Continuous Strategy Spaces

So far, all the games considered have had finite (discrete) strategy spaces S_i. Sometimes it is preferable to model strategic interactions with infinite (continuous) strategy spaces. e.g.

Pricing. Two firms selling identical products must choose their prices. The firm with the lower price gains the entire market, but firms would rather charge high prices.

Production. Two profit-maximizing firms must choose the scale of their output. Increasing output increases sales, but depresses the market price (which affects both firms).

Electoral Competition. Two political parties choose their advertising budgets. They dislike campaign spending, but the electoral outcome depends on the relative advertising of the parties.

The concepts from earlier lectures are applicable to games with continuous strategy spaces…

1. Represent such "continuous" interactions in strategic-form games.
2. Iteratively delete strictly dominated strategies where possible.
3. Construct best-response functions for the players (and plot them).
4. Find pure-strategy and mixed-strategy Nash equilibria.

The Nash Demand Game

"Two individuals argue over the division of a pot of money. They simultaneously make irrevocable demands, and receive their demands if and only if they can jointly be met from the money pot."

Players. Two players labelled $i \in N = \{1, 2\}$.

Strategies. The players choose a demand $s_i \in S_i = [0, \infty)$.

Payoffs. i's payoff is $u_i(s_i)$ if $\sum_{j \in N} s_j \leq 1$, u_i increasing. Otherwise its $u_i(0)$.

- Suppose that $s_i \geq 1$. Then $B_j(s_i) = [0, \infty)$: any demand is a best response!
- Suppose that $s_i < 1$. Then $B_j(s_i) = 1 - s_i$.
- There is a continuum of equilibria, since the best-response functions coincide.
- These include all $s = \{s_1, s_2\}$ such that $s_1 + s_2 = 1$, and many more.

More structure is required for more precise predictions.

Cournot Competition with Linear Demand

"Two profit-maximizing firms simultaneously choose production quantities of a homogeneous good. Market price is decreasing in total quantity Q, with linear demand, so that $p = a - bQ$. There are constant unit production costs of c for each firm."

Players. Two firms labelled $i \in \{1, 2\}$.

Strategies. Player 1 chooses quantity $x \in [0, \infty)$ and player 2 chooses quantity $y \in [0, \infty)$.

Payoffs. Payoffs are profits. That is, for players 1 and 2 respectively:

$$\pi_1 = x\,[a - b(x + y) - c] \quad \text{and} \quad \pi_2 = y\,[a - b(x + y) - c]\,.$$

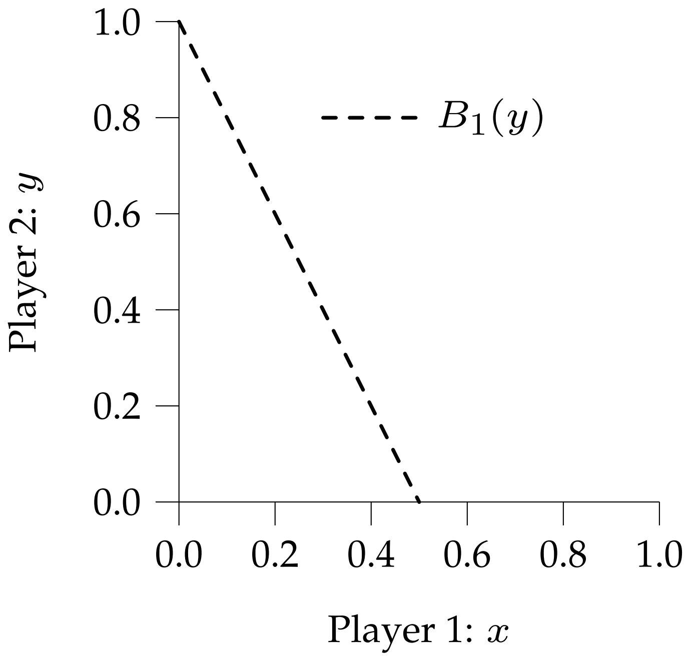
1. Fix firm 2's strategy. Calculate a best response for firm 1, yielding a best-response function.
2. Fix firm 1's strategy. Calculate a best response for firm 2: a second best-response function.
3. Combine the two best-response functions. Solve to find a Nash equilibrium.

Cournot Best-Response Functions

Fixing y, profits for player 1 are $\pi_1 = x\,[a - b(x+y) - c]$. This is strictly concave in x, so can calculate first-order conditions for a solution:

$$\frac{\partial \pi_1}{\partial x} = [a - b(x+y) - c] - bx = a - 2bx - by - c = 0.$$

Re-arrange this to obtain: $2bx = a - by - c$ which implies $B_1(y) = (a - by - c)/2b$.



- Plot of reaction function for $a{=}b{=}1$, $c{=}0$.
- This is downward sloping:

$$\frac{\partial B_1(y)}{\partial y} = -\frac{1}{2} < 0$$

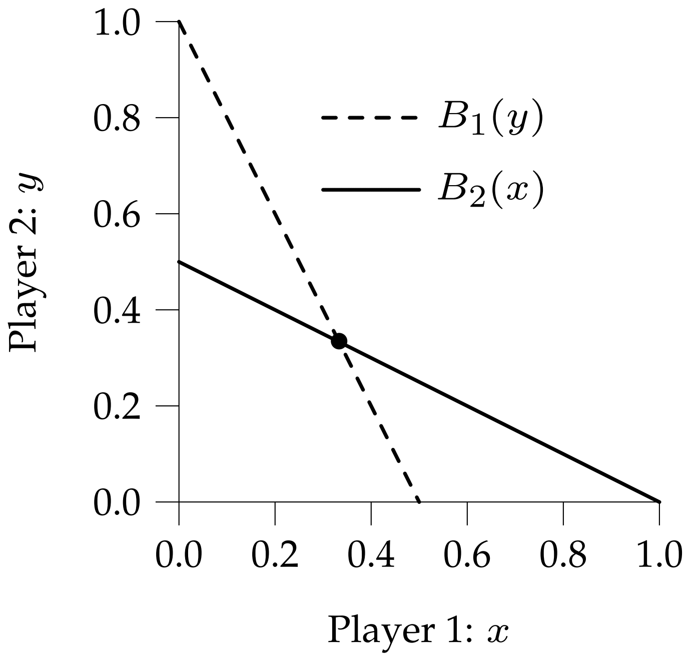
- Quantities are *strategic substitutes*.
- This is a *submodular* game.

Cournot-Nash Equilibrium

At a Nash equilibrium, players mutually best-respond: $y = B_2(x)$ and $x = B_1(y)$. So,

$$x = \frac{a - by - c}{2b} \quad \text{and} \quad y = \frac{a - bx - c}{2b}.$$

Solve these two equations simultaneously. The solution will be symmetric since the first-order conditions (e.g. $bx = a - b(x + y) - c = a - bQ - c$) depend only upon Q.



- From symmetry: $x = (a - bx - c)/2b$.
- Multiplying up: $2bx = a - bx - c$.
- Adding a term to both sides: $3bx = a - c$.
- So, infamously: $x^* = y^* = (a - c)/3b$.
- Plot the B_i functions and equilibrium.

Iterative Deletion of Strictly Dominated Strategies

The Cournot-Nash equilibrium strategies are the only survivors from the iterated deletion of strictly dominated strategies. To see this, simplify the game $a = b = 1$ and $c = 0$, so $\pi_1 = x(1 - x - y)$.

Consider the strategies $x \in (\frac{1}{2}, \infty)$. These are strictly dominated by $x = \frac{1}{2}$. The profits from playing these strategies are respectively, $x(1 - x - y)$ and $\frac{1}{2}(\frac{1}{2} - y)$. Suppose to the contrary

$$x(1 - x - y) > (1/2 - y)/2 \quad \Leftrightarrow \quad (1/2 - x)y > 1/4 - x + x^2.$$

Since $y \geq 0$, and $x \in (\frac{1}{2}, \infty)$, the left-hand side is less-than-or-equal-to zero. The right-hand side is minimised at zero when $x = \frac{1}{2}$, and therefore is positive, a contradiction. Similarly for $y > \frac{1}{2}$.

Now consider $x \in [0, \frac{1}{4})$. These strategies are strictly dominated by $x = \frac{1}{4}$. The payoffs are $x(1 - x - y)$ and $\frac{1}{4}(\frac{3}{4} - y)$. Suppose again, to the contrary, that

$$x(1 - x - y) > (3/4 - y)/4 \quad \Leftrightarrow \quad (1/4 - x)y > 3/16 - x + x^2.$$

Now $y \leq \frac{1}{2}$, so this is true $\forall y$ if and only if $(\frac{1}{4} - x)\frac{1}{2} > \frac{3}{16} - x + x^2 \Leftrightarrow 0 > \frac{1}{16} - \frac{1}{2}x + x^2$. Which is a contradiction. This process continues until $x = y = \frac{1}{3}$ remains.

Equilibrium in General Cournot Games

- General: n firms, firm i has constant marginal cost c_i, inverse demand $P(Q)$.
- Maximise profits for firm i. If $P(Q) < c_i$ then $q_i = 0$. Otherwise:

$$\pi_i = q_i[P(Q) - c_i] \quad \Rightarrow \quad \frac{\partial \pi_i}{\partial q_i} = P(Q) - c_i + q_i P'(Q) = 0 \quad \Leftrightarrow \quad q_i = -\frac{P(Q) - c_i}{P'(Q)}.$$

- Individual quantities are uniquely defined by industry supply Q.
- Thus, if $c_i = c$ for all i, then any equilibrium is symmetric.
- Sum the first-order conditions for all n firms, divide by n to obtain:

$$\frac{nP(Q) - [\sum_{i=1}^{n} c_i]}{P(Q)} + \frac{QP'(Q)}{P(Q)} = 0 \quad \Leftrightarrow \quad \frac{P(Q) - \frac{1}{n}\sum_{i=1}^{n} c_i}{P(Q)} = \frac{1}{n\varepsilon}$$

- Hence outcome determined by industry-average of marginal cost.
- In games where there is a single "state variable" (here, Q) determining equilibria…
- …the solution boils down to a single fixed-point equation.

Bertrand Competition

"Two firms selling identical products must simultaneously choose what price to charge. The firm that charges the lower price gains the entire market, but firms would rather charge high prices. A group of consumers will only buy if the price is less than $\bar{p}$. For simplicity, and without loss of generality, the marginal cost of production is zero."

Players. Two firms labelled $i \in N = \{1, 2\}$.

Strategies. Player i chooses price $p_i \in [0, \infty)$.

Payoffs. Payoffs are profits. There is a unit mass of consumers. If $p_1 = p_2$ the market is split 50:50.

$$\pi_i = \begin{cases} p_i & p_i < \min\{\bar{p}, p_j\}, \\ p_i/2 & p_i = p_j < \bar{p}, \\ 0 & p_i \geq \bar{p} \text{ and/or } p_i > p_j. \end{cases}$$

Bertrand-Nash Equilibrium

- There is a unique pure-strategy Nash equilibrium at $p_1 = p_2 = 0$:
  - If the lowest price were negative, then that firm will make a loss.
  - If the lowest price were strictly positive, then opponent should undercut.
  - If one price is zero, e.g. $0 = p_i < p_j$ then firm i should raise its price.
  - Hence only possibility is $p_1 = p_2 = 0$, where there is no better response.
- Notice that best-response functions are not well-defined everywhere:
  - Suppose, for example, that $0 < p_j < \overline{p}$.
  - Always a best-response for player i to undercut player j.
  - But if player i undercuts by ε, then ε as small as possible without $\varepsilon = 0$.
  - Mathematically, the set of feasible payoffs is open above, cannot attain a maximum.
- The Bertrand specification is *degenerate* — due to the discontinuity in payoffs.
- Be careful when using continuous action sets!

Differentiated Products

"Two firms selling differentiated products simultaneously choose prices. Total market size is a single unit mass. Suppose that each consumer is willing to pay a large amount to obtain a product. They do not necessarily buy from the cheapest firm, however."

- If $p_j - p_i > t$ then firm i captures the whole market: $q_i = 1$ and $q_j = 0$.
- If $|p_2 - p_1| \leq t$ then the split depends on the price difference:

$$q_i = \frac{1}{2} + \frac{p_j - p_i}{2t}.$$

Players. Two firms labelled $i \in N = \{1, 2\}$.

Strategies. Player i chooses $p_i \in [0, \infty)$.

Payoffs. If $p_i - p_j > t$, then $\pi_i = 0$. If $p_j - p_i > t$, then $\pi_i = p_i - c$. Otherwise

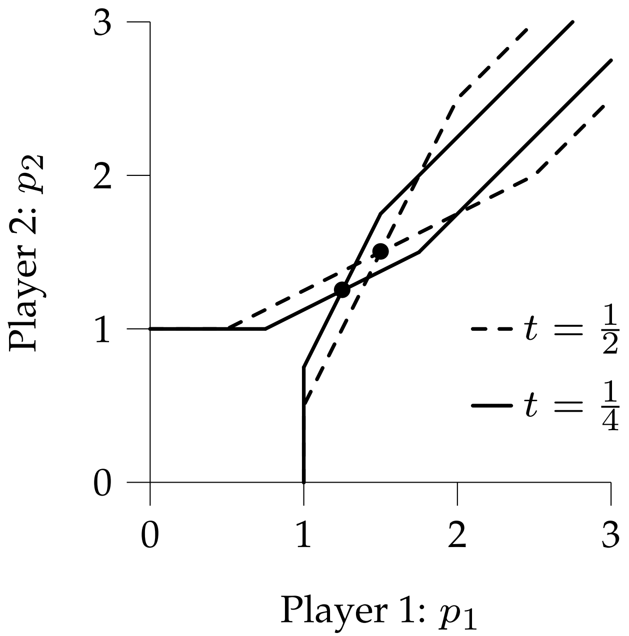
$$\pi_i = (p_i - c)\left(\frac{1}{2} + \frac{p_j - p_i}{2t}\right).$$

Differentiated Products Best-Response Functions

Profit is concave in price. Differentiate to obtain the first-order condition

$$\frac{\partial \pi_i}{\partial p_i} = \frac{t + p_j - 2p_i + c}{2t} = 0.$$

Solving for p_i yields the best-response function $B_i(p_j) = (t + c + p_j)/2$. This is upward sloping since $\partial B_i/\partial p_j > 0$. Prices are *strategic complements* — the game is *supermodular*.



- This solution applies when $|p_2 - p_1| \leq t$. In fact,

$$B_i(p_j) = \begin{cases} c & p_j < c - t, \\ (t + c + p_j)/2 & c - t \leq p_j \leq 3t + c, \\ p_j - t & 3t + c < p_j. \end{cases}$$

- For an interior equilibrium, $p_i = (t + c + p_j)/2$.
- Symmetry ensures $p_i = p_j = p^*$.
- So $p^* = (t + c + p^*)/2$, and so $p^* = t + c$.

Submodular and Supermodular Games

Players. Two players labelled $i \in N = \{1, 2\}$.

Strategies. Player 1 chooses $x \in X \subseteq \mathcal{R}$, player 2 chooses $y \in Y \subseteq \mathcal{R}$.

Payoffs. Payoffs are $u_1(x, y)$ and $u_2(y, x)$, with symmetry $u_1 = u_2 = u$.

Calculate the slope of a player's best-response function:

$$\begin{aligned} x = B_1(y) \qquad &\Rightarrow \qquad \frac{\partial u(x,y)}{\partial x} = 0, \\ &\Rightarrow \qquad \frac{\partial^2 u(x,y)}{\partial x \partial y} + \frac{\partial^2 u(x,y)}{\partial x^2}\frac{dx}{dy} = 0, \\ &\Rightarrow \qquad \frac{dx}{dy} = -\frac{\partial^2 u(x,y)/\partial x \partial y}{\partial^2 u(x,y)/\partial x^2}. \end{aligned}$$

Denominator is negative from second-order conditions. Sign determined by numerator, i.e.

$$\frac{\partial^2 u(x,y)}{\partial x^2} < 0 \quad \Rightarrow \quad \text{sign}\left\{\frac{dx}{dy}\right\} = \text{sign}\left\{\frac{\partial^2 u(x,y)}{\partial x \partial y}\right\}.$$

Informal Definitions

Definition 14. The game $\mathcal{G} = \langle \{1,2\}, \{X, Y\}, \{u, u\} \rangle$ is *supermodular* if $\partial^2 u(x,y)/\partial x \partial y > 0$, and so $B_1(y)$ is upward sloping. X and Y are said to be *strategic complements*.

Definition 15. The game $\mathcal{G} = \langle \{1,2\}, \{X, Y\}, \{u, u\} \rangle$ is *submodular* if $\partial^2 u(x,y)/\partial x \partial y < 0$, and so $B_1(y)$ is downward sloping. X and Y are said to be *strategic substitutes*.

These are only informal definitions. True definitions are a little bit more general. Some examples:

- Cournot competition is typically submodular. Quantities are strategic substitutes: an increase in an opponent's quantity reduces the incentive a firm has to raise quantity.
- Bertrand competition is typically supermodular. Prices are strategic complements: an increase in an opponent's price increases the incentive a firm has to raise price.
- Sometimes best-response functions can be non-monotonic…

Political Advertising

"The Labour and Conservative parties choose their advertising budgets for an election, denoted by x and y respectively. They dislike campaign spending, but wish to obtain a higher share of the vote. Advertising is the sole determinant of the election outcome, yielding vote shares of $x/(x+y)$ and $y/(x+y)$ respectively."

Players. The Labour and Conservative Parties ($N = \{L, C\}$).

Strategies. Labour chooses $x \in [0, \infty)$ and Conservatives choose $y \in [0, \infty)$.

Payoffs. Suitable payoff functions might be

$$u_L(x, y) = \frac{x}{x+y} - x \quad \text{and} \quad u_C(y, x) = \frac{y}{x+y} - y.$$

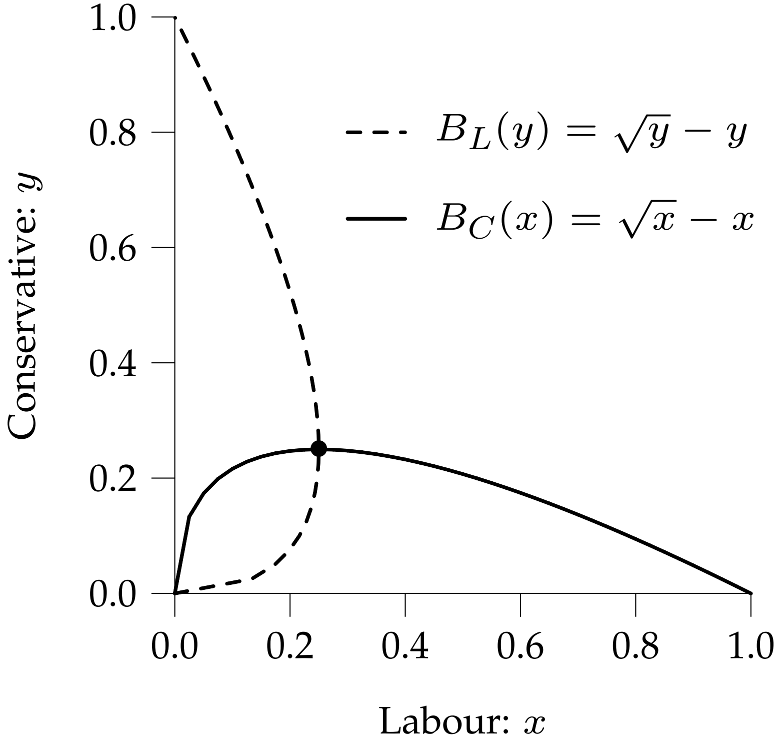
To calculate best-response functions, consider Labour's maximisation problem:

$$\begin{aligned} \frac{\partial u_L}{\partial x} = \frac{1}{x+y} - \frac{x}{(x+y)^2} - 1 = 0 \qquad &\Leftrightarrow \qquad \frac{1}{x+y} = \frac{x}{(x+y)^2} + 1, \\ &\Leftrightarrow \qquad x + y = x + (x+y)^2, \\ &\Leftrightarrow \qquad \sqrt{y} = x + y. \end{aligned}$$

Political Advertising Best-Responses

This yields the best-response functions $B_L(y) = \sqrt{y} - y$ and $B_C(x) = \sqrt{x} - x$.

At a Nash equilibrium $y = B_C(x)$ and $x = B_L(y)$. Hence $x^* = y^* = \frac{1}{4}$.



- The best-response functions for this game slope upward initially, then downward.
- The game is neither sub- nor supermodular.
- Variables are both strategic complements and substitutes, depending on the region.
- Equilibrium at $(0, 0)$? It would seem not…
- Since $u_L(0, 0) = u_C(0, 0) = \frac{1}{2}$ seems apt.

Note. B_L and B_C only make sense for $y > 0$ and $x > 0$. Setting $u_L(0, 0) = u_C(0, 0) = 1$ generates an equilibrium at $(0, 0)$, but doesn't make much sense. In any case it would not be a very "stable" equilibrium, whereas $(\frac{1}{4}, \frac{1}{4})$ is.

The Investment Race

"Two firms choose investment levels from the unit interval. The firm with the highest investment wins the market, which has unit value. If the same level is chosen, they split the market 50:50."

Players. Two firms $i \in N = \{1, 2\}$.

Strategies. Investment levels $x \in X = [0, 1]$ for player 1 and $y \in Y = [0, 1]$ for player 2.

Mixed Strategies. Distributions on $[0, 1]$; $F(x)$ and $G(y)$ respectively.

Payoffs. The payoffs are expected profit flows. For instance:

$$\pi_1(x, y) = \begin{cases} 1 - x & x > y \\ \frac{1}{2} - x & x = y \\ -x & x < y \end{cases}$$

- Notice that there are no pure-strategy Nash equilibria:
  - If $y < 1$, then player 1 would do better with $x = y + \varepsilon < 1$.
  - If $x = y = 1$, then firm 1 would do better to choose $x = 0$.
- There is, however, a mixed strategy Nash equilibrium…

A Mixed-Strategy Nash Equilibrium

Recall the *indifference* property of mixed equilibria. Argue that there is an equilibrium:

1. In mixed equilibria a player must be indifferent across all the pure strategies they mix over.
2. Suppose $x \in X$ is in the support of player 1's strategy $F(\cdot)$.
3. Then this strategy must yield a *constant* amount in expectation.
4. The probability player 1 wins with x is $\Pr[x > y] = G(x)$.
5. The probability player 1 loses with x is $\Pr[x < y] = 1 - G(x)$.
6. The probability player 1 draws with x is $\Pr[x = y] = 0$.

$$\mathrm{E}(\pi_1) = -x[1 - G(x)] + [1 - x]G(x) = k \quad \Leftrightarrow \quad G(x) = x + k.$$

7. But $G(0) = 0$ since this is a cdf, and so $k = 0$. Thus $G(x) = x$, and $\mathrm{E}(\pi_1) = 0$.

Symmetry implies there is a Nash equilibrium where both players mix *uniformly* over $[0, 1]$.

Uniqueness and Existence

In fact, it can be shown that this is the *only* Nash equilibrium of this game. More complicated…

1. Show that there are no *atoms* in the distribution.
2. Show that there are no *gaps* in the distribution.
3. Show that any mixed equilibrium strategy must have full support on $[0, 1]$.
4. Show (by the argument on the previous slide) that there is only one such distribution.

There are games that have no Nash equilibria at all (pure or mixed). Recall that in finite games there was always at least one (possibly mixed) equilibrium. No longer! Does this matter?

- Peculiar games with peculiar properties…
- A result of poor modelling choice? The world is not continuous…
- Even so, a theoretical question remains: what happens in games with no equilibria?

However, usually faced with multiplicity rather than non-existence — e.g. Nash Demands.

A Return to Nash Demand

"Two individuals argue over the division of a pot of money. They simultaneously make irrevocable demands, and receive their demands if and only if they can jointly be met from the money pot. The money pot, however, is of indeterminate size."

Players. Two players labelled $i \in N = \{1, 2\}$.

Strategies. Player 1 chooses $x \geq 0$, player 2 chooses $y \geq 0$.

Payoffs. Players receive payoffs of $u_1(x)$ and $u_2(y)$ with probability $1 - F(x + y)$.

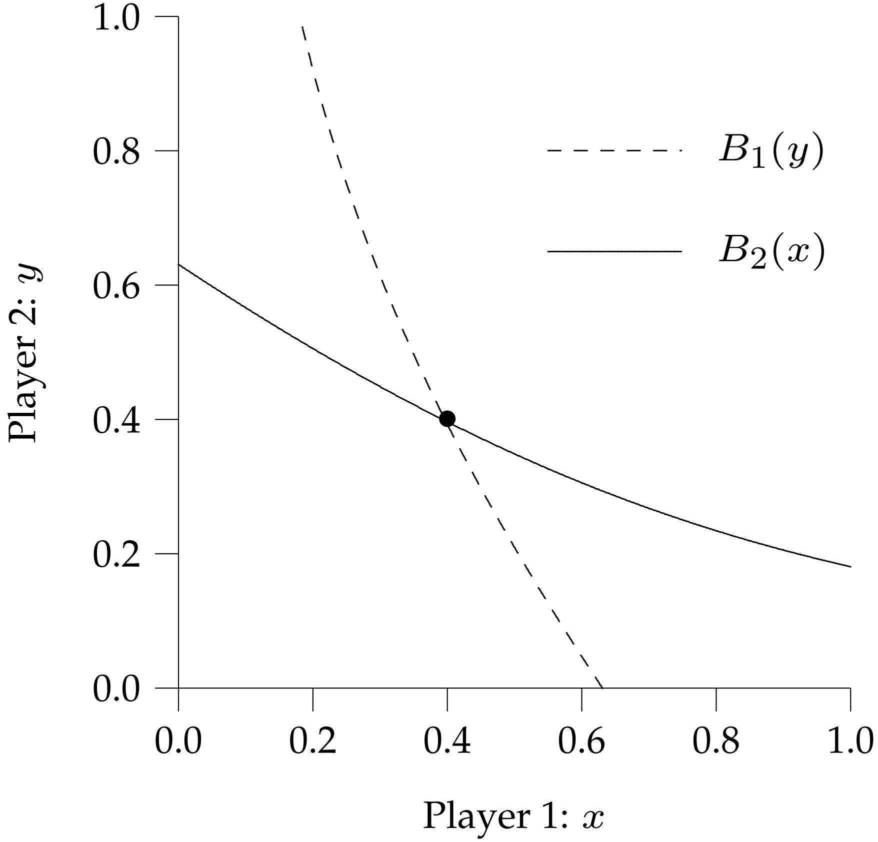
- The interpretation is that the money pot is of size z, where $z \sim F(z)$.
- Thus $\Pr[x + y \leq z] = 1 - F(x + y)$.
- Calculating the best-response function for player 1,

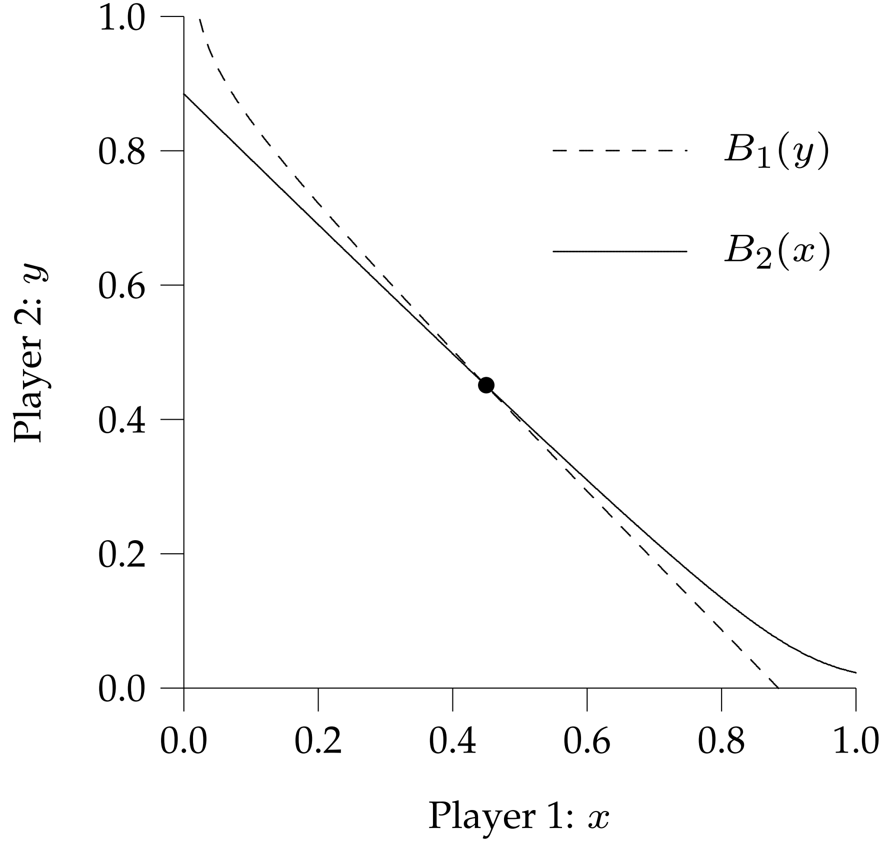
$$\max_{x \geq 0} u_1(x)[1 - F(x + y)] \quad \Rightarrow \quad u_1'(x)[1 - F(x + y)] = f(x + y)u_1(x)$$

- For example, set $u(x) = \sqrt{x}$ and $z \sim N(1, \sigma^2)$, and plot best-response functions.

Nash Demand Best-Responses

With more structure, the prediction becomes (much) tighter.





$\sigma = 0.4 \Rightarrow x^* = y^* \approx 0.4$

$\sigma = 0.05 \Rightarrow x^* = y^* \approx 0.45$