Proof for Linearity of Expectations

To prove:- $E[X_1 + X_2] = E[X_1] + E[X_2]$

Proof:- (Assuming X_1 & X_2 to be continuous random variables, similar proof applies for discrete case as well.)

$$X_3 = X_1 + X_2$$

We know that, PDF of X_3 is joint distribution of X_1 & X_2

$$f_{X_3}(x_3) = f_{X_1, X_2}(x_1, x_2)$$

$$E[X_3] = \int_{-\infty}^{\infty} x_3 \, f_{X_3}(x_3) \, dx_3$$

(but $x_3 = x_1 + x_2$)

$$= \int_{x_1 = -\infty}^{\infty} \int_{x_2 = -\infty}^{\infty} (x_1 + x_2) \, f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

$$= \int_{x_1} \int_{x_2} x_1 \, f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \; + \int_{x_1} \int_{x_2} x_2 \, f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

$$= \int_{x_1} x_1 \int_{x_2} f_{X_1, X_2}(x_1, x_2) \, dx_2 \, dx_1 \; + \int_{x_2} x_2 \int_{x_1} f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

(Now, $\int_{x_2} f_{X_1, X_2}(x_1, x_2) \, dx_2$ = Marginal distribution of X_1

$= f_{X_1}(x_1)$)

Similarly, $\int_{x_1} f_{X_1,X_2}(x_1,x_2)\,dx_1 = f_{X_2}(x_2)$

So, $E[X_3] = \int_{x_1} x_1 f_{X_1}(x_1)\,dx_1 + \int_{x_2} x_2 f_{X_2}(x_2)\,dx_2$

$= E[X_1] + E[X_2]$ Proved.

Proof when X_1 & X_2 are discrete :-

$X_3 = X_1 + X_2$

Probability mass function (PMF) of X_3 is given by the joint PMF of X_1 & X_2

$$P_{X_3}(x_3) = P_{X_1,X_2}(x_1,x_2)$$

$$E[X_3] = \sum_{x_3=-\infty}^{\infty} x_3 P_{X_3}(x_3)$$

$$= \sum_{x_1}\sum_{x_2}(x_1+x_2)\,P_{X_1,X_2}(x_1,x_2)$$

$$= \sum_{x_1}\sum_{x_2} x_1 P_{X_1,X_2}(x_1,x_2) + \sum_{x_1}\sum_{x_2} x_2 P_{X_1,X_2}(x_1,x_2)$$

$$= \cancel{\sum_{x_2}\sum_{x_1} x_1 P_{X_1,X_2}(x_1,x_2)} + \cancel{\sum_{x_1}\sum_{x_2} x_2 P_{X_1,X_2}}$$

$$= \sum_{x_1} x_1 \sum_{x_2} P_{X_1,X_2}(x_1,x_2) + \sum_{x_2} x_2 \sum_{x_1} P_{X_1,X_2}(x_1,x_2)$$

$\sum_{x_2} P_{X_1,X_2}(x_1,x_2) = P_{X_1}(x_1)$ = Marginal distribution of x_1

$\sum_{x_1} P_{X_1,X_2}(x_1,x_2) = P_{X_2}(x_2)$

So, $E[X_3] = \sum_{x_1} x_1 P_{X_1}(x_1) + \sum_{x_2} x_2 P_{X_2}(x_2)$

$= E[X_1] + E[X_2]$ Proved.