M.PHIL. MICROECONOMICS IV: GAME THEORY

Lecture Course Notebook

Chris Wallace, Michaelmas Term 2009

1. Course Objective

The course is intended for those taking the M.Phil. in Economics at Oxford University, and forms part of the Microeconomics Core Paper in the first year. The course will make use of elementary mathematics. Knowledge of basic calculus, some set theory, and some probability theory is required. The relevant part of the rubric in the Green Book is:

> Game Theory: The main focus is on non-cooperative games, Nash equilibrium, and its refinements. Nash equilibrium; existence of equilibrium; mixed strategies; Bayesian-Nash equilibrium; examples; Cournot and Bertrand. Other non-cooperative solution concepts: correlated equilibrium; iterated dominance; rationalisability. Refinements of Nash equilibrium: perfection; perfect Bayesian, and sequential equilibrium; intuitive criterion; forward induction. Cooperative game theory: Nash bargaining solution; Shapley value; the core.

2. Teaching

There will be eight 90-minute lectures during Michaelmas Term on Mondays, Tuesdays, and Thursdays in weeks 6 and 7 and on Tuesday and Thursday of week 8. The lectures take place in the Lecture Theatre in the Manor Road Building at 9.30am. The lectures are given by Chris Wallace (Trinity College). The Microeconomics classes in week 8 of Michaelmas Term and week 1 of Hilary Term accompany this part of the course.

3. The Exam

Sample exam papers (for the Microeconomics course) can be obtained from

`http://www.economics.ox.ac.uk/index.php/graduate...`
`.../intranet/mphil_past_exam_papers/`,

or `http://www.oxam.ox.ac.uk/pls/oxam/main`. Candidates must answer three questions, of which at least one must be an essay section, and at least one a problem set.

4. Contact

I can be reached by email at `christopher.wallace@economics.ox.ac.uk`, in the department in room 2133, or at Trinity College. Further copies of this document, all the lecture slides, and other resources can be found at my website:

`http://malroy.econ.ox.ac.uk/ccw/`.

5. Textbooks

There are many textbooks on Game Theory. I list a few here. The reading list contained in the Lecture Outline below references some of these books, along with some important/classic papers.

- Osborne, M. J. (2004) *An Introduction to Game Theory*, OUP, Oxford.
- Rasmusen, E. (2007) *Games and Information*, 4th edition, Blackwell, Oxford.
- Myerson, R. B. (1991) *Game Theory: Analysis of Conflict*, Harvard, Cambridge MA.
- Binmore, K. G. (2007) *Playing for Real: A Text on Game Theory*, OUP, Oxford.
- Fudenberg, D. and Tirole, J. (1991) *Game Theory*, MIT Press, Cambridge MA.
- Osborne, M. J. and Rubinstein, A. (1994) *A Course in Game Theory*, MIT Press, Cambridge MA.

There are, of course, many others. I think the first one above is excellent, at an introductory level; the last one is also excellent, at a more advanced level. The following reading list refers to chapter numbers in these two texts as O and O&R respectively.

6. Lecture Outline and Readings

There are eight lectures. A brief guide follows, with contents and suggested readings from the textbooks mentioned above, as well as important/classic papers.

(1) **Strategic-Form Games.** Games, dominance, Nash equilibria, rationalisability.
   - See O2 and O12, O&R2 and O&R4: Classic books and papers are:
   - Nash, J.F. (1950) "Equilibrium Points in N-Person Games", *Proceedings of the National Academy of Sciences* **36**, 48-49.
   - von Neumann, J. and Morgernstern, O. (1944) *Theory of Games and Economic Behavior*, Princeton University Press, Princeton NJ.
   - Bernheim, B. D. (1984) "Rationalisable Strategic Behavior", *Econometrica* **52**, 1007–1028.
   - Pearce, D. G. (1984) "Rationalisable Strategic Behavior and the Problem of Perfection", *Econometrica* **52**, 1029–1050.

(2) **Mixed Strategies.** Mixed extensions and equilibria, correlated equilibria, ESS.

- See O4 and O13, O&R3: Classic books and papers are:
- Aumann, R. J. (1974) "Subjectivity and Correlation in Randomized Strategies", *Journal of Mathematical Economics* **1**, 67-96.
- Aumann, R. J. (1987) "Correlated Equilibrium as an Expression of Bayesian Rationality", *Econometrica* **55**, 1-18.
- Maynard Smith, J. and Price, G. R. (1973) "The Logic of Animal Conflict", *Nature* **246**, 15-18.
- Maynard Smith, J. (1974) "The Theory of Games and the Evolution of Animal Conflicts", *Journal of Theoretical Biology* **47**, 209-221.

(3) **Continuous Games.** Best-response functions, equilibria, and mixed equilibria.

- See O3, O&R3: Classic books and papers are:
- Cournot, A. A. (1838) *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, Hachette, Paris.
- Bertrand, J. (1883) "Review of 'Théorie Mathématique de la Richesse Sociale' by Léon Walras and 'Recherches sur les Principes Mathématiques de la Théorie des Richesses' by Augustin Cournot", *Journal des Savants*, 499-508.
- Nash, J. F. (1953) "Two-Person Cooperative Games", *Econometrica* **21**, 128-140.

(4) **Bayesian Games.** Games of incomplete information, Bayesian-Nash equilibria, purification, and global games.

- See O9, O&R2: Classic books and papers are:
- Harsanyi, J. C. (1967/68) "Games with Incomplete Information Played by 'Bayesian' Players, Parts I, II, and III", *Management Science* **14**, 159-182, 320-334, and 486-502.
- Harsanyi, J. C. (1973) "Games with Randomly Disturbed Payoffs: A New Rationale for Mixed-Strategy Equilibrium Points", *International Journal of Game Theory* **2**, 1-23.
- Morris, S. and Shin, H. S. (2002) "Global Games: Theory and Applications", *Advances in Economics and Econometrics: Theory and Applications*, ed. by Dewatripont, M., Hansen, L. P. and Turnovsky, S. J., Cambridge University Press, London.
- Carlsson, H. and van Damme, E. (1993) "Global Games and Equilibrium Selection", *Econometrica* **61**, 989-1018.

(5) **Extensive-Form Games.** Information sets, subgame-perfect equilibria, and backward induction.

- See O5, O6, and O7; and O&R6: Classic books and papers are:
- Kuhn, H. W. (1953) "Extensive Games and the Problem of Information", *Contributions to the Theory of Games* Volume II, ed. by Kuhn, H. W. and Tucker, A. W., Princeton University Press, Princeton, NJ.

- Selten, R. (1965) "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit", *Zeitschrift für die Gesamte Staatswissenschaft* **121**, 301-324 and 667-689.
- Selten, R. (1975) "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games", *International Journal of Game Theory* **4**, 25-55.

(6) **Perfect-Bayesian Equilibrium.** Forward induction, beliefs, perfect-Bayesian equilibrium, and selection criteria.

- See O10; O&R11 and O&R12: Classic books and papers are:
- Kreps, D. M. and Wilson, R. B. (1982) "Sequential Equilibria", *Econometrica* **50**, 863-894.
- Kohlberg, E. and Mertens, J.-F. (1986) "On the Strategic Stability of Equilibria", *Econometrica* **54**, 1003-1037.
- Cho, I.-K. and Kreps, D. M. (1987) "Signaling Games and Stable Equilibria", *Quarterly Journal of Economics* **102**, 179-221.
- Fudenberg, D. and Tirole, J. (1991) "Perfect Bayesian Equilibrium and Sequential Equilibrium", *Journal of Economic Theory* **53**, 236-260.

(7) **Repeated Games.** Finite and infinite repetition, the one-deviation principle, folk theorems.

- See O14 and O15, and O&R8: Classic books and papers are:
- Mailath, G. J. and Samuelson, L. (2006) *Repeated Games and Reputations: Long-Run Relationships*, OUP, Oxford.
- Friedman, J. W. (1971) "A Non-Cooperative Equilibrium for Supergames", *Review of Economic Studies* **38**, 1-12.
- Abreu, D. (1988) "On the Theory of Infinitely Repeated Games with Discounting", *Econometrica* **56**, 383-396.
- Luce, R. D. and Raiffa, H. (1957) *Games and Decisions*, John Wiley and Sons, New York.

(8) **Cooperative Games.** TU/NTU games, the core, and the Shapley value.

- See O8; O&R13, O&R14, and O&R15: Classic books and papers are:
- Shapley, L. S. (1953) "A Value for n-Person Games", *Contributions to the Theory of Games* Volume II, ed. by Kuhn, H. W. and Tucker, A. W., Princeton University Press, Princeton, NJ.
- Gillies, D. B. (1959) "Solutions to General Non-Zero-Sum Games", *Contributions to the Theory of Games* Volume IV, ed. by Tucker, A. W. and Luce, R. D., Princeton University Press, Princeton, NJ.
- Nash, J. F. (1950) "The Bargaining Problem", *Econometrica* **18**, 155-162.

M.PHIL. GAME THEORY: PROBLEMS

These problems are designed for discussion in the classes of Week 8 of Michaelmas Term (full class) and Week 1 of Hilary Term (half class):

> **Week 8 (MT):** Problems 1–8.
> **Week 1 (HT):** Problems 9–12.

1. A Simple Strategic Form Game. Consider the following symmetric 2×2 game:

	L	R
U	9, 9	0, 8
D	8, 0	7, 7

(1) By inspection, what are the pure-strategy Nash equilibria?

(2) Find the mixed-strategy equilibrium by using the fact that if players are willing to mix between two or more strategies, they must be indifferent between them.

(3) Draw the best-response correspondences. Where do they intersect?

2. Deletion of Dominated Strategies. Consider the following strategic-form game:

	L	C	R
T	2, 0	1, 1	4, 2
M	3, 4	1, 2	2, 3
B	1, 3	0, 2	3, 0

(1) What strategies survive iterated elimination of strictly dominated strategies?

(2) What are the pure-strategy Nash equilibria?

3. Cournot Competition. Consider an n firm homogeneous product industry where firm i produces output q_i at cost cq_i. Price is

$$p = \alpha - \beta Q \quad \text{where} \quad Q = \sum_{i=1}^{n} q_i.$$

(1) What are the firms' outputs, prices and profits in the Cournot equilibrium? What happens as $n \to \infty$?

(2) Two firms merge. The merged firm has marginal costs of c, just as before. What happens to the merged firms' profits? What happens to the remaining firms' profits? Comment.

(3) Now suppose that any firm producing positive output incurs a fixed cost of F:

$$c_i(q_i) = F + cq_i \quad \text{if} \quad q_i > 0$$

and $c_i(0) = 0$. Let $n = 4$. Suppose F satisfies:

$$F = \frac{1}{\beta}\left[\frac{2(\alpha - c)}{9}\right]^2.$$

(a) What are the pure strategy equilibria?[1]

(b) Without calculations, do you think there may be any mixed equilibria?

4. Switching Costs. In many markets, consumers have *switching costs*.[2] Consider the following simple model of such a market: Two firms A and B simultaneously and non-cooperatively set prices in a single period for a commodity that they can each produce at zero cost. there are $n + s$ customers, where $n > 0$ and $s > 0$ all with reservation price R. Because of switching costs, $s/2$ customers can only buy from A and $s/2$ can only buy from B. The n "new" customers buy from the cheapest firm, if at all.

(1) Show that there are no pure-strategy equilibria.

(2) Find a mixed-strategy Nash equilibrium in which each firm chooses price p according to the distribution $F(p)$.[3]

[1]Hint: There may be equilibria with only $m \leq n$ active firms, so try each case $m = 1, \ldots, 4$. Start by looking at the case where one firm is producing the monopoly output and the other firms are producing nothing. Can this be a Nash equilibrium? Does a firm not producing in this situation have an incentive to deviate? Then look at the case when two firms are producing, and so on.

[2]Examples include the transactions costs of closing an account with one bank and opening another with a competitor, learning costs incurred by switching to a new make of computer after having learnt to use one make, the artificial switching costs caused by frequent flyer programmes, etc.

[3]Hint: First check that F is continuous, with no atoms. Now, for a given p_i, calculate expected profit. What can you say about all prices in the support of the mix? What value does p take when $F(p) = 1$? You should now be able to find $F(p)$. Over what interval of prices do the firms mix?

5. Interpretation of Nash Equilibria.

> "Ordinary people are not familiar with the Nash concept. There is no reason to expect them to play such equilibrium strategies."

(1) Do you agree?

(2) When should the Nash concept be applied?

(3) Is there any alternative?

(4) Are there any good reasons for predicting that players will play a mixed-strategy Nash equilibrium in a strategic-form game?

6. Private Provision of Public Goods. Consider the following *public goods* game:

	Contribute	Don't
Contribute	$1-c_2$ $1-c_1$	1 $1-c_1$
Don't	$1-c_2$ 1	0 0

The benefits (1 to each player if at least one player contributes) are commonly known, but the costs of contributing c_i is private information to player i. A *strategy* for player i in this game specifies an action ("Contribute" or "Don't") for each possible value of c_i.

(1) Suppose it is common knowledge that $c_1 = \frac{1}{4}$, but Player 1 does not know c_2. Player 1 believes $c_2 = \frac{1}{4}$ with probability $\frac{1}{2}$ and $c_2 = 2$ with probability $\frac{1}{2}$.

  (a) If $c_2 = 2$, does Player 2 have a dominant strategy? If so, what is it? Does Player 2 have a dominant strategy if $c_2 = \frac{1}{4}$?

  (b) Let z denote the probability that Player 2 contributes if the cost is $c_2 = \frac{1}{4}$. What is Player 1's expected payoff from "Don't", given his beliefs, in terms of z? Does Player 1 have a dominant strategy?

  (c) What is the Bayesian-Nash equilibrium?

(2) Suppose that Player i believes $\Pr[c_j = \frac{1}{4}] = \Pr[c_j = 2] = \frac{1}{2}$ for $i = 1, 2$ and $j = 1, 2$, $j \neq i$. What is the Bayesian-Nash equilibrium of the game?

(3) Suppose that both players believe costs are drawn independently from a uniform distribution on the interval $[0, 2]$. What is the Bayesian-Nash Equilibrium now?[4]

[4] Hint: Show that an equilibrium strategy has the trigger form of contributing whenever $c_i \leq c_i^*$, and note the uniform distribution of costs.

7. An Entry Game. Consider the following two-period game with no discounting.

- In period 1, four firms simultaneously and independently decide whether or not to pay 1 to enter an industry.
- In period 2, all firms that chose to enter now simultaneously and independently choose production levels, with fixed cost $F = 5$ and zero marginal cost $c = 0$. (That is, a firm in the industry can either produce no output and incur no costs in period 2, or can produce any positive output and incur a total cost of 5 in period 2.) All production is sold at price $10 - Q$ where Q is total industry output.

Consider the five possible post-period-1 outcomes: n firms enter, where $n = 0, 1, 2, 3, 4$.

(1) Consider the Nash equilibria of the five different possible period-2 subgames corresponding to $n = 0, 1, 2, 3, 4$ entrants. (You should have a good understanding of these from the earlier question.) Which period-1 outcomes are consistent with all firms choosing pure strategies in the Nash equilibrium of the whole game that are subgame perfect (i.e., consistent with backwards induction logic)? Explain.

(2) One of the outcomes you found in part (1) (you should have found more than one) is inconsistent with forwards induction logic. Which is it? Explain.

(3) In addition to the subgame-perfect outcomes, there is one outcome consistent with all firms choosing pure strategies in a Nash equilibrium of the whole game that is imperfect. Which is it? Explain.

(4) Of the remaining period-1 outcomes, which are consistent with Nash equilibrium behaviour (perhaps including mixed strategies).

8. A Bayesian Trading Game. Suppose that a buyer has a valuation for a good v_b, uniformly distributed on $[0, 1]$. The seller has a valuation v_s independently and identically distributed on $[0, 1]$. They each observe their own valuation, but not that of the other player. Simultaneously they each announce a price, p_b and p_s respectively. If $p_b \geq p_s$ a sale takes place at a price halfway between their announcements, $p = (p_b + p_s)/2$, and the buyer receives the good, yielding payoffs of

$$u_b(p; v_b) = v_b - p \quad \text{and} \quad u_s(p; v_s) = p - v_s.$$

Otherwise, there is no sale and both players receive 0.[5]

(1) Suppose that the seller uses a linear strategy of the form $p_s(v_s) = \alpha + \beta v_s$. Show that the buyer's expected payoff may be written

$$\frac{p_b - \alpha}{\beta}\left[v_b - \frac{1}{2}\left\{p_b + \frac{\alpha + p_b}{2}\right\}\right].$$

(2) Hence calculate the buyer's best reply to the seller's linear strategy, and show that it is also linear.

(3) Now suppose the buyer uses a linear strategy of the form $p_b(v_b) = \gamma + \delta v_b$. By calculating the seller's expected payoff, find the best reply of the seller to this strategy, and show that it is also linear.

(4) Calculate values of α, β, γ, and δ such that these strategies constitute a linear Bayesian-Nash equilibrium of the trading game.

(5) For what values of v_b and v_s is trade mutually advantageous? For what values of v_b and v_s does trade take place? Comment.

(6) Now suppose the buyer and seller use the following strategies:

$$p_b = \tfrac{1}{2} \quad \text{if} \quad v_b \geq \tfrac{1}{2} \quad \text{and} \quad p_b = 0 \quad \text{otherwise,}$$
$$p_s = \tfrac{1}{2} \quad \text{if} \quad v_s \leq \tfrac{1}{2} \quad \text{and} \quad p_s = 1 \quad \text{otherwise.}$$

Argue that these strategies constitute a Bayesian-Nash equilibrium of the trading game. Without doing any further calculations, are there any other Bayesian-Nash equilibria of this game? Are any of these efficient?

[5] It would be equivalent to assume $u_s = p$ in the case of a sale and $u_s = v_s$ when there is no sale. Why?

9. Purification and Forward Induction. Chris and David attend a dinner at Morris College. They need to decide how to divide the remaining dessert wine and chocolates available. Both are particularly partial to the dessert wine (which is known to be very expensive), whereas the chocolates are known to be less good. They have one opportunity to choose either wine or chocolate before the end of dinner. If they both select the same product, neither get any satisfaction from the tiny amount they receive. Hence their preferences could be summarised in the following game:

	Wine	Choc
Choc	1, $3+\varepsilon_d$	0, 0
Wine	0, 0	$3+\varepsilon_c$, 1

where Chris is the row player (gaining the payoffs in the bottom left corners of the cells) and David is the column player (with payoffs in the top right). Chris and David have slightly different preferences over the wine types which differ from dinner to dinner: $\varepsilon_c \geq 0$ and $\varepsilon_d \geq 0$ are variables which reflect this.

(1) Suppose ε_c and ε_d are known. Calculate the Nash equilibria of this game.

(2) Suppose ε_c and ε_d are private information to Chris and David respectively, and that it is commonly known that ε_c and ε_d are independently drawn from an identical uniform distribution on $[0, \bar{\varepsilon}]$. Find a Bayesian Nash equilibrium in which neither player simply ignores their private information.

(3) What happens to this equilibrium as $\bar{\varepsilon} \to 0$?

(4) Does purification provide a justification for mixed strategy Nash equilibria?

(5) Suppose that $\varepsilon_c = \varepsilon_d = 0$. Before the game is played, David has an opportunity to bellow loudly for the wine to be passed to him. If he does so, he will receive the wine, and get a payoff of 2 whilst Chris gets a payoff of 0 (the lower payoffs are due to the shame attached to such boorish behaviour). Draw this new game in extensive form. What are the Nash equilibria? Use a forward induction argument to generate a prediction of play.

(6) Suppose again that $\varepsilon_c = \varepsilon_d = 0$. Before the 2×2 subgame is played, David has the opportunity to be rude to another guest. Doing so is simply bad; it lowers everyone's payoffs in every situation by 1. Once again, use a forward induction argument to generate a prediction of play.

10. Monetary Policy Game. Consider the following two-period game between an economy's monetary authority and its labour force. The timing of the game is:

- Nature picks the monetary authority's type. The monetary authority is *weak* with probability 0.4 and *strong* with probability 0.6.
- The monetary authority picks first period inflation to be either HIGH or LOW.
- The labour force forms *high* or *low* expectations of second period inflation.
- The authority picks second period inflation: HIGH if weak; LOW if strong.

Strong monetary authorities prefer low inflation; weak monetary authorities are less prepared to adopt policies that they believe may compromise national income in the short-run, and prefer a situation that results in higher inflation. The authority receives a payoff of 100 if it chooses its preferred first period inflation level, zero otherwise. In addition, the authority receives a bonus payoff if labour force expectations of second period inflation are low. These benefit the strong authority by keeping down inflationary pressures, gaining it a bonus payoff of 200; while the weak authority benefits from surprise inflation in the second period, gaining it a bonus of 50. (The (present value) of the bonus to the weak authority is lower since it cares less about the future.) The labour force simply wants to get its expectations of second period inflation correct. It receives a payoff of zero if it is correct, -100 otherwise.

(1) Draw this game in extensive form.

(2) Show that there is no equilibrium in which a weak monetary authority chooses LOW first period inflation.

(3) Is there a separating (perfect Bayesian) equilibrium in which a strong monetary authority chooses LOW first period inflation, and a weak monetary authority chooses HIGH first period inflation?

(4) What out-of-equilibrium beliefs would the labour force have to hold to support a pooling (perfect Bayesian) equilibrium in which both types of monetary authority chose HIGH first period inflation?

(5) Are these beliefs compatible with the intuitive criterion? If not, then why not?

11. Two Stage Game. Consider the following simultaneous-move stage game:

	L	C	R
T	3, 1	0, 0	5, 0
M	2, 1	1, 2	3, 1
B	1, 2	0, 1	4, 4

This stage game is played twice, with the outcome from the first stage observed before the second stage begins. There is no discounting.

Can the payoff (4,4) be achieved in the first stage in a pure-strategy subgame-perfect Nash equilibrium? If so, describe a strategy profile that does so and prove that it is a subgame perfect Nash equilibrium. If not, prove why not.

12. Repeated Game. Consider an infinitely repeated game where the stage game is:

	L	R
U	9, 9	1, 10
D	10, 1	7, 7

Players discount the future using the common discount factor δ.

(1) What outcomes in the stage-game are consistent with Nash equilibrium play?

(2) Let v_R and v_C be the repeated game payoffs to Row and Column respectively. Draw the set of feasible payoffs from the repeated game, explaining any normalisation you use.

(3) Are all the payoffs in the feasible set obtainable from mixed-strategy combinations in the stage-game? (That is, for every point in the feasible set, can you find a p such that $0 \leq p \leq 1$ and a q such that $0 \leq q \leq 1$ that will give those expected payoffs from a single play?)

(4) What are the players' minmax values? Show the individually rational feasible set.

(5) Find a Nash equilibrium in which the players obtain the $(9, 9)$ payoff each period forever. What restrictions on δ are necessary?