Some Imp. Integrals

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Gamma function:-

- $\Gamma n = (n-1)!$ if n is integer
- $\Gamma n = (n-1)\Gamma(n-1)$
- $\Gamma \frac{1}{2} = \sqrt{\pi}$
- $\Gamma 1 = \Gamma 2 = 1$
- $\int_0^\infty e^{-x} \cdot x^{n-1} dx = \Gamma n$
- $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$
- $\int_0^\infty e^{-ax} \cdot \cos bx \, dx = \frac{a}{a^2+b^2}$
- $\int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx).$$

- $L\{x^n\} = \frac{\Gamma n+1}{p^{(n+1)}}$
- $L\{e^{ax}\} = \frac{1}{(p-a)}$
- $L\{1\} = \frac{1}{p}$
- $L\{e^{-ax}\} = \frac{1}{p+a}$
- $L\{\sin ax\} = \frac{a}{p^2+a^2}$
- $L\{\cos ax\} = \frac{p}{p^2+a^2}$
- $L\{\sinh ax\} = \frac{a}{p^2-a^2}$
- $L\{\cosh ax\} = \frac{p}{p^2-a^2}$

Integral Transform

$$I\{f(x)\} = \int_{-\infty}^{\infty} f(x)\cdot K(p,x)\,dx$$

where $K(p,x)$ is called kernel of the transform & p is the parameter.

Laplace Transform →

$$K(p,x) = \begin{cases} e^{-px}, & x>0 \\ 0, & x<0 \end{cases}$$

$$\boxed{L\{f(x)\} = f(p) = \int_0^{\infty} e^{-px} f(x)\,dx}$$

This is laplace transform of function $f(x)$.

eg $$L\{x^n\} = \int_0^{\infty} e^{-px}\cdot x^n\,dx$$

$$= \frac{\Gamma(n+1)}{p^{n+1}}$$

$$L\{e^{ax}\} = \int_0^{\infty} e^{-px}\cdot e^{-ax}\,dx = \int_0^{\infty} e^{(a-p)x}\,dx = \int_0^{\infty} e^{-(p-a)x}\,dx$$

$$= \left[\frac{e^{-(p-a)}}{-(p-a)}\right]_0^{\infty} = \frac{0+1}{(p-a)} = \frac{1}{(p-a)}$$

Qu find laplace transform of $f(x) = \begin{cases} x+1 & \text{if } x<1 \\ 0 & \text{if } x>1 \end{cases}$

$$L\{f(x)\} = \int_0^{1} e^{-px}\cdot(x+1)\,dx + \int_1^{\infty} e^{-px}\cdot 0\,dx$$

$$= \int_0^{1} e^{-px}(x+1)\,dx$$

$\int e^{ax} \sin$

$$= (x+1)\int_0^1 e^{-px}dx - \int_0^1\left(\frac{d}{dx}(x+1)\int e^{-px}dx\right)dx$$

$$= (x+1)\left[\frac{e^{-px}}{-p}\right]_0^1 - \int \frac{e^{-px}}{-p}dx$$

$$= \left[(x+1)\left[\frac{e^{-px}}{-p}\right] + \frac{1}{p}\left(\frac{e^{-px}}{-p}\right)\right]_0^1$$

$$= \left(2\cdot\frac{e^{-p}}{-p} - \frac{1}{p^2}e^{-p}\right) - \left(\frac{1}{-p} - \frac{1}{p^2}\right)$$

$$= \frac{-2e^{-p}}{p} - \frac{e^{-p}}{p^2} + \frac{1}{p} + \frac{1}{p^2}$$

$$= \frac{1}{p}(-2e^{-p}+1) + \frac{1}{p^2}(1-e^{-p})$$

Properties of Laplace Transform →

① Linear Property →

$$L\{c_1 f_1(x) \pm c_2 f_2(x) \pm \cdots\} = c_1\cdot L\{f_1(x)\} \pm c_2\cdot L\{f_2(x)\} \pm \cdots$$

eg. $L\{e^{2t} - \sin 3t + t^4\} = L\{e^{2t}\} - L\{\sin 3t\} + L\{t^4\}$

$$= \frac{1}{p-2} - \frac{3}{p^2+9} + \frac{\lfloor 5}{p^5}$$

Qu. $L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\}$

$$= \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\}$$

$$= \frac{1}{2}\left[\frac{1}{p} + \frac{p}{p^2+4}\right]$$

② Scale Change Property→ if $L\{F(t)\} = f(p)$ then

$$L\{F(at)\} = \frac{1}{a} f\left(\frac{p}{a}\right)$$

wkt, $L\{f(t)\} = f(p) = \int_0^\infty e^{-pt} f(t)\, dt$. —— ①

$$L\{f(at)\} = \int_0^\infty e^{-pt} \cdot F(at)\, dt$$

put $at = u$

$dt = \frac{1}{a} du$

Substitute $L\{F(u)\} = \frac{1}{a}\int_0^\infty e^{-p\left(\frac{u}{a}\right)} \cdot F(u)\, du$

$$\Rightarrow L\{F(u)\} = \frac{1}{a}\int_0^\infty e^{-\left(\frac{p}{a}\right)u} F(u)\, du$$

$$= \cancel{\frac{1}{a} L\{F(\tfrac{p}{a})\}}\ \frac{1}{a} f\left(\frac{p}{a}\right).$$

③ **First Shifting theorem →** If $L\{F(t)\} = f(p)$ then

$$\boxed{L\{e^{at} \cdot F(t)\} = f(p-a)}$$

wkt $L\{F(t)\} = f(p) = \int_0^\infty e^{-pt} F(t)\, dt$

$$\therefore L\{e^{at} F(t)\} = \int_0^\infty e^{-pt} \cdot e^{at} F(t)\, dt$$

$$= \int_0^\infty e^{-(p-a)t} F(t)\, dt$$

$$= \underline{\underline{f(p-a).}}$$

eg → Find, $L\{e^{2t} \sin 3t\}$

$$L\{\sin 3t\} = \frac{3}{p^2+9}$$

By first shifting theorem -

$$\Rightarrow L\{e^{2t} \sin 3t\} = \frac{3}{(p-2)^2+9} = \frac{3}{p^2-4t+13}$$

eg Find $L(e^{\cancel{8}-t} \cdot t^3)$

$$\Rightarrow L\{t^3\} = \frac{\lceil 4}{p^4} = \frac{6}{p^4}$$

by FST

$$L\{e^{-t} t^3\} = \frac{6}{(p+1)^4}$$

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④ Second shifting theorem →

If $L\{F(t)\} = f(P)$ and $G(t) = \begin{cases} F(t-a) & , t > a \\ 0 & , t < a \end{cases}$

then $L\{G(t)\} = e^{-ap} f(P)$

⑤ Laplace transform of derivative →

If $L\{F(t)\} = f(P)$ then

$$\boxed{L\{F'(t)\} = pf(P) - F(0)}$$

Similarly

* $L\{F''(t)\} = P^2 f(P) - P F(0) - F'(0)$

* $L\{F'''(t)\} = P^3 f(P) - P^2 F(0) - P F'(0) - F''(0)$

* In general

$$L\{F^{n\text{th}}(t)\} = P^n f(P) - P^{(n-1)} F(0) - P^{(n-2)} F'(0) - \cdots F^{(n-1)}(0)$$

⑥ Laplace transform of integral →

If $L\{F(t)\} = f(P)$ then,

$$L\left\{\int_0^t F(t)\,dt\right\} = \frac{f(P)}{P}$$

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⑦ Laplace transform of multiplication by powers of $t \Rightarrow$

If $L\{F(t)\} = f(p)$ then

$$L\{t^n \cdot F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p).$$

eg $\Rightarrow$ $L\{t^2 \cos t\} =$

$$L\{\cos t\} = \frac{p}{p^2+1}.$$

$$L\{t^2 \cos t\} = (-1)^2 \cdot \frac{d^2}{dp^2} \cdot \left(\frac{p}{p^2+1}\right)$$

$$= \frac{d^2}{dp^2}\left(\frac{p}{p^2+1}\right).$$

$$= \frac{d}{dp}\left\{\frac{(p^2+1) - 2p^2}{(p^2+1)^2}\right\} = \frac{d}{dp}\left\{\frac{1-p^2}{(1+p^2)^2}\right\}$$

$$= \frac{(p^2+1)^2\{-2p\} - (1-p^2) \cdot 2(1+p^2) \cdot 2p}{(1+p^2)^4}$$

$$= \frac{(p^4+1+2p^2)(-2p) - (1-p^2)(4p+4p^3)}{(1+p^2)^4}$$

$$= \frac{-2p^5 - 2p - 4p^3 - 4p - 4p^3 + 4p^3 + 4p^5}{(1+p^2)^4}$$

$$= \frac{2p^5 - 4p^3 - 6p}{(1+p^2)^4}$$

(8) Laplace transform of divident by ~~power of~~ t →

If $L\{F(~~p~~t)\} = f(p)$ then,

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p)\,dp$$

eg find ~~to~~ $L\left\{\frac{e^{at} - e^{bt}}{t}\right\}$

$$L\{e^{at} - e^{bt}\} = L\{e^{at}\} - L\{e^{bt}\}$$

$$= \frac{1}{p-a} - \frac{1}{p-b}$$

$$L\left\{\frac{e^{at} - e^{bt}}{t}\right\} = \int_p^\infty \left(\frac{1}{p-a} - \frac{1}{p-b}\right)dp$$

$$= \left|\ln(p-a) - \ln(p-b)\right|_p^\infty$$

$$= \left|\ln\left(\frac{(p-a)}{(p-b)}\right)\right|_p^\infty$$

$$= \left[\ln\frac{(1-\frac{a}{p})}{(1-\frac{b}{p})}\right]_0^\infty$$

$$= \ln 1 - \ln\frac{(p-a)}{(p-b)} = -\ln\left(\frac{p-a}{p-b}\right)$$

$$= \ln\left(\frac{p-b}{p-a}\right).$$

(9) Laplace transform of periodic function →

Let $F(t)$ is a periodic function with period ω then

$$L\{F(t)\} = \frac{1}{1-e^{-p\omega}} \int_0^{\omega} e^{-pt} F(t)\, dt.$$

eg Find ~~of~~ Laplace transform of $F(t) = t^2 \quad 0<t<2$

$\& \; F(t+2) = F(t)$

Here the given function $F(t)$ is a periodic function with period 2 so its Laplace transform

$$L\{F(t)\} = \frac{1}{1-e^{-2p}} \int_0^{2} e^{-pt} F(t)\, dt$$

$$= \frac{1}{1-e^{-2p}} \int_0^{2} e^{-pt}\, t^2\, dt$$

$$= \frac{1}{1-e^{-2p}} \left\{ e^{-pt} \cdot \frac{t^3}{3} - \int \frac{e^{-pt}}{-p} \cdot \frac{t^3}{3}\, dt \right\}_0^2$$

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Laplace transform of some special function→

Unit Step function→ Heavy side unit step function, generally denoted by $H(t-a)$ or $u(t-a)$ and defined as

$$H(t-a) = U(t-a) = \begin{cases} 1, & \text{if } t > a \\ 0, & t < a \end{cases}$$

Now $L\{H(t-a)\} = \int_0^\infty e^{-pt} H(t-a)\,dt$

$$= \int_0^a e^{-pt}\cdot 0\,dt + \int_a^\infty e^{-pt}\cdot 1\,dt$$

$$= 0 + \left[\frac{[e^{-pt}]}{-p}\right]_a^\infty = 0 + \frac{e^{-pa}}{p}$$

Dirac delta function→ (unit impulse function)

$$\delta_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & 0 < t \leq \epsilon \\ 0, & t > \epsilon \end{cases}$$

$$L\{\delta_\epsilon(t)\} = \int_0^\infty e^{-pt}\cdot \delta_\epsilon(t)\,dt$$

$$= \int_0^\epsilon e^{-pt}\cdot \frac{1}{\epsilon}\,dt + \int_\epsilon^\infty e^{-pt}\cdot 0\,dt = \int_0^\epsilon e^{-pt}\cdot\frac{1}{\epsilon}\,dt$$

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$$= \frac{1}{\epsilon}\left[\frac{e^{-pt}}{-p}\right]_0^{\epsilon} = \frac{1}{\epsilon}\left[\frac{e^{-p\epsilon}}{-p} - \frac{e^{-0}}{-p}\right]$$

$$= \frac{1}{\epsilon}\left[\frac{e^{-p\epsilon}}{-p} + \frac{1}{p}\right] = \frac{1}{\epsilon p}\left(1 - e^{-\epsilon p}\right)$$

if $\epsilon \to 0$.

$$L\{\delta_\epsilon(t)\} = \lim_{\epsilon \to 0} \frac{1 - e^{-p\epsilon}}{\epsilon p} = \underline{1}$$

$$\boxed{L\{\delta(t)\} = 1}$$ ⊛

Error function →

$$\boxed{\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx.}$$

pt

$$L\{\operatorname{erf}(\sqrt{t})\} = \frac{1}{p\sqrt{p+1}}$$

we have $\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) dx$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots\right]_0^{\sqrt{t}}$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}x^2 = 1 - \frac{x}{2} - \frac{3}{8}x^2 + \cdots$$

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$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\left[\sqrt{t} - \frac{t\sqrt{t}}{3} + \frac{t^2\sqrt{t}}{5\cdot 2!} - \frac{t^3\sqrt{t}}{7\cdot 3!} + \cdots\right]$$

$$\therefore L\{\operatorname{erf}(\sqrt{t})\} = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{p^{3/2}} - \frac{\Gamma 5/2}{3p^{5/2}} + \frac{\Gamma 7/2}{5\cdot 2!\, p^{7/2}} - \cdots\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}}{p^{3/2}} - \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma 1/2}{3\cdot p^{5/2}} + \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}{5\,2!\; p^{7/2}} + \cdots\right]$$

$$= 2\left[\frac{1}{2p^{3/2}} - \frac{1}{4p^{3/2}} + \frac{3}{16\, p^{7/2}} - \cdots\right]$$

$$= \frac{1}{p^{3/2}}\left[1 - \frac{1}{2p} + \frac{3}{8p^2} - \cdots\right]$$

$$= \frac{1}{p^{3/2}}\cdot\left(1+\frac{1}{p}\right)^{-\frac{1}{2}} = \frac{1}{p^{3/2}}\cdot\left(\frac{p+1}{p}\right)^{-\frac{1}{2}} = \frac{1}{p^{3/2}}\left(\frac{p}{(p+1)}\right)^{1/2}$$

$$\boxed{L\{\operatorname{erf}(\sqrt{t})\} = \frac{1}{p\sqrt{p+1}}}$$

Q. $L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{\lfloor 3} + \frac{(\sqrt{t})^5}{\lfloor 5} - \cdots\right\}$

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Que- Find laplace transform of $t^2 \cdot e^{3t}$.

$$L\{t^2 \cdot e^{3t}\}$$

$$\cancel{L\{e^{3t}\} = \frac{1}{(p-3)}} \quad L\{t^2\} = \frac{\Gamma 3}{p^3} = \frac{2}{p^3}$$

$$\cancel{L\{t^2 \cdot e^{3t}\} =}$$

By first shifting theorem

$$L\{e^{3t} \cdot t^2\} = \frac{2}{(p-3)^3}$$

Qu $L\left\{\frac{\sin t}{t}\right\}$

$$L\{\sin t\} = \frac{1}{p^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{p^2+1}\,dp = \left[\tan^{-1} p\right]_p^\infty$$

$$= \left[\frac{\pi}{2} - \tan^{-1} p\right] = \cot^{-1} p$$

Que $L\{t^2 \cdot e^{-t} \cdot \sin 3t\}$

$$= L\{\sin 3t\} = \frac{3}{p^2+9}$$

$$L\{e^{-t} \cdot \sin 3t\} = \frac{3}{(p+1)^2+9} = \frac{3}{p^2+2p+10}$$

$$L\{t^2 \cdot e^{-t} \cdot \sin 3t\} = (-1)^2 \cdot \frac{d^2}{dp^2}\left(\frac{3}{p^2+2p+10}\right)$$

$$= \frac{d^2}{dp^2}\left(\frac{3}{p^2+2p+10}\right)$$

$$= 3 \cdot \frac{d^2}{dp^2}\left[\frac{1}{(p+1)^2+9}\right]$$

$$\Rightarrow 3 \cdot \frac{d}{dp}\left\{\frac{\left((p+1)^2+9\right)\cdot 0 - 1 \cdot 2(p+1)}{\left((p+1)^2+9\right)^2}\right\}$$

$$= 3\frac{d}{dp}\left\{-\frac{2(p+1)}{(p^2+2p+10)^2}\right\}$$

$$= -6\left\{\frac{(p^2+2p+10)^2 \cdot 1 - (p+1)\cdot 2(p^2+2p+10)(2p+2)}{(p^2+2p+10)^4}\right\}$$

$$= -6\left\{\frac{(p^2+2p+10) - 4p^2+4p+4p+4}{(p^2+2p+10)^3}\right\}$$

$$= -6\left\{\frac{\overset{-3p^2-6p+6}{\cancel{-p^2-2p-8}}}{(p^2+2p+10)^2}\right\} = 18\left\{\frac{p^2+2p-2}{(p^2+2p+10)^2}\right\}$$

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Qu find laplace transform of f(t)

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

f(t) is a periodic function

$$L\{f(t)\} = \frac{1}{1-e^{-p\omega}} \int_0^{\omega} e^{-pt} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi p}} \int_0^{2\pi} e^{-pt} \cdot f(t)\, dt$$

$$= \frac{1}{1-e^{-2\pi p}} \int_0^{\pi} e^{-pt} \sin t\, dt + 0.$$

$$= \frac{1}{1-e^{-2\pi p}} \left[\frac{e^{-pt}}{p^2+1} (-p\sin t - \cos t) \right]_0^{\pi}$$

$$= \frac{1}{(1-e^{-2\pi p})} \cdot \left[\frac{e^{-p\pi}}{p^2+1} (0+1) - \frac{1}{p^2+1} (0-1) \right.$$

$$= \frac{1}{(p^2+1)(1-e^{-2\pi p})} \left[e^{-p\pi} + 1 \right]$$

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Qu find $L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\}$

$$L\left\{\int_0^t \frac{\sin t}{t}\right\} = \frac{1}{p} L\left\{\frac{\sin t}{t}\right\} \quad \text{——— (1)}$$

WKT $L\left\{\frac{f(t)}{t}\right\} = \int_p^\infty L\{f(t)\}\,dp$

$$= \int_p^\infty \frac{1}{p^2+1}\,dp = \left[\tan^{-1} p\right]_p^\infty = \frac{\pi}{2} - \tan^{-1} p$$

$$= \underline{\underline{\cot^{-1} p}}$$

from (1)

$$L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \underline{\underline{\frac{1}{p}\cot^{-1} p}}.$$

Qu $L\left\{\frac{\cos t}{t}\right\} = \int_p^\infty L\{\cos t\}\,dp = \int_p^\infty \frac{p}{p^2+1}\,dp$

$$= \left[\frac{1}{2}\log_e|p^2+1|\right]_p^\infty \longrightarrow \text{does not exist}$$

$\because \lim_{p\to\infty} \log|p^2+1| \to$ does not exist.

(*) The second shifting theorem can be also stated as

$$\boxed{L\{f(t-a)u(t-a)\} = e^{-ap} \cdot L\{f(t)\}}$$

eg find $L\{(t-2)^2 u(t-2)\}$

$a = 2$
$f(t) = t^2$

$$L\{(t-2)^2 u(t-2)\} = e^{-2p} \cdot L\{t^2\}$$

$$= e^{-2p} \cdot \frac{2}{p^3}$$

Qu Find $L\{t^2 \cdot u(t-3)\}$

$$= L\{(t-3+3)^2 \cdot u(t-3)\}$$

$a = 3$
$f(t) = (t+3)^2$

$$L\{(t+3-3)^2 u(t-3)\} = e^{-3p} \cdot L\{(t+3)^2\}$$

$$= e^{-3p} L\{t^2 + 9 + 6t\}$$

$$= e^{-3p}\left[L\{t^2\} + 6L\{t\} + L\{9\}\right]$$

$$= e^{-3p}\left[\frac{2}{p^3} + 6\frac{1}{p^2} + \frac{9}{p}\right]$$

Qu Find $L\{\sin t \cdot u(t-\pi)\}$

$$L\{\sin(t+\pi-\pi) \cdot u(t-\pi)\}$$

$$= e^{-\pi p} L\{\sin(t+\pi)\} = -e^{-\pi p} L\{\sin t\}$$

$$= -e^{-\pi p} \cdot \frac{1}{1+p^2}$$

INVERSE LAPLACE TRANSFORM →

(*) $L\{1\} = \frac{1}{p} \Rightarrow L^{-1}\left\{\frac{1}{p}\right\} = 1$

(*) $L\{e^{at}\} = \frac{1}{p-a} \Rightarrow L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$

(*) $L\{t^n\} = \frac{\lceil n+1}{p^{n+1}} \Rightarrow L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{\lceil n+1}$ $\left\{L^{-1}\left\{\frac{1}{p^n}\right\} = \frac{t^{n-1}}{\lceil n}\right\}$

(*) $L\{\sin at\} = \frac{a}{p^2+a^2} \Rightarrow L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{\sin at}{a}$

(*) $L\{\cos at\} = \frac{p}{p^2+a^2} \Rightarrow L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

(*) $L\{\delta(t)\} = 1 \Rightarrow L^{-1}\{1\} = \delta(t)$

① Linear Property →

$$L^{-1}\{c_1 f_1(p) \pm c_2 f_2(p) \pm c_3 f_3(p) \pm \cdots\} =$$

$$c_1 L^{-1}\{f_1(p)\} \pm c_2 L^{-1}\{f_2(p)\} \pm c_3 L^{-1}\{f_3(p)\}$$

Qu find $L^{-1}\left\{\frac{1}{p^2} - \frac{p+1}{p^2+3p+2} + 3\right\}$

$$L^{-1}\left\{\frac{1}{p^2}\right\} - L^{-1}\left\{\frac{p+1}{p^2+3p+2}\right\} + 3\,L^{-1}\{1\}$$

$$t - L^{-1}\left\{\frac{p+1}{(p+1)(p+2)}\right\} + 3\delta(t)$$

$$t - e^{-2t} + 3\delta(t)$$

② First Shifting theorem →

if $L^{-1}\{f(p)\} = F(t)$

then $L^{-1}\{f(p-a)\} = e^{at} \cdot F(t)$

$$L^{-1}\{f(p+a)\} = e^{-at} F(t)$$

Qu Find $L^{-1}\left\{\frac{1}{p^2+2p+5}\right\} = L^{-1}\left\{\frac{1}{(p+1)^2+4}\right\}$

$$= e^{-t}\,\frac{\sin 2t}{2}$$

Qu → Find $L^{-1}\left\{\frac{p+1}{p^2+4p+8}\right\}$

$$L^{-1}\left\{\frac{p+1}{(p+2)^2+4}\right\} = L^{-1}\left\{\frac{(p+2)-1}{(p+2)^2+4}\right\}$$

$$e^{-2t}\,L^{-1}\left\{\frac{p-1}{p^2+4}\right\}$$

$$e^{-2t}\left[L^{-1}\left\{\frac{p}{p^2+4}\right\} - L^{-1}\left\{\frac{1}{p^2+4}\right\}\right]$$

$$e^{-2t}\left[\cos 2t - \frac{\sin 2t}{2}\right]$$

③ Second shifting theorem →

$$L\{F(t-a)\,u(t-a)\} = e^{-ap} \cdot f(p)$$

$$L^{-1}\{e^{-ap} \cdot f(p)\} = F(t-a) \cdot u(t-a)$$

where $F(t) = L^{-1}\{f(p)\}$

eg $L^{-1}\left\{\frac{e^{-2p}}{p^3}\right\} =$

$a = 2$

$f(p) = \frac{1}{p^3} \Rightarrow F(t) = L^{-1}\left\{\frac{1}{p^3}\right\} = \frac{t^2}{2}$

$$L^{-1}\left\{\frac{e^{-2p}}{p^3}\right\} = \frac{(t-2)^2}{2} \cdot u(t-2)$$

Qu find $L^{-1}\left\{\frac{1}{p^3-1}\right\} = L^{-1}\left\{\frac{1}{(p-1)(p^2+p+1)}\right\}$

$$\frac{1}{(p-1)(p^2+p+1)} = \frac{A}{(p-1)} + \frac{Bp+C}{p^2+p+1}$$

$$= \frac{A(p^2+p+1) + (Bp+C)(p-1)}{(p-1)(p^2+p+1)}$$

$A+B=0 \Rightarrow \boxed{B=-A}$

$$= \frac{Ap^2+Ap+A + Bp^2 - Bp + Cp - C}{(p-1)(p^2+p+1)}$$

$A-B+C=0$, $\cancel{A-C=1}$

$\Downarrow$ $\boxed{A=1+C}$

$$= \frac{(A+B)p^2 + (A+C-B)p + (A-C)}{(p-1)(p^2+p+1)}.$$

$A+A+A-1=0$

$\boxed{A=\frac{1}{3}}$

$\boxed{B=\frac{-1}{3}}$

$\boxed{C=\frac{-2}{3}}$

$\Rightarrow \cancel{A=-B}$

$A+B=0 \Rightarrow \cancel{2B+C=0} \Rightarrow \cancel{2B+C=0}$

$A+C=0 \Rightarrow \cancel{B+2C=0} \Rightarrow \cancel{2B+4C=0}$

$A-B-C=1$ $\quad \frac{1}{3}$ $\quad \cancel{-3C=0}$

$\cancel{A=B+C}$ $\quad \boxed{C=0}$

$\cancel{A+B}+A+A+A=1$

$\boxed{\cancel{A=\frac{1}{3}}} \Rightarrow \boxed{\cancel{B=\frac{-1}{3}}, \cancel{C=\frac{-1}{3}}}$

On solving

$$L^{-1}\left\{\frac{1}{p^3-1}\right\} = \frac{1}{3}e^{t} - \frac{1}{3}e^{-t/2}\left(-\cos\frac{\sqrt{3}}{2}t + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$

Inverse laplace transform of derivative →

$\because L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} L\{F(t)\}$

$\Rightarrow L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\} = (-1)^n t^n L^{-1}\{f(p)\}$

$= (-1)^n t^n F(t)$

eg $L^{-1}\left\{\frac{2p}{(p^2+1)^2}\right\} = L^{-1}\left\{\frac{d}{dp}\frac{1}{p^2+1}\right\}$

$= (-1)^1 t^1 \cdot L^{-1}\left\{\frac{1}{p^2+1}\right\}$

$= -t \sin t$

~~eg $L^{-1}\{\log(1+p)\} = L^{-1}\left\{\frac{d}{dp}\frac{1}{1+p}\right\} = (-1)t\, L^{-1}\left\{\frac{1}{1+p}\right\}$~~

~~$= -t \cdot e^{t}$~~

eg $L^{-1}\{\log(1+p)\}$

$$L^{-1}\{f(p)\} = \frac{(-1)^n}{t^n} L^{-1}\left\{\frac{d^n}{dp^n} f(p)\right\}$$

$$= \frac{(-1)^1}{t} L^{-1}\left\{\frac{d}{dp}\log(1+p)\right\}$$

$$= \frac{-1}{t} e^{-t}$$

for $n=1$ $\boxed{L^{-1}\{f(p)\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{dp} f(p)\right\}}$

eg $L^{-1}\{\tan^{-1} p\}$

$\Rightarrow -\frac{1}{t} L^{-1}\left\{\frac{d}{dp} \tan^{-1} p\right\} = -\frac{1}{t} L^{-1}\left\{\frac{1}{1+p^2}\right\}$

$= -\frac{1}{t} \sin t$

Multiplication by $p \rightarrow$

If $L^{-1}\{f(p)\} = F(t)$ & $F(0) = 0$

then $\boxed{L^{-1}\{p f(p)\} = \frac{d}{dt} F(t)}$

Division by $p \rightarrow$

$L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t F(t)\, dt$

$\Rightarrow \boxed{L^{-1}\left\{\frac{f(p)}{p}\right\} = \int_0^t L^{-1}\{f(p)\}\, dt}$

eg $L^{-1}\left\{\frac{1}{p(p^2+4)}\right\} = L^{-1}\left\{\frac{\left(\frac{1}{p^2+4}\right)}{p}\right\}$

$= \int_0^t \frac{1}{2} \sin 2t\, dt = \frac{-1}{2}\left[\frac{\cos 2t}{2}\right]_0^t$

$= \frac{-1}{4}(\cos 2t - 1)$

Qu Find $L^{-1}\left\{\frac{1}{(p+1)(p^2+4)}\right\}$

$\frac{1}{(p+1)(p^2+4)} = \frac{A}{(p+1)} + \frac{Bp+C}{(p^2+4)}$

$\Rightarrow \frac{A(p^2+4) + (Bp+C)(p+1)}{(p+1)(p^2+4)}$

$\frac{1}{(p+1)(p^2+4)} = \frac{(A+B)p^2 + (B+C)p + (4A+C)}{(p+1)(p^2+4)}$

$A+B=0 \Rightarrow \boxed{B=-A}$

$B+C=0 \Rightarrow \boxed{C=-B=A}$

$4A+C=1$

$4A+A=1$

$\boxed{A=\frac{1}{5}} \Rightarrow \boxed{C=\frac{1}{5}}$ $\boxed{B=\frac{-1}{5}}$

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Qu find $L\{\sin t\, u(t-4)\}$

$L\{\sin(t-4+4)\cdot u(t-4)\}$

$= e^{-4p}\cdot L\{\sin(t+4)\}$

$= e^{-4p}\, L\{\sin t\cos 4 + \cos t \sin 4\}$

$= e^{-4p}\left[\cos 4\, L\{\sin t\} + \sin 4\, L\{\cos t\}\right]$

$= e^{-4p}\left[\cos 4\cdot \frac{1}{1+p^2} + \sin 4\cdot \frac{p}{1+p^2}\right]$

Qu ~~F(t)~~ =

$L\{K(t-2)[u(t-2)-u(t-3)]\}$

$K[L\{(t-2)\,u(t-2)\} - L\{(t-2)\,u(t-3)\}]$

$K[e^{-2p}\cdot L\{t\} - L\{(t-3+1)\,u(t-3)\}]$

$K[e^{-2p}\cdot \frac{1}{p^2} - e^{-3p}\cdot L\{t+1\}]$

$K\left[e^{-2p}\cdot\frac{1}{p^2} - e^{-3p}\left[\frac{1}{p^2}+\frac{1}{p}\right]\right]$

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Qu $L\left\{\frac{K\sin \pi t}{T}\left[u(t-2T) - u(t-3T)\right]\right\}$

$\frac{K}{T}L\{\sin\pi t\cdot u(t-2T) - \sin\pi t\cdot u(t-3T)\}$

$\frac{K}{T}\left[L\{\sin\pi t\cdot u(t-2T)\} - L\{\sin\pi t\cdot u(t-3T)\}\right]$

$\frac{K}{T}\left[L\{\sin(\pi t-2T+2T)\cdot u(t-2T)\} - L\{\sin(\pi t-3T+3T)\cdot u(t-3T)\}\right]$

$\frac{K}{T}\left[e^{-2Tp}\, L\{\sin(\pi t+2T)\} - e^{-3Tp}\, L\{\sin(\pi t+3T)\}\right]$

Convolution Theorem→

Convolution of two function $F(t)$ & $G(t)$ is

$$\boxed{F * G = \int_0^t F(u) \cdot G(t-u)\,du = \int_0^t F(t-u) \cdot G(u)\,du}$$

Statement: If $L^{-1}\{f(p)\} = F(t)$ & $L^{-1}\{g(p)\} = G(t)$

then $$L^{-1}\{f(p) \cdot g(p)\} = F * G$$

$$= \int_0^t F(u) \cdot G(t-u)\,du$$

$$= \int_0^t F(t-u) \cdot G(u)\,du$$

Proof→

Que $L^{-1}\left\{\frac{p}{(p^2+a^2)(p^2+b^2)}\right\}$

$= L^{-1}\left\{\frac{p}{(p^2+a^2)}\cdot\frac{1}{(p^2+b^2)}\right\}$

take $f(p) = \frac{p}{p^2+a^2} = \cos at$

$g(p) = \frac{1}{p^2+b^2} = \frac{\sin bt}{b}$.

Now by convolution theorem,

$L^{-1}\{f(p)\cdot g(p)\} = \int_0^t F(u)\cdot G(t-u)\,du$

$\Rightarrow L^{-1}\left\{\frac{p}{p^2+a^2}\cdot\frac{1}{p^2+b^2}\right\} = \cancel{\frac{1}{b}\int_0^t \cos u \sin}$

$= \frac{1}{2b}\int_0^t 2\cos au \cdot \sin b(t-u)\,du.$

$= \frac{1}{2b}\int_0^t \left[\sin(au+bt-bu) - \sin(au-bt+bu)\right] du$

$= \frac{1}{2b}\int_0^t \left[\sin(bt+(a-b)u) - \sin((a+b)u-bt)\right] du.$

$= \frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{(a-b)} + \frac{\cos((a+b)u-bt)}{(a+b)}\right]_0^t$

$= \frac{1}{2b}\left[\frac{-\cos at}{(a-b)} + \frac{\cos at}{(a+b)} + \frac{\cos bt}{(a-b)} - \frac{\cos bt}{(a+b)}\right]$

$= \frac{1}{2b}\left[\frac{\cos bt - \cos at}{(a-b)} + \frac{\cos at - \cos bt}{(a+b)}\right]$

$= \frac{1}{2b}\left[\cos bt\left(\frac{1}{a-b} - \frac{1}{a+b}\right) + \cos at\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right]$

$= \frac{1}{2b}\left[\cos bt \cdot \frac{2b}{a^2-b^2} + \frac{\cos at(-2b)}{a^2-b^2}\right]$

$= \frac{\cos bt - \cos at}{a^2-b^2}$

Que Find inverse laplas transform of

$L^{-1}\left\{\frac{p}{(p+2)(p^2+4)}\right\} = L^{-1}\left\{\frac{p}{(p^2+4)}\cdot\frac{1}{(p+2)}\right\}$

$g(p) = \frac{p}{p^2+4} \Rightarrow G(t) = \cos 2t.$

$f(p) = \frac{1}{p+2} \Rightarrow F(t) = e^{-2t}$

$L^{-1}\left\{\frac{1}{(p+2)}\cdot\frac{p}{p^2+4}\right\} = e^{-2t} * \cos 2t.$

$= \int_0^t e^{-2(t-u)}\cdot \cos 2u\,du$

$$= \int_0^t e^{-2t+2u} \cdot \cos 2u \, du$$

$$= \frac{1}{e^{2t}} \int_0^t e^{2u} \cdot \cos 2u \, du$$

$$= \frac{1}{e^{2t}} \left[\frac{e^{2u}}{4+4} (2\cos 2u + 2\sin 2u) \right]_0^t$$

$$= \frac{1}{4e^{2t}} \left[e^{2u} (\cos 2u + \sin 2u) \right]_0^t$$

$$= \frac{1}{4e^{2t}} \left[e^{2t} (\cos 2t + \sin 2t) - 1 \cdot (1) \right]$$

$$= \frac{1}{4e^{2t}} \left[e^{2t} (\cos 2t + \sin 2t) - 1 \right]$$

$$= \frac{1}{4} \left[\cos 2t + \sin 2t - e^{-2t} \right]$$

Qu. $L^{-1}\left\{ \frac{1}{p^3(p^2+1)} \right\}$

$f(p) = \frac{1}{p^3} \Rightarrow F(t) = \frac{t^2}{2}$

$g(p) = \frac{1}{p^2+1} \Rightarrow G(t) = \sin t$

$$L^{-1}\left\{ \frac{1}{p^3} \cdot \frac{1}{p^2+1} \right\} = \int_0^t \frac{u^2}{2} \sin(t-u) \, du$$

$$= \frac{1}{2} \left[u^2 \cdot \cos(t-u) + 2u \sin(t-u) - 2\cos(t-u) \right]_0^t$$

$$= \frac{1}{2} \left[t^2 + 2t \times 0 - 2 \right] - \frac{1}{2} \left[-2\cos t \right]$$

$$= \frac{t^2}{2} - 1 + \cancel{2}\cos t$$

Heavi side's expansion formula →

Let $f(p)$ & $g(p)$ are two polynomials in p where degree of $f(p)$ is less than degree of $g(p)$ and $g(p)$ can be factorised in ~~two~~ linear factors as –

$$g(p) = (p-\alpha_1)(p-\alpha_2)(p-\alpha_3) \cdots (p-\alpha_n)$$

then

$$L^{-1}\left\{ \frac{f(p)}{g(p)} \right\} = e^{\alpha_1 t} \frac{f(\alpha_1)}{g'(\alpha_1)} + e^{\alpha_2 t} \frac{f(\alpha_2)}{g'(\alpha_2)} + \cdots \cdots e^{\alpha_n t} \cdot \frac{f(\alpha_n)}{g'(\alpha_n)}$$

Qu find $L^{-1}\left\{\frac{p^2+2}{p^3-6p^2+11p-6}\right\}$

$f(p) = p^2+2$

$g(p) = p^3-6p^2+11p-6 = (p-1)(p-2)(p-3)$

$g'(p) = 3p^2-12p+11$ $\boxed{\alpha = 1, 2, 3}$

$L^{-1}\left\{\frac{p^2+2}{p^3-6p^2+11p-6}\right\} = e^{t}\cdot\frac{3}{2} + e^{2t}\cdot\frac{6}{(-1)} + e^{3t}\frac{11}{2}$

$= \boxed{\frac{3}{2}e^{t} - 6e^{2t} + \frac{11}{2}e^{3t}}$

Qu $L^{-1}\left\{\frac{2p^2+5p-4}{p^3+p^2-2p}\right\}$

$f(p) = 2p^2+5p-4$

$g(p) = p^3+p^2-2p = (p-0)(p^2+p-2)$

$= p(p+2)(p-1)$

$\Rightarrow \boxed{\alpha = 0, 1, -2}$

$g'(p) = 3p^2+2p-2.$

$f(0) = -4,\ f(1) = 3,\ f(-2) = -6$

$g'(0) = -2\ \ g'(1) = 3\ \ g'(-2) = 6$

$L^{-1}\left\{\frac{2p^2+5p-4}{p^3+p^2-2p}\right\} = e^{0t}\cdot\frac{-4}{-2} + e^{t}\cdot\frac{3}{3} + e^{-2t}\cdot\frac{-6}{6}$

$= 2 + e^{t} - e^{-2t}$

Qu Evaluate $\int_0^\infty e^{-3t}\cdot t\sin t\, dt$

take $t\sin t = F(t)$

$L\{F(t)\} = L\{t\cdot\sin t\} = (-1)\frac{d}{dp}\frac{1}{p^2+1} = \frac{2p}{(p^2+1)^2}$

$\Rightarrow \int_0^\infty e^{-pt}\cdot t\sin t\, dt = \frac{2p}{(p^2+1)^2}$

$\int_0^\infty e^{-3t}\cdot t\sin t\, dt = \frac{2\times 3}{(3^2+1)^2} = \frac{3}{50}$

Qu Evaluate $\int_0^\infty \frac{\sin t}{t}\, dt$

$L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty \frac{1}{1+p^2}\, dp = \frac{\pi}{2} - \tan^{-1}p$

$\Rightarrow \int_0^\infty e^{-pt}\cdot\frac{\sin t}{t}\, dt = \frac{\pi}{2} - \tan^{-1}p$

put $p=0$

$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$

then $L\{y'\} = p\bar{y} - y(0)$

$L\{y''\} = p^2\bar{y} - py(0) - y'(0)$

Solution of Differential Eq^n →

Qu Using laplase transform solve

$$y'' - 2y' + 3y = e^t \quad \text{given } y(0) = y'(0) = 0.$$

taking laplace transform on both side

$$L\{y''\} - 2L\{y'\} + 3L\{y\} = L\{e^t\}$$

$$[p^2\bar{y} - py(0) - y'(0)] - 2[p\bar{y} - y(0)] + 3\bar{y} = \frac{1}{(p-1)}$$

using given^n cond^n

$$p^2\bar{y} - 2p\bar{y} + 3\bar{y} = \frac{1}{(p-1)}$$

$$\bar{y}(p^2 - 2p + 3) = \frac{1}{(p-1)}$$

$$\bar{y} = \frac{1}{(p-1)(p^2-2p+3)}.$$

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{1}{(p-1)(p^2-2p+3)}\right\}$$

$$y = L^{-1}\left\{\frac{1}{(p-1)(p^2-2p+3)}\right\}$$

$$\frac{1}{(p-1)(p^2-2p+3)} = \frac{A}{(p-1)} + \frac{Bp+C}{p^2-2p+3}$$

$$= \frac{Ap^2 - 2pA + 3A + Bp^2 + Cp - Bp - C}{(p-1)(p^2-2p+3)}$$

$$\frac{1}{(p-1)(p^2-2p+3)} = \frac{p^2(A+B) + p(-2A-B+C) + (3A-C)}{(p-1)(p^2-2p+3)}$$

$A + B = 0 \Rightarrow \boxed{A = -B}$

$3A - C = 1$

$\cancel{A=}\ C = 3A - 1$

$-2A - B + C = 0$

$-2A + A + 3A - 1 = 0$

$2A - 1 = 0$

$\boxed{A = \frac{1}{2}}$ $\boxed{B = -\frac{1}{2}}$ $\boxed{C = \frac{1}{2}}$

$$\Rightarrow \frac{1}{(p-1)(p^2-2p+3)} = \frac{1}{2}\left\{\frac{1}{(p-1)} + \frac{-p+1}{p^2-2p+3}\right\}$$

$$L^{-1}\left\{\frac{1}{(p-1)(p^2-2p+3)}\right\} = \frac{1}{2}\left\{L^{-1}\left\{\frac{1}{p-1}\right\} + L^{-1}\left\{\frac{-p+1}{p^2-2p+3}\right\}\right\}$$

$$= \frac{1}{2}\left\{e^t + L^{-1}\left\{\frac{-p+1}{(p-1)^2+2}\right\}\right\}$$

$$= \frac{1}{2}\left\{e^t - L^{-1}\left\{\frac{(p-1)}{(p-1)^2+2}\right\}\right\}$$

$$= \frac{1}{2}\left\{e^t - e^t L^{-1}\left\{\frac{p}{p^2+2}\right\}\right\}$$

$$= \frac{1}{2}\left\{e^t - e^t \cos\sqrt{2}t\right\}$$

Que→ $(D^3 - D^2 + 4D - 4)x = 68 e^t \sin 2t$

given that $x(0)=1,\ x'(0)=x''(0)=0$

taking laplace transform both side

$\cancel{L\{D^3\}-L\{D^2\}} - x''' - x'' + 4x' - 4x = 68 e^t \sin 2t$

$L\{x'''\} - L\{x''\} + 4L\{x'\} - 4L\{x\} = 68\, L\{e^t \sin 2t\}$

$[p^3\bar{x} - p^2 x(0) - p(x'(0)) - x''(0)] - [p^2\bar{x} - p\,x(0) - x'(0)]$

$+ 4[p(\bar{x}) - x(0)] - 4\bar{x} = 68 \cdot \frac{1}{(p-1)^2+4} \times 2$

$\Rightarrow p^3\bar{x} - p^2 - p^2\bar{x} + p + 4p\bar{x} - 4 - 4\bar{x} = \frac{136}{(p-1)^2+4}$

$\Rightarrow \bar{x}(p^3 - p^2 + 4p - 4) = \frac{136}{(p-1)^2+4} + p^2 - p + 4$

$\Rightarrow \bar{x} = \frac{136}{((p-1)^2+4)(p^3-p^2+4p-4)} + \frac{p^2-p+4}{(p^3-p^2+4p-4)}$

taking inverse laplace,

$x = L^{-1}\{\bar{x}\} = L^{-1}\left\{\frac{136}{((p-1)^2+4)(p^3-p^2+4p-4)}\right\} + L^{-1}\left\{\frac{p^2-p+4}{p^3-p^2+4p-4}\right\}$

Q.u → $\frac{d^2y}{dt^2} + y = 6\cos 2t$

given $y = 3$, $\frac{dy}{dt} = 1$ when $t = 0$.

$y(0) = 3$, $y'(0) = 1$

solve using laplace transform

$(D^2 + 1)y = 6\cos 2t$

taking laplace on both side

$L\{y''\} + L\{y\} = 6L\{\cos 2t\}$

$p^2\bar{y} - py(0) - y'(0) + \bar{y} = 6 \cdot \frac{p}{p^2+4}$

put given values -

$p^2\bar{y} - 3p - 1 + \bar{y} = 6 \cdot \frac{p}{p^2+4}$

$\bar{y}(p^2+1) - 3p - 1 = \frac{6p}{p^2+4}$

$\bar{y}(p^2+1) = \frac{6p}{p^2+4} + 3p + 1$

$\bar{y} = \frac{6p}{(p^2+4)(p^2+1)} + \frac{3p}{(p^2+1)} + \frac{1}{(p^2+1)}$

taking inverse laplace transform

$y = L^{-1}\left\{\frac{6p}{(p^2+4)(p^2+1)}\right\} + L^{-1}\left\{\frac{3p}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p^2+1}\right\}$

$y = 6L^{-1}\left\{\frac{p}{(p^2+4)} \cdot \frac{1}{(p^2+1)}\right\} + 3L^{-1}\left\{\frac{p}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p^2+1}\right\}$

$= 6\int_0^t \cancel{\cos 2u \sin(t-u)\,du} + 3\cos t + \sin t$

$L^{-1}\left\{\frac{p}{(p^2+4)(p^2+1)}\right\}$

$\frac{p}{(p^2+4)(p^2+1)} = \frac{Ap+B}{p^2+4} + \frac{Cp+D}{(p^2+1)}$

$= \frac{Ap^3 + Bp^2 + Ap + B + Cp^3 + Dp^2 + 4Cp + 4D}{(p^2+4)(p^2+1)}$

$= \frac{(A+C)p^3 + (B+D)p^2 + (A+4C)p + (B+4D)}{(p^2+4)(p^2+1)}$

⇒

⇒ $A + C = 0$ — (i)

$B + D = 0$ — (ii)

$A + 4C = 1$ — (iii)

$B + 4D = 0$ — (iv)

$A + C = 0$

$-A - 4C = -1$

$-3C = -1$

$\boxed{C = \frac{1}{3}}$ $\boxed{A = -\frac{1}{3}}$

$B + D = 0$

$-B - 4D = 0$

$-3D = 0$

$\boxed{D = 0}$, $\boxed{B = 0}$

$$\frac{p}{(p^2+4)(p^2+1)} = \frac{-\frac{1}{3}p}{(p^2+4)} + \frac{\frac{1}{3}p}{(p^2+1)}$$

$$L^{-1}\left\{\frac{-\frac{1}{3}p}{p^2+4}\right\} + L^{-1}\left\{\frac{\frac{1}{3}p}{p^2+1}\right\}$$

$$-\frac{1}{3}\cos 2t + \frac{1}{3}\cos t$$

Ans
$$\boxed{\cancel{y = -\frac{6}{3}\cos 2t + \frac{6}{3}\cos t + 3\cos t + \sin t}}$$

$$y = -2\cos 2t + 2\cos t + 3\cos t + \sin t$$

$$\boxed{y = 5\cos t + \sin t - 2\cos 2t}$$

Que→ Solve $t\cdot\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty = 0$ given $y(0) = y'(0) = 0$ ①

let $L\{y\} = \bar{y}$

taking laplace transform in ①

$$L\left\{t\cdot\frac{d^2y}{dt^2}\right\} + L\left\{\frac{dy}{dt}\right\} + L\{ty\} = 0$$

$$-\frac{d}{dp}L\left\{\frac{d^2y}{dt^2}\right\} + L\left\{\frac{dy}{dt}\right\} - \frac{d}{dp}L\{y\} = 0.$$

$$-\left[\frac{d}{dp}\left(p^2\bar{y} - py(0) - y'(0)\right) - \left(p\bar{y} - y(0)\right) + \frac{d}{dp}\bar{y}\right] = 0$$

using given values

$$+\left[\frac{d}{dp}(p^2\bar{y}) - p\bar{y} + \frac{d}{dp}\bar{y}\right] = 0$$

$$p^2\cdot\frac{d}{dp}\bar{y} + 2p\bar{y} - p\bar{y} + \frac{d}{dp}\bar{y} = 0$$

$$(1+p^2)\frac{d}{dp}\bar{y} + \bar{y}(2p-p) = 0.$$

$$(1+p^2)\frac{d}{dp}\bar{y} + p\bar{y} = 0.$$

$$\frac{1}{\bar{y}}d\bar{y} = \frac{-p}{(1+p^2)}dp$$

Integrating both side

$$\log\bar{y} = -\frac{1}{2}\log(1+p^2) + \log c$$

$$\bar{y} = \frac{c}{\sqrt{1+p^2}}$$

taking inverse laplace both side

$$y = c L^{-1}\left\{\frac{1}{\sqrt{1+p^2}}\right\}$$

$$\boxed{y = c J_0(t)} \qquad \left\{ L^{-1}\left\{\frac{1}{\sqrt{1+p^2}}\right\} = J_0(t) \right\}$$

Que By using laplace transform solve

$$\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0 \quad \text{given } y(0)=0,\ y'(0)=1$$

taking laplace both side

$$L\left\{\frac{d^2y}{dt^2}\right\} + L\left\{t\frac{dy}{dt}\right\} + L\{y\} = 0.$$

$$L\left\{\frac{d^2y}{dt^2}\right\} - \frac{d}{dp} L\left\{\frac{dy}{dt}\right\} - L\{y\} = 0$$

$$\left(p^2\bar{y} - py(0) - y'(0)\right) - \frac{d}{dp}\left(p\bar{y} - y(0)\right) - \bar{y} = 0$$

put given value

$$(p^2\bar{y} - 1) - \frac{d}{dp}(p\bar{y}) - \bar{y} = 0.$$

$$(p^2-1)\bar{y} - \frac{d}{dp}(p\bar{y}) = 1$$

$$(p^2-1)\bar{y} - \bar{y} - p\frac{d}{dp}\bar{y} = 1$$

$$(p^2-2)\bar{y} - p\frac{d}{dp}\bar{y} = 1$$

$$\frac{d\bar{y}}{dp} - \frac{(p^2-2)\bar{y}}{p} = -\frac{1}{p}$$

$$~~IF = e^{-\int (p^2-2)dp} = e^{-\left[\frac{p^3}{3} - 2p\right]}~~$$

$$~~sol^n \quad \bar{y} \cdot e^{-\left(\frac{p^3}{3}-2p\right)} = -\int e^{-\left(\frac{p^3}{3}-2p\right)} dp + C~~$$

$$\frac{d\bar{y}}{dp} - \left(\frac{p^2-2}{p}\right)\bar{y} = \frac{-1}{p}$$

$$IF = e^{-\int\left(p - \frac{2}{p}\right)dp} = e^{-\left(\frac{p^2}{2} - 2\log p\right)} = p^2 \cdot e^{-p^2/2}$$

$$sol^n \quad \bar{y} \cdot p^2 e^{\left(-\frac{p^2}{2} + \cancel{2\log p}\right)} = -\int \frac{p^2}{p} e^{\left(-\frac{p^2}{2} + \cancel{2\log p}\right)} dp + C$$

$$\bar{y} \cdot p^2 e^{-p^2/2} = \int -p e^{\left(-p^2/2\right)} dp + C$$

$$\bar{y} \cdot p^2 e^{-p^2/2} = e^{-p^2/2} + C.$$

$$\bar{y} = \frac{1}{p^2} + \frac{C}{p^2 \cdot e^{-p^2/2}}$$

taking inverse laplace both side

$$y = L^{-1}\left\{\frac{1}{p^2}\right\} + c\,L^{-1}\left\{\frac{1}{p^2 \cdot e^{-p^2/2}}\right\}$$

$$y = t + c\,L^{-1}\left\{\frac{e^{p^2/2}}{p^2}\right\}$$

Que→ By using laplace transform solve

$$\frac{d^2}{dt^2}y + y = t \quad \text{given } \bar{y}(\pi) = 0,\ y'(0) = 1$$

taking laplace both side,

$$L\left\{\frac{d^2y}{dt^2}\right\} + L\{y\} = L\{t\}$$

$$p^2\bar{y} - py(0) - y'(0) + \bar{y} = \frac{1}{p^2}$$

let $y(0) = K$

$$p^2\bar{y} - pK - 1 + \bar{y} = \frac{1}{p^2}$$

$$\bar{y}(p^2+1) = \frac{1}{p^2} + Kp + 1$$

$$\bar{y} = \frac{1}{p^2(p^2+1)} + \frac{Kp}{(p^2+1)} + \frac{1}{(p^2+1)}$$

taking inverse laplace

$$y = L^{-1}\left\{\frac{1}{p^2(p^2+1)}\right\} + KL^{-1}\left\{\frac{p}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= \int_0^t (t-u)\sin(u)\,du + K\cos t + \sin t$$

$$= -\left|(t-u)\cdot\cos u\right|_0^t - \int_0^t \cos u\,du + K\cos t + \sin t$$

$$= t - \left|\sin u\right|_0^t + K\cos t + \sin t$$

$$= t - \sin t + K\cos t + \sin t$$

$$\boxed{y = t + K\cos t}$$

using given cond^n $y(\pi) = 0$

$$0 = \pi + K\cos\pi$$

$$\boxed{K = \pi}$$

$$\Rightarrow \boxed{y = t + \pi\cos t}$$

Solution of Simultaneous Linear Equation by using Laplace transforms

Q4 Using laplace transform solve

$$\frac{dx}{dt} + \frac{dy}{dt} = t \quad \text{—— (1)}$$

$$\frac{d^2x}{dt^2} - y = e^t \quad \text{—— (2)}$$

given $x(0) = 3$, $x'(0) = -2$, $y(0) = 0$.

taking laplace transform of (1) & (2) we get,

let $L\{x\} = \bar{x}$, $L\{y\} = \bar{y}$

$$p\bar{x} - x(0) + p\bar{y} - y(0) = \frac{1}{p^2}$$

$$p^2\bar{x} - px(0) - x'(0) - \bar{y} = \frac{1}{(p-1)}$$

using given values

$$p\bar{x} - 3 + p\bar{y} = \frac{1}{p^2}$$

$$p^2\bar{x} - 3p + 2 - \bar{y} = \frac{1}{p-1}$$

these can be written as

$$p\bar{x} + p\bar{y} = \frac{1}{p^2} + 3 \quad \text{—— (3)}$$

$$p^2\bar{x} - \bar{y} = \frac{1}{(p-1)} + 3p - 2 \Rightarrow p^3\bar{x} - p\bar{y} = \frac{p}{(p-1)} + 3p^2 - 2p \quad \text{—— (4)}$$

adding eq^n (3) & (4)

$$p\bar{x} + p^3\bar{x} = \frac{1}{p^2} + 3 + \frac{p}{(p-1)} + 3p^2 - 2p$$

$$(p + p^3)\bar{x} = \frac{1}{p^2} + 3 + \frac{p}{(p-1)} + 3p^2 - 2p$$

$$\bar{x} = \frac{1}{p^3(p^2+1)} + \frac{3}{p(p^2+1)} + \frac{1}{(p-1)(p^2+1)} + \frac{3p}{(p^2+1)} - \frac{2p}{(p^2+1)}$$

taking inverse Laplace

$$x = L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\} + 3L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} + L^{-1}\left\{\frac{1}{(p-1)(p^2+1)}\right\} + 3L^{-1}\left\{\frac{p}{p^2+1}\right\} - 2L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= \frac{t^2}{2} - 1 + \cos t - 3(\cos t - 1) + L^{-1}\left\{\frac{1}{(p-1)(p^2+1)}\right\}$$

$$+ 3\cos t - 2\sin t$$

Now

$$\frac{1}{(p-1)(p^2+1)} = \frac{A}{(p-1)} + \frac{Bp+C}{(p^2+1)}$$

$$\frac{1}{(p-1)(p^2+1)} = \frac{Ap^2 + A + Bp^2 + Cp - Bp - C}{(p-1)(p^2+1)}$$

$$\frac{1}{(p-1)(p^2+1)} = \frac{(A+B)p^2+(C-B)p+(A-C)}{(p-1)(p^2+1)}$$

$A+B=0$

$C-B=0 \Rightarrow \boxed{B=C}$

$A-C=1$

$A=C+1$

$A-B=1$

$A+B=0$

$2A=1$

$\boxed{A=\frac{1}{2}}$ $\boxed{B=\frac{-1}{2}}$ $\boxed{C=\frac{-1}{2}}$

$$\frac{1}{(p-1)(p^2+1)} = \frac{\frac{1}{2}}{(p-1)} - \frac{\frac{1}{2}p+\frac{1}{2}}{(p^2+1)}$$

$$L^{-1}\left\{\frac{1}{(p-1)(p^2+1)}\right\} = \frac{1}{2}L^{-1}\left\{\frac{1}{p-1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{p}{p^2+1}\right\} - \frac{1}{2}\left\{\frac{1}{p^2+1}\right\}$$

$$= \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

Solⁿ

$$x = \frac{t^2}{2} - 1 + \cos t - 3\cos t + 3 + \frac{1}{2}e^t - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$
$$+3\cos t - 2\sin t$$

$$\boxed{x = \frac{t^2}{2} + \frac{e^t}{2} + \frac{1}{2}\cos t - \frac{5}{2}\sin t + 2}$$

eqⁿ (3) & (4)

$$p\bar{x} + p\bar{y} = \frac{1}{p^2} + 3 \Rightarrow p^2\bar{x} + p^2\bar{y} = \frac{1}{p} + 3p$$

$$p^2\bar{x} - \bar{y} = \frac{1}{p-1} + 3p - 2 \Rightarrow p^2\bar{x} - \bar{y} = \frac{1}{(p-1)} + 3p - 2$$

$- \quad + \quad - \quad - \quad +$

$$p^2\bar{y} + \bar{y} = \frac{1}{p} + 3p - \frac{1}{p-1} - 3p + 2$$

$$(p^2+1)\bar{y} = \frac{1}{p} + \frac{1}{(p-1)} + 2$$

$$\bar{y} = \frac{1}{p(p^2+1)} + \frac{1}{(p-1)(p^2+1)} + \frac{2}{(p^2+1)}$$

$$y = L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} + L^{-1}\left\{\frac{1}{(p-1)(p^2+1)}\right\} + 2L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= -(\cos t - 1) + \frac{e^t}{2} - \frac{\cos t}{2} - \frac{\sin t}{2} + 2\sin t$$

$$\boxed{y = \frac{-3}{2}\cos t + 1 + \frac{e^t}{2} + \frac{3}{2}\sin t}$$

Ques- $\cancel{\frac{dx}{dy}}$ $\cancel{dx}$

$$Dx + Dy = t$$

$$D^2x - y = e^{-t}$$

given $x(0)=3$, $x'(0)=-2$, $y(0)=0$.

$$\frac{dx}{dt} + \frac{dy}{dt} = t \quad \text{——— (1)}$$

$$\frac{d^2x}{dt^2} - y = e^{-t} \quad \text{——— (2)}$$

taking laplace of eq^n (1) & (2)

$$L\left\{\frac{dx}{dt}\right\} + L\left\{\frac{dy}{dt}\right\} = L\{t\}$$

$$= p\bar{x} - x(0) + p\bar{y} - y(0) = \frac{1}{p^2} \quad \text{——— (3)}$$

$$L\left\{\frac{d^2x}{dt^2}\right\} - L\{y\} = L\{e^{-t}\}$$

$$p^2\bar{x} - px(0) - x'(0) - \bar{y} = \frac{1}{p+1} \quad \text{——— (4)}$$

using given cond in (3) & (4)

$$p\bar{x} - 3 + p\bar{y} = \frac{1}{p^2} \Rightarrow p\bar{x} + p\bar{y} = \frac{1}{p^2} + 3 \quad \text{——— (5)}$$

$$p^2\bar{x} - 3p + 2 - \bar{y} = \frac{1}{p+1} \Rightarrow p^2\bar{x} - \bar{y} = \frac{1}{p+1} + 3p - 2 \quad \text{——— (6)}$$

multiply by p in (6) & adding with (5)

$$p\bar{x} + p^3\bar{x} = \frac{1}{p^2} + 3 + \frac{p}{p+1} + 3p^2 - 2p$$

$$\bar{x} = \frac{1}{p^3(p^2+1)} + \frac{3}{p(p^2+1)} + \frac{1}{(p^2+1)(p+1)} + \frac{3p}{(p^2+1)} - \frac{2}{(p^2+1)}$$

$$\Rightarrow x = L^{-1}\left\{\frac{1}{p^3(p^2+1)}\right\} + 3L^{-1}\left\{\frac{1}{p(p^2+1)}\right\} + L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$$

$$+ 3L^{-1}\left\{\frac{p}{p^2+1}\right\} - 2L^{-1}\left\{\frac{1}{p^2+1}\right\}$$

$$= \frac{t^2}{2} - 1 + \cos t + 3 - 3\cos t + 3 + L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\}$$

$$+ 3\cos t - 2\sin t.$$

$$\frac{1}{(p+1)(p^2+1)} = \frac{A}{p+1} + \frac{Bp+C}{p^2+1}$$

$$\frac{1}{(p+1)(p^2+1)} = \frac{Ap^2 + A + Bp^2 + Bp + Cp + C}{(p+1)(p^2+1)}$$

$A+B=0 \Rightarrow$ ~~A~~

$B+C=0 \Rightarrow B=-C$

$A+C=1 \Rightarrow A-B=1$

$A+B=0$

$A-B=1$

$\boxed{A=\frac{1}{2}}$ $\boxed{B=-\frac{1}{2}}$ $\boxed{C=\frac{1}{2}}$

$$\frac{1}{(p+1)(p^2+1)} = \frac{1}{2}\frac{1}{(p+1)} - \frac{1}{2}\frac{p}{p^2+1} + \frac{1}{2}\frac{1}{p^2+1}$$

$$L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\} = \frac{1}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

So

$$x = \frac{t^2}{2} + 5 + \cos t - 2\sin t + \frac{e^{-t}}{2} - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

$$\boxed{x = \frac{t^2}{2} + \frac{e^{-t}}{2} + \frac{1}{2}\cos t - \frac{3}{2}\sin t + 2}$$

Multiplying by p in (5) & subtracting from (6)

$$p^2\bar{x} - \bar{y} = \frac{1}{(p+1)} + 3p - 2$$

$$p^2\bar{x} + p^2\bar{y} = \frac{1}{p} + 3p$$

$$-\bar{y} - p^2\bar{y} = \frac{1}{(p+1)} + 3p - 2 - \frac{1}{p} - 3p$$

$$\bar{y} + p^2\bar{y} = \frac{-1}{(p+1)} + 2 + \frac{1}{p}$$

$$\bar{y} = -\frac{1}{(p+1)(p^2+1)} + \frac{2}{p^2+1} + \frac{1}{p(p^2+1)}$$

$$\Rightarrow y = -L^{-1}\left\{\frac{1}{(p+1)(p^2+1)}\right\} + 2L^{-1}\left\{\frac{1}{p^2+1}\right\} + L^{-1}\left\{\frac{1}{p(p^2+1)}\right\}$$

$$y = -\left[\frac{e^{-t}}{2} - \frac{1}{2}\cos t + \frac{1}{2}\sin t\right] + 2\sin t - \cos t + 1$$

$$\boxed{y = -\frac{e^{-t}}{2} - \frac{\cos t}{2} + \frac{3}{2}\sin t + 1}$$

<u>Function of Exponential Order</u>: A function $F(t)$ is said to be of exponential order α as t tends to infinity if

$$\boxed{\lim_{t\to\infty} e^{-\alpha t} F(t) = \text{finite}}$$

In other words function $F(t)$ is called of exponential order α if for a given +ve integer n_0 there exist a real no. $M (>0)$ s.t.

$$\boxed{|F(t)| \le Me^{\alpha t}} \quad \forall\, t \ge n_0$$

eg- Show that function $F(t) = t^n$ is of exponential order as $t \to \infty$

$$\lim_{t\to\infty} e^{-\alpha t}\, t^n = \lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} = 0 = \text{finite}$$

therefore t^n is of exponential order.

Existense of laplace Transformation →

A function $F(t)$ is a function of class A if –

(i) it is piecewise continuous in every subinterval

(ii) it is of exponential order as $t \to \infty$.

If a function $F(t)$ is of class A then its laplace transform exists.

Solution of Partial Differential Equation →

Qu Find the bounded sol^n of $\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y$ if $y(x,0) = 6e^{-3x}$

$$\frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y \qquad \text{let } L\{y\} = \bar{y}$$

taking laplace both side

$$L\left\{\frac{\partial y}{\partial x}\right\} = 2L\left\{\frac{\partial y}{\partial t}\right\} + L\{y\}$$

$$\frac{d\bar{y}}{dx} = 2[p\bar{y} - y(x,0)] + \bar{y}$$

put given values

$$\frac{d\bar{y}}{dx} = 2[p\bar{y} - 6e^{-3x}] + \bar{y}$$

$$\frac{d\bar{y}}{dx} = (2p+1)\bar{y} - 12e^{-3x}$$

$$\frac{d\bar{y}}{dx} - (2p+1)\bar{y} = -12e^{-3x}$$

$$IF = e^{\int -(2p+1)dx} = e^{-(2p+1)x}$$

Solution

$$\bar{y} \cdot e^{-(2p+1)x} = \int e^{-(2p+1)} \cdot (-12e^{-3x})\,dx + c$$

$$\bar{y} \cdot e^{-(2p+1)x} = -12\int e^{-(2p+\cancel{3}4)x}\,dx + c$$

$$\bar{y} \cdot e^{-(2p+1)x} = \frac{12e^{-(2p+4)x}}{(2p+4)} + c$$

$$\bar{y} \cdot e^{-(2p+1)x} = \frac{6e^{-(2p+4)x}}{(p+2)} + c$$

$$\bar{y} = \frac{6e^{-3x}}{(p+2)} + ce^{(2p+1)x}$$

for bounded solution, we have to take c such that sol^n is finite for any value of x. It is possible only when $\boxed{c=0}$

So, $\bar{y} = \frac{6e^{-3x}}{(p+2)}$

taking inverse laplace

$$y = 6e^{-3x} L^{-1}\left\{\frac{1}{p+2}\right\} = 6e^{-3x} \cdot e^{-2t}$$

Que- Solve $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ if $y(x,0) = 3\sin 2\pi x$, $y(0,t) = 0$, $y(1,t) = 0$

$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$

taking laplace both side

$L\left\{\frac{\partial y}{\partial t}\right\} = L\left\{\frac{\partial^2 y}{\partial x^2}\right\}$

$p\bar{y} - y(x,0) = \frac{d^2\bar{y}}{dx^2}$

using given values

$p\bar{y} - 3\sin 2\pi x = \frac{d^2\bar{y}}{dx^2}$

$\frac{d^2\bar{y}}{dx^2} - p\bar{y} = -3\sin 2\pi x$

$(D^2 - p)\bar{y} = -3\sin 2\pi x$

AE- $m^2 - p = 0$

$m = \pm\sqrt{p}$

$CF = c_1 e^{\sqrt{p}x} + c_2 e^{-\sqrt{p}x}$

$PI = -3\left[\frac{1}{(D^2-p)}\sin 2\pi x\right] = -3\left[\frac{1}{-4\pi^2 - p}\sin 2\pi x\right]$

$PI = \frac{3\sin 2\pi x}{4\pi^2 + p}$

It's solⁿ is

$\bar{y} = c_1 e^{\sqrt{p}x} + c_2 e^{-\sqrt{p}x} + \frac{3\sin 2\pi x}{4\pi^2 + p}$

put $x=0$ then $y=0$

$\Rightarrow 0 = c_1 + c_2 \Rightarrow c_1 = -c_2$

put $x=1$, $y=0$.

then $0 = c_1 e^{\sqrt{p}} + c_2 e^{-\sqrt{p}}$

$\Rightarrow c_1 e^{\sqrt{p}} - c_1 e^{-\sqrt{p}} = 0$

$\Rightarrow c_1 (e^{\sqrt{p}} - e^{-\sqrt{p}}) = 0$

$\Rightarrow \boxed{c_1 = 0} \Rightarrow \boxed{c_2 = 0}$

So,

$\bar{y} = \frac{3\sin 2\pi x}{4\pi^2 + p}$

taking inverse laplace

$y = 3\sin 2\pi x\, L^{-1}\left\{\frac{1}{p + 4\pi^2}\right\} = 3\sin 2\pi x \cdot e^{-4\pi^2 t}$