$$\hat{\beta} = \beta + \sum z_i \varepsilon_i$$

$$Var(\hat{\beta}) = Var(\beta) + Var(\sum z_i \varepsilon_i) + 2\,Cov(\beta, \sum z_i \varepsilon_i)$$

$$= 0 + Var(\sum z_i \varepsilon_i) + 0$$

$$= Var(z_1 \varepsilon_1 + z_2 \varepsilon_2 + \dots + z_n \varepsilon_n)$$

$$\sum_{i=1}^{n} Var\, z_i \varepsilon_i + 2 \sum_{i \neq j} Cov(z_i \varepsilon_i, z_j \varepsilon_j)$$

$$\sum_{i=1}^{n} z_i^2 Var(\varepsilon_i) + 2 \sum_{i \neq j} z_i z_j Cov(\varepsilon_i, \varepsilon_j)$$

$$\sum_{i=1}^{n} z_i^2 \sigma^2 + 2 \sum_{i \neq j} 0$$

$$\cancel{\sigma^2 \sum_{i=1}^{n} x_i^2} + 0$$

$$= \sigma^2 \sum_{i=1}^{n} z_i^2 + 0 = \sigma^2 \sum_{i=1}^{n} z_i^2$$

where $z_i = \dfrac{x_i}{\sum_{j=1}^{n} x_j^2}$

Note:- ~~$Cov[z_i, z_j$~~
$Cov(\varepsilon_i, \varepsilon_j) = 0$ since $\varepsilon_i, \varepsilon_j$ are iids.

β is a constant (unknown) so
$Var[\beta] = 0$, $E[\beta] = \beta \Rightarrow \beta - E[\beta] = 0$

$$Cov(\beta, \sum z_i \varepsilon_i) = E[(\beta - E[\beta]) \cdot (\sum z_i \varepsilon_i - E[z_i \varepsilon_i])]$$

$$= E[0] = 0.$$

$$\hat{\beta} = \beta + (\cancel{\Sigma z_i}) \Sigma z_i \varepsilon_i$$

$$\beta = \Sigma z_i \varepsilon_i$$

$$= (z_1^2 + z_2^2 + z_n^2)\, \sigma^2$$

$$E\{(\varepsilon_i - 0)(\varepsilon_j)\}$$

$$\beta$$

$$E(\varepsilon_i \varepsilon_j)$$

$$\mathrm{Cov}(\varepsilon_i, \varepsilon_j) = E\{(\varepsilon_i - E[\varepsilon_i])(\varepsilon_j - E[\varepsilon_j])\}$$

$$= E[\varepsilon_i \varepsilon_j] = E[\varepsilon_i]\, E[\varepsilon_j]$$

$$= 0 \times 0 = 0$$

since iids