EE22BTECH11049 - Shivansh Kirar

Question EC 31 2023

The signal-to-noise ratio (SNR) of an ADC with a full-scale sinusoidal input is given to be 61.96 dB. The resolution of the ADC is   GATE EC 2023

Solution:

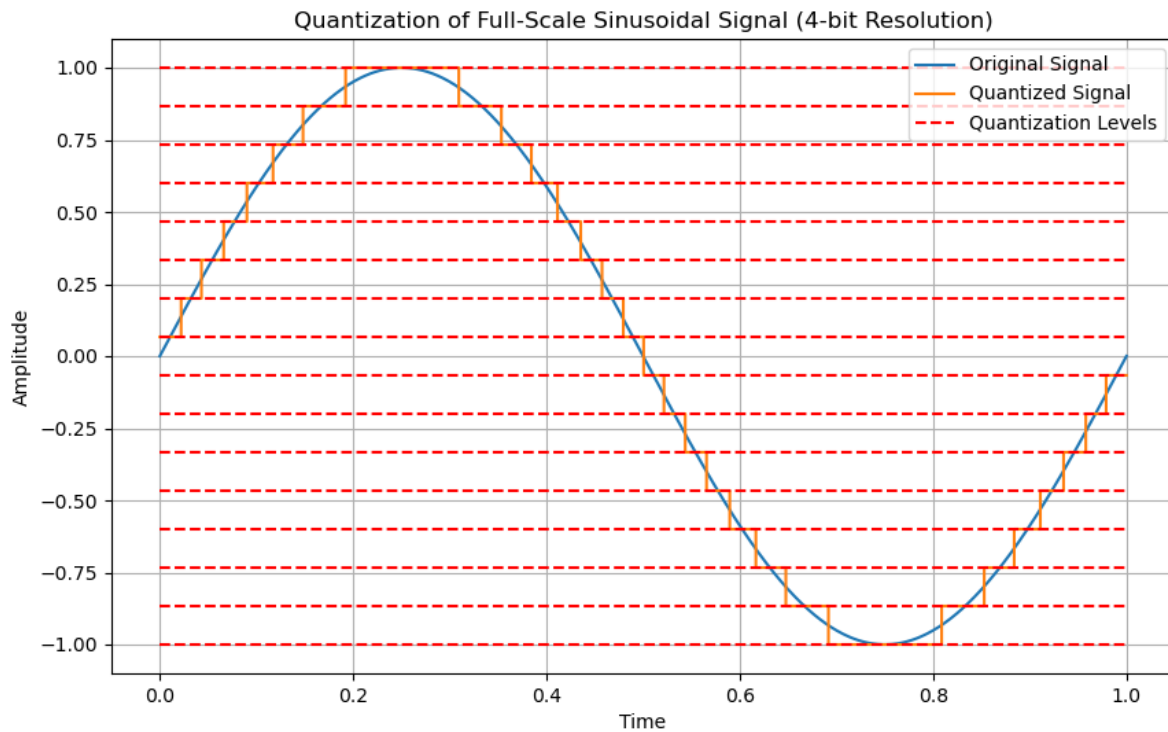| Variable	Defination
T	Time Period
f	Frequency
A	Amplitude
P_S	Signal Power
P_N	Noise Power
q	Quantization step size
n	No of bits
Y	Quantization Error

Fig. 1: Quantization of Sinusoidal Signal

1) Signal Power:

   The power of a continuous-time signal is defined as the average value of the square of the signal over a certain time interval. For a sinusoidal signal $x(t) = A \sin(2\pi f t + \phi)$, the power ($P_S$) is calculated as:

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt \tag{1}$$

where ϕ is the phase of the signal.

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |A \sin(2\pi ft + \phi)|^2 dt \tag{2}$$

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \sin^2(2\pi ft + \phi) dt \tag{3}$$

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cdot \frac{1 - \cos(4\pi ft + 2\phi)}{2} dt \tag{4}$$

$$P_S = \frac{1}{2} \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt - \frac{1}{2} \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cos(4\pi ft + 2\phi) dt \tag{5}$$

$$P_S = \frac{1}{2} \cdot A^2 - 0 \tag{6}$$

$$P_S = \frac{A^2}{2} \tag{7}$$

$$\tag{8}$$

Here, A= 1, so:

$$P_S = \frac{1}{2} \tag{9}$$

2) Noise Power:
No of Quantization levels is given by 2^n, Where n is resolution or no of bits.
Distance between any two Quantization levels = Quantization step
(No of Quantization levels)*(Quantization step) = Peak Distance (Refer to Fig 1)

$$q = \frac{\text{Peak distance}}{2^n} = \frac{1 - (-1)}{2^n} = \frac{2}{2^n} = 2^{-(n-1)} \tag{10}$$

We know,quantization error has a maximum value of plus or minus half the step size, so

$$|Y| \leq \frac{q}{2} \text{ and therefore, } |Y| \leq 2^{-n} \tag{11}$$

For a large enough number of quantization steps, the probability density function of the quantization error tends toward being flat 1. Pdf of error (Y) of quantization is defined as

$$p_Y(y) = \begin{cases} \frac{1}{q}, & \text{if } -\frac{q}{2} \leq y \leq \frac{q}{2} \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

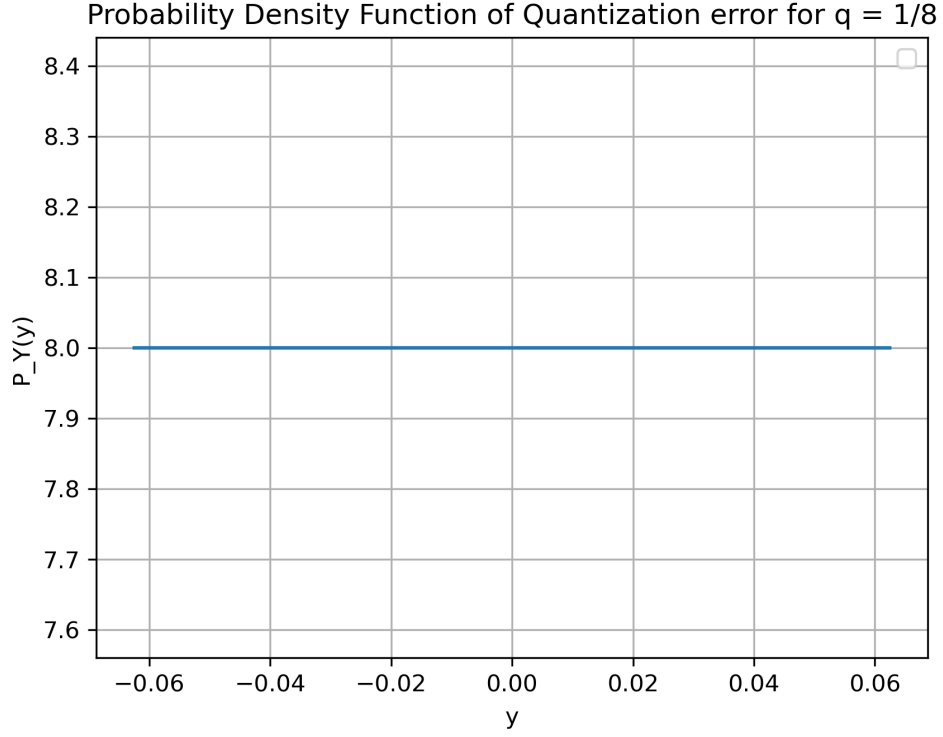## Probability Density Function of Quantization error for q = 1/8

Fig. 2: plot of pdf of Quantization Error

So, we can calculate its mean power or variance as the 2nd moment of its distribution. Since, the distribution of error is uniform hence E[Y]=0.

$$E(Y) = \int_{-\frac{q}{2}}^{\frac{q}{2}} p_Y(y)y \, de \tag{13}$$

$$= \frac{1}{q} \cdot \frac{1}{2} \left(\left(\frac{q}{2}\right)^2 - \left(-\frac{q}{2}\right)^2 \right) 10 \tag{14}$$

$$= \frac{1}{q} \cdot \frac{1}{2} (0 - 0) \tag{15}$$

$$= 0 \tag{16}$$

$$E[Y^2] = \int_{-\frac{q}{2}}^{\frac{q}{2}} p_Y(y)y^2 \, de \tag{17}$$

$$= \frac{1}{q} \cdot \frac{1}{3} \left(\left(\frac{q}{2}\right)^3 - \left(-\frac{q}{2}\right)^3 \right) 10 \tag{18}$$

$$= \frac{1}{3q} \cdot \frac{q^3}{4} \tag{19}$$

$$= \frac{q^2}{12} \tag{20}$$

$$\tag{21}$$

On putting $q = 2^{-(n-1)}$, we have:

$$E[Y^2] \approx \frac{2^{-2n}}{3} 10 \tag{22}$$

$$P_Y \approx \frac{2^{-2n}}{3} \tag{23}$$

$$P_N \approx \frac{2^{-2n}}{3} \tag{24}$$

$$\tag{25}$$

Thus, an ideal ADC would have a signal-to-noise ratio

$$SNR = \frac{P_S}{P_N} = 1.5 \cdot 2^{2n} \tag{26}$$

or, expressed in decibels,

$$SNR = 10 \left(\log_{10}(1.5 \cdot 2^n) \right) \tag{27}$$

$$= 10 \left(\log_{10}(1.5) + \log_{10}(2^{2n}) \right) \tag{28}$$

$$= 10 \left(0.176 + 2n \cdot 0.3010 \right) \tag{29}$$

$$= 1.76 + 6.02n \tag{30}$$

Substituting value of SNR

$$61.96 = 1.76 + 6.02n \tag{31}$$

$$n = \frac{61.96 - 1.76}{6.02} \tag{32}$$

$$= \frac{60.2}{6.02} \tag{33}$$

$$= 10 \tag{34}$$

So, the resolution of the ADC is approximately 10 bits.

Simulation Steps :-

a) Initialize the simulation parameters, including the given SNR in dB (which is 61.96 dB) and signal power (P_s) (which is 0.5), and create an array of quantization error values (q-values) to test.

b) Generate random variables uniformly distributed over $-\frac{q}{2}$ to $\frac{q}{2}$. For each quantization error value, calculate mean error, variance error, and average power (which is equivalent to noise power), and then use these values to calculate the SNR.

c) The number of bits required for quantization is calculated as $1 - \log_2(q)$.

d) Compare the calculated SNR to the target SNR_dB, and when it's within a specified tolerance, print the corresponding number of bits required for quantization.

e) The program breaks the loop and exits after finding the number of bits that achieves the target SNR_dB.