i \geq 0$  for all  $i = 1, \dots, m$ .

Note that if  $u_{is} \leq 0$ , for some  $i = 1, \dots, m$ , then for all  $x'_s \geq 0$ ,  $x'_i \geq 0$  for that  $i$ .

So in order that  $x'_i \geq 0$ , for all  $i = 1, \dots, m$ ,

we have to choose  $x'_s \geq 0$ , such that  $x'_s \leq \frac{x_i}{u_{is}}$  for which  $u_{is} > 0$  (if at all there is one such  $i$ ).

**Case 1a:** For some (atleast one)  $i = 1, \dots, m$ ,  $u_{is} > 0$  (where  $s$  is as defined in Case 1).

Let  $\frac{x_r}{u_{rs}} = \min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$ .

(Note that there could exist  $r, t$  such that for both  $r, t \in \{m + 1, \dots, n\}$

$\frac{x_r}{u_{rs}} = \frac{x_t}{u_{ts}} = \min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$ ).

Here  $\frac{x_r}{u_{rs}}$  is called the **minimum ratio**.

Then in order that  $\mathbf{x}' \geq \mathbf{0}$ ,  $x'_s \leq \frac{x_r}{u_{rs}}$ .

Then it can be easily checked that  $A\mathbf{x}' = \mathbf{b}$  or  $\mathbf{x}' \in \text{Fea}(LPP)$ , where  $\mathbf{x}'$  is as defined in Case 1.

To see this, note that  $A\mathbf{x}' = \sum_{i=1}^m \tilde{\mathbf{a}}_i x'_i + \tilde{\mathbf{a}}_s x'_s$ , which implies

$A\mathbf{x}' = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i - \sum_{i=1}^m \tilde{\mathbf{a}}_i u_{is} x'_s + \tilde{\mathbf{a}}_s x'_s = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i + x'_s (\tilde{\mathbf{a}}_s - \sum_{i=1}^m \tilde{\mathbf{a}}_i u_{is}) = \sum_{i=1}^m \tilde{\mathbf{a}}_i x_i = \mathbf{b}$ .

Also  $\mathbf{c}^T \mathbf{x}' = \sum_{i=1}^m c_i x'_i + c_s x'_s$ , which implies

$\mathbf{c}^T \mathbf{x}' = \sum_{i=1}^m c_i x_i - \sum_{i=1}^m c_i u_{is} x'_s + c_s x'_s = \sum_{i=1}^m c_i x_i + x'_s (c_s - \sum_{i=1}^m c_i u_{is}) = \mathbf{c}^T \mathbf{x} + x'_s (c_s - z_s) \leq \mathbf{c}^T \mathbf{x}$ .

Note that  $\mathbf{c}^T \mathbf{x}' < \mathbf{c}^T \mathbf{x}$  if  $x'_s > 0$ , which is possible **only if** the **minimum ratio**  $\frac{x_r}{u_{rs}} > 0$ .

In order to reduce cost (or the value of the objective function) as much as possible,  $x'_s$  is given its maximum possible value, which is equal to  $\frac{x_r}{u_{rs}}$ .

Then note that  $x'_r = x_r - u_{rs} \frac{x_r}{u_{rs}} = 0$ .

The variable  $x_s$  is called the **entering variable** for the new basis (choose any one if there are more than one choice for the **entering variable**), and the variable  $x_r$  is called a **leaving variable**.

If there exists  $r, t$  such that  $r, t \in \{m + 1, \dots, n\}$

$\frac{x_r}{u_{rs}} = \frac{x_t}{u_{ts}} = \min\{\frac{x_i}{u_{is}} : u_{is} > 0\}$ ,

then take any **one** of  $r, t$  as the **leaving variable**.

So  $\mathbf{x}' \in \text{Fea}(LPP)$  again has at most  $m$  nonzero components.

To prove  $\mathbf{x}'$  is a BFS of the LPP we need to show that the set of columns  $\{\tilde{\mathbf{a}}_1, \dots, \mathbf{a}_{r-1}, \tilde{\mathbf{a}}_s, \mathbf{a}_{r+1}, \dots, \tilde{\mathbf{a}}_m\}$  of  $A$  is LI and hence forms a basis of  $\mathbb{R}^m$ .

If suppose not, then since the collection  $\{\tilde{\mathbf{a}}_1, \dots, \mathbf{a}_{r-1}, \mathbf{a}_{r+1}, \dots, \tilde{\mathbf{a}}_m\}$  is LI, it implies that  $\tilde{\mathbf{a}}_s$  can be written as a linear combination of the  $(m-1)$  columns,  $\{\tilde{\mathbf{a}}_1, \dots, \mathbf{a}_{r-1}, \mathbf{a}_{r+1}, \dots, \tilde{\mathbf{a}}_m\}$ , but this would imply that  $u_{rs} = 0$ , which is a contradiction.

Hence  $\mathbf{x}'$  is an improved BFS (or at least as good as) compared to the BFS  $\mathbf{x}$  with respect to the value of the objective function).

Let us denote the new basis matrix corresponding to  $\mathbf{x}'$  as  $B'$  where  $B' = [\tilde{\mathbf{a}}_1 \dots \mathbf{a}_{r-1} \tilde{\mathbf{a}}_s \mathbf{a}_{r+1} \dots \tilde{\mathbf{a}}_m]$ .

The simplex table corresponding to the BFS  $\mathbf{x}'$  will be obtained again by expressing each of the vectors  $\tilde{\mathbf{a}}_i$ ,  $i = 1, \dots, n$ , and  $\mathbf{b}$  in terms of the new set of basis vectors  $\{\tilde{\mathbf{a}}_1, \dots, \mathbf{a}_{r-1}, \mathbf{a}_{r+1}, \dots, \tilde{\mathbf{a}}_m, \tilde{\mathbf{a}}_s\}$ .

Since  $\tilde{\mathbf{a}}_s = \sum_{i=1, i \neq r}^m u_{is} \tilde{\mathbf{a}}_i + u_{rs} \tilde{\mathbf{a}}_r$ ,

$$\text{so } \tilde{\mathbf{a}}_r = \frac{\tilde{\mathbf{a}}_s}{u_{rs}} - \sum_{i=1, i \neq r}^m \frac{u_{is}}{u_{rs}} \tilde{\mathbf{a}}_i. \quad (*)$$

Also since for any  $k = 1, \dots, n$ ,

$$\tilde{\mathbf{a}}_k = \sum_{i=1}^m u_{ik} \tilde{\mathbf{a}}_i = \sum_{i=1, i \neq r}^m u_{ik} \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}}_r u_{rk}. \quad (**)$$

By substituting the expression for  $\tilde{\mathbf{a}}_r$  given in  $(*)$  in the equation  $(**)$ , we get

$$\tilde{\mathbf{a}}_k = \sum_{i=1, i \neq r}^m u_{ik} \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}}_r u_{rk} = \sum_{i=1, i \neq r}^m (u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk}) \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}}_s \frac{u_{rk}}{u_{rs}}.$$

Let us denote the new coefficients corresponding to  $\tilde{\mathbf{a}}_i$ , in the above expression as  $u'_{ik}$ , for  $i = 1, \dots, r-1, r+1, m, s$ , and  $k = 1, \dots, n$ .

Hence for all  $k = 1, \dots, n$ ,

$$u'_{ik} = u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk} \text{ for } i = 1, \dots, r-1, r+1, m$$

$$\text{and } u'_{sk} = \frac{u_{rk}}{u_{rs}}.$$

Note that the coefficients corresponding to  $\mathbf{b}$ , when  $\mathbf{b}$  is expressed as a linear combination of the new basis of  $\mathbb{R}^m$ , ( which was already obtained and the philosophy behind explained ) can be recalculated as follows:

$$\text{Recall that } \mathbf{b} = \sum_{i=1, i \neq r}^m \tilde{\mathbf{a}}_i x_i + \tilde{\mathbf{a}}_r x_r.$$

Now again substituting the expression for  $\tilde{\mathbf{a}}_i$  given in  $(*)$  in the above equation we get

$$\mathbf{b} = \sum_{i=1, i \neq r}^m \tilde{\mathbf{a}}_i x_i + x_r \left( \frac{\tilde{\mathbf{a}}_s}{u_{rs}} - \sum_{i=1, i \neq r}^m \frac{u_{is}}{u_{rs}} \tilde{\mathbf{a}}_i \right).$$

On simplifying the above expression we get

$$\mathbf{b} = \sum_{i=1, i \neq r}^m \left( x_i - \frac{u_{is}}{u_{rs}} x_r \right) \tilde{\mathbf{a}}_i + \tilde{\mathbf{a}}_s \left( \frac{x_r}{u_{rs}} \right).$$

Hence Let us denote the new values of  $z_k$  now corresponding to  $\mathbf{x}'$ , by  $z'_k$ . Then we get

$$\begin{aligned} z'_k &= \sum_{i=1, i \neq r}^m u'_{ik} c_i + u'_{sk} c_s \\ &= \sum_{i=1, i \neq r}^m \left( u_{ik} - \frac{u_{is}}{u_{rs}} u_{rk} \right) c_i + \frac{u_{rk}}{u_{rs}} c_s = \sum_{i=1, i \neq r}^m u_{ik} c_i - \sum_{i=1, i \neq r}^m \left( \frac{u_{is}}{u_{rs}} u_{rk} \right) c_i + \frac{u_{rk}}{u_{rs}} c_s \\ &= \left( \sum_{i=1}^m u_{ik} c_i - u_{rk} c_r \right) - \frac{u_{rk}}{u_{rs}} \left( \sum_{i=1}^m u_{is} c_i - u_{rs} c_r \right) + \frac{u_{rk}}{u_{rs}} c_s \\ &= z_k - u_{rk} c_r - \frac{u_{rk}}{u_{rs}} z_s - \frac{u_{rk}}{u_{rs}} (-u_{rs} c_r) + \frac{u_{rk}}{u_{rs}} c_s = z_k + \frac{(c_s - z_s)}{u_{rs}} u_{rk}. \end{aligned}$$

From this we get the new values of  $c_k - z'_k$  as

$$c_k - z'_k = (c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}.$$

Hence the simplex table corresponding to the new BFS  $\mathbf{x}'$  is given by

	$c_1 - z'_1$	$\dots$	$c_m - z'_m$	$\dots$	$c_s - z'_s$	$\dots$	$c_k - z'_k$	$\dots$	$c_n - z'_n$	
	0	$\dots$	0	$\dots$	$c_s - z_s - \frac{(c_s - z_s)}{u_{rs}} u_{rs} = 0$	$\dots$	$(c_k - z_k) - \frac{(c_s - z_s)}{u_{rs}} u_{rk}$	$\dots$	$\dots$	
	$B'^{-1} \tilde{\mathbf{a}}_1$	$\dots$	$B'^{-1} \tilde{\mathbf{a}}_m$	$\dots$	$B'^{-1} \tilde{\mathbf{a}}_s$	$\dots$	$B'^{-1} \tilde{\mathbf{a}}_k$	$\dots$	$B'^{-1} \tilde{\mathbf{a}}_n$	$B'^{-1} \mathbf{b}$
$\tilde{\mathbf{a}}_1$	1	$\dots$	0	$\dots$	$u_{1s} - \frac{u_{1s}}{u_{rs}} u_{rs} = 0$	$\dots$	$u_{1k} - \frac{u_{1s}}{u_{rs}} u_{rk}$	$\dots$	$\dots$	$x_1 - \frac{u_{1s}}{u_{rs}} x_r$
$\tilde{\mathbf{a}}_2$	0	$\dots$	0	$\dots$	$u_{2s} - \frac{u_{2s}}{u_{rs}} u_{rs} = 0$	$\dots$	$u_{2k} - \frac{u_{2s}}{u_{rs}} u_{rk}$	$\dots$	$\dots$	$x_2 - \frac{u_{2s}}{u_{rs}} x_r$
$\vdots$	0	$\dots$	0	$\dots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_r$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\frac{u_{rs}}{u_{rs}} = 1$	$\dots$	$\frac{u_{rk}}{u_{rs}}$	$\dots$	$\frac{u_{rn}}{u_{rs}}$	$\frac{x_r}{u_{rs}}$
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\dots$	$\vdots$	$\vdots$
$\tilde{\mathbf{a}}_m$	0	$\dots$	1	$\dots$	$u_{ms} - \frac{u_{ms}}{u_{rs}} u_{rs} = 0$	$\dots$	$u_{mk} - \frac{u_{ms}}{u_{rs}} u_{rk}$	$\dots$	$\dots$	$x_m - \frac{u_{ms}}{u_{rs}} x_r$

The entry  $u_{rs}$  of the previous table which is made 1 (by dividing) in this table is called the **pivot element**.

**Case 1b:** For all  $i = 1, \dots, m$ ,  $u_{is} \leq 0$  (where  $s$  is as defined in **Case 1**).

Then  $\mathbf{x}' \geq \mathbf{0}$ , for all  $x'_s \geq 0$ , hence  $\mathbf{x}' \in \text{Fea}(LPP)$  for all  $x'_s \geq 0$ .

Also since  $\mathbf{c}^T \mathbf{x}' = \mathbf{c}^T \mathbf{x} + x'_s (c_s - z_s)$ , and  $(c_s - z_s) < 0$ , so given any real number  $M$ , by taking  $x'_s$  sufficiently large we can make  $\mathbf{c}^T \mathbf{x}'$  smaller than  $M$ , hence it does not have an optimal solution.

$$\text{Also note that } \mathbf{x}' = \mathbf{x} + x'_s \begin{bmatrix} -u_{1s} \\ \vdots \\ -u_{ms} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{x} + x'_s \begin{bmatrix} -B^{-1} \tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

where the 1 in the above vector occurs at the  $s$ th position.

$$\text{Let us call this vector } \begin{bmatrix} -B^{-1} \tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ as } \mathbf{d}, \text{ then since } \mathbf{x} + x'_s \mathbf{d} \in \text{Fea}(LPP) \text{ for all } x'_s \geq 0, \text{ from definition,}$$

$\mathbf{d}$  should be a direction of  $\text{Fea}(LPP)$ .

**Result 1:** The set of all directions of  $S = \text{Fea}(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m \times n} \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \text{rank}(A) = m\}$ , is given by

$$D = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \neq \mathbf{0}, A_{m \times n} \mathbf{d} = \mathbf{0}, \mathbf{d} \geq \mathbf{0}\}.$$

**Proof:** Exercise.

**Result 2:** If for some basis matrix  $B$  and a column  $\tilde{\mathbf{a}}_s$  of  $A$ ,  $B^{-1} \tilde{\mathbf{a}}_s \leq \mathbf{0}$  then

$$\mathbf{d} = \begin{bmatrix} -B^{-1} \tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is an extreme direction of  $S = \{\mathbf{x} \in \mathbb{R}^n : A_{m \times n} \mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ , where the entry 1 in the above vector is at the  $s$ th position.

**Proof:** That it is a direction of  $S$ , follows from the definition of a set.

It can also be checked that

$$A\mathbf{d} = [\tilde{\mathbf{a}}_1 \dots \tilde{\mathbf{a}}_m | \mathbf{a}_{m+1} \dots \tilde{\mathbf{a}}_s \dots \tilde{\mathbf{a}}_n] \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [B | \mathbf{a}_{m+1} \dots \tilde{\mathbf{a}}_s \dots \tilde{\mathbf{a}}_n] \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= -\tilde{\mathbf{a}}_s + \tilde{\mathbf{a}}_s = \mathbf{0}.$$

To check that  $\mathbf{d}$  is an extreme direction, let there exist directions  $\mathbf{d}_1, \mathbf{d}_2$  of  $S$  and  $\alpha_1, \alpha_2 > 0$  such that  $\mathbf{d} = \alpha_1 \mathbf{d}_1 + \alpha_2 \mathbf{d}_2$ .

Since  $\alpha_1, \alpha_2 > 0$  and  $\mathbf{d}_1, \mathbf{d}_2 \geq \mathbf{0}$

$$\mathbf{d}_1 = \begin{bmatrix} \mathbf{c}_{m \times 1} \\ 0 \\ \vdots \\ 0 \\ u \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

for some  $\mathbf{c}_{m \times 1} \geq \mathbf{0}$  and  $u \geq 0$ , where  $u$  is at the  $s$ th position.

$$\text{But from result 1, } A\mathbf{d}_1 = [B : \mathbf{a}_{m+1} \dots \tilde{\mathbf{a}}_s \dots \tilde{\mathbf{a}}_n] \begin{bmatrix} \mathbf{c}_{m \times 1} \\ 0 \\ \vdots \\ 0 \\ u \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

$$\Rightarrow B\mathbf{c} = -u\tilde{\mathbf{a}}_s.$$

Hence  $\mathbf{c} = -uB^{-1}\tilde{\mathbf{a}}_s$  or  $\mathbf{d}_1 = u\mathbf{d}$ , and  $u > 0$ .

Similarly we get  $\mathbf{d}_2 = v\mathbf{d}$  for some  $v > 0$ .

Hence  $\mathbf{d}$  is an extreme direction.

**Alternatively** note that this  $\mathbf{d}$  lies on  $(n-1)$  LI hyperplanes (check this) of the  $m+n$  hyperplanes defining  $D$ .

It lies on  $m$  LI hyperplanes given by  $A\mathbf{d} = \mathbf{0}$  and also on the  $(n-m-1)$  LI hyperplanes given by  $d_i = 0$  for  $i = m+1, \dots, n, i \neq s$ .

It is easy to see that the  $(n-1)$  hyperplanes are LI.

(Hint: The normals corresponding to these  $(n-1)$  hyperplanes written together as a matrix with  $(n-1)$  rows look somewhat like

$$\begin{bmatrix} B & \mathbf{a}_{m+1} & \dots & \tilde{\mathbf{a}}_s & \dots & \tilde{\mathbf{a}}_n \\ \mathbf{0} & 1 & \dots & 0 & \dots & 0 \\ \mathbf{0} & 0 & \dots & 0 & \dots & 0 \\ \mathbf{0} & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & 0 & \dots & 0 & \dots & 1 \end{bmatrix}.$$

Note that the column below the vector  $\tilde{\mathbf{a}}_s$  in the above matrix is the zero column, since  $d_s \neq 0$ .)

Since the LPP does not have an optimal solution, so there should exist at least one extreme direction  $\mathbf{d}$  of the LPP such that  $\mathbf{c}^T \mathbf{d} < 0$ .

Check that if  $\mathbf{d}$  is as defined above then

$$\mathbf{c}^T \mathbf{d} = [\mathbf{c}_B^T \mid c_{m+1} \dots c_s \dots c_n] \begin{bmatrix} -B^{-1} \tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -\mathbf{c}_B^T B^{-1} \tilde{\mathbf{a}}_s + c_s = c_s - z_s < 0.$$

**Case 2:(Optimality Condition)**  $c_k - z_k \geq 0$  for all  $k = 1, \dots, n$ .  
Then  $\mathbf{x}$  is optimal for the LPP.

Among the various ways to check this, one way is produce a feasible solution of the dual of this problem say  $\mathbf{y}$  such that  $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}$ . Yet another way (which is actually same as the previous ) is to produce a feasible solution  $\mathbf{y}$  of the dual and to show that this  $\mathbf{y}$  satisfies the complementary slackness property with this BFS  $\mathbf{x}$  of the primal.

Since  $\mathbf{c}^T \mathbf{x} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T B^{-1} \mathbf{b}$ , and since this has to be equal to  $\mathbf{b}^T \mathbf{y} = \mathbf{y}^T \mathbf{b}$  for some feasible solution  $\mathbf{y}$  of the dual, so we can start by checking if  $\mathbf{y}_0^T = \mathbf{c}_B^T B^{-1}$  gives a feasible solution of the dual.

Recall that the problem (LPP) is given by,

Min  $\mathbf{c}^T \mathbf{x}$   
subject to  
 $A_{m \times n} \mathbf{x} = \mathbf{b}$ ,  
 $\mathbf{x} \geq \mathbf{0}$ .

The dual of the above problem is given by,

Max  $\mathbf{b}^T \mathbf{y}$   
subject to  
 $A_{n \times m}^T \mathbf{y} \leq \mathbf{c}$ .

**Proof :** In order to see this note that,  
if  $\mathbf{x}$  satisfies  $A\mathbf{x} = \mathbf{b}$  then

$$\mathbf{a}_i^T \mathbf{x} \geq b_i \text{ for } i = 1, \dots, m \quad (*)$$

$$\text{and } -\mathbf{a}_i^T \mathbf{x} \geq -b_i \text{ for } i = 1, \dots, m \quad (**),$$

where  $\mathbf{a}_i^T$  is the  $i$  th row of the matrix  $A$ .

The above inequalities can be written as

$$\begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}.$$

Hence the LPP (primal problem) can be written as:

Min  $\mathbf{c}^T \mathbf{x}$   
subject to  
 $\begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \geq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}, \mathbf{x} \geq \mathbf{0}$ .

Since each constraint of the primal corresponds to a variable of the dual, let  $u_i$  correspond to the  $i$ th constraint of the type  $(*)$ ,  $i = 1, \dots, n$ , and let  $w_i$  correspond to the  $i$ th constraint of the type  $(**)$ ,  $i = 1, \dots, n$ .

Then the dual of the above LPP problem is given as follows:

Given an LPP  
Max  $\mathbf{b}^T \mathbf{u} - \mathbf{b}^T \mathbf{w}$   
subject to



$$\begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = A^T \mathbf{u} - A^T \mathbf{w} \leq \mathbf{c}$$

$$\mathbf{u} \geq \mathbf{0}, \mathbf{w} \geq \mathbf{0}.$$

Hence if we let  $\mathbf{y} = \mathbf{u} - \mathbf{w}$  then the above dual reduces to  
Max  $\mathbf{b}^T \mathbf{y}$   
subject to  
 $A^T \mathbf{y} \leq \mathbf{c}$ ,  
where  $\mathbf{y}$  is unrestricted in sign.

**Case 2 (continued):**

If  $c_k - z_k \geq 0$  for all  $k = 1, \dots, n$ , then  $z_k \leq c_k$  for all  $k = 1, \dots, n$ .  
Then note that  $z_k = \mathbf{c}_B^T B^{-1} \tilde{\mathbf{a}}_k = \mathbf{y}_0^T \tilde{\mathbf{a}}_k \leq c_k$  for all  $k = 1, \dots, n$ , or  $\mathbf{y}_0$  satisfies the condition:  
 $A^T \mathbf{y}_0 \leq \mathbf{c}$ , which implies that  $\mathbf{y}_0$  is a feasible solution of the dual satisfying  $\mathbf{c}^T \mathbf{x} = \mathbf{b}^T \mathbf{y}_0$ , hence  $\mathbf{x}$  and  $\mathbf{y}_0$  are optimal solutions of the LPP (primal problem) and its dual, respectively.

Also note that if  $(A^T \mathbf{y}_0)_k = \mathbf{y}_0^T \tilde{\mathbf{a}}_k = \mathbf{c}_B^T B^{-1} \tilde{\mathbf{a}}_k = z_k < c_k$ , then  $x_k$  has to be a nonbasic variable or  $x_k = 0$ . Hence  $\mathbf{y}_0 \in \text{Fea}(D)$  satisfies the **complementary slackness property** with this  $\mathbf{x}$  as expected.  
(The **complementary slackness condition** for this LPP and its dual reduces to only one condition:  
 $x_i = 0$ , whenever  $(A^T \mathbf{y})_k < c_k$ .)

Note that the above optimality condition is sufficient but not necessary for an optimal solution.

**Remark:** If instead the LPP would have been a maximization problem as given below:

Max  $\mathbf{c}^T \mathbf{x}$   
subject to  
 $A_{m \times n} \mathbf{x} = \mathbf{b}$ ,  
 $\mathbf{x} \geq \mathbf{0}$ ,  $\text{rank}(A) = m$ ,  
then **Case 1** and **Case 2** conditions would change accordingly, as given below:

**Case 1:**  $c_k - z_k > 0$  for at least one  $k$ ,  $k = m + 1, \dots, n$ .

And the condition for the **entering variable** becomes,  
 $s$  th variable will enter the basis if  $c_s - z_s = \max\{c_k - z_k : c_k - z_k > 0, k = m + 1, \dots, n\}$ .

**Case 2: (Optimality condition (sufficient condition))**  $c_k - z_k \leq 0$  for all  $k = 1, \dots, n$ .

Note that the above optimality condition is sufficient but not necessary for an optimal solution.

**Exercise:** Give an example of a Linear Programming Problem for which an optimal table does not satisfy the sufficient condition for optimality (given above).

It remains to justify that a LPP with nonempty feasible region has at least one BFS.