Numerical Integration ⇒ The process of evaluating a definite integral from a set of tabulated values of the integrand $y=f(x)$ is called numerical integration.

Newton-Cotes Quadrature Formula ⇒

Let $y_0, y_1, y_2, \ldots, y_n$ are given at $a=x_0, x_1, x_2, \ldots, x_n=b$ (equally spaced) where $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$

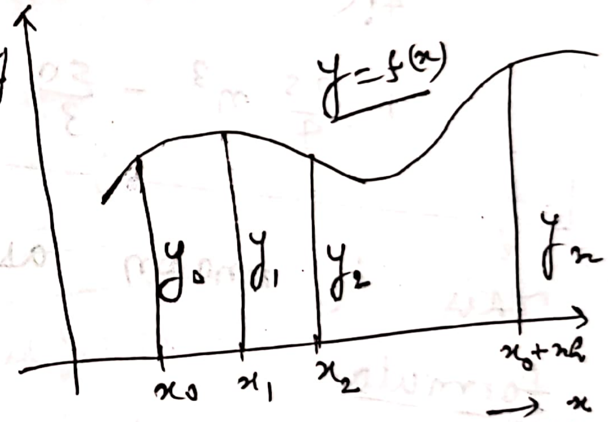
and we have to find

$$I = \int_{x_0}^{x_n} y(x)\,dx = \int_a^b y(x)\,dx$$

$$I = \int_{x_0}^{x_n} y(x)\,dx = \int_{x_0}^{x_0+nh} y(x)\,dx$$

$$= h \times \int_0^n y(x_0 + ph)\,dp$$

where $p = \dfrac{x - x_0}{h}$



$$= h \times \int_0^n E^p y(x_0)\,dp \qquad (y(x_0) = y_0)$$

$$= h \int_0^n (1+\Delta)^p y_0\,dp \qquad E = 1+\Delta$$

$$= h \int_0^n \Big[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 y_0 + \cdots \Big]\,dp \qquad \text{—— (i)}$$

Integrating (i) w.r.t. p we get

$$I = h\Big[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\Big(\frac{p^3}{3} - \frac{p^2}{2}\Big)\Delta^2 y_0 + \frac{1}{6}\Big\{\frac{p^4}{4} - p^3 + p^2\Big\}\Delta^3 y_0$$

$$+\frac{1}{24}\left\{\frac{p^5}{5}-\frac{3p^4}{2}+\frac{11p^3}{3}-3p^2\right\}\Delta^4 y_0+\frac{1}{120}\left\{\frac{p^6}{6}-2p^5+\frac{35}{4}p^4-\frac{50}{3}p^3+12p^2\right\}\Delta^5 y_0+\cdots\Big]_0^n$$

$$=h\Big[ny_0+\frac{n^2}{2}\Delta y_0+\frac{1}{2}\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\Delta^2 y_0+\frac{1}{6}\left(\frac{n^4}{4}-n^3+n^2\right)\Delta^3 y_0$$
$$+\frac{1}{24}\left(\frac{n^5}{5}-\frac{3}{2}n^4+\frac{11}{3}n^3-3n^2\right)\Delta^4 y_0+\frac{1}{120}\left(\frac{n^6}{6}-2n^5+\frac{35}{4}n^4-\frac{50}{3}n^3+12n^2\right)\Delta^5 y_0+\cdots\Big]$$

$$\boxed{I=nh\Big[y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0+\frac{1}{4!}\left(\frac{n^4}{5}-\frac{3n^3}{2}+\frac{11}{3}n^2-3n\right)\Delta^4 y_0+\left(\frac{n^5}{6}-2n^4+\frac{35}{4}n^3-\frac{50}{3}n^2+12n\right)\frac{\Delta^5 y_0}{5!}+\cdots\Big]} \quad \text{—(ii)}$$

This is known as Newton-cotes quadrature formula.

* We can deduce the following important quadrature rules by taking $n=1, 2, 3, \ldots$

① Trapezoidal rule

Putting $n=1$, in eq^n (ii) [i.e. taking curve through (x_0, y_0) & (x_1, y_1) as a straight line i.e a polynomial of first order so that differences of order higher than first becomes zero,

So, we get

$$\int_{x_0}^{x_0+h} y(x)\,dx = h\left[y_0+\frac{\Delta y_0}{2}\right]=\frac{h}{2}(y_0+y_1)$$

Similarly $\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{\Delta y_1}{2}\right) = \frac{h}{2}(y_1+y_2)$ (8)

$\vdots$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Adding these n integrals, we get

$$\cancel{\int_{x_0}^{x_0+nh} y(x)\,dx} \; \int_{x_0}^{x_0+h} y(x)\,dx + \int_{x_0+h}^{x_0+2h} y(x)\,dx$$

$$+ \int_{x_0+2h}^{x_0+3h} y(x)\,dx + \cdots + \int_{x_0+(n-1)h}^{x_0+nh} y(x)\,dx$$

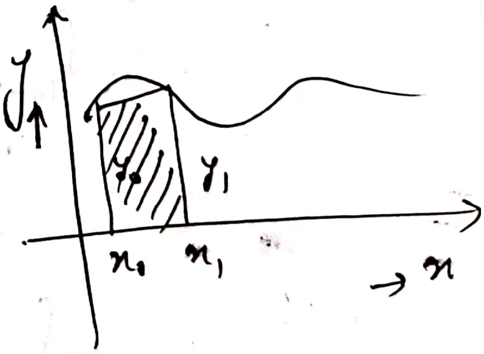
$$= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1}) + y_n\right]$$

$$\Rightarrow \boxed{\int_{x_0}^{x_n} y(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+\cdots+y_{n-1}) + y_n\right]}$$

(or)

$$\boxed{\int_a^b y(x)\,dx = \frac{h}{2}\left[y_0 + 2(y_1+y_2+\cdots+y_{n-1}) + y_n\right]} \quad \text{(iii)}$$

Trapezoidal rule



* The area of each strip (Trapezium) is found separately. Then the area under the curve and the ordinates x_0 & x_n is approximately equal to the sum of the areas of the n-trapeziums.