Indian Institute of Technology Mandi

भारतीय प्रौद्योगिकी संस्थान मण्डी

IC-252

Theory Assignment - 4

1. Let X and Y be two independent standard normal random variables, and let $Z = 2X - Y$ and $W = -X + Y$. Find the Jacobian J.

2. Suppose X and Y are independent exponential random variables with parameter λ. Find the joint density of $V = \frac{X}{Y}$ and $W = X + Y$. Use the joint density to find the marginal distributions.

3. Let X and Y have joint density $f(x, y)$. Let (R, θ) be the polar coordinates of (X, Y).
   (a) Give a general expression for the joint density of R and θ.
   (b) Suppose X and Y are independent with $f(x) = 2x$ for $0 < x < 1$ and $f(y) = 2y$ for $0 < y < 1$. Use the result from part a to find the probability that (X, Y) lies inside the circle of radius 1 centered at the origin.

4. Let X and Y be independent random variables, each having probability density function
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$
   and let $U = X + Y$ and $V = X - Y$.
   (a) Find the joint probability density function of U and V.
   (b) Derive the marginal probability density functions of U and V.
   (c) Are U and V independent?

5. If the random variables X and Y have joint density function
$$f_{XY}(x, y) = \begin{cases} \frac{xy}{96} & 0 < x < 4, 1 < y < 5 \\ 0 & \text{otherwise,} \end{cases}$$
   find the density function of $U = X + 2Y$.

6. Let the joint pdf of random variables X and Y is given by:
$$f_{XY}(x, y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$
   Then, find the density function of $U = XY$.

7. Let the random variable X have three possible outcomes $\{a, b, c\}$. Consider two distributions on this random variable:

Symbol	$p(x)$	$q(x)$
a	1/2	1/3
b	1/4	1/3
c	1/4	1/3

   (a) Calculate entropy of $p(x)$ and $q(x)$, and their cross-entropy.
   (b) Calculate the KL-divergence $D(p||q)$ and $D(q||p)$.
   (c) Is $D(p||q) = D(q||p)$?

8. Derive the formula for the KL divergence $KL(p(x)||q(x))$ between two univariate Gaussians distributions:
$$p(x) = N(\mu_1, \sigma^2), \quad q(x) = N(\mu_2, 1).$$
   For fixed μ_2 and σ, what value of μ_1 minimizes $KL(p(x)||q(x))$? At the minimum, what is the value of $KL(p(x)||q(x))$? (Hint: Your answers should depend only on μ_2 and/or σ).

9. Use the inverse transform method to generate a sample from the distribution with probability densiy function:

$$f_X(x) = \begin{cases} \left(\frac{2}{x}\right)^3 & x > 2 \\ 0 & \text{otherwise.} \end{cases}$$

10. Devise an algorithm to simulate a random number from Truncated Normal distribution. Using this algorithm, generate a sample of 5 random numbers.