EE22BTECH11049 - Shivansh Kirar

Question EC 31 2023

The signal-to-noise ratio (SNR) of an ADC with a full-scale sinusoidal input is given to be 61.96 dB. The resolution of the ADC is   GATE EC 2023

Solution:

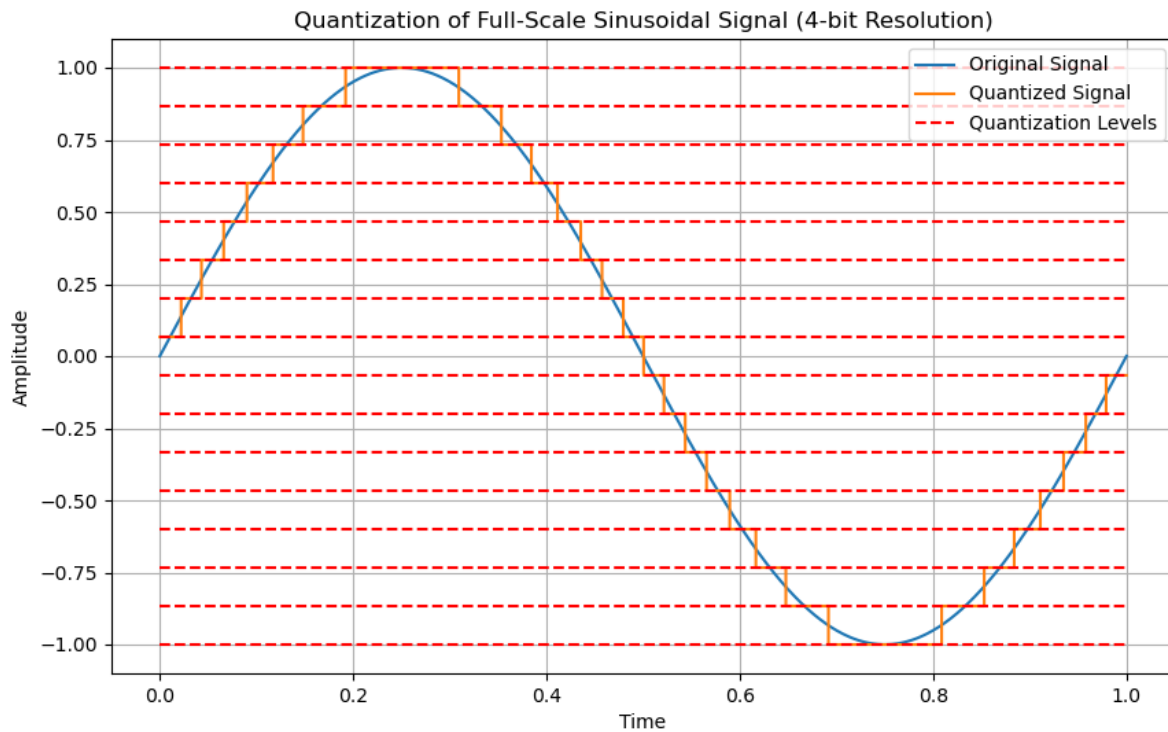| Variable	Defination
T	Time Period
f	Frequency
A	Amplitude
P_S	Signal Power
P_N	Noise Power
q	Quantization step size
n	No of bits
Y	Quantization Error

Fig. 1: Quantization of Sinusoidal Signal

1) Signal Power:
   The power of a continuous-time signal is defined as the average value of the square of the signal over a certain time interval. For a sinusoidal signal $x(t) = A\sin(2\pi f t + \phi)$, the power ($P_S$) is calculated as:

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt \tag{1}$$

where ϕ is the phase of the signal.

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |A \sin(2\pi f t + \phi)|^2 dt \tag{2}$$

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \sin^2(2\pi f t + \phi) dt \tag{3}$$

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cdot \frac{1 - \cos(4\pi f t + 2\phi)}{2} dt \tag{4}$$

$$P_S = \frac{1}{2} \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 dt - \frac{1}{2} \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cos(4\pi f t + 2\phi) dt \tag{5}$$

$$P_S = \frac{1}{2} \cdot A^2 - 0 \tag{6}$$

$$P_S = \frac{A^2}{2} \tag{7}$$

$$\tag{8}$$

Here, A= 1, so:

$$P_S = \frac{1}{2} \tag{9}$$

2) Noise Power:
No of Quantization levels is given by 2^n, Where n is resolution or no of bits.
Distance between any two Quantization levels = Quantization step
(No of Quantization levels)*(Quantization step) = Peak Distance (Refer to Fig 1)

$$q = \frac{\text{Peak distance}}{2^n} = \frac{1 - (-1)}{2^n} = \frac{2}{2^n} = 2^{-(n-1)} \tag{10}$$

We know,quantization error has a maximum value of plus or minus half the step size, so

$$|Y| \le \frac{q}{2} \text{ and therefore, } |Y| \le 2^{-n} \tag{11}$$

For a large enough number of quantization steps, the probability density function of the quantization error tends toward being flat 1. Pdf of error (Y) of quantization is defined as

$$p_Y(y) = \begin{cases} \frac{1}{q}, & \text{if } -\frac{q}{2} \le y \le \frac{q}{2} \\ 0, & \text{otherwise} \end{cases} \tag{12}$$
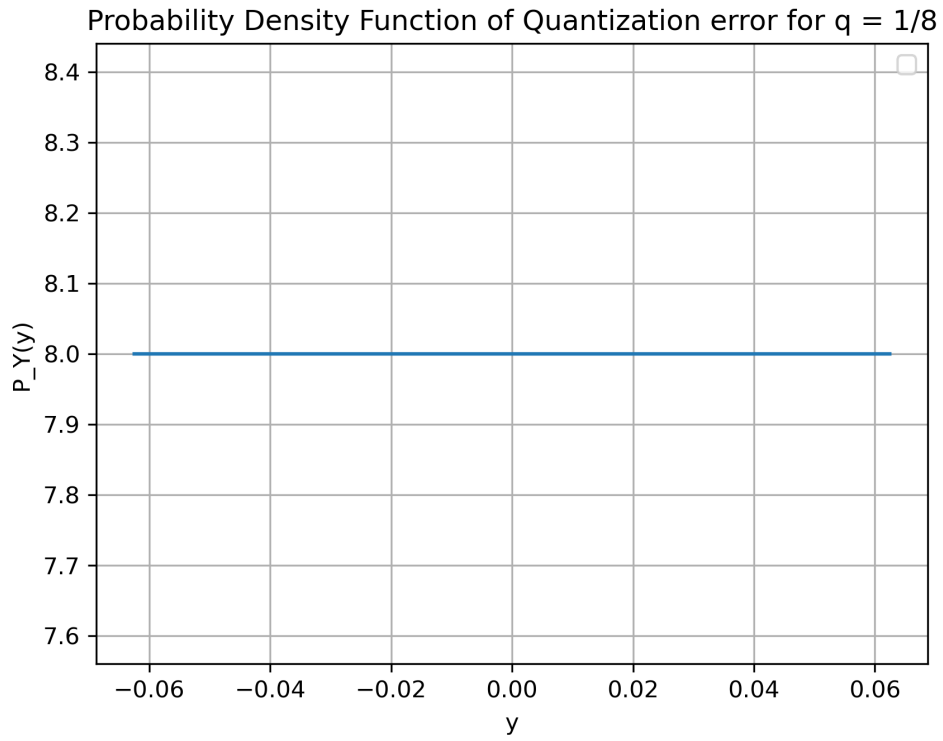
Fig. 2: plot of pdf of Quantization Error

So, we can calculate its mean power or variance as the 2nd moment of its distribution. Since, the distribution of error is uniform hence E[Y]=0.

$$E\,(Y) = \int_{-\frac{q}{2}}^{\frac{q}{2}} p_Y(y)y\,de \tag{13}$$

$$= \frac{1}{q} \cdot \frac{1}{2}\left(\left(\frac{q}{2}\right)^2 - \left(-\frac{q}{2}\right)^2\right)10 \tag{14}$$

$$= \frac{1}{q} \cdot \frac{1}{2}(0-0) \tag{15}$$

$$= 0 \tag{16}$$

$$E[Y^2] = \int_{-\frac{q}{2}}^{\frac{q}{2}} p_Y(y)y^2\,de \tag{17}$$

$$= \frac{1}{q} \cdot \frac{1}{3}\left(\left(\frac{q}{2}\right)^3 - \left(-\frac{q}{2}\right)^3\right)10 \tag{18}$$

$$= \frac{1}{3q} \cdot \frac{q^3}{4} \tag{19}$$

$$= \frac{q^2}{12} \tag{20}$$

$$\tag{21}$$

On putting $q = 2^{-(n-1)}$, we have:

$$E[Y^2] \approx \frac{2^{-2n}}{3} 10 \tag{22}$$

$$P_Y \approx \frac{2^{-2n}}{3} \tag{23}$$

$$P_N \approx \frac{2^{-2n}}{3} \tag{24}$$

$$\tag{25}$$

Thus, an ideal ADC would have a signal-to-noise ratio

$$SNR = \frac{P_S}{P_N} = 1.5 \cdot 2^{2n} \tag{26}$$

or, expressed in decibels,

$$SNR = 10 \left(\log_{10}(1.5 \cdot 2^n) \right) \tag{27}$$

$$= 10 \left(\log_{10}(1.5) + \log_{10}(2^{2n}) \right) \tag{28}$$

$$= 10 \left(0.176 + 2n \cdot 0.3010 \right) \tag{29}$$

$$= 1.76 + 6.02n \tag{30}$$

Substituting value of SNR

$$61.96 = 1.76 + 6.02n \tag{31}$$

$$n = \frac{61.96 - 1.76}{6.02} \tag{32}$$

$$= \frac{60.2}{6.02} \tag{33}$$

$$= 10 \tag{34}$$

So, the resolution of the ADC is approximately 10 bits.