Extensive Form Games

Writing a game in its strategic form may omit important information:

- In a strategic form, only players, strategies, and payoffs are defined.
- Many games have a dynamic structure. Players might select their actions at different times.
- An extensive-form games explicitly models this additional feature.

The analysis of extensive-form games in this lecture will focus upon:

- Representing dynamic interactions as extensive-form games: game trees.
- The connection between the extensive form and the strategic form.
- Representing a player's information in the extensive form.
- The iterated deletion of weakly dominated strategies, and backward induction.
- Nash equilibrium *refinements*: subgame perfection.

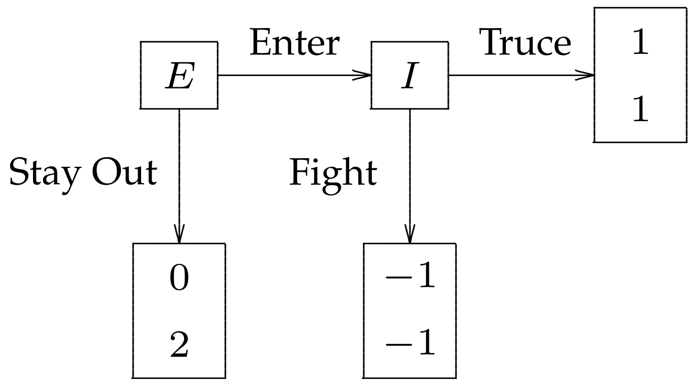
Entry Deterrence

"An incumbent monopolist faces the threat of entry from a new firm. The new firm first chooses whether to enter the industry or not. The incumbent observes this decision, and, upon entry, may decide whether to fight the entrant or acquiesce (call a truce). Fighting involves lowering prices below marginal cost in an attempt to deter entry, acquiescing involves setting prices at their one-shot Bertrand (differentiated-product) level."

Players. The entrant, E; and the incumbent, I. So $N = \{E, I\}$.

Strategies. E can choose to "Enter" or "Stay Out". I can choose to "Fight" or "Truce".

Payoffs. Payoffs (profits), and timing, are represented in the below *game tree*:



Extensive-Form Games

Formally, the definition of an extensive-form game includes some additional elements:

Definition 18. An extensive-form game with perfect information consists of:

1. *Players.* A set of players with typical member $i \in N$.
2. *Histories.* A set of histories with typical member $h \in H$. h is a sequence of actions by individual players. $\emptyset \in H$ is the start of the game. If $h \in H$, but there is no $(h, a) \in H$ where a is an action for some player, then h is "terminal". Denote the set of terminal histories as $Z \subset H$.
3. *Player Function.* A function $P : H \backslash Z \mapsto N$, assigning a player to each non-terminal history.
4. *Payoffs.* vNM payoffs for each $i \in N$ are defined over terminal histories, $u_i : Z \mapsto \mathcal{R}$.

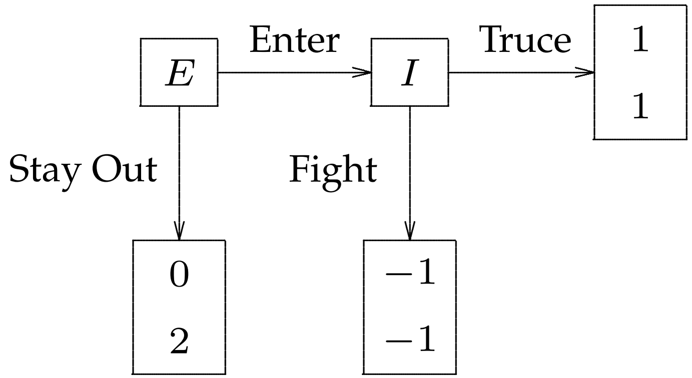
Anything with these four features can be written as an extensive-form game:

$$\mathcal{G} = \langle N, H, P, \{u_i\}_{i \in N} \rangle.$$

Note. (h, a) is the history (of length $t + 1$) which consists of h (of length t), followed by a. Actions and strategies have yet to be defined! But notice that they require nothing further than what is implicitly contained in the definition of $\mathcal{G}$.

Entry Deterrence Revisited

Recall the earlier game-tree for the entry-deterrence game:



Players. The players are the entrant the incumbent: $N = \{E, I\}$.

Histories. The set of histories is $H = \{(\emptyset), (\text{Stay Out}), (\text{Enter}), (\text{Enter}, \text{Fight}), (\text{Enter}, \text{Truce})\}$. The set of terminal histories is $Z = \{(\text{Stay Out}), (\text{Enter}, \text{Fight}), (\text{Enter}, \text{Truce})\}$.

Player Function. $P(\emptyset) = E$ and $P(\text{Enter}) = I$.

Payoffs. Defined over Z, and shown in the tree, e.g. $u_E(\text{Stay Out}) = 0$ and $u_I(\text{Stay Out}) = 2$.

Strategies and Outcomes

Every extensive-form game can be represented as a strategic-form game.

Definition 19. A strategy for player $i \in N$ in the extensive-form game $\mathcal{G} = \langle N, H, P, \{u_i\}_{i \in N} \rangle$ is a function that assigns an action $A(h)$ to each $h \in H \backslash Z$ where $P(h) = i$, so that $(h, A(h)) \in H$.

A strategy is a *complete plan of action* specifying a move after every history where the player makes a choice: moves are specified even for histories that are never reached if that strategy is played!

Definition 20. An outcome $z(s)$ is the terminal history (in Z) that results when each player $i \in N$ plays strategy s_i and $s = (s_i)_{i \in N}$ is the strategy profile.

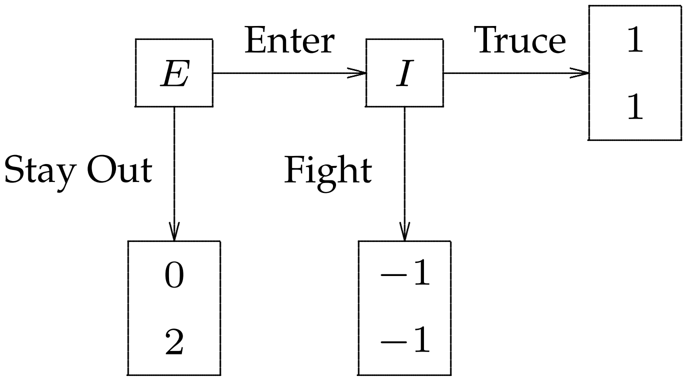
- In the entry-deterrence game, $S_I = \{\text{Fight}, \text{Truce}\}$ and $S_E = \{\text{Stay Out}, \text{Enter}\}$.
- The outcomes are, e.g. $z(\text{Stay Out}, \text{Fight}) = (\text{Stay Out}) \in Z$.

So payoffs are defined over outcomes, and extensive games can be written as strategic games…

Extensive Forms and Strategic Forms

Definition 21. The strategic-form of the extensive-form game $\mathcal{G} = \langle N, H, P, \{u_i\}_{i \in N} \rangle$ is $\Gamma = \langle N, \{S_i\}_{i \in N}, \{v_i\}_{i \in N} \rangle$ where:

1. S_j is the set of strategies in $\mathcal{G}$ for $j \in N$ as in Definition 19.
2. $v_j(s) = u_j(z(s))$, where $z(\cdot)$ is an outcome as in Definition 20, and $s \in S = \times_{i \in N} S_i$.



	Fight	Truce
Stay Out	2 0	2 0
Enter	-1 -1	1 1

The strategic form of the entry-deterrence example is represented in the payoff matrix above.

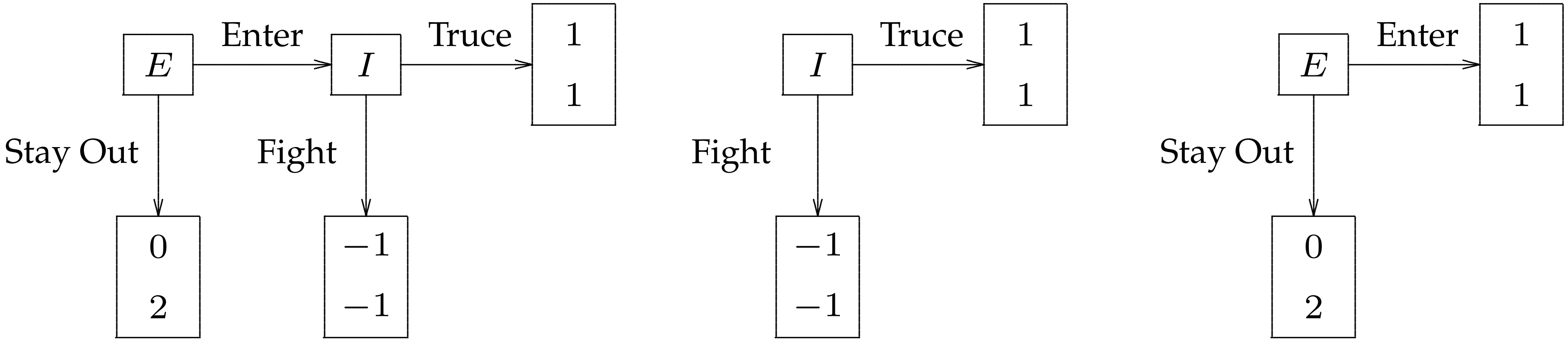
Nash Equilibrium

Once an extensive-form game has been represented in its strategic form, it is straightforward to find its Nash equilibria (pure and mixed). The Nash equilibria of the entry-deterrence game are:

Pure. {Stay Out, Fight} and {Enter, Truce}.

Mixed. E plays Stay Out with probability 1 and I plays Fight with probability $p \geq \frac{1}{2}$.

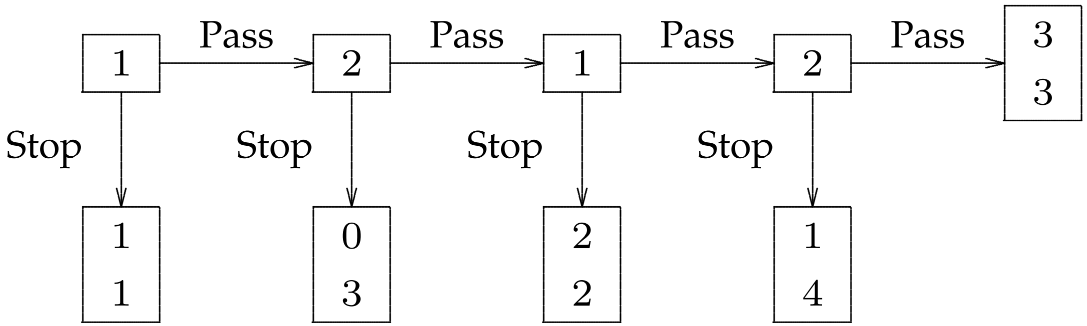
But this ignores the dynamic aspect of the game — is placing positive probability on Fight *credible*?



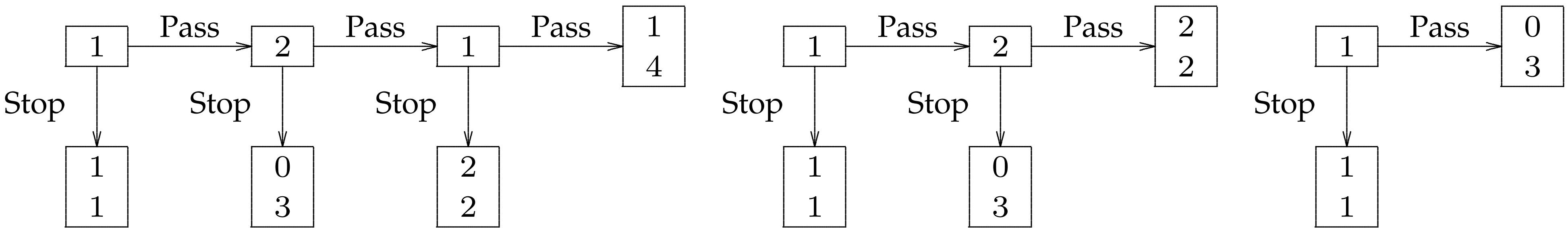
When called upon to choose, what would I do? The second diagram examines this choice: Truce. Now it is as if E faces the choice in the third diagram. This process is called *backward induction*.

Backward Induction and the Centipede

Starting from (all of) the ends of the game, work out best responses. Move back a step, bringing the payoffs associated with the initial best responses along. Work out best responses…



This is a centipede game. Applying the notion of backward induction…



So the unique strategy profile that survives backward induction is $\{(\text{Stop}, \text{Stop}), (\text{Stop}, \text{Stop})\}$.

Iterated Deletion of Weakly Dominated Strategies

There is a relationship between the strategy profile that survives backward induction and the one that survives the iterated deletion of *weakly* dominated strategies...

	Fight	Truce
Stay Out	0, 2	0, 2
Enter	-1, -1	1, 1

→

	Truce
Stay Out	0, 2
Enter	1, 1

→

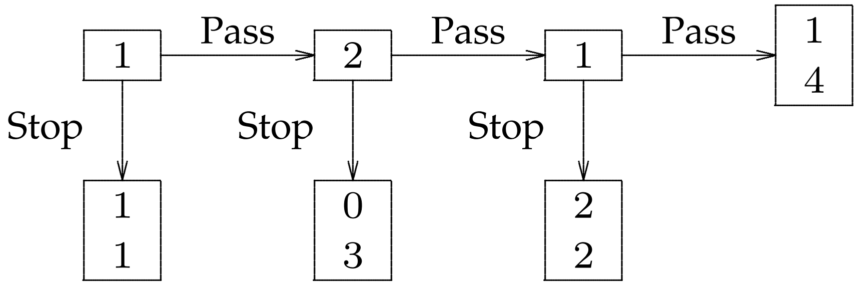
	Truce
Enter	1, 1

- Only {Enter, Truce} remains — exactly the strategy profile that survived backward induction.
- However, the order in which weakly dominated strategies are deleted can matter...
- Backward induction implies survives iterated deletion, but not necessarily the other way.

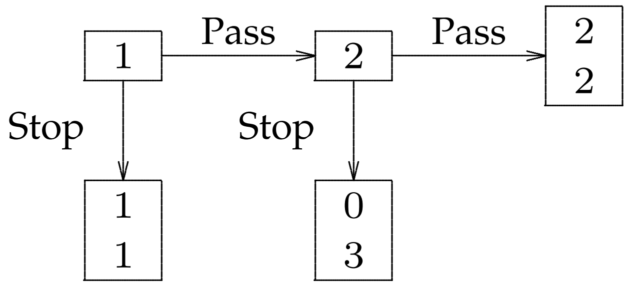
The strategy profile that survives backward induction is always a Nash equilibrium.

Deletion in the Centipede Game

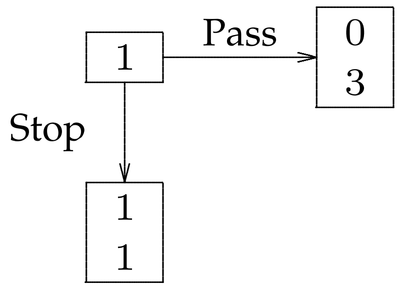
Starting with a slightly smaller centipede, notice the order of deletion matters…



	Stop	Pass
Stop,Stop	1 1	1 1
Stop,Pass	1 1	1 1
Pass,Stop	0 3	2 2
Pass,Pass	0 3	1 4



	Stop	Pass
Stop,Stop	1 1	1 1
Pass,Stop	0 3	2 2



	Stop
Stop,Stop	1 1
Pass,Stop	0 3

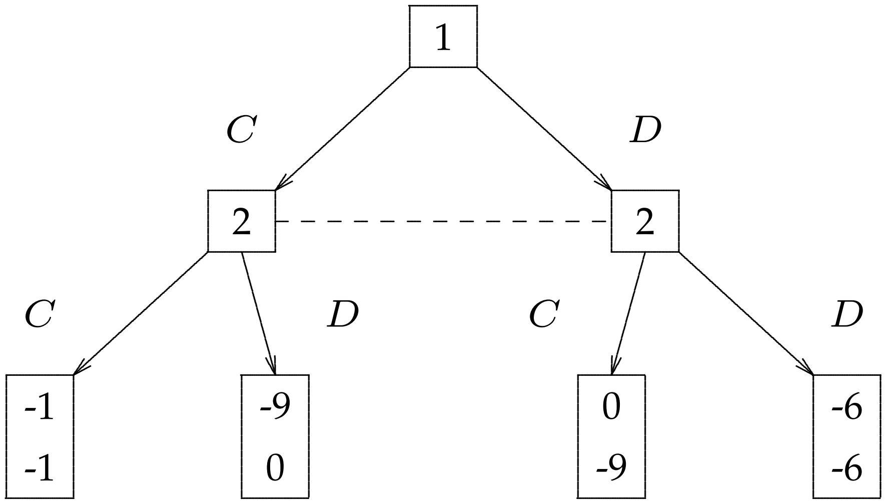
Information Sets

Generalise notion of extensive-form games to allow for *imperfect* information.

Consider the Prisoners' Dilemma game introduced in the first lecture. Representing the game in extensive form requires the introduction of *information sets*.

	C	D
C	-1, -1	-9, 0
D	0, -9	-6, -6

$\Longrightarrow$



Player 2 does not observe player 1's action and hence does not know whether the game has reached the first "node" or the second. The actions available to player 2 are the same at either node.

Note. This is not the only extensive-form representation of the Prisoners' Dilemma. e.g. player 2 could move first.

Extensive-Form Games with Imperfect Information

Definition 22. An extensive-form game consists of:

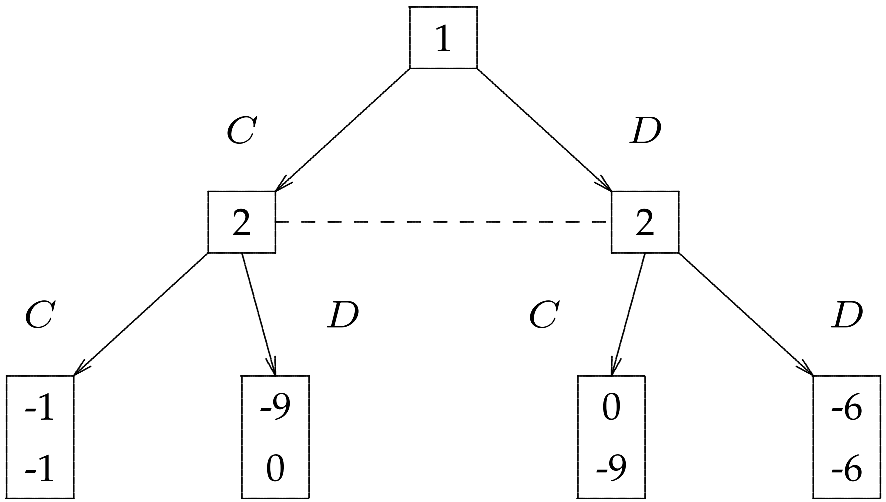
1. *Players.* A set of players with typical member $i \in N$.
2. *Histories.* A set of histories with typical member $h \in H$. h is a sequence of actions by individual players. $\emptyset \in H$ is the start of the game. If $h \in H$, but there is no $(h, a) \in H$ where a is an action for some player, then h is "terminal". Denote the set of terminal histories as $Z \subset H$.
3. *Player Function.* A function $P : H \backslash Z \mapsto N \cup \{c\}$, assigning a player or "chance" (sometimes called nature) to each non-terminal history.
4. *Nature.* For each $h \in H$ such that $P(h) = c$, $f(a|h)$ is the probability that $(h, a) \in H$ occurs.
5. *Information.* For each player $i \in N$ an information partition $\mathcal{I}_i$ of $\{h \in H : P(h) = i\}$. $(h, a) \in H \Leftrightarrow (h', a) \in H$ for all histories $h, h' \in H$ in the information set $I_i \in \mathcal{I}_i$.
6. *Payoffs.* vNM payoffs for each $i \in N$ are defined over terminal histories, $u_i : Z \mapsto \mathcal{R}$.

Anything with these features can be written as an extensive-form game:

$$\mathcal{G} = \langle N, H, P, f, \{\mathcal{I}_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle.$$

Applying the Definition

This definition is quite formal. . . but again, simple to apply.



- The set of players is $N = \{1, 2\}$. Nature does not move.
- $H = \{(\emptyset), (C), (D), (C, C), (C, D), (D, C), (D, D)\}$, and $Z = H\backslash\{(\emptyset), (C), (D)\}$.
- The player function is $P(\emptyset) = 1$, $P(C) = P(D) = 2$.
- There is a single information set for each player, $I_1 = \emptyset$, and $I_2 = \{(C), (D)\}$.
- The payoffs assigned to the terminal nodes are as given in the tree (e.g. $u_1(C, C) = -1$).

Strategies and Subgames

With this more general definition of extensive-form, backward induction may not be possible. Strategies, however, are a straightforward generalisation of Definition 19:

Definition 23. A strategy for $i \in N$ in the game $\mathcal{G} = \langle N, H, P, f, \{\mathcal{I}_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ is a function that assigns an action $A(h)$ to each $h \in H\backslash Z$ where $P(h) = i$ with $A(h) = A(h')$ whenever $h, h' \in I_i$, so that $(h, A(h)) \in H$. This extends to mixed strategies in a natural way.

A subgame of $\mathcal{G}$ is the extensive-form game given when all but the tree branches which follow a singleton information set are removed, so long as this sub-tree does not break any information set.

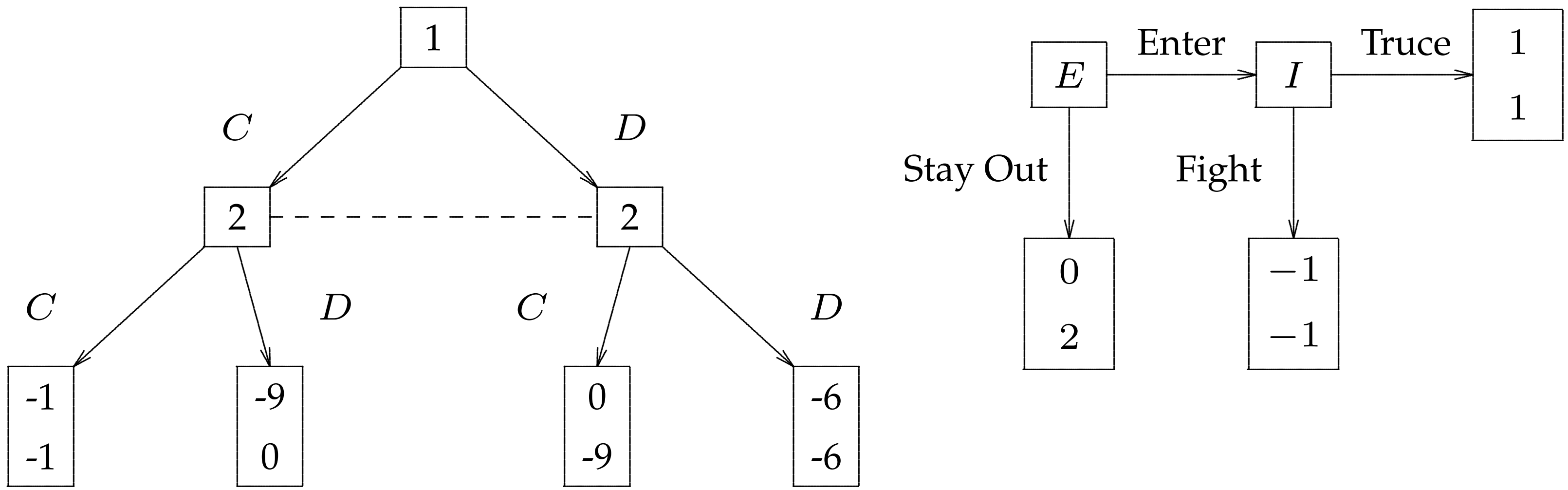
Definition 24. The subgame of $\mathcal{G} = \langle N, H, P, f, \{\mathcal{I}_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ following history $h \in I_j$ is the game $\mathcal{G}^h = \langle N, H^h, P^h, f^h, \{\mathcal{I}_i^h\}_{i\in N}, \{u_i^h\}_{i\in N}\rangle$ where I_j is singleton for $j = P(h)$, and

1. Histories are those following this node: $h' \in H^h$ if $(h, h') \in H$.
2. Players are assigned as before: $P^h(h') = P(h, h')$.
3. Nature's moves follow the same probability distributions: $f^h(\cdot \mid h') = f(\cdot \mid h, h')$.
4. Information partitions unbroken: $\mathcal{I}_i^h = \mathcal{I}_i$ defined over H^h, and $h' \in I_i^h \in \mathcal{I}_i^h \Rightarrow h' \in H^h$.
5. Payoffs defined over the remaining terminal histories: $u_i^h(h') = u_i(h, h')$.

Some Subgames

The formal definition is (again) *very* difficult. But the application is (again) *very* simple.

The Prisoners' Dilemma has one subgame, the whole game. The entry deterrence game has two:



e.g. in the Prisoners' Dilemma, there is no subgame starting from history (C) since this breaks an information set — but there is a subgame starting from the history (Enter) in the deterrence game.

Subgame Perfection and Backward Induction

Definition 25. A *subgame-perfect equilibrium* induces a Nash equilibrium in every subgame.

At last, a simple definition! Clearly, such an equilibrium will also be a Nash equilibrium (as every game has at least one subgame — itself). This is a *refinement* of Nash equilibrium.

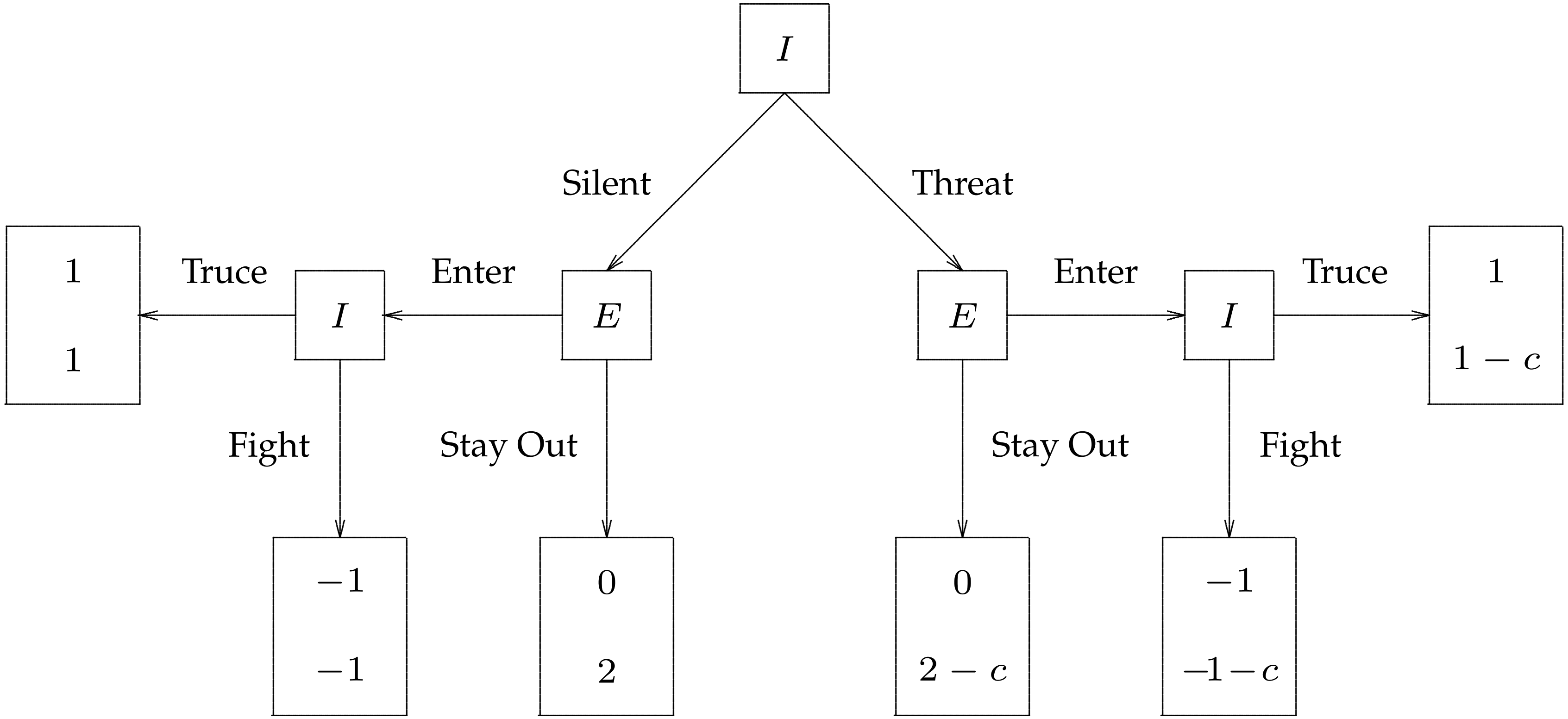
- The Prisoners' Dilemma has one subgame-perfect equilibrium, $\{D, D\}$.
- The entry deterrence game has one subgame-perfect equilibrium, $\{\text{Enter}, \text{Truce}\}$.
- The centipede game has one subgame-perfect equilibrium, $\{\text{Always Stop}, \text{Always Stop}\}$.

Notice the connection between backward induction and subgame perfection. In fact, in perfect-information extensive-form games, subgame perfection and backward induction coincide.

So subgame-perfect equilibria are easy to find — use backward induction when possible.

Entry Deterrence with an Explicit Threat

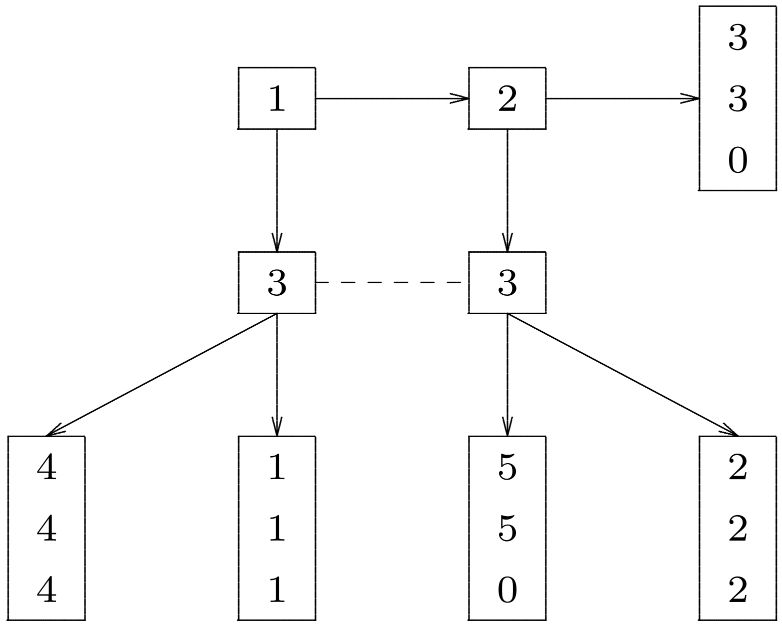
Modify the entry deterrence game to allow for an explicit threat at a cost of c...



There are 5 subgames. Need to find a Nash equilibrium of each, or use backward-induction... there are many Nash equilibria, but only $\{$(Silent,Truce,Truce), (Enter,Enter)$\}$ is subgame perfect.

Subgames and Imperfect Information

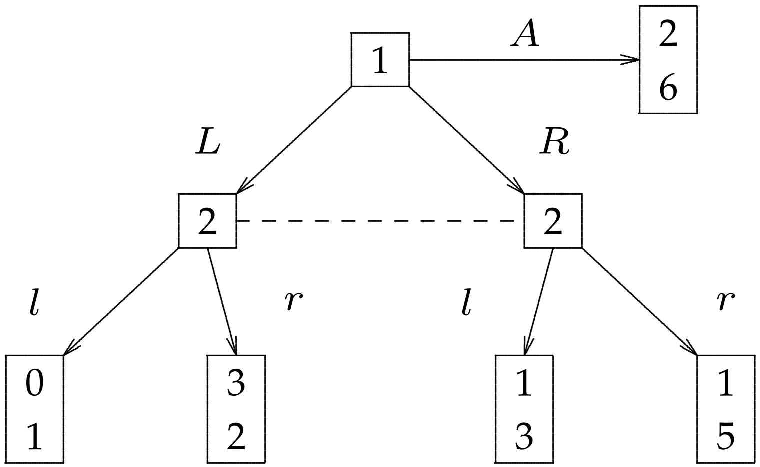
A subgame must always start at a singleton information set. It must also never break an information set. So, for example, there is no subgame starting at player 2's move in the game below:



Nor is there any subgame starting from player 3's move. In fact, this game has only one subgame — the whole game. Backward induction and subgame perfection do not help refine the equilibria.

Subgame Perfection and Imperfect Information

Imperfect-information games may have no "proper" subgames: all equilibria are subgame perfect.



$\Longrightarrow$

	l		r	
A	2	6	2	6
L	0	1	3	2
R	1	3	1	5

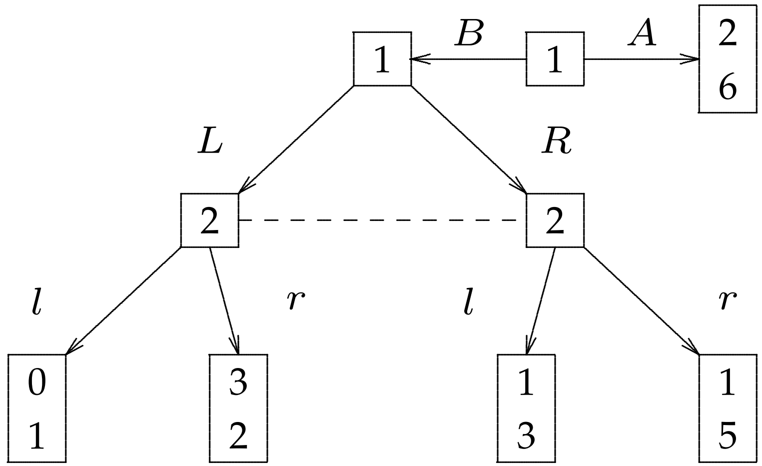
The above game has no proper subgames. All Nash equilibria are subgame perfect:

- There is a subgame-perfect equilibrium at $\{A, l\}$. This yields payoffs of $(2, 6)$.
- There is a subgame-perfect equilibrium at $\{L, r\}$. This yields payoffs of $(3, 2)$.
- There are also mixed equilibria (A with probability 1 and l with probability $p \geq \frac{1}{3}$).

Surely player 2 will play r if called upon to move? 1 should play L! $\Rightarrow$ sequential rationality.

Modifying the Extensive Form

Rewrite the extensive-form game from the last slide in the following way:



This has (almost) the same strategic form. But there is now a subgame starting from history (B).

- There is a unique Nash equilibrium of this subgame: $\{L, r\}$.
- A subgame-perfect equilibrium must induce a Nash equilibrium in each subgame.
- Hence r must be played in equilibrium. $\{(B, L), r\}$ is the unique subgame-perfect equilibrium.

Extend notion of perfection to games with imperfect/incomplete information next week.