Name	Interval (n)	Formula	Error
Trapezoidal	1	$\frac{h}{2}[y_0+y_1]$	$-\frac{h^2}{12}y''(\theta)$
Simpson's 1/3	2	$\frac{h}{3}[y_0+4y_1+y_2]$	$-\frac{h^5}{12}y^{(iv)}(\theta)$
Simpson's 3/8	3	$\frac{3h}{8}[y_0+3y_1+3y_2+y_3]$	$-\frac{3h^5}{80}y^{(iv)}(\theta)$
Boole's Rule	4	$\frac{2h}{45}[7y_0+32y_1+12y_2+32y_3+7y_4]$	$-\frac{8h^7}{945}y^{(vi)}(\theta)$

Romberg Integration ⇒

* The accuracy of a numerical integration process can be improved in two ways

(i) By increasing the number of subintervals (i.e. by decreasing h), this decreases the magnitude of error terms. Here, the order of the method is fixed.

(ii) By using higher-order methods - this eliminates the lower-order error term's. Here the order of method is varied and therefore this method is know as variable-order approach.

* Variable order technique involves combining two estimates of a given order to obtain a third estimate of higher order. The method that incorporates this process (i.e Richardson's extrapolation) to the trapezoidal rule is called Romberg integration.

formula

Let us improve upon the value of the integral

$$I = \int_a^b y(x)\,dx$$

by the Trapezoidal rule. If I_1 & I_2 are the values of I with subintervals of width h_1 & h_2 and E_1 & E_2 are their corresponding errors, respectively then

$$E_1 = -\frac{(b-a)\,h_1^2}{12}\, y''(\theta)$$

$$E_2 = -\frac{(b-a)\,h_2^2}{12}\, y''(\bar{\theta})$$

Since $y''(\bar{\theta})$ is also the largest value of $y''(\theta)$, we can reasonably assume that $y''(\theta)$ & $y''(\bar{\theta})$ are very nearly equal.

$$\Rightarrow \quad \frac{E_1}{E_2} = \frac{h_1^2}{h_2^2} \quad \text{or} \quad \frac{E_1}{E_2 - E_1} = \frac{h_1^2}{h_2^2 - h_1^2} \quad \text{--- (1)}$$

Now, since $I = I_1 + E_1 = I_2 + E_2$

$$\Rightarrow \quad E_2 - E_1 = I_1 - I_2$$

now (1) $\Rightarrow$

$$E_1 = \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

Hence

$$I = I_1 + E_1 = I_1 + \frac{h_1^2}{h_2^2 - h_1^2}(I_1 - I_2)$$

$$= \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2} \quad \text{--- (2)}$$

which is better approximation of I.

To evaluate I systematically, we take $h_1 = h$ & $h_2 = \frac{h}{2}$

(3) $\Rightarrow$ $$I = \frac{I_1 \left(\frac{h}{2}\right)^2 - I_2 (h)^2}{\left(\frac{h}{2}\right)^2 - h^2} = \frac{4I_2 - I_1}{3}$$

i.e. $$\boxed{I(h, h/2) = \frac{1}{3}\left[4I(h/2) - I(h)\right]} \quad \text{—} (x_1)$$

Now, we use the Trapezoidal rule several times successively, halving h and applying (x_1) to each pair of values as per the following scheme-

$I(h)$

$I(h, h/2)$

$I(h, h/2, h/4)$

$I(h/2)$

$I(h, h/2, h/4, h/8)$

$I(h/2, h/4)$

$I(h/4)$

$I(h/2, h/4, h/8)$

$I(h/4, h/8)$

$I(h/8)$

The computation is continued until successive values are close to each other.

example ⇒ Evaluate $\int_0^1 \frac{dx}{1+x}$ correct to three decimal places using Romberg's method.

Hence find the value of $\log_e 2$.

Solution ⇒ Taking $h = 0.5, 0.25, 0.125$, let us evaluate the given integral by the Trapezoidal rule.

(i) when $h = 0.5$, value of $y(x) = (1+x)^{-1}$ are

x	0	0.5	1
y	1	0.6666	0.5

Therefore $I = \frac{h}{2}[y_0 + 2y_1 + y_2] = \frac{0.5}{2}[1 + 2\times .6666 + 0.5]$ (90)

(By Trapezoidal rule)

$= 0.7083$

(ii) When $h = 0.25$, the values of $y = (1+x)^{-1}$ are

x	0	0.25	0.5	0.75	1
y	1	0.8	0.6666	0.5714	0.5

$$I = \frac{0.25}{2}[1 + 2(0.8 + 0.6666 + 0.5714) + 0.5]$$

$$= 0.697$$

(iii) When $h = 0.125$, the values of $y = (1+x)^{-1}$ are

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.8889	0.80	0.7272	0.6667	0.6153	0.5714	0.5333	0.5

$$I = \frac{0.125}{2}[1 + 2(0.8889 + 0.80 + 0.7272 + 0.6667 + 0.6153 + 0.5714 + 0.5333) + 0.5]$$

$$I = 0.6941$$

using Romberg's formula, we obtain

$$I(h, h/2) = \frac{1}{3}[4I(h/2) - I(h)] = \frac{1}{3}[4\times 0.697 - 0.7083]$$

$$= 0.6932$$

$$I(h/2, h/4) = \frac{1}{3}[4I(h/4) - I(h/2)] = \frac{1}{3}[4\times 0.6941 - 0.697]$$

$$= 0.6931$$

$$I(h, h/2, h/4) = \frac{1}{3}[4I(h/2, h/4) - I(h, h/2)]$$

$$= \frac{1}{3}[4\times 0.6931 - 0.6932]$$

$$= 0.6931$$

$\boxed{\int_0^1 \frac{dx}{1+x} = \log_e 2 = 0.693 \quad \text{Ans}}$

$$\Rightarrow \int_0^1 \frac{dx}{1+x} = 0.693$$ Ans

Example ⇒ Use Romberg's method to compute $\int_0^1 \frac{dx}{1+x^2}$ correct to four decimal places.

Solution ⇒ We take $h=0.5$, 0.25, and 0.125 successively and evaluate the given integral using the Trapezoidal rule.

(i) $h=0.5$, the values of $y=(1+x^2)^{-1}$ are

x	0	0.5	1
y	1	0.8	0.5

$I = \frac{0.5}{2}[y_0 + 2y_1 + y_2] = \frac{0.5}{2}[1 + 2(0.8) + 0.5]$

(using Trapezoidal rule) $= 0.775$

(ii) When $h=0.25$, the value of $y(x) = (1+x^2)^{-1}$ are

x	0	0.25	0.5	0.75	1
y	1	0.9412	0.8	0.64	0.5

$I = \frac{0.25}{2}[1 + 2(0.9412 + 0.8 + 0.64) + 0.5]$

$= 0.7828$

(iii) When $h = 0.125$, we find that $I = 0.7848$

Thus, we have

$I(h) = 0.775$, $I(h/2) = 0.7828$, $I(h/4) = 0.7848$

Now, we have

$I(h, h/2) = \frac{1}{3}[4\, I(h/2) - I(h)] = \frac{1}{3}[4 \times 0.7828 - 0.775]$

$= \frac{1}{3}(3.1312 - 0.775) = 0.7854$

$I(h/2, h/4) = \frac{1}{3}[4\, I(h/4) - I(h/2)] = \frac{1}{3}[4 \times 0.7848 - 0.7828]$

$= 0.7855$

$I(h, h/2, h/4) = \frac{1}{3}[4\, I(h/2, h/4) - I(h, h/2)]$

$= \frac{1}{3}[3.142 - 0.7854] = 0.7855$

The table of these values are

0.7750

0.7854

0.7828

0.7855

0.7855

0.7848

Hence the value of the integral = 0.7855

Ans