Extensive-Form Games with Imperfect Information

When there is imperfect information, there may be no proper subgames — so subgame perfection may coincide with Nash equilibrium and hence not provide tighter predictions.

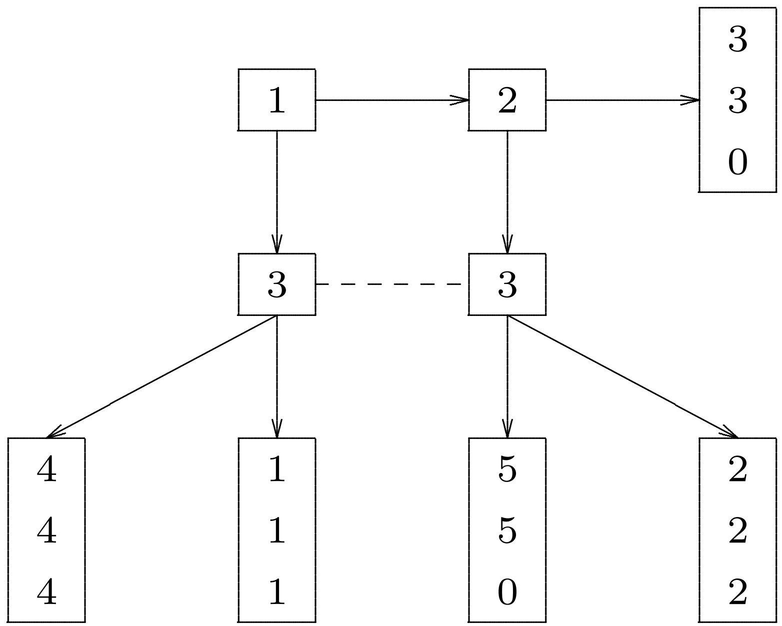
- Sometimes it will be possible to modify the extensive form…
- …and subgame perfection may once again rule out some "incredible" threats.
- Often modification of the extensive form will still leave multiple subgame-perfect equilibria.

In this case, further refinements are possible — motivated by players' reasoning.

- Forward Induction may reduce the number of subgame-perfect equilibria…
- …via the iterated deletion of weakly dominated strategies.
- Perfect-Bayesian equilibrium involves the introduction of "beliefs" for players…
- …and can itself be refined: for example, via the intuitive criterion.

Games with no Proper Subgames

Recall the following game. It has no proper subgames. All Nash equilibria are subgame perfect.

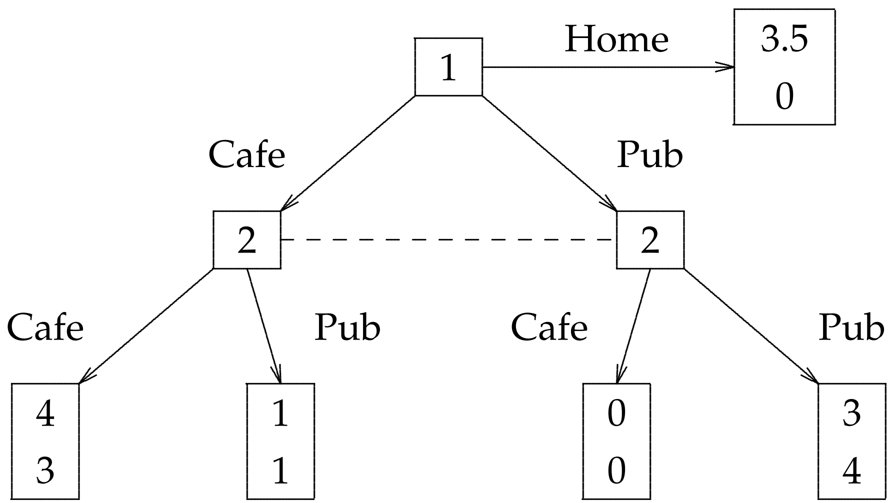


If player 3 is called upon to move, cannot observe what player 1 has done, and hence whether or not player 2 has had a move. But 3's optimal action depends upon what has happened.

- Does modifying the extensive form help make a prediction? If not...
- What does player 3 believe has happened? Will this help predict?

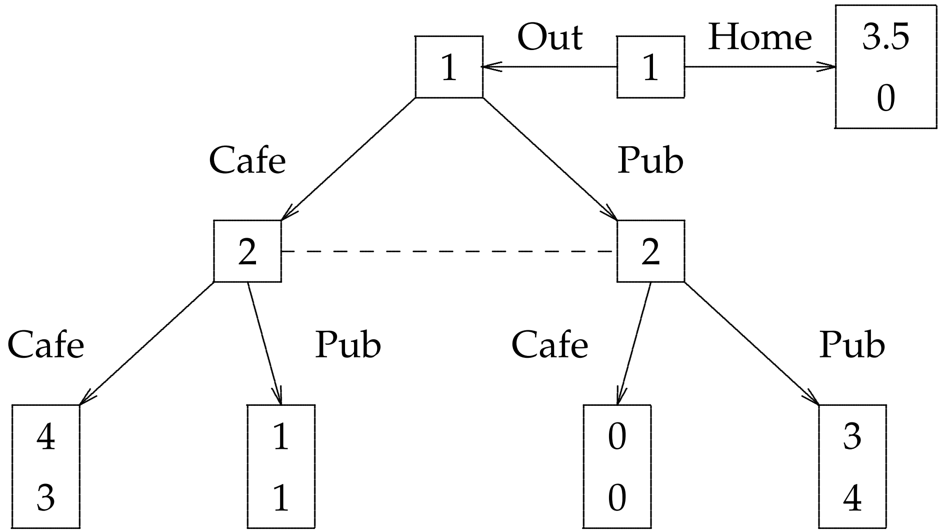
Battle of the Sexes Revisited

"Two M.Phil. students need to meet up (again) to discuss their love for economics. They can meet in either the pub or the cafe. The first likes coffee, and prefers the cafe. The other is a big fan of beer — and prefers the pub. They would both rather meet (wherever it may be) than miss each other. The first student, however, always has the option not to go at all. A deep passion for childish PlayStation-type activities leads them to obtain a bigger payoff staying at home than they would receive from going to the pub."



Multiple Subgame-Perfect Equilibria

This game has no proper subgames, so all Nash are subgame perfect. Modify extensive form:



$\Longrightarrow$

	Cafe	Pub
Home, Pub	3.5 0	3.5 0
Home, Cafe	3.5 0	3.5 0
Out, Cafe	4 3	1 1
Out, Pub	0 0	3 4

This modification still has multiple pure-strategy subgame-perfect equilibria…

- Two (pure) Nash equilibria of the subgame starting at (Out): {Cafe, Cafe} and {Pub, Pub}.
- Every strategy combination of the subgame at (Home) is a Nash equilibrium.
- Thus {(Out, Cafe), Cafe} and {(Home, Pub), Pub} are subgame perfect.

Forward Induction

But is there something fishy about the equilibrium {(Home, Pub), Pub}? Why would player 1 ever choose to go to the Pub? They could do better by staying at Home.

- Hence, given that they choose Out, surely they intend to choose Cafe?
- {(Out, Cafe), Cafe} is the only equilibrium that survives *forward induction*.
- This requires a consideration of player 2's *beliefs* about player 1's action.

Equilibria that survive forward induction will survive the iterated deletion of weakly dominated strategies (again, the order of deletion can matter).

- Notice that (Out, Pub) is *strictly* dominated by (Home, Pub) for player 1.
- In the reduced game (Pub) is *weakly* dominated by (Cafe) for player 2.
- (Home, Pub) and (Home, Cafe) are now strictly dominated for player 1.

More formally: forward induction requires an equilibrium to remain an equilibrium even when strategies dominated in that equilibrium are removed from the game, and this procedure is iterated.

Beliefs

Recall Definition 23: a strategy for player $i \in N$ is a function mapping each history where i is called upon to move (that is, $P(h) = i$) to an action $A(h)$. Recall that $A(h) = A(h')$ if $h, h' \in I_i$.

Add to this the notion of a *belief*. Each player assigns probabilities to each possible history within a particular information set, and does so for every one of their information sets.

Definition 26. Beliefs for $i \in N$ are probability distributions $\mu_{i,I_i} : I_i \mapsto [0, 1]$ for each $I_i \in \mathcal{I}_i$.

An *assessment* or *prospect* is a strategy profile and a set of beliefs for each player (s, μ).

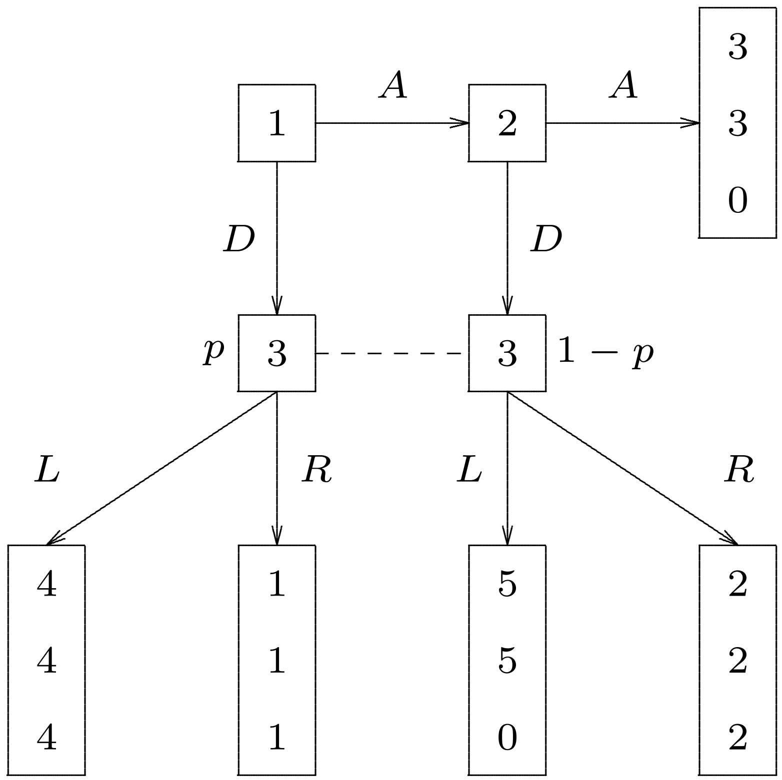
Rather than equilibria being particular strategy profiles, they will now be particular assessments.

Beliefs are part of any equilibrium.

These may include beliefs over information sets that are never reached. It is vitally important, when calculating and writing down equilibria, to remember this.

Applying Beliefs

Defining player 3's beliefs in the example from earlier (a version of "Selten's Horse") yields:

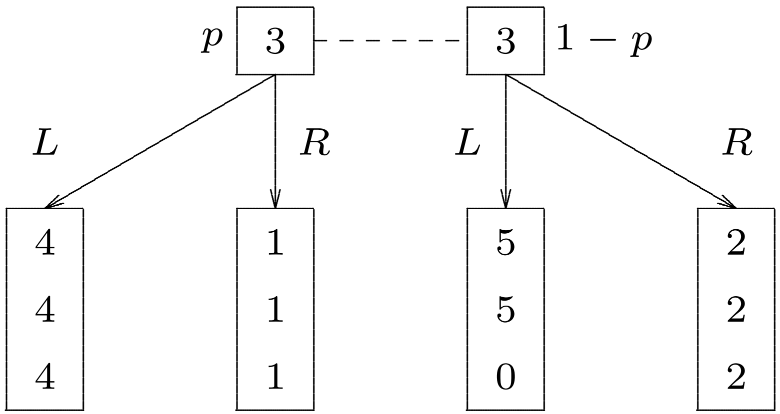


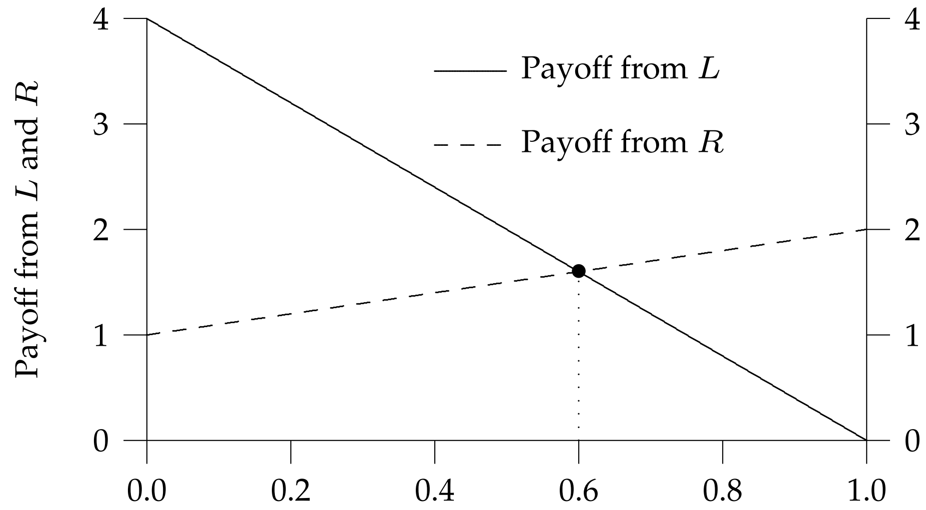
A strategy profile is an action for each of the three players, e.g. $\{A, A, L\}$.

Beliefs are given by the probability distribution $(p, 1-p)$ over player 3's information set.

Optimality and Beliefs

Suppose the belief of player 3 that history (D) has occurred is set at p.





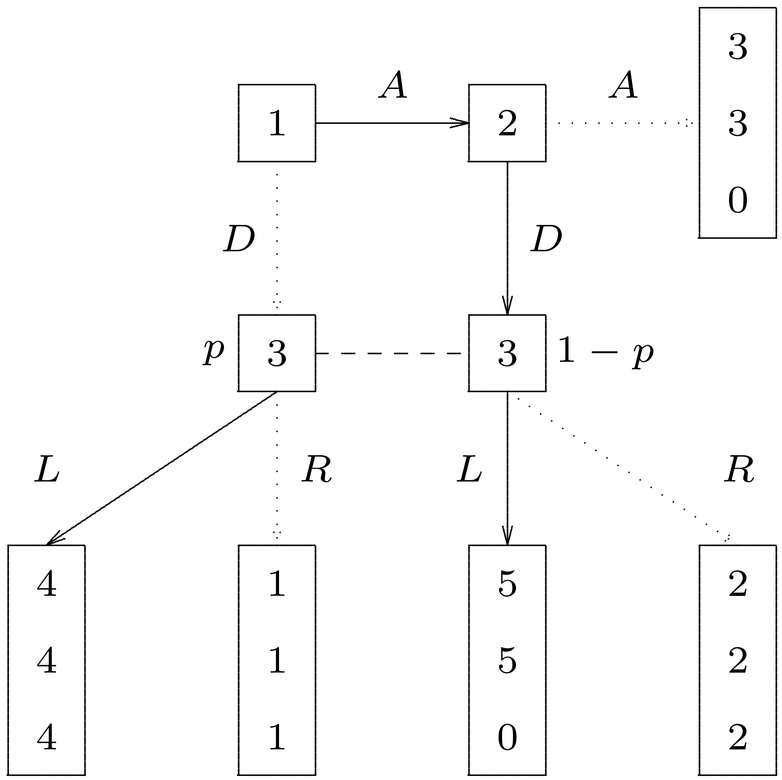
Probability $(1-p)$ that history (A, D) occurs.

The payoffs for L and R are $L = 4p$ and $R = p + 2(1-p)$. L is a best response whenever:

$$L \geq R \quad \Leftrightarrow \quad 4p \geq p + 2(1-p) \quad \Leftrightarrow \quad p \geq \frac{2}{5} \quad \left[\text{or } 1 - p \leq \frac{3}{5}\right].$$

Inconsistent Beliefs

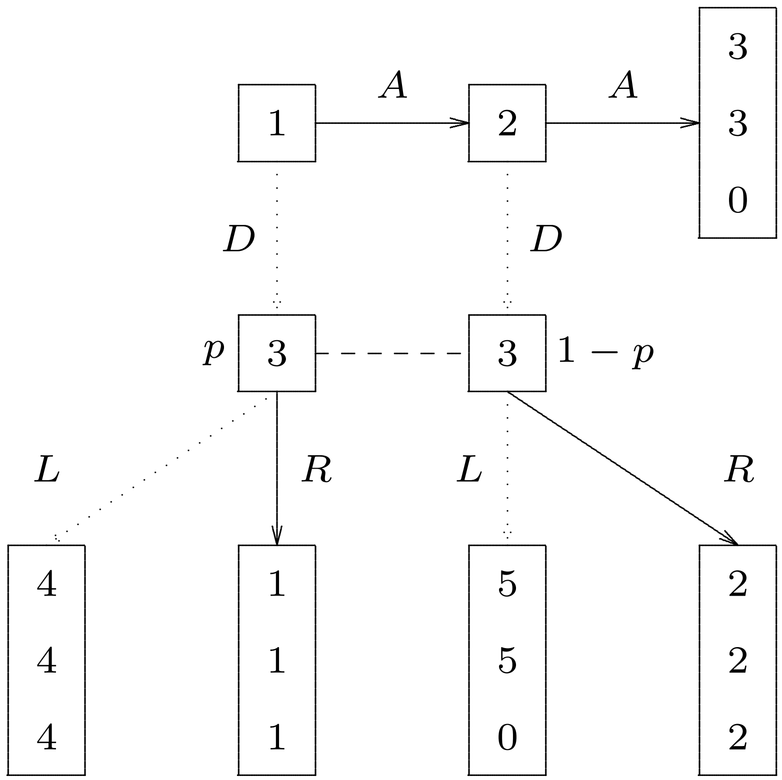
Set $p > \frac{2}{5}$. Compute best responses, player 3 strictly prefers L. So…



Player 2 will play D, so player 1 will play A. But now the belief p is inconsistent with the play of the game: history (A, D) occurs with probability one, not with probability $1 - p < \frac{3}{5}$.

Consistent Beliefs

Set $p \leq \frac{2}{5}$. Now player 3 (weakly) prefers to play R, so…



Player 2 will play A, and player 1 will play A. The belief p is *not* inconsistent with the play of the game. It is not however, uniquely defined. For any such p there is a "perfect-Bayesian equilibrium".

Perfect-Bayesian Equilibrium

Bringing together ideas of the optimality of strategies *given* beliefs and the consistency of beliefs *given* equilibrium play motivates the notion of perfect-Bayesian equilibrium.

Definition 27. A perfect-Bayesian equilibrium is a strategy profile (see definition 23) and a set of beliefs for each player (see definition 26), (s^*, μ^*) such that:

1. At every information set I_i player i's strategy maximises their payoff, given the actions of all the other players, and player i's beliefs.
2. At information sets reached with positive probability when s^* is played, beliefs are formed according to s^* and Bayes' rule when necessary.
3. At information sets that are reached with probability zero when s^* is played, beliefs may be arbitrary but must be formed according to Bayes' rule when possible.

Intuitively: optimal actions given beliefs and consistent beliefs in equilibrium.

Note. A formal definition will not be given here, it takes three pages. See Fudenberg and Tirole (1991), pp. 331-333.

The Beer-Quiche Game

"Two M.Phil. students go to the pub. The first is a bully, and likes picking on wimps, by starting fights. But, if their opponent is stronger, they would rather not fight. The second would rather not fight at all. The first does not know whether the second is *strong* or *weak*, but believes the second to be strong with probability 0.8. Some extra information is available to the first student before deciding whether to pick a fight with the second: they observe the second's order at the bar. The second M.Phil. student can order either *beer* or *quiche*. As one might imagine, strong people prefer beer and wimps prefer quiche. The second student (whatever their type) would rather pick their preferred refreshment, but would also like to avoid unnecessary conflict with the first."

Players. The two students and nature (or chance) $N = \{A, B, c\}$.

Histories. Play proceeds as follows:

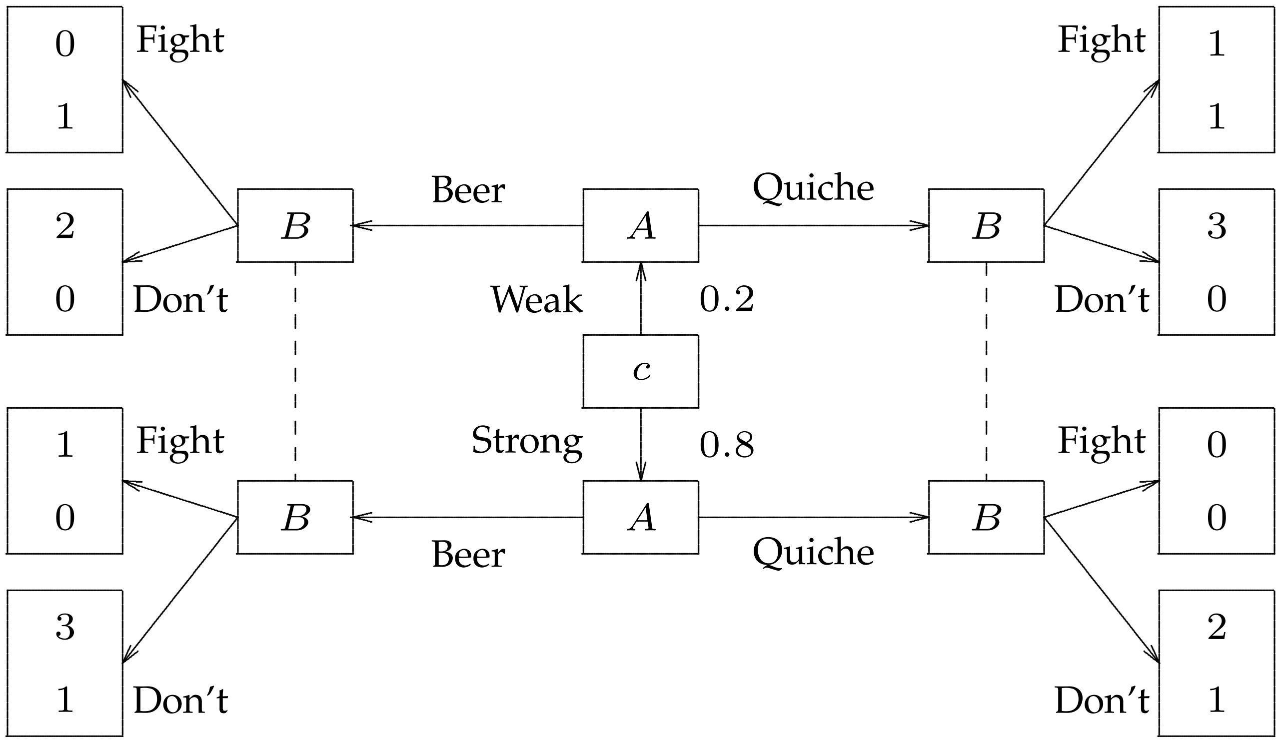
1. Chance decides whether A is weak ($p = 0.2$) or strong ($p = 0.8$).
2. A observes their type, and chooses either "Beer" or "Quiche".
3. B observes A's choice, but not their type, and chooses "Fight" or "Don't Fight".

Payoffs. These are as follows:

1. Player A gets 1 when consuming a favoured refreshment, and 2 for avoiding a fight.
2. Player B gets 1 if fighting *weak*, or avoiding a fight with *strong*, and 0 otherwise.

The Extensive-Form Game

This game has incomplete information (there are types). By incorporating nature's move into the extensive form, it is represented as a game of imperfect information.



What are the Bayesian-Nash equilibria of this game? Are they all perfect-Bayesian equilibria?

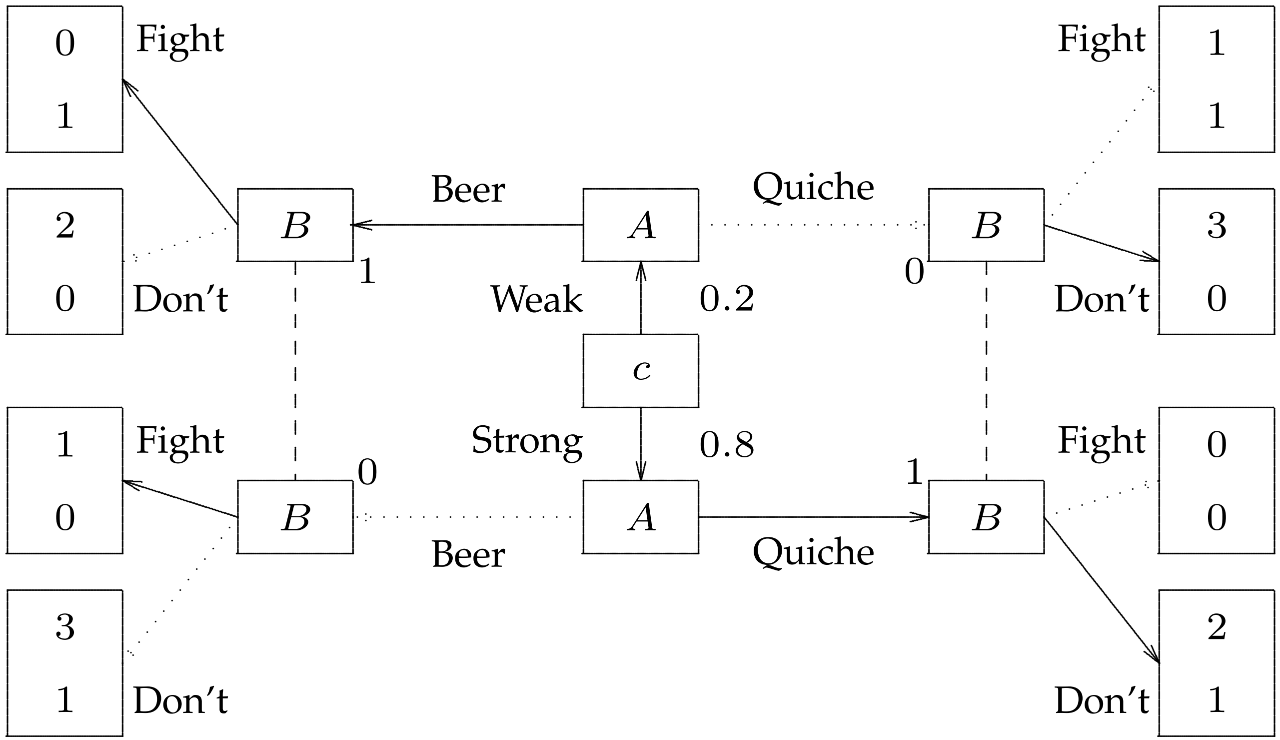
The Strategic-Form Game

	B_w, B_s	B_w, Q_s	Q_w, B_s	Q_w, Q_s
F, F	0 $\underline{1}$ 0.2	0 0 0.2	$\underline{1}$ $\underline{1}$ 0.2	$\underline{1}$ 0 0.2
F, D	0 1 0.2	0 $\underline{2}$ $\underline{1.0}$	$\underline{3}$ 1 0.0	$\underline{3}$ $\underline{2}$ $\underline{0.8}$
D, F	$\underline{2}$ $\underline{3}$ $\underline{0.8}$	$\underline{2}$ 0 0.0	1 $\underline{3}$ $\underline{1.0}$	1 0 0.2
D, D	2 $\underline{3}$ $\underline{0.8}$	2 2 0.8	$\underline{3}$ $\underline{3}$ 0.8	$\underline{3}$ 2 $\underline{0.8}$

Three players in the strategic-form representation, A_w (payoffs in top right), A_s (payoffs in middle), and B (payoffs in bottom left). B has four strategies, e.g. (F, D) is “fight if beer, don’t if quiche”. A_w and A_s have two strategies each: $\{B_w, Q_w\}$ and $\{B_s, Q_s\}$. Two pure Bayesian-Nash equilibria.

Separating Equilibrium Candidate

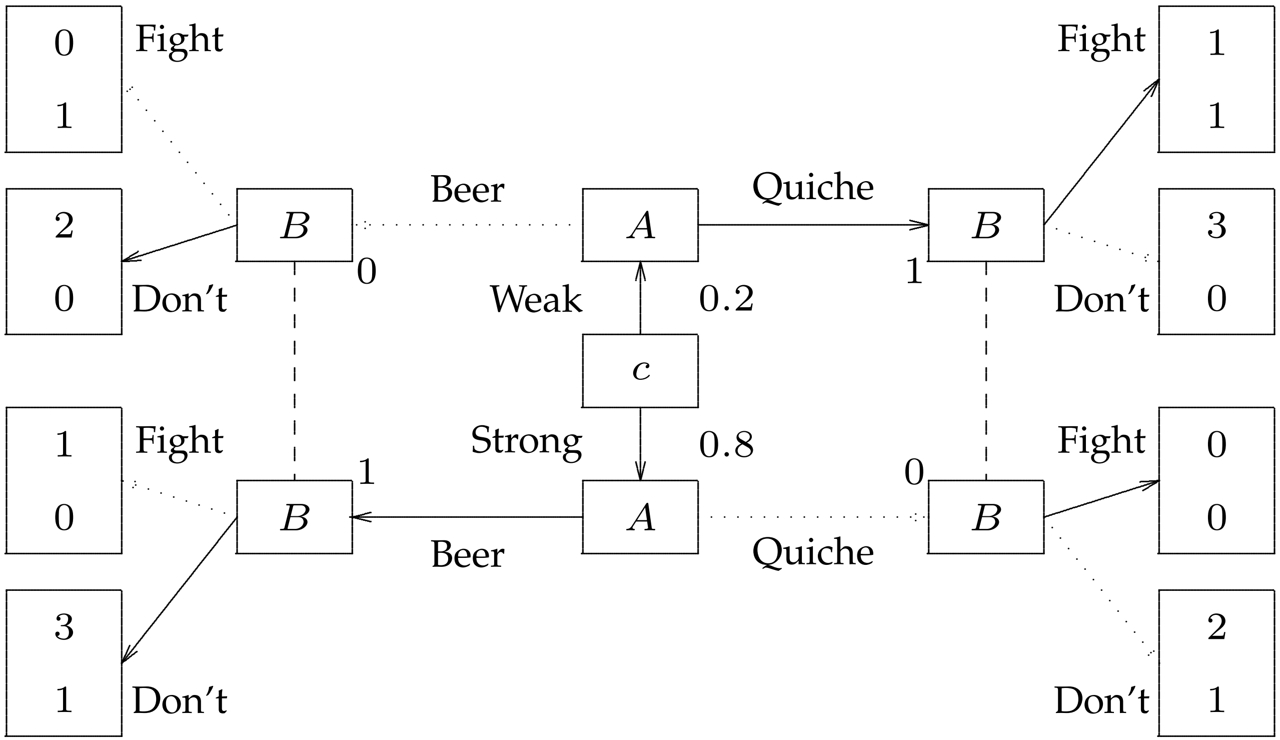
A *separating equilibrium* involves each player-type choosing a different action. For example:



Suppose A plays Beer if weak and Quiche if strong (B_w, Q_s). PBE requires consistent beliefs: as shown. So B will Fight if Beer, and Don't if Quiche. But then weak A would do better to eat quiche.

Alternative Separating Equilibrium Candidate

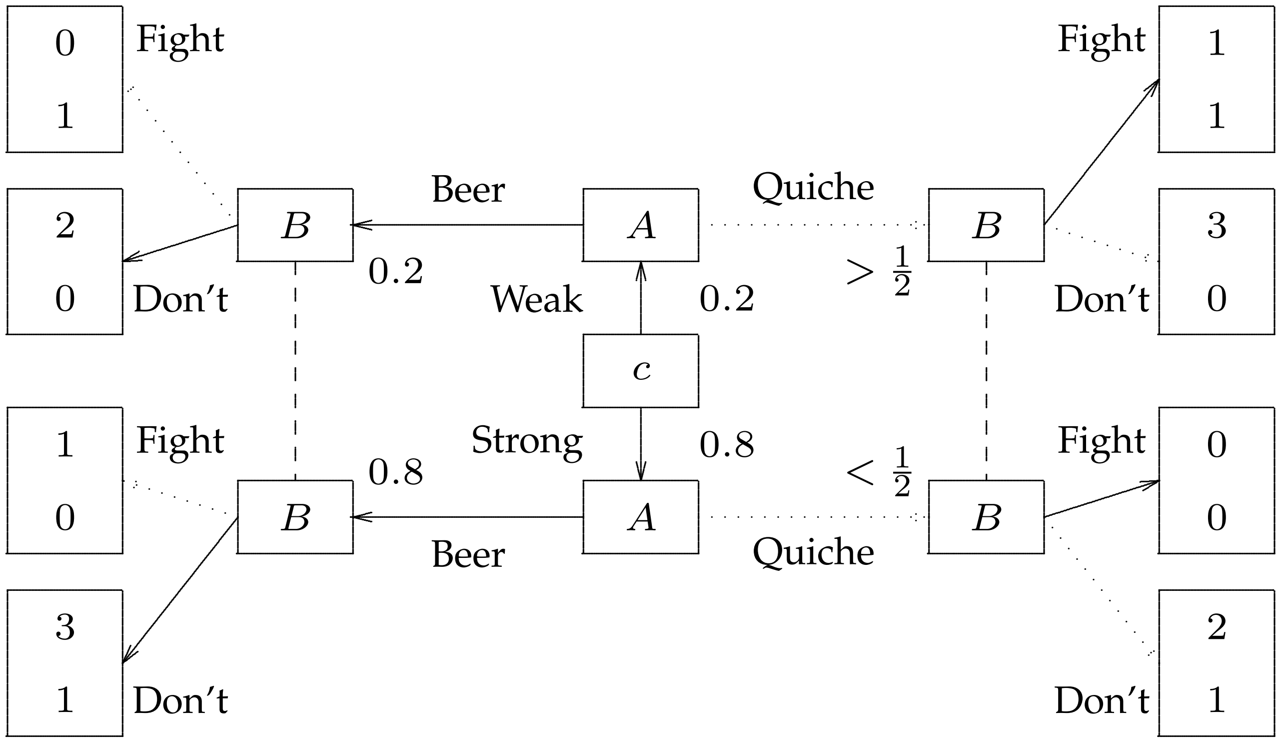
The alternative is for A to play Beer if strong and Quiche if weak.



Again, given consistent beliefs, B will play Fight if quiche and Don't if beer. But then weak A wishes to imitate a strong A and play Beer to avoid a fight. There are no separating equilibria.

Pooling Equilibrium: Everyone Drinks Beer

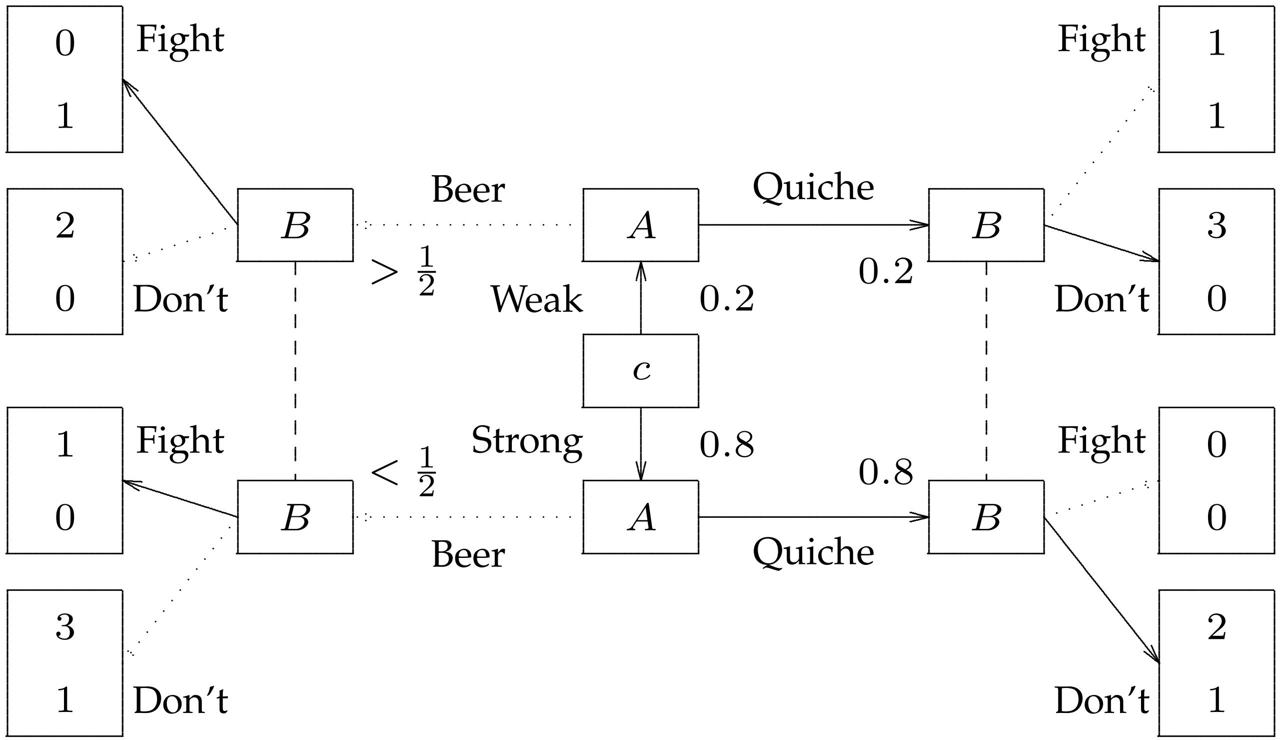
There is a *pooling equilibrium* in which A always plays Beer.



PBE involves consistent beliefs: so if beer is played, must believe with probability 0.2 that A is weak. Specify beliefs if quiche played: $p > \frac{1}{2}$ (probability that A is weak) ensures A doesn't deviate.

Pooling Equilibrium: Everyone Eats Quiche

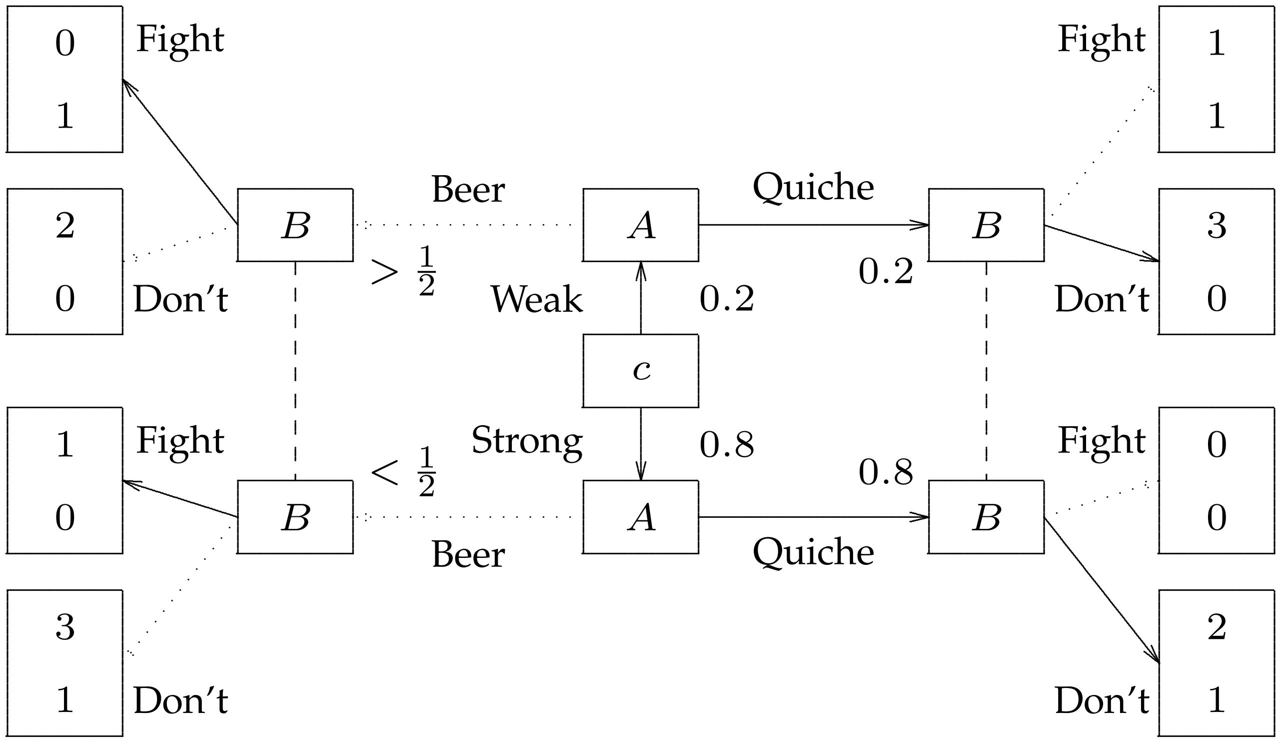
Is the other Bayesian-Nash equilibrium part of a perfect-Bayesian equilibrium?



There are pooling equilibria in which A always plays Quiche. B must believe with probability $p > \frac{1}{2}$ that A is weak when beer is observed. Is this a reasonable belief?!

The Intuitive Criterion

If A is weak they cannot possibly benefit from deviating from Quiche to Beer…



They receive 3 in equilibrium, and all payoffs following the deviation are lower — surely B must place probability zero on such a deviation? Only $\{B_w, B_s, D\}$ survives the "intuitive criterion".

Further Refinements, Hybrid Equilibria, and Signalling Games

Loosely speaking, only if zero probability is placed on deviations that are equilibrium dominated does a perfect-Bayesian equilibrium survive the intuitive criterion. Other refinements include:

1. *Sequential Equilibria*: Contained in the set of perfect-Bayesian equilibria.
2. *Trembling-Hand-Perfect Equilibria*: Contained in the set of sequential equilibria.
3. *Passive Beliefs*: Prior beliefs maintained off the equilibrium path.

None of these yield uniqueness, and so the search went on…"divine equilibria", "universally divine equilibria"…Perfect Bayesian equilibrium is usually sufficient at this stage.

- There may also be mixed equilibria, often called hybrid, semi-separating, or semi-pooling.
- For example, find the equilibria in the Beer-Quiche game when 0.2 and 0.8 are switched.

These kinds of signalling games are common in economics. For example, education signalling and screening, entry deterrence by price signalling, and so on.