EE 342: HW 6

Spr 19
Due Fri Mar 1

We will continue working on the principle of total probability and Bayes rule some more in this set, as well as begin working a little bit on random variables.

Prob 1 Let X_1, X_2 be random variables defined on a sample space Ω. Here $X_1 : \Omega \to \{1, \ldots, 6\}$ with the probability of each event $\{\omega \in \Omega : X_1(\omega) = i\}$, where $i \in \{1, \ldots, 6\}$, being $\frac{7}{6i(i+1)}$. Henceforth, as in class, we will abbreviate $\{\omega \in \Omega : X_1(\omega) = i\}$ by $\{X_1 = 1\}$. Given X_1,

$$X_2 = \begin{cases} (X_1 + 1) \mod 6 & \text{w.p. } 2/3 \\ (X_1 - 1) \mod 6 & \text{w.p. } 1/3. \end{cases}$$

Here $a \mod 6$ refers to the remainder when a is divided by 6. Therefore, we will have $6 \mod 6 = 0$, $8 \mod 6 = 2$, and $3 \mod 6 = 3$.

1. What is the probability of each of the events $\{X_2 = i\}$, for $i \in \{0, \ldots, 5\}$?
2. Given $X_2 = 0$, what is the conditional probability of each of the events $\{X_1 = i\}$, for $i \in \{1, \ldots, 6\}$.

Prob 2 Considering generating permutations of the set $N = \{1, \ldots, n\}$ as follows. First we pick a number from N at random, call it P_1. In the next step, we pick a number from $N \backslash \{P_1\}$ at random[1], call it P_2. And so on. Recall the example we did in class on Wed Feb 20.

1. What is the probability of each of the events $P_2 = j$, $j \in N$?
2. Given that $P_2 = 1$, what is the probability $P_1 = i$, $i \in N$?

Prob 3 There are two coins. The first has a bias p (namely probability of heads is p), and the second has a bias q. You pick one of the two coins at random, and toss the coin twice (conditioned on your choice of coin, you can assume that the tosses are independent).

[1] $A \backslash B$ refers to the set of all elements in A that are not in B

1. What is the probability of each of the events—2 heads, 2 tails, first is heads and second is tails, and vice versa?

2. Suppose $p \neq q$. Show that the event "heads in the first toss" is *not* independent of "heads in the second toss"? Can you explain why you get this result?

Prob 4 We follow up on HW 5, prob 5. Read the solutions given for HW 5 first. We wrote the sample space with events K to denote the probability you knew the answer and R to denote the probability the answer was right. You guess if you do not know, ie, the event you guess is K^c. We saw that the conditional probability $\mathbb{P}(K|R)$ depends on $\mathbb{P}(K)$.

As mentioned in the solutions, the quantity $\mathbb{P}(K)$ is usually called the *prior* probability (as in prior to the observation of whether the answer was right/wrong), and the quantity $\mathbb{P}(K|R)$ is called the posterior (after we make the observation).

With this in mind, answer the following questions.

1. If you guess all the time (whether or not you know the answer), do you still need $\mathbb{P}(K)$ to compute $\mathbb{P}(K|R)$?

2. Do you need $\mathbb{P}(K)$ to compute $\mathbb{P}(K|R^c)$? If not, what is the numerical value of $\mathbb{P}(K|R^c)$? Does it make sense?

3. Suppose there are two students, Focused (F) and Distracted (D). The student F has $\mathbb{P}(K) = 3/4$ and D has probability $\mathbb{P}(K) = 1/3$ (and their probabilities are independent). But F is so focused on the problem and D is so distracted by other things that they both forget to write their names on their exams!

   Suppose you find one exam with the correct answer and one with the wrong one. What is the conditional probability that F got it wrong and D got it correct?

Prob 5 We will continue with HW 5, Prob 6 here. New research has now concluded that there is a visual marker that can help us guess if a person is infected or not—a person with a visual marker has a 3/4 chance the person of being actually infected and a person without one has a 1/4 chance of being actually infected.

You do the same test as before. But this time, you keep picking people at random till you find someone who has the visual marker. Only then will you administer the test as before. The results of the test are independent of the presence or absence of the visual marker.

Read the solutions to HW 5 Prob 6. Again, you do the test twice. Should you administer the test on the same person twice or on two different people?