# Topology Reconstruction of a Circular Planar Resistor Network

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***Abstract*—The reconstruction problem is an inverse problem and, in general, has no unique solution. We consider the problem of reconstructing all possible topologies with their edge resistance values of an unknown circular planar passive resistive network, whose response matrix is known apriori. The response matrix is used to deduce a set of all possible connections (such as $1$-connection, $2$-connection,. . ., $k$-connection [6]) in the unknown circular planar network. Since number of such connections are large, a reduced set of connections is derived. For each connection in reduced connection set, we generate all possible path permutations along with there graph representations. The graph representations of all path permutations are used to generate several candidate planar graphs using union and edge deletion operations. A method is proposed wherein, the candidate planar graphs are posed as a set of non-linear multivariate polynomials. We then use the Gröbner basis to simultaneously reconstruct all possible topologies and the edge resistance values of an electrical network enclosed inside a black box. Numerical simulation establishes the effectiveness of the proposed strategy.**

## I. INTRODUCTION

The network reconstruction problem is ill-posed and, in general, has multiple non-unique solutions. This problem involves identification of an electrical network structure and the underlying edge resistor values of an unknown circular planar electrical network contained in the black box, using the available measurements. This area of research has witnessed significant interest among researchers due to its widespread application in areas such as system biology [1], geology [2], medical imaging [3], phylogenetics [9] and reconfiguration of VLSI arrays [5]. Two principal objectives in electrical network topology reconstruction are $i$) to determine the topology of a circular planar electrical network enclosed inside a black box, using the boundary voltage and current measurements, and $ii$) to estimate the edge conductance values of an electrical network [6]. In this paper, we propose a novel strategy to reconstruct all possible electrical network topologies and edge resistance values simultaneously using graph-based methods and the *Gröbner basis*.

Consider a black box with $n$ exposed boundary terminals enclosing a circular planar passive resistive electrical network $\Gamma$. The $n$ exposed boundary terminal is labeled $[n] = \{1, 2, ..., n\}$, and are connected through $d$ interior nodes (placed inside the black box), labelled $[d] = \{n + 1, n + 2, ..., n + d\}$. The connections between the nodes form the edges, the boundary terminal and the interior nodes are the vertices of a graph $G$. The boundary terminals are used to conduct experiments and collect boundary data. Unlike the inverse problem, the forward problem assumes that network topology and the edge $(\sigma)$ conductances, $\gamma(\sigma)$, are known. A response matrix $\Lambda$ is defined on an electrical network that maps the applied boundary voltages $v_b$ to response boundary currents $\phi_b$. A $k$-connection in $\Gamma$ is a set of $k$-disjoint paths connecting two disjoint sets of $k$-boundary nodes arranged in a particular fashion.

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**Related work.** Electrical network reconstruction is a challenging problem and has invoked much interest among researchers in physics, VLSI, geology, electrical impedance tomography, and control communities. Existing studies on electrical network reconstruction relate the response matrix $\Lambda$ to the possible $k$-connections in an unknown circular planar resistive network, which helps realize the network's structure. In [6], it has been shown that the response matrix $(\Lambda)$ corresponding to a circular planar electrical network with positive circular minors enumerates all possible $k$-connections between the $2k$ boundary terminals. In [7], the authors present an approach for computing the values of the conductances in a circular planar passive resistor network using boundary voltages and currents. The network structure is assumed to be known and is, in particular, a circular network. A $\gamma$-harmonic function is defined on the circular network. The process of harmonic continuation, along with the response matrix $\Lambda$, is used to compute the conductances in a circular network. In [8], it is shown that for any critical circular planar graph $G$, the conductance values can be computed using the response matrix $\Lambda$.

The Gröbner basis [10] is a set of nonlinear multivariate polynomials; which allows simple algorithmic solutions to an otherwise huge system of nonlinear multivariate polynomial equations. With the increase in computational capabilities in recent years, there has been a significant increase in applications of Gröbner basis. Some of the significant applications are automatic theorem proving [11], graph coloring [12], Integer programming [13], signal and image processing [14], testing controllability and observability, determination of equilibrium points,and computing time-optimal feedback control [15]. Our work presents a new application of Gröbner basis in characterizing all possible circuit topologies corresponding to a known response matrix $\Lambda$. A related work that uses the Gröbner basis for identifying all the resistive networks (with only 1-ohm resistances) satisfying the limitedly available boundary measurements has been reported [16].

**Contributions.** This paper makes three main contributions,

- First, we formulate an algorithm to obtain a reduced connection set (say $\hat{1}$-connection, $\hat{2}$-connection,. . ., $\hat{k}$-connection) from the set of all possible connections (such as 1-connection, 2-connection,. . ., $k$-connection [6]) obtained using the response matrix $\Lambda$.

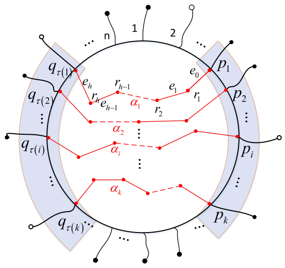


Fig. 1. $k$-connection between $\mathcal{P}$ and $\mathcal{Q}$

- Second, we provide a procedure for obtaining candidate planar graphs from the reduced connection set. Here, each connection is used to generate a set of path permutations along with their graph representations. These graph representations of path permutations are combined using edge deletion and graph union operations to generate candidate planar graphs.
- Finally, each of the generated candidate planar graphs in which the original network is assumed to be embedded is posed as a set of nonlinear multivariate polynomials. Gröbner basis is then computed corresponding to the set of nonlinear multivariate polynomials. The edge conductance values and the resulting network topology are the solutions to the Gröbner basis.

Hence, the network topology and edge conductance value corresponding to the response matrix $\Lambda$ are recovered.

**Mathematical Notations.** The set $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{Z}_{\geq}0$ is the set of non-negative natural numbers. Cardinality of the set is denoted as $|\cdot|$. An arc of a circle is represented by $\widehat{ab}$, where points $a$ and $b$ are on the circle. Let $R = \{a_1, a_2, \ldots, a_r\}$ be row indices and $D = \{b_1, b_2, \ldots, b_s\}$ column indices, the submatrix $\Lambda(R; D)$ is formed from row indices set $R$ and column indices set $D$.

## II. Preliminaries and Problem Setup

### A. Preliminaries:

In this paper, we characterize the set of all possible electrical network topologies with their corresponding edge resistance values using the response matrix $\Lambda$. Towards this objective, consider a passive resistive circular planar electrical network $\Gamma = (\mathcal{G}, \gamma)$ housed inside a black box with $n$ exposed boundary terminals. A weighted, undirected, simple graph is a triple $\mathcal{G} = (\mathcal{V}_\mathcal{I}, \mathcal{V}_\mathcal{B}, \mathcal{E})$, where $\mathcal{V}_\mathcal{B}$ is the set of boundary nodes and $|\mathcal{V}_\mathcal{B}| = n$. The $n$ boundary nodes are the $n$ exposed boundary terminals. $\mathcal{V}_\mathcal{I}$ is the set of interior nodes housed entirely inside the black box, and $|\mathcal{V}_\mathcal{I}| = d$. $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of undirected edges, where $\mathcal{V} = \mathcal{V}_\mathcal{B} \cup \mathcal{V}_\mathcal{I}$. A circular planar graph $\mathcal{G}$ of $\Gamma$ is a graph with a boundary embedded in disc $D$ on the plane, such that boundary nodes lie on the circle $C$ which bounds $D$, rest of $\Gamma$ is entirely inside $D$ [9]. The boundary nodes in $\mathcal{V}_\mathcal{B}$ are placed in circular order in a clockwise direction on the boundary circle $C$. The boundary nodes say $r_1, r_2, ..., r_m$ on $C$, are in circular order, if

- the points $r_2, \ldots, r_{m-1}$ lie on the arc $\widehat{r_1 r_m}$ on $C$,
- the order $r_1 < r_2 < \ldots < r_m$ is induced by the angles of the arc, measured clockwise from $r_1$.

A conductivity function $\gamma : \mathcal{E} \to \mathbb{R}^+$, assigns to each edge $ij \in \mathcal{E}$ a positive real number $\gamma_{ij}$, known as the edge conductance. Moving forward, we will define some basic terminologies for better understanding.

**Definition 1** (Path and Disjoint Paths)**.** A path $\beta$ is sequence of edges $e_0 = pi_1, e_1 = i_1 i_2, \ldots, e_r = i_r q$ from nodes $p$ to $q$, such that $i_1, i_2, \ldots, i_r \in \mathcal{V}_\mathcal{I}$. The sequences of edges $e_0, e_1, \ldots, e_r$ form a path $\beta = e_0 e_1 \ldots e_r$.
A finite set of paths say $\{\alpha_1, \alpha_2, ..., \alpha_k\}$ are said to be disjoint if for every $1 \leq l, m \leq k$ and $l \neq m$, paths $\alpha_l$ and $\alpha_m$ have no common vertex.

**Definition 2** (Connected nodes)**.** Two nodes $p$ and $q$ are said to be connected, $p \overset{\beta}{\leftrightarrow} q$, through $\mathcal{G}$, if there exist a path $\beta$ between $p$ and $q$.

Consider two sequences of boundary nodes $\mathcal{P} = (p_1, p_2, \ldots, p_k)$ and $\mathcal{Q} = (q_{\tau(1)}, q_{\tau(2)}, \ldots, q_{\tau(k)})$, where each $1 \leq p_l, q_{\tau(l)} \leq n$, $1 \leq l \leq k$ and $\tau$ is an element of the permutation group $\mathcal{S}_k$ and is the permutation on the indices $1, 2, ..., k$.

**Definition 3** (Connected sequence of nodes)**.** Two sequences of boundary nodes $\mathcal{P}$ and $\mathcal{Q}$ are said to be connected, $\mathcal{P} \overset{\alpha}{\leftrightarrow} \mathcal{Q}$, through $\mathcal{G}$, if there exist $k$ disjoint paths $\alpha_1, \alpha_2, ..., \alpha_k$ in $\mathcal{G}$, such that for each $l \in \{1, 2, \ldots, k\}$, the path $\alpha_l$ starts at $p_l$ and terminates at $q_{\tau(l)}$, and $\alpha = \{\alpha_1, \alpha_2, ..., \alpha_k\}$.
Such sequences of nodes are termed as a $k$-connection, since there are $k$ disjoint paths connecting two sequence of boundary nodes $\mathcal{P}$ and $\mathcal{Q}$ , as shown in Figure. 1.

**Definition 4** (Circular pair)**.** The pair of sequences of boundary nodes $(\mathcal{P}, \mathcal{Q}) = (p_1, p_2, \ldots, p_k; q_{\tau(1)}, q_{\tau(2)}, \ldots, q_{\tau(k)})$ are considered to be a circular pair if $\tau(l) = l \; \forall \; 1 \leq l \leq k$ and the sequence $(p_1, p_2, \ldots, p_k, q_k, q_{k-1}, \ldots, q_1)$ is in clockwise circular order on the boundary circle $C$.

Let us consider the permutation groups $\mathcal{S}_k$ and $\mathcal{S}_{k+d}$, and also let a permutation $\tau \in \mathcal{S}_k$ and $\varphi \in \mathcal{S}_{k+d}$. $\tau$ is the permutation on $k$ boundary nodes, and $\varphi$ is the permutation on $k + d$ nodes. We make use of a fixed $\tau$ and a permutation $\varphi \in \mathcal{S}_{k+d}$ to generate Path Permutation Diagram's (PPD's) as shown generally for two sequence of boundary nodes $\mathcal{P}$ and $\mathcal{Q}$ in Figure 2. In the PPD in Figure 2, the indices $l$ and $\tau(l)$ of the boundary nodes $p_l$ and $q_{\tau(l)}$ respectively are denoted as $a(l, 0)$ and $a(l, n_l)$, where $n_l \in \mathbb{Z}_{\geq 0}$ and $l \in \{1, \ldots, k\}$,

$$1 = a(1,0) \xrightarrow{\varphi} a(1,1) \xrightarrow{\varphi} a(1,2) \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} a(1,n_1) = \tau(1)$$
$$2 = a(2,0) \xrightarrow{\varphi} a(2,1) \xrightarrow{\varphi} a(2,2) \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} a(2,n_2) = \tau(2)$$
$$\cdots$$
$$k = a(k,0) \xrightarrow{\varphi} a(k,1) \xrightarrow{\varphi} a(k,2) \xrightarrow{\varphi} \cdots \xrightarrow{\varphi} a(k,n_k) = \tau(k)$$

Fig. 2. Path diagram of $k$-connection corresponding to a fixed $\tau$ and a permutation $\varphi$

whereas $a(l,t) \in \mathcal{V}_I \ \forall \ 1 \leq t \leq n_l - 1$. One of the path among $k$-disjoint paths starting at a boundary node say $p_1$, denoted by its index $1(= a(1,0))$, goes through several permutations and reaches another boundary node $q_{\tau(l)}$ denoted by its index $\tau(l)(= a(1, n_l))$. The permutation $\varphi$ on index $1(= a(1,0))$ is $\varphi(1) = a(1,1)$ which generates an edge $a(1,0)a(1,1)$, here $a(1,1) \in \mathcal{V}_I$, such permutations is continued till $\tau(1)$ is reached. Similarly $k$ such disjoint paths are formed.

Let us consider a circular planar resistive network $\Gamma$ with $n$ boundary nodes, labeled $\{1, 2, ..., n\}$ in circular order clockwise direction, and $d$ interior nodes labeled $\{n+1, n+2, ..., n+d\}$ arbitrarily. We define the planarity of $\Gamma$ in disc $D$ as,

**Definition 5.** The circuit $\Gamma$ inside the disc $D$ is said to be *planar in $D$* iff 1) all the paths are contained in the interior of circle $C$ which bounds disc $D$, 2) there exists planar embedding of $\Gamma$ in the interior of $D$.

We define the current injections $I \in \mathbb{R}^{(n+d)\times 1}$ and the node voltages $V \in \mathbb{R}^{(n+d)\times 1}$. The network variables $I$ and $V$ are related by the current balance equation $I = \mathcal{K}V$, where $\mathcal{K}$ is called the Kirchhoff's matrix. The Kirchhoff's matrix is a symmetric $(n+d) \times (n+d)$ matrix, where $\mathcal{K} = [\mathcal{K}_{ij}], 1 \leq i, j \leq n+d$, defined as follows:

$$\mathcal{K} = [\mathcal{K}_{ij}] \begin{cases} = -\gamma_{ij}, & \text{if } (i,j) \in \mathcal{E}, \\ = \sum\limits_{j \in \mathcal{N}(i)} \gamma_{ij}, & \text{if } i = j, \\ = 0, & \text{otherwise.} \end{cases}$$

Here, $\gamma_{ij}$ is the conductance of edge $ij \in \mathcal{E}$. In line with the labeling of nodes, we define $\phi_b = [i_1, i_2, ..., i_n]^T$ as the boundary response current and $v_b = [v_1, v_2, ..., v_n]^T$ is the applied boundary voltages. The boundary voltages $v_b$ is mapped to the response boundary currents $\phi_b$ through the response matrix $\Lambda$ and is given as,

$$\phi_b = \Lambda v_b. \tag{1}$$

The response matrix $\Lambda$ is given as Schur complement of $\mathcal{K}$ with respect to $\mathcal{K}([d];[d])$ i.e.,

$$\Lambda = \mathcal{K}([n];[n]) - \mathcal{K}([n];[d])\mathcal{K}([d];[d])^{-1}\mathcal{K}([d];[n]). \tag{2}$$

Let us for a moment draw our attention towards important terminologies related to Gröbner basis. The ideal of the set of polynomials $\mathbf{F}$, denoted as $\langle \mathbf{F} \rangle$, is defined as:

**Definition 6** (Ideal of **F**).

$$\langle \mathbf{F} \rangle = \left\{ \begin{array}{r} \sum\limits_{i=1}^{n} h_i f_i : \forall h_i \in \mathbb{R}\left[\gamma_{12}, \ldots, \gamma_{ij}, \ldots, \gamma_{(m-1)m}\right] \\ \& \forall f_i \in \mathbf{F}, m = n+d \end{array} \right\}$$

We define a solution set $\mathbf{V}$, generated by set of polynomials $\mathbf{F}$ as:

**Definition 7** (**Variety of F**). $\mathbf{V}(\mathbf{F})$ is called the affine variety generated by the set of polynomials $\mathbf{F}$ and is defined as

$$\mathbf{V}(\mathbf{F}) = \left\{ \mathbf{w} \in \mathbb{R}_{\geq 0}^{\frac{(n+d)(n+d-1)}{2}} : f(\mathbf{w}) = 0, \forall f \in \mathbf{F} \right\}. \tag{3}$$

*B. Problem Setup:*

In this paper we characterize a set of all the Kirchhoff's matrix corresponding to the known response matrix $\Lambda$, hence reconstructing the unknown $\Gamma$. Consider a problem setup as shown in Figure. 3. We start with building a set of all connections $\Pi_\Lambda$ in $\Gamma$. Therefore, let us consider $l\pi_r$ to be two sequence of $l$ boundary nodes $(\mathcal{P}^{l\pi_r}, \mathcal{Q}^{l\pi_r})$, where $\mathcal{P}^{l\pi_r} = (p_1^{l\pi_r}, p_2^{l\pi_r}, \ldots, p_l^{l\pi_r})$, $\mathcal{Q}^{l\pi_r} = \left(q_{\tau^{l\pi_r}(1)}^{l\pi_r}, q_{\tau^{l\pi_r}(2)}^{l\pi_r}, \ldots, q_{\tau^{l\pi_r}(l)}^{l\pi_r}\right)$ which are circular pairs and hence $\tau^{l\pi_r} \in \mathcal{S}_l$ is an identity permutation i.e. $\tau^{l\pi_r}(h) = h$ and $1 \leq p_h^{l\pi_r}, q_h^{l\pi_r} \leq n$, where $\forall \ 1 \leq h \leq l$. The connection $l\pi_r$ is assumed to be $l$-connected i.e., $\mathcal{P}^{l\pi_r} \xleftrightarrow{\alpha^{l\pi_r}} \mathcal{Q}^{l\pi_r}$. With no knowledge of the internal structure of $\Gamma$, the $l$-connectedness in $\Gamma$ can also be observed from $\Lambda$ using the following theorem,

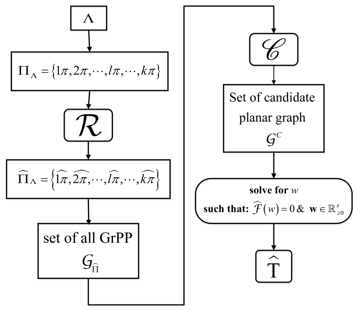


Fig. 3. Problem setup considered in this paper.

**Theorem 1.** *[6] Let $\Gamma = (\mathcal{G}, \gamma)$ be a resistive circular planar network and $\Lambda \in \mathbb{R}^{n \times n}$ be the response matrix. Suppose $\mathcal{P} = (p_1, p_2, \ldots, p_l)$ and $\mathcal{Q} = (q_1, q_2, \ldots, q_l)$ be two sequence of $l$ boundary nodes, and the sequence $(P, Q) = (p_1, p_2, \ldots, p_l; q_1, q_2, \ldots, q_l)$ is a circular pair. $\mathcal{P}$ and $\mathcal{Q}$ is said to*

- *$l$-connected through $\mathcal{G}$: If $(-1)^l det[\Lambda(P;Q)] > 0$,*
- *Not $l$-connected through $\mathcal{G}$: If $det[\Lambda(P;Q)] = 0$.*

For ease of visualisation and understanding we will in future write $(\mathcal{P}^{l\pi_r}; \mathcal{Q}^{l\pi_r}) = \begin{pmatrix} p_1^{l\pi_r} & p_2^{l\pi_r} & \cdots & p_l^{l\pi_r} \\ q_1^{l\pi_r} & q_2^{l\pi_r} & \cdots & q_l^{l\pi_r} \end{pmatrix}$ wherever required. Further, in this paper we consider only circular pairs for analysis.

We collect all such $l$-connections, using Theorem 1 from $\Lambda$, in a set $l\pi = \{l\pi_1, l\pi_2, \ldots, l\pi_{|l\pi|}\}$. Similarly, we gather all other possible connections, say $1\pi, 2\pi, \ldots, k\pi$, using Theorem 1, in a set $\Pi_\Lambda = \{1\pi, \ldots, l\pi, \ldots, k\pi\}$. The number of possible circular pairs in $\Gamma$ is generally large and hence needs to be reduced such that the overall connection remains intact. We devise a reduction algorithm $\mathcal{R}$, such that $\mathcal{R} : \Pi_\Lambda \rightarrow \widehat{\Pi}_\Lambda$, $\widehat{\Pi}_\Lambda$ is the set of reduced connections. Let

$\widehat{\Pi}_\Lambda = \left\{\widehat{1\pi}, \dots, \widehat{l\pi}, \dots, \widehat{k\pi}\right\}$, such that for every $\widehat{l\pi}$, where $l \in \{1, \dots, k\}$, $|\widehat{l\pi}| \leq |l\pi|$ and also $\widehat{l\pi} \subseteq l\pi$ is true, corresponding to $\widehat{\Pi}_\Lambda$. Now we use the $\widehat{\Pi}_\Lambda$ along with the permutation's $\varphi \in \mathcal{S}_{l+d}$ to generates all the possible path diagrams corresponding to each circular pair. For better understanding let us consider an example of an unknown circuit $\Gamma$ with 6 boundary nodes $(n)$ labelled as $\{1, 2, 3, 4, 5, 6\}$ and 2 interior nodes $(d)$ labelled as $\{7, 8\}$, as shown in the Figure 4.

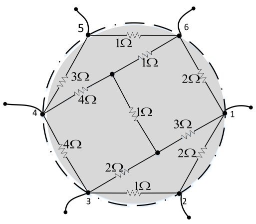

Fig. 4. $\Gamma$ with 6 boundary nodes and 2 interior nodes.

Let us consider a $r^{th}$ 3-connection, $3\pi_r \in 3\pi$, which represents 2-sequences of three boundary nodes $\mathcal{P}^{3\pi_r} = \left(p_1^{3\pi_r} = 2, p_2^{3\pi_r} = 3, p_3^{3\pi_r} = 4\right)$ and $\mathcal{Q}^{3\pi_r} = \left(q^{3\pi_r}_{\tau^{3\pi_r}(1)} = 1, q^{3\pi_r}_{\tau^{3\pi_r}(2)} = 6, q^{3\pi_r}_{\tau^{3\pi_r}(3)} = 5\right)$, which are 3-connected and circular pairs. Now, using $\left(\mathcal{P}^{3\pi_r}; \mathcal{Q}^{3\pi_r}\right) = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 6 & 5 \end{pmatrix}$, and the permutations $\varphi^{3\pi_r} \in \mathcal{S}_5$ we list all possible valid path permutation diagrams. For this, we start with listing all valid path permutations (PP) by defining a permutation $\varphi^{3\pi_r} \in \mathcal{S}_5$ on $\{1, 2, 3, 7, 8\}$, here $1, 2, 3$ are indices of boundary nodes $p_1, p_2, p_3$ respectively and $7, 8 \in \mathcal{V}_I$, for a fixed $\tau^{3\pi_r}$ i.e., $\tau^{3\pi_r}(1) = 1, \tau^{3\pi_r}(2) = 2$ and $\tau^{3\pi_r}(3) = 4$ which implies $q_1 = 1, q_2 = 6, q_3 = 5$, as shown in Figure 5. Each of the path permutation describes a path linking $\mathcal{P}^{3\pi_r}$ and $\mathcal{Q}^{3\pi_r}$ and has an equivalent graph representation. In Path Permutation Diagram (PPD) in Figure 6, corresponding to the underlined path permutation in Figure 5, the 3 disjoint paths $\alpha_1^{3\pi_r}, \alpha_2^{3\pi_r}, \alpha_3^{3\pi_r}$ connects $\mathcal{P}^{3\pi_r}$ and $\mathcal{Q}^{3\pi_r}$ through $\mathcal{G}$, as shown in a Graph representation of PP (GrPP) in Figure 6. We list all the GrPP for all possible path permutations $\varphi^{3\pi_r}$ as a set $\mathcal{G}^{3\pi_r} = \left\{\mathcal{G}_1^{3\pi_r}, \mathcal{G}_2^{3\pi_r}, \dots, \mathcal{G}^{3\pi_r}_{|\mathcal{G}^{3\pi_r}|}\right\}$. Similarly, for $r^{th}$ reduced connection $\widehat{l\pi}_r \in \widehat{l\pi}$ we get a set of GrPP $\mathcal{G}^{\widehat{l\pi}_r} = \left\{\mathcal{G}_1^{\widehat{l\pi}_r}, \mathcal{G}_2^{\widehat{l\pi}_r}, \dots, \mathcal{G}^{\widehat{l\pi}_r}_{|\mathcal{G}^{\widehat{l\pi}_r}|}\right\}$, likewise for all the $|\widehat{l\pi}|$ $l$-connections in the set $\widehat{l\pi}$, we group all the GrPP to get a set $\mathcal{G}^{\widehat{l\pi}} = \left\{\mathcal{G}^{\widehat{l\pi}_1}, \dots \mathcal{G}^{\widehat{l\pi}_r}, \dots, \mathcal{G}^{\widehat{l\pi}_{|\widehat{l\pi}|}}\right\}$. For all the connections in $\widehat{\Pi}_\Lambda$, we get $\mathcal{G}_{\widehat{\Pi}} = \left\{\mathcal{G}^{\widehat{1\pi}}, \dots \mathcal{G}^{\widehat{l\pi}}, \dots, \mathcal{G}^{\widehat{k\pi}}\right\}$ which is the set of all GrPP's corresponding to the reduced connections $\widehat{1\pi}, \dots, \widehat{l\pi}, \dots, \widehat{k\pi}$.

Now, we use the set $\mathcal{G}_{\widehat{\Pi}}$ to construct a set of candidate planar graphs $\mathcal{G}^C$, the candidate graphs in the set $\mathcal{G}^C$ are the possible skeletons in which the original graph, say $\mathcal{G}^*$, is embedded. To construct the set $\mathcal{G}^C$ we formulate a construction algorithm $\mathscr{C}$,

| $p_{\boxed{1}}$ | $p_{\boxed{2}}$ | $p_{\boxed{3}}$ | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 7 | 8 |
| 1 | 2 | 3 | 8 | 7 |
| 1 | 2 | 8 | 3 | 7 |
| | | ⋮ | | |
| $\underline{1}$ | $\underline{8}$ | $\underline{7}$ | $\underline{3}$ | $\underline{2}$ |
| | | ⋮ | | |
| 8 | 7 | 3 | 2 | 1 |

Fig. 5. All valid permutations $\varphi^{3\pi_r} \in \mathcal{S}_d$

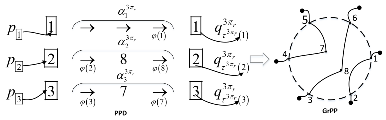


Fig. 6. Path diagram corresponding to the $\varphi(1)$=1, $\varphi(2)$=8, $\varphi(3)$=7, $\varphi(7)$=3, $\varphi(8)$=2 and the corresponding graph permutation.

such that $\mathscr{C} : \mathcal{G}_{\widehat{\Pi}} \to \mathcal{G}^C$. Once the set of all candidate planar graph $\mathcal{G}^C$ is obtained, each candidate planar graph $\mathcal{G}_w^C \in \mathcal{G}^C$, $\forall\ 1 \leq w \leq |\mathcal{G}^C|$, is posed as a set of nonlinear multivariate polynomial, this is done using the equation 2. For instance, consider $w^{th}$ candidate planar graph $\mathcal{G}_w^C \in \mathcal{G}^C$, corresponding to $\mathcal{G}_w^C$ there exist a Kirchhoff's matrix $\mathcal{K}_w^C$ whose structure, i.e. position of zeros and unknown non-zero elements in matrix, is derived from the known $\mathcal{G}^C$ as explained in equation **??**. Let $\mathcal{K}_w^C \in \mathbb{R}^{(n+d)\times(n+d)}$, a sparse symmetric matrix be defined as

$$\mathcal{K}_w^C = \left[\begin{array}{c|c} \mathcal{K}_w^C([n];[n]) & \mathcal{K}_w^C([n];[d]) \\ \hline \mathcal{K}_w^C([d];[n]) & \mathcal{K}_w^C([d];[d]) \end{array}\right]. \tag{4}$$

Here, $\mathcal{K}_w^C([n];[n]) \in \mathbb{R}^{n\times n}$ describes connections between $n$ boundary nodes only, $\mathcal{K}_w^C([n];[d]) \in \mathbb{R}^{n\times d}$ stores the information on connections between $n$ boundary nodes and $d$ interior nodes and correspondingly $\mathcal{K}_w^C([d];[d]) \in \mathbb{R}^{d\times d}$ tells us about the internal connections between interior nodes only. The non zero elements, say some $\gamma_{ij}$'s, in $\mathcal{K}_w^C$ are the unknown's and are to be found to reconstruct the network $\Gamma$. Hence, let $\mathbf{w} = [\dots\gamma_{ij},\dots]^T \in \mathbb{R}^x$ be the vector of unknown elements in $\mathcal{K}_w^C$. To compute the unknown $\mathbf{w}$, we relate $\mathcal{K}_w^C$ to the known response matrix $\Lambda$ using equation 2 and hence, we get a following relation,

$$\Lambda = \mathcal{K}_w^C([n];[n]) - \mathcal{K}_w^C([n];[d])\mathcal{K}_w^C([d];[d])^{-1}\mathcal{K}_w^C([d];[n]), \tag{5}$$

Using

$$\mathcal{K}_w^C([d];[d])^{-1} = adj\left(\mathcal{K}_w^C([d];[d])\right)\det\left(\mathcal{K}_w^C([d];[d])\right)^{-1}$$

in equation 5, we get the following symmetric matrix polynomial equation

$$\begin{aligned} &\Lambda \det\left(\mathcal{K}_w^C([d];[d])\right) \\ &+\mathcal{K}_w^C([n];[d])\, adj\left(\mathcal{K}_w^C([d];[d])\right)\mathcal{K}_w^C([d];[n]) \\ &-\mathcal{K}_w^C([n];[n])\det\left(\mathcal{K}_w^C([d];[d])\right) = \mathbf{0}, \end{aligned} \tag{6}$$

here $adj(\cdot)$ is the adjoint of the matrix and $det(\cdot)$ is the determinant of the matrix. Let equation 6 be written as

$$\mathcal{P}(\mathcal{K}_w^C) = \mathbf{0} \tag{7}$$

where, $\mathcal{P}(\mathcal{K}_w^C) \in \mathbb{R}[\ldots\gamma_{ij},\ldots]^{n\times n}$, $\mathcal{K}_w^C \in \mathbb{R}^{(n+d)\times(n+d)}$. From the matrix polynomial $\mathcal{P}(\mathcal{K}_w^C)$ in equation 7, we form a set of nonlinear multivariate polynomials $\mathbf{F} = \{f_{11},\ldots,f_{ij},\ldots,f_{nn}\}$ and $|\mathbf{F}| = \frac{n(n+1)}{2}$. Consider a system of polynomial equations $\mathcal{F}$ created using the polynomials in set $\mathbf{F}$, i.e.,

$$\mathcal{F}(\mathbf{w}) = 0 \Rightarrow f_{km}(\ldots\gamma_{ij},\ldots) = 0, \forall f_{km} \in \mathbf{F} \ . \tag{8}$$

Our objective now is to

$$\begin{array}{c} solve \ \ for \ \ \mathbf{w} \\ such \ \ that : \mathcal{F}(\mathbf{w}) = 0 \ \& \ \mathbf{w} \in \mathbb{R}^x_{\geq 0} \end{array} . \tag{9}$$

Let $\mathbf{K}$ be a linear transformation that transforms a vector $\bar{\mathbf{w}} \in \mathbb{R}^{(n+d)}_{\geq 0}$ to a symmetric matrix $\mathbf{K}_{\bar{\mathbf{w}} \in \mathbb{R}^{(n+d)\times(n+d)}}$. Vector $\bar{\mathbf{w}}$ contains the elements of unknown vector $\mathbf{w}$ at appropriate locations and other elements are zero.

**Definition 8.** The linear transformation $\mathbf{K} : \mathbb{R}^{(n+d)}_{\geq 0} \rightarrow \mathbb{R}^{(n+d)\times(n+d)}$ is defined as

$$[\mathbf{K}_\mathbf{w}]_{ij} = \begin{cases} -w_{i+q_j} & i > j, \\ [\mathbf{K}_\mathbf{w}]_{ji} & i < j, \\ \sum_{i\neq j} [\mathbf{K}_\mathbf{w}]_{ij} & i = j, \end{cases} \tag{10}$$

where $q_j = -j + \frac{j-1}{2}(2(n+d) - j)$. The matrix $\mathbf{K}_{\bar{\mathbf{w}}}$ satisfies the constraints $[\mathbf{K}_{\bar{\mathbf{w}}}]_{ij} = [\mathbf{K}_{\bar{\mathbf{w}}}]_{ji}$ and $\mathbf{1} \cdot [\mathbf{K}_{\bar{\mathbf{w}}}] = [\mathbf{K}_{\bar{\mathbf{w}}}] \cdot \mathbf{1} = 0$.

There are many solutions to the objective defined in (10) and hence let the solution set $\mathbf{T}$ be defined as

$$\mathbf{T} = \left\{\mathbf{K}_{\bar{\mathbf{w}}} \in \mathbb{R}^{(n+d)\times(n+d)} : \mathbf{w} \in \mathbf{V}(\mathbf{F}) \ \& \ \bar{\mathbf{w}} \in \mathbb{R}^{(n+d)}_{\geq 0}\right\}. \tag{11}$$

Equivalently, a special set of basis polynomials corresponding to $\mathbf{F}$ can be constructed known as the *Gröbner* basis $\widehat{\mathbf{F}} \in \mathbb{R}[\gamma_{12},\ldots\gamma_{ij},\ldots,\gamma_{(n+d-1)(n+d)}]$, which allows simple algorithmic solutions to system of polynomials equations $\mathcal{F}(\mathbf{w}) = 0$. The variety of the set $\mathbf{F}$ and the set $\widehat{\mathbf{F}}$ are related, this relation is stated below

**Lemma 1.** *[10] Let $\mathbf{F}$ and $\widehat{\mathbf{F}}$ be a set of polynomials as defined above, then, $\langle\mathbf{F}\rangle = \left\langle\widehat{\mathbf{F}}\right\rangle$.*

**Lemma 2.** *[10] Let $\mathbf{F}$ and $\widehat{\mathbf{F}}$ be polynomials in $\mathbb{R}\left[\gamma_{12},\ldots\gamma_{ij},\ldots,\gamma_{(n+d-1)(n+d)}\right]$, such that $\langle\mathbf{F}\rangle = \left\langle\widehat{\mathbf{F}}\right\rangle$, then $\mathbf{V}(\mathbf{F}) = \mathbf{V}\left(\widehat{\mathbf{F}}\right)$.*

Now, let $\widehat{\mathcal{F}}(\mathbf{w}) = 0$ be the system of polynomial equations constructed using the Gröbner basis polynomials in $\widehat{\mathbf{F}}$. The objective defined in (10) can be reformulated as

$$\begin{array}{c} solve \ \ for \ \ \mathbf{w} \\ such \ \ that : \widehat{\mathcal{F}}(\mathbf{w}) = 0 \ \& \ \mathbf{w} \in \mathbb{R}^x_{\geq 0} \end{array} , \tag{12}$$

corresponding to the reformulated objective in (13), we define a solution set $\widehat{\mathbf{T}}$, which is given as

$$\widehat{\mathbf{T}} = \left\{\mathbf{K}_{\bar{\mathbf{w}}} \in \mathbb{R}^{(n+d)\times(n+d)} : \mathbf{w} \in \mathbf{V}\left(\widehat{\mathbf{F}}\right) \ \& \ \bar{\mathbf{w}} \in \mathbb{R}^{(n+d)}_{\geq 0}\right\}. \tag{13}$$

The solution sets $\mathbf{T}$ and $\widehat{\mathbf{T}}$ are equal from lemma 1 and lemma 2.

***Problem:*** Find the set $\widehat{\mathbf{T}}$ corresponding to the Gröbner basis polynomials $\hat{\mathbf{F}}$.

$\widehat{\mathbf{T}}$ is the set of all the network structure corresponding to the given network response matrix $\Lambda$.

## III. Reconstruction Algorithm

We aim to characterize the set of all possible electrical network topology $\widehat{\mathbf{T}}$ along with there edge resistances values using the given response matrix $\Lambda$. To achieve this we begin with the construction of set $\Pi_\Lambda$ from $\Lambda$ using Theorem 1. Since the population of such circular pairs is large, we formulate a reduction algorithm $\mathcal{R}$, such that $\mathcal{R} : \Pi_\Lambda \rightarrow \widehat{\Pi}_\Lambda$, $\widehat{\Pi}_\Lambda$ is the set of reduced connections. For properly understanding the reduction algorithm $\mathcal{R}$ we define some terms i.e.

**Definition 9** (**Union of connections**)**.** The union of some arbitrary connected, circular pair $l$-connections say, $\left\{\begin{array}{l} l\pi_i = \left(P^{l\pi_i}, Q^{l\pi_i}\right), l\pi_j = \left(P^{l\pi_j}, Q^{l\pi_j}\right), \\ \ldots, l\pi_h = \left(P^{l\pi_h}, Q^{l\pi_h}\right) \end{array}\right\}$ is defined as $\bigcup\{l\pi_i, l\pi_j, \ldots, l\pi_h\} = \bigcup\{\left(P^{l\pi_i}, Q^{l\pi_i}\right), \ldots, \left(P^{l\pi_h}, Q^{l\pi_h}\right)\}$, resulting connection is connected and not necessarily a circular pair.

**Definition 10** (**Redundant connection**)**.** A connected circular pair $l$-connection is said to be redundant if and only if this $l$-connection exists already in some union of connections or exists in other connections say a $\bar{l}$-connection, where $\bar{l} > l$.

With these definitions, algorithm $\mathcal{R}$ is as follows. The

**Algorithm 1** Reduction Algorithm $\mathcal{R}$

**Input:** $\Pi_\Lambda = \{k\pi,\ldots,l\pi,\ldots,2\pi,1\pi\}$ **(place connections in descending order)**
**Output:** $\widehat{\Pi}_\Lambda = \{\widehat{k\pi},\ldots,\widehat{l\pi},\ldots,\widehat{2\pi},\widehat{1\pi}\}$
**Initialisation:** $\widehat{k\pi} = k\pi, \widehat{\Pi}_\Lambda = \emptyset$

```
for l = k − 1 to 1 do
  for r = 1 to |lπ| do
    l̂π ← lπ
    if lπ_r ∈ some union of connections in Π_Λ.
      l̂π ← l̂π − {lπ_r}
    else if lπ_r is a redundant connection.
      l̂π ← l̂π − {lπ_r}
    end if
  end for
  Π̂_Λ ← Π̂_Λ ⋃ l̂π
end for
Return Π̂_Λ
```

obtained set of reduced connection $\widehat{\Pi}_\Lambda$ is used to get the set of all GrPP $\mathcal{G}_{\widehat{\Pi}}$ corresponding to $\widehat{\Pi}_\Lambda$ as explained in section II. The GrPP's in $\mathcal{G}_{\widehat{\Pi}}$ are to be processed such that a set of all candidate planar graph $\mathcal{G}^C$ is obtained. Therefore, we design a strategy $\mathscr{C}$ to obtain $\mathcal{G}^C$ from $\mathcal{G}_{\widehat{\Pi}}$. The strategy is a two phase procedure. In the first phase, $\forall\ 1 \leq l \leq k$ we construct set's $\mathcal{G}_p^{\widehat{l\pi_r}}$, which is the set of all the planar graphs obtained from the set $\mathcal{G}^{\widehat{l\pi_r}}$. To achieve this objective, we define an operation $\bigcup_p$ defined as,

**Definition 11** ($\bigcup_p$)**.** An operation $\bigcup_p$ on a set $\mathcal{G}^{\widehat{l\pi_r}}$ is a mapping $\bigcup_p : \mathcal{G}^{\widehat{l\pi_r}} \rightarrow \mathcal{G}_p^{\widehat{l\pi_r}}$. $\bigcup_p$ is essentially an union operation defined on a set $\mathcal{G}^{\widehat{l\pi_r}} \in \mathcal{G}^{\widehat{l\pi}}$, such that unions of certain combinations of GrPP's in $\mathcal{G}^{\widehat{l\pi_r}}$ generate several planar graphs. The operation $\bigcup_p$ on $\mathcal{G}^{\widehat{l\pi_r}}$ list's all such planar graphs as a set $\mathcal{G}_p^{\widehat{l\pi_r}}$.

The algorithm corresponding to the first phase is given below for finer understanding

**Algorithm 2** First Phase of Algorithm $\mathscr{C}$

**Input:** $\mathcal{G}_{\widehat{\Pi}} = \left\{\mathcal{G}^{\widehat{1\pi}}, \dots \mathcal{G}^{\widehat{l\pi}}, \dots, \mathcal{G}^{\widehat{k\pi}}\right\}$
**Output:** $\mathcal{G}_p$
**Initialisation:** $\mathcal{G}_p = \emptyset$

```
for l = 1 to k do
    G_p^{l̂π} = ∅
    for r = 1 to |l̂π| do
        G_p^{l̂π_r} ← ⋃_p G^{l̂π_r}
        G_p^{l̂π} ← G_p^{l̂π} ⋃ G_p^{l̂π_r}
    end for
    G_p ← G_p ⋃ G_p^{l̂π}
end for
Return G_p
```

In the second phase of algorithm, in general, we sequentially take union's of the planar graphs in $\mathcal{G}_p$ and check if resultant graph is planar in $D$ or not, if the resulting graph is non planar in $D$, we conduct an edge deletion operation to generate planar graph/s in $D$. This sequential union and edge deletion operations subsequently stabilises to a set of candidate planar graphs $\mathcal{G}^C$. The second phase of algorithm $\mathscr{C}$ is as follows. In this phase, we conduct four major operations i.e. • taking union of GrPP's, • Checking whether union generates a non-planar graphs in $D$, • perform edge deletion operation to generate planar graphs in $D$, • checking for redundancy. Here, we start with selecting a graph with more edge population from $\mathcal{G}_p^{\widehat{1\pi}}$, say $\mathcal{G}_p\left(\widehat{1\pi}, *\right)$, now we take union of $\mathcal{G}_p\left(\widehat{1\pi}, *\right)$ with each GrPP's in $\mathcal{G}_p^{\widehat{2\pi}}$ and simultaneously check whether union operation gives birth to a non-planar graph in $D$, if yes, we conduct an edge deletion operation to generate planar graphs in $D$. Hence, we get host of planar graphs in $D$, which are then passed through redundancy check i.e. we check whether a graph is already embedded in other graph/s, if yes, we consider that graph to be redundant, otherwise it is kept. With this group of planar, non-redundant graphs, we again select a graph which is relatively more populated in terms of edges and carry out the same sequence of four major operations as described above to get a set of candidate planar graph $\mathcal{G}^C$.

## IV. Numerical example

Let us consider an unknown circuit $\Gamma_x$ as shown in Figure 4 with 6 boundary nodes arranged in circular order, and 2 interior nodes. The response matrix is assumed to be known. Using the response matrix we list out all possible connections. We have set of all 1-connections $1\pi$, set of all 2-connections $2\pi$ and set of all 3-connections $3\pi$, from these sets we construct a set of all connections $\Pi_\Lambda = \{1\pi, 2\pi, 3\pi\}$. Since there are large number of circular pair connections in $\Pi_\Lambda$, we derive a reduced set of connections using algorithm $\mathcal{R}$. The reduced set of 1-connections is $\widehat{1\pi}$, the reduced set of $2-$connections is $\widehat{2\pi}$ and the reduced set of 3-connection is $\widehat{3\pi}$ . Therefore, we get $\widehat{\Pi}_\Lambda = \{\widehat{1\pi}, \widehat{2\pi}, \widehat{3\pi}\}$, the set of all reduced connections. For each circular pair connection in $\widehat{\Pi}_\Lambda$ we get a set of GrPP's, for example, for a 2-connection $\widehat{2\pi}_1$, we have following GrPP's as shown in Figure 7 Therefore, for $2\pi_1$ we get a set of GrPP

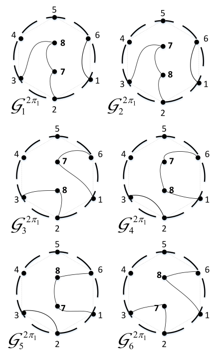

Fig. 7. GrPP corresponding to circular pair $2\pi_1$.

$\mathcal{G}^{\widehat{2\pi_1}} = \left\{\mathcal{G}_1^{\widehat{2\pi_1}}, \mathcal{G}_2^{\widehat{2\pi_1}}, \mathcal{G}_3^{\widehat{2\pi_1}}, \mathcal{G}_4^{\widehat{2\pi_1}}, \mathcal{G}_5^{\widehat{2\pi_1}}, \mathcal{G}_6^{\widehat{2\pi_1}}\right\}$, similarly we list all GrPP for connections in $2\pi$ and arrive at a set $\mathcal{G}^{\widehat{2\pi}} = \{\mathcal{G}^{\widehat{2\pi_1}}, \mathcal{G}^{\widehat{2\pi_2}}, \mathcal{G}^{\widehat{2\pi_3}}, \mathcal{G}^{\widehat{2\pi_4}}, \mathcal{G}^{\widehat{2\pi_5}}, \mathcal{G}^{\widehat{2\pi_6}}\}$. This is done for all the connections in $\widehat{\Pi}_\Lambda$ and hence we construct a set $\mathcal{G}_{\widehat{\Pi}} = \{\mathcal{G}^{\widehat{2\pi}}, \mathcal{G}^{\widehat{3\pi}}\}$, which is the set of all GrPP for $\Pi_\Lambda$. The set $\mathcal{G}_{\widehat{\Pi}}$ is to be processed to get a set of candidate planar graphs $\mathcal{G}^C$. The first phase of algorithm $\mathscr{C}$ gives us set of planar graphs $\mathcal{G}_p = \{\mathcal{G}_p^{\widehat{2\pi}}, \mathcal{G}_p^{\widehat{3\pi}}\}$ generated from $\mathcal{G}_{\widehat{\Pi}}$. For example, from $\bigcup_p \mathcal{G}^{\widehat{2\pi_1}}$ we extract two planar graphs as shown in Figure 8, this result in a set $\mathcal{G}_p^{\widehat{2\pi_1}} = \{\mathcal{G}_{p,1}^{\widehat{2\pi_1}}, \mathcal{G}_{p,2}^{\widehat{2\pi_1}}\}$ of planar graphs obtained from $\mathcal{G}^{\widehat{2\pi_1}}$, we similarly obtain sets $\mathcal{G}_p^{\widehat{2\pi_2}}, \mathcal{G}_p^{\widehat{2\pi_3}}, \mathcal{G}_p^{\widehat{2\pi_4}}, \mathcal{G}_p^{\widehat{2\pi_5}}, \mathcal{G}_p^{\widehat{2\pi_6}}$ corresponding to set $\mathcal{G}^{\widehat{2\pi}}$ and set $\mathcal{G}_p^{\widehat{3\pi_1}}, \mathcal{G}_p^{\widehat{3\pi_2}}$ corresponding to $\mathcal{G}^{\widehat{3\pi}}$. Therefore, we construct a set $\mathcal{G}_p = \{\mathcal{G}_p^{\widehat{2\pi}}, \mathcal{G}_p^{\widehat{3\pi}}\}$. The second phase of algorithm $\mathscr{C}$, uses $\mathcal{G}_p$ to generates a set $\mathcal{G}^C$, which are given below in Figure 9. Therefore, a set of candidate planar graphs is $\mathcal{G}^C = \{\mathcal{G}_1^C, \mathcal{G}_2^C, \mathcal{G}_3^C, \mathcal{G}_4^C, \mathcal{G}_5^C, \mathcal{G}_6^C, \mathcal{G}_7^C, \mathcal{G}_8^C, \mathcal{G}_9^C\}$, select a candidate planar graph which is relatively more

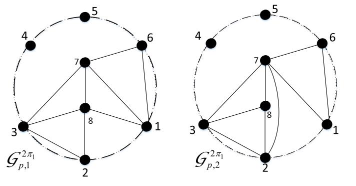

Fig. 8. Two planar graphs extracted from $\mathcal{G}^{2\pi_1}$

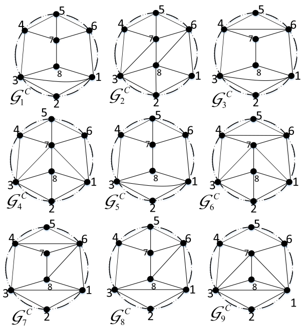

Fig. 9. Candidate planar graphs

populated with respect to edges, here we select $\mathcal{G}_9^C$. Express the selected graph as a set of non-linear multivariate polynomials $\mathbf{F}$, as described in section II.B, further obtain the set of Gröbner basis polynomials $\widehat{\mathbf{F}}$ using Buchberger's algorithm [10] form a set $\mathbf{F}$. We now solve for unknown vector $\mathbf{w}$ as discussed in (12) to get a set as given in (13). We use Matlab's symbolic toolbox [1] to implement the above objective and the reconstruction algorithm. There are several solutions to the objective as defined in (12), which satisfies the response matrix $\Lambda$ and one of them is $\mathbf{w} = \begin{bmatrix} \frac{1}{2} & \sigma_1 & \sigma_2 & 1 & \sigma_3 & \frac{1}{3} & 1 & \sigma_4 & \sigma_5 & z_1 & z & \sigma_6 & z_2 & 1 \end{bmatrix}^T$, where

$$\sigma_1 = \frac{\begin{pmatrix} 253z + 253z_1 + 231z_2 + 253zz_1 - 669zz_2 \\ -825z_1z_2 - 825zz_1z_2 - 23 \end{pmatrix}}{(825z_2 - 253)}$$

$$\sigma_2 = \frac{17}{26} - \frac{2z_2}{14}$$

$$\sigma_3 = \frac{87z_2 - 35}{300z_2 - 92}$$

$$\sigma_4 = \frac{-\begin{pmatrix} 253z + 253z_1 + 156z_2 + 253zz_1 - 715zz_2 \\ -825z_1z_2 + 150zz_2^2 - 825zz_1z_2 \end{pmatrix}}{253z - 825zz_2}$$

$$\sigma_5 = \frac{-\begin{pmatrix} 253z + 253z_1 + 156z_2 + 253zz_1 - 669zz_2 \\ -825z_1z_2 - 825zz_1z_2 \end{pmatrix}}{253z_1 + 156z_2 - 825z_1z_2}$$

$$\sigma_6 = \frac{13z_2}{75z_2 - 23}$$

Here, $z, z_1, z_2$ are free variables, if we choose $\begin{bmatrix} z & z_1 & z_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 1 \end{bmatrix}$, we get $\mathbf{w} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{4} & 1 & 1 \end{bmatrix}^T$. The corresponding network $\Gamma^*$ is given in Figure 10, . The network $\Gamma^* = (\mathcal{G}^*, \gamma)$ exactly matches the original network $\Gamma$ as shown in Figure 4. Since only boundary information is available to us we get infinitely many solutions, if more

[1]Toolbox, Symbolic Math. "Matlab." Mathworks Inc (1993).

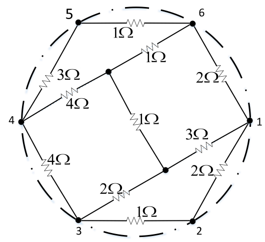

Fig. 10. The network $\Gamma^*$

information from network's interior is available the number of possible solutions can be reduced.

## V. Conclusion

This article proposes novel algorithms which uses the response matrix to arrive at a set of circular pairs which forms a basis for the construction of the set of candidate planar graphs. We then use Gröbner basis which help's characterize the set of all possible network topologies satisfying the given response matrix. The numerical example demonstrate the effectiveness of the proposed algorithm. The proposed algorithm is restricted to smaller passive resistive networks due to its high worst case computational complexity which also involves computing solutions to nonlinear multivariate polynomials. Future study will focus on developing techniques for reconstructing large passive resistive electrical networks with low computing cost and developing techniques to find the original solution.

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