Mixing and Matching

Recall the following game from the previous lectures: Matching Pennies.

	H	T
H	-1, $\underline{1}$	$\underline{1}$, -1
T	$\underline{1}$, -1	-1, $\underline{1}$

There is no "pure-strategy" Nash equilibrium. How might a prediction be made? Neither rationalisability nor iteratively eliminating strictly dominated strategies helps.

Suppose row player tossed their (fair) coin…

Column player (equipped with vNM utility) would receive $\frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0$ upon playing H. Column receives $\frac{1}{2} \times (-1) + \frac{1}{2} \times 1 = 0$ upon playing T. Column is indifferent!

…and might decide by tossing a (fair) coin…

Mixed Strategies

This is referred to as mixing, or playing *mixed strategies*. Extend Definition 1 to incorporate this:

Definition 10. The *mixed extension* of a game $\mathcal{G} = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is the game Γ, where:

1. $\Gamma = \langle N, \{\Delta(S_i)\}_{i \in N}, \{U_i\}_{i \in N} \rangle$.
2. $\Delta(S_i)$ is the set of probability distributions over S_i, and $\Delta(S) = \times_{i \in N} \Delta(S_i)$.
3. $U_i : \Delta(S) \mapsto \mathcal{R}$ is a vNM expected utility function that assigns to each $\sigma \in \Delta(S)$ the expected value under u_i of the lottery over S induced by σ.

Consider part (3) for finite games. Suppose player i plays mixed strategy $\sigma_i \in \Delta(S_i)$. Denote the probability that this places on pure strategy $s_i \in S_i$ as $\sigma_i(s_i)$. Then,

$$U_i(\sigma) = \sum_{s \in S} u_i(s) \prod_{j \in N} \sigma_j(s_j).$$

Notation. Define $\sigma_{-i} \in \Delta(S_{-i}) = \times_{j \neq i} \Delta(S_j)$ analogously to the pure strategy case.

Nash Equilibrium

Definition 7a can be immediately brought to bear: A mixed-strategy Nash equilibrium of a game $\mathcal{G}$ is a Nash equilibrium of its mixed extension Γ. Formally…

Definition 11a. A *mixed-strategy Nash equilibrium* is a strategy profile $\sigma^* \in \Delta(S)$ such that,

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*) \qquad \forall \sigma_i \in \Delta(S_i) \quad \text{and} \quad \forall i \in N.$$

Definition 11b. $\sigma^* \in \Delta(S)$ is a Nash equilibrium if and only if $\sigma_i^* \in B_i(\sigma_{-i}^*)$ for all $i \in N$.

Definitions for rationalisability and dominance can be extended to mixed strategies similarly.

Definition 11b gives a useful insight into the properties of mixed-strategy Nash equilibria. For a mixed strategy σ_i to be a best response to a given combination of opponents' strategies σ_{-i}, every pure strategy in its support must also be a best response to σ_{-i}. Therefore all pure strategies in the support of the equilibrium strategy for a given player must yield the same payoff to that player…

Use this *indifference property* to find equilibria.

Equilibrium in Matching Pennies

Consider, once again, the matching pennies game:

	H	T
H	-1, $\underline{1}$	$\underline{1}$, -1
T	$\underline{1}$, -1	-1, $\underline{1}$

Now suppose row mixes with probability x and $1 - x$ on H and T respectively. Abusing notation,

$$U_C(H) = x \times 1 + (1 - x) \times (-1) = 2x - 1,$$

$$U_C(T) = x \times (-1) + (1 - x) \times 1 = 1 - 2x.$$

Column is indifferent only when $2x - 1 = 1 - 2x \Leftrightarrow x = \frac{1}{2}$. Similarly for row. The mixed-strategy Nash equilibrium involves both players mixing with probability $\frac{1}{2}$.

Notice row player's equilibrium strategy is determined by column's payoffs and vice versa!

Battle of the Sexes Revisited

Players. The players are M.Phil. student 1 (row) and 2 (column).

Pure Strategies. The strategies available to both players are Cafe and Pub.

Mixed Strategies. Row chooses Cafe w.p. $x \in [0, 1]$, column chooses Cafe w.p. $y \in [0, 1]$.

Payoffs. Represent the payoffs in the strategic form matrix:

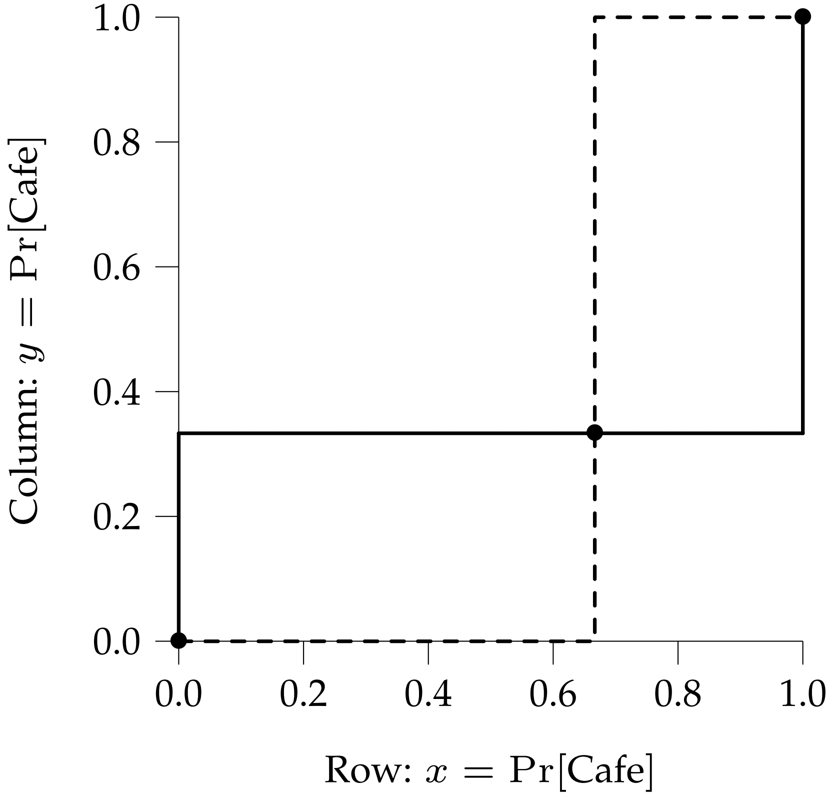
	Cafe (y)	Pub ($1 - y$)	Expected
Cafe (x)	3 4	1 1	$4y + (1 - y)$
Pub ($1 - x$)	0 0	4 3	$3(1 - y)$
Expected	$3x$	$x + 4(1 - x)$	

$$\text{Column chooses } y = 1 \text{ whenever} \quad 3x > x + 4(1 - x) \quad \Leftrightarrow \quad 6x > 4 \quad \Leftrightarrow \quad x > 2/3.$$

$$\text{Row chooses } x = 1 \text{ whenever} \quad 4y + (1 - y) > 3(1 - y) \quad \Leftrightarrow \quad 6y > 2 \quad \Leftrightarrow \quad y > 1/3.$$

Plotting Best-Response Functions

Best-responses are now: $B_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) \mid U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}),\ \forall \sigma_i' \in \Delta(S_i)\}$.



- Plot functions for Battle of the Sexes.
- Equilibria occur where $\sigma_i \in B_i(\sigma_{-i}),\ \forall i$.
- Two pure equilibria: $x{=}y{=}1$ and $x{=}y{=}0$.
- One mixed equilibrium: $x = \frac{2}{3}$ and $y = \frac{1}{3}$.
- Three equilibria: An odd number…
- Generically, odd numbers of equilibria.

Note that, in mixed extensions of finite games, the best-response functions will be continuous.

Hence they must always intersect: a "fixed point" theorem yields existence of Nash equilibrium.

Dominance and Mixed Strategies

In the following game, no strategy is strictly dominated by another pure strategy:

	L	M	R
T	4 10	3 0	1 3
B	0 0	2 10	10 3

Suppose column plays L with probability $\frac{1}{2}$ and M with probability $\frac{1}{2}$, so that $\sigma_C = (\frac{1}{2}, \frac{1}{2}, 0)$.

$U_C(\sigma_C, \sigma_R) = 5$, whatever row plays (for all σ_R). This is better than a payoff of 3 from playing R.

R is strictly dominated by a mixed strategy placing probability $\frac{1}{2}$ on L and M, and therefore:

	L	M
T	4 10	3 0
B	0 0	2 10

$\longrightarrow$

	L	M
T	4 10	3 0

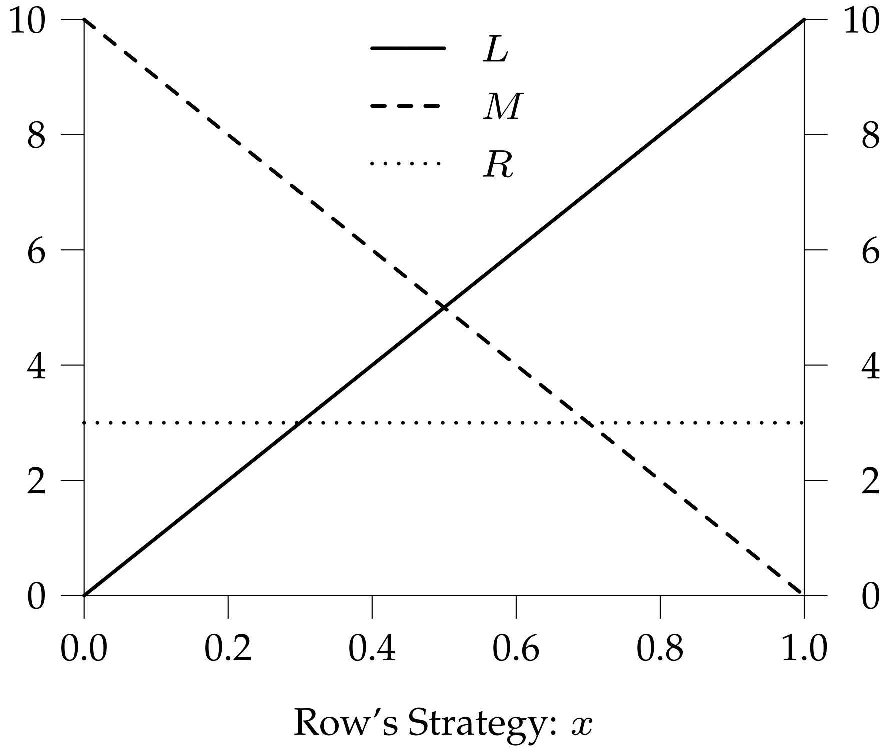
$\longrightarrow$

	L
T	4 10

A single strategy profile survives iterated deletion of strictly dominated strategies: $\{T, L\}$.

Never-Best-Responses

	L		M		R	
$T \quad (x)$	4	10	3	0	1	3
$B \quad (1-x)$	0	0	2	10	10	3
Column's Payoff	$10x$		$10(1-x)$		3	

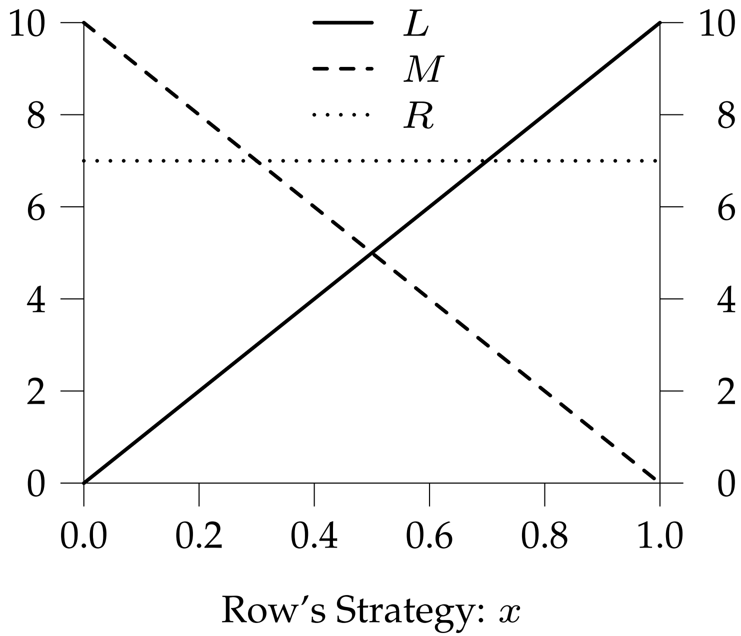


- Plot column's payoffs to each strategy.
- Irrespective of the value of the belief x, R is never-a-best-response.
- In fact, a strategy is strictly dominated if and only if it is never-a-best-response.
- Rational players would always play a best response given some beliefs.
- The equilibrium is at $\{T, L\}$.

Dominated Mixed Strategies

A mixed strategy with positive weight on a strictly dominated pure strategy is strictly dominated. But a mixed can be dominated by a pure even if all strategies in its support are undominated.

	L		M		R	
T	3	10	0	0	1	7
B	0	0	3	10	1	7



- Plot column's payoffs to each strategy. x is the weight attached to T by row.
- Note that neither the pure strategy L nor M is strictly dominated by R.
- The strategy which places probability $\frac{1}{2}$ on each of L and M earns 5.
- This *is* strictly dominated by R.
- Nash: $\{T, L\}$, $\{B, M\}$, $\{x \in [\frac{3}{10}, \frac{7}{10}], R\}$.

A Return to Rationalisability

	L	M	R
T	4 10	3 0	1 3
B	0 0	2 10	10 3

Only T and L are rationalisable. Consider R: Not rationalisable as there is no belief such that R would be a best response. Thus B is not rationalisable, row would need to believe R is to be played by column. Thus M is not rationalisable, as this requires column to believe that row will play B.

Restricting to pure strategies for a moment — no strategies are strictly dominated:

- Rationalisable strategies $\subset$ Iterated deletion of dominated strategies survivors.

Allowing for mixed strategies again, R is strictly dominated by a 50:50 mix over L and M. Then B is strictly dominated by T, and then M is strictly dominated by L. So now:

- Rationalisable strategies $=$ Iterated deletion of dominated strategies survivors.

Rationalisability and Dominance

Recall $R = \times_{i \in N} R_i$ where R_i is the set of rationalisable strategies for player i, and A is the set that survives iterated deletion of dominated strategies. In textbooks, one often sees:

$$R \subseteq A \quad \text{and} \quad R = A \quad \text{if} \quad n = 2.$$

However, under the definition of rationalisability from the last lecture, $R = A$ generally. Why?

The classic definition (and the one most textbooks follow) restricts the kinds of beliefs players can have about their opponents' play. In particular, opponents must mix *independently*.

Here, a player can rationalise a strategy based on a belief which requires opponents to *correlate* their strategies. An example (might) help to enlighten. Row, column, and matrix player…

$$\begin{array}{c|c|c|}
 & L & R \\ \hline
U & 0 & 2 \\ \hline
D & 0 & 3 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & L & R \\ \hline
U & 3 & 0 \\ \hline
D & 2 & 0 \\ \hline
\end{array}
\qquad
\begin{array}{c|c|c|}
 & L & R \\ \hline
U & 2 & 0 \\ \hline
D & 0 & 2 \\ \hline
\end{array}$$

$$X \qquad\qquad\qquad Y \qquad\qquad\qquad Z$$

Rationalisability and Correlation

Is Z dominated? Is it rationalisable when beliefs are "independent"? When they can "correlate"?

$$\begin{array}{c|c|c|} & L & R \\ \hline U & 0 & 2 \\ \hline D & 0 & 3 \\ \hline \end{array} \qquad \begin{array}{c|c|c|} & L & R \\ \hline U & 3 & 0 \\ \hline D & 2 & 0 \\ \hline \end{array} \qquad \begin{array}{c|c|c|} & L & R \\ \hline U & 2 & 0 \\ \hline D & 0 & 2 \\ \hline \end{array}$$

$$X \qquad\qquad\qquad Y \qquad\qquad\qquad Z$$

Suppose row plays U with probability p, column plays L with probability q. For Z to be strictly dominated there must exist a (mixed) strategy σ_M which does strictly better for all p and q.

In particular, σ_M would have to do better than Z for $p = q = 1$ and for $p = q = 0$. Then:

$$\begin{aligned} 3\sigma_M(X) + 2\left[1 - \sigma_M(X) - \sigma_M(Y)\right] &> 2, \\ 3\sigma_M(Y) + 2\left[1 - \sigma_M(X) - \sigma_M(Y)\right] &> 2. \end{aligned}$$

Which immediately yields a contradiction. Can Z be rationalised?

Independence and Correlation

So Z survives iterated deletion of strictly dominated strategies. Is it rationalisable?

To what pair of *independent* mixed strategies is it a best response? None! Suppose it were a best response to the mixed strategy pair p (row plays U: $\sigma_R(U) = p$) and q (i.e. $\sigma_C(L) = q$). Then

$$\begin{aligned} 2pq + 2(1-p)(1-q) &> 3pq + 2(1-p)q, \quad \text{and} \\ 2pq + 2(1-p)(1-q) &> 2p(1-q) + 3(1-p)(1-q). \end{aligned}$$

Some elementary algebra confirms that these inequalities are mutually inconsistent.

$$\begin{array}{c|c|c|} & L & R \\ \hline U & 0 & 2 \\ \hline D & 0 & 3 \\ \hline \end{array} \qquad \begin{array}{c|c|c|} & L & R \\ \hline U & 3 & 0 \\ \hline D & 2 & 0 \\ \hline \end{array} \qquad \begin{array}{c|c|c|} & L & R \\ \hline U & 2 & 0 \\ \hline D & 0 & 2 \\ \hline \end{array}$$

$$X \qquad\qquad\qquad Y \qquad\qquad\qquad Z$$

But Z is a best response to a belief that assigns probability $\frac{1}{2}$ to $\{U, L\}$ and probability $\frac{1}{2}$ to $\{D, R\}$. Is such a belief *rational*? Some say no…some say yes…why should correlated play be ruled out?

Information Partitions

A mixed strategy might be interpreted as a pure strategy of an extended game in which a player receives a private, independently distributed signal, and conditions behaviour upon this signal. e.g. in matching pennies, toss a coin and play H if it comes up heads and T if it comes up tails.

What if these signals were not independent, but rather (possibly imperfectly) correlated?

A "state of nature" is $\omega \in \Omega$. The probability of ω occurring is $\pi(\omega)$. Hence $\sum_{\omega \in \Omega} \pi(\omega) = 1$.

An information partition $\mathcal{P}_i$ for player $i \in N$ is a partition of Ω. It has typical element $P_i \in \mathcal{P}_i$.

$$\text{e.g. } \Omega = \{x, y, z\}, \quad \mathcal{P}_1 = \{\{x, y\}, z\}, \quad \mathcal{P}_2 = \{x, \{y, z\}\}.$$

So, for example, if the state is z player 1 knows this, but player 2 only knows it is either y or z. If the state is y player 1 knows it is either x or y and player 2 knows it is either y or z, etc.

So this would allow players to correlate their actions. A strategy tells each player what to do given their signal. e.g. if the coin-toss was publicly seen, both players might choose to play H if heads and T if tails (although this would be very bad for row player!)

Raising Payoffs by Correlating Actions

Suppose $\Omega = \{x, y, z\}$, $\mathcal{P}_1 = \{\{x, y\}, z\}$, $\mathcal{P}_2 = \{x, \{y, z\}\}$, and $\pi(x) = \pi(y) = \pi(z) = \frac{1}{3}$.

	L	R
T	6 6	2 7
B	7 2	0 0

	L	R
T	y	x
B	z	—

Assign strategies for player 1 (row) as T if $\{x, y\}$ is observed, and B if z is observed. Player 2 (column) plays L if $\{y, z\}$ is observed and R if x is observed. Can player 1 do any better? No:

By playing T if $\{x, y\}$ gets 4, playing B would yield 3.5 (the state is x with probability $\frac{1}{2}$ and player 2 plays L, the state is y with probability $\frac{1}{2}$ and 2 plays R). By playing B if z, gets 7, T yields 6.

Symmetrically, player 2 can do no better—this is a *correlated equilibrium*.

The second matrix illustrates what happens in each state. Each player gets 5 in equilibrium!

Note the symmetric mixed-strategy equilibrium, play T (and L) with probability $\frac{2}{3}$, yields $4\frac{2}{3}$ to each player, whilst pure strategy equilibria yield 7 and 2 respectively.

Correlated Equilibrium

Putting these ideas together yields the following formal definition.

Definition 12. A *correlated equilibrium* of a game $\mathcal{G} = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ consists of:

1. *Probability Space.* (Ω, π). State space Ω with probability measure π,
2. *Information Partition.* Partition of Ω, $\mathcal{P}_i$ for each player $i \in N$,
3. *Strategy.* $\beta_i : \Omega \mapsto S_i$ where $\beta_i(\omega) = \beta_i(\omega')$ whenever $\omega \in P_i$ and $\omega' \in P_i$, some $P_i \in \mathcal{P}_i$,

such that, for all $i \in N$, and each strategy for i, $\beta_i' : \Omega \mapsto S_i$,

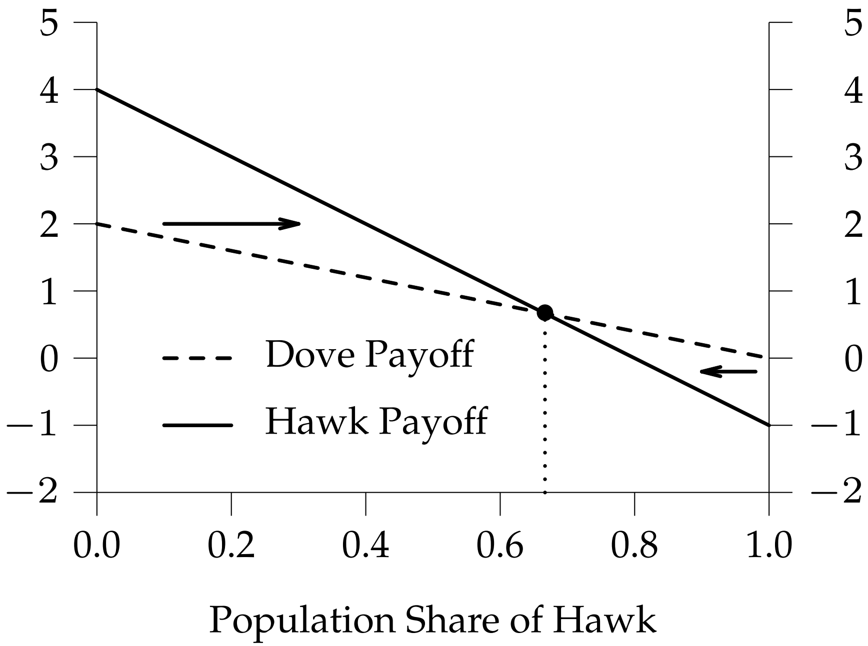
$$\sum_{\omega \in \Omega} \pi(\omega) u_i(\beta_i(\omega), \beta_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \pi(\omega) u_i(\beta_i'(\omega), \beta_{-i}(\omega)).$$

- Notice that the probability distribution, and the state space itself are endogenous.
- It is apparent that the set of correlated equilibria contains the set of Nash equilibria.

Evolutionary Games and Hawk-Dove

Recall Hawk-Dove from the first lecture. It has a mixed equilibrium with $\sigma_i(\text{Hawk}) = y = \frac{2}{3}$.

	Hawk (y)	Dove ($1-y$)	Row's Payoff
Hawk	$-1 \quad -1$	$4 \quad 0$	$4(1-y)-y$
Dove	$0 \quad 4$	$2 \quad 2$	$2(1-y)$



- Think of a population of players, each playing the game against the rest.
- e.g. if half the population are Hawks, Hawk yields $1\frac{1}{2}$, playing Dove yields 1.
- As before, plot payoffs against y (now the population proportion playing Hawk).
- If y is small, higher payoff from Hawk. Dove better when y is large.

Move in direction of higher payoff—mixed equilibrium is "evolutionarily stable".

Evolutionarily Stable Strategies

To formalise this notion, consider a single population of agents playing a *symmetric* game $\mathcal{G} = \langle\{1,2\}, \{\Delta(S), \Delta(S)\}, \{U_1, U_2\}\rangle$, where $U_1(\sigma, \sigma') = U_2(\sigma', \sigma) = U(\sigma, \sigma')$ for $\sigma, \sigma' \in \Delta(S)$.

A strategy is evolutionarily stable if a "small move away" from it (called a *mutation*) within the population results in pressure to "move back toward" it. That is, σ^* is evolutionarily stable if

$$(1-\varepsilon)U(\sigma, \sigma^*) + \varepsilon U(\sigma, \sigma) < (1-\varepsilon)U(\sigma^*, \sigma^*) + \varepsilon U(\sigma^*, \sigma).$$

In words: the payoff the "mutant" strategy σ gets in the mutated population is lower than the payoff the original strategy σ^* gets in the mutated population. Or equivalently:

Definition 13. An *evolutionarily stable strategy* (ESS) of $\mathcal{G}$ is a $\sigma^* \in \Delta(S)$ where

1. (σ^*, σ^*) is a Nash equilibrium and,
2. $U(\sigma, \sigma) < U(\sigma^*, \sigma)$ for all $\sigma \neq \sigma^*$ such that $\sigma \in B(\sigma^*)$.

Note that an ESS is always Nash, but the reverse is not true...

ESS in Hawk-Dove

In particular, only symmetric Nash-equilibrium strategies are candidates to be ESSs. It is straightforward to check that the symmetric mixed in Hawk Dove is an ESS.

	Hawk (y)	Dove ($1-y$)
Hawk	-1 -1	4 0
Dove	0 4	2 2

Notice that $U(\frac{2}{3},\frac{2}{3}) = \frac{2}{3}$, and that $\sigma(\text{Hawk}) = \frac{2}{3}$ constitutes a Nash equilibrium strategy. Now consider a mutation to Hawk. Checking condition 2, $U(1,1) = -1 < -\frac{2}{3} = U(\frac{2}{3},1)$.

What about a mutation to Dove? Checking condition 2, $U(0,0) = 2 < \frac{10}{3} = U(\frac{2}{3},0)$. It is easy to check other mutations and conclude that $\sigma(\text{Hawk}) = \frac{2}{3}$ is indeed an ESS.

- Does this provide a neat justification for mixed-strategy Nash equilibria?
- Note this definition is the *single* population version of ESS…
- Under the multiple-population definition, the Hawk-Dove mixed equilibrium is not stable.
- Also note that an ESS may not exist—consider Rock-Scissors-Paper…

Rock-Scissors-Paper Revisited

The unique mixed-strategy Nash equilibrium involves $\sigma(R) = \sigma(S) = \sigma(P) = \frac{1}{3}$. Is this an ESS?

	R	S	P
R	$0,\ 0$	$\underline{1},\ -1$	$-1,\ \underline{1}$
S	$-1,\ \underline{1}$	$0,\ 0$	$\underline{1},\ -1$
P	$\underline{1},\ -1$	$-1,\ \underline{1}$	$0,\ 0$

No! Playing R against R yields 0. Playing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ against R yields 0. Condition 2 fails.

To summarise: call E the set of equilibria with evolutionarily stable strategies. Then…

$$E \subseteq Z \subseteq R = A \subseteq \Delta(S).$$

If C is the set of correlated equilibria, $Z \subseteq C$. "$=$" is replaced by "$\subseteq$" when the more common notion of rationalisability is used (the one here is sometimes called *correlated rationalisability*).