What is Game Theory?

Briefly put, game and economic theory are concerned with the interactive behavior of *Homo rationalis*—rational man. . . [An] important function of game theory is the classification of interactive decision situations.

Robert J. Aumann, 1985

Game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers. Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare.

Roger B. Myerson, 1991

Game theory is a mathematical method for analysing strategic interaction.

Nobel Prize Citation, 1994

Course Outline

There are eight lectures, all held in the Lecture Theatre at 9.30-11.00: three in week 6 and 7 (Monday, Tuesday, Thursday) and two in week 8 (Tuesday, Thursday only). They cover the following topics:

1. **Strategic-Form Games.** Games, dominance, Nash equilibria, and rationalisability.
2. **Mixed Strategies.** Mixed extensions, Nash equilibria, correlated equilibria, and ESS.
3. **Continuous Games.** Best-response functions, equilibria, and mixed equilibria.
4. **Bayesian Games.** Incomplete-information games, Bayesian-Nash equilibria, purification.
5. **Extensive-Form Games.** Information sets, subgame-perfect equilibria, and backward induction.
6. **Perfect-Bayesian Equilibrium.** Forward induction, beliefs, PBE, and selection criteria.
7. **Repeated Games.** Finite and infinite repetition, the one-deviation principle, folk theorems.
8. **Cooperative Games.** TU/NTU games, the core, and the Shapley value.

Useful Information

All lecture course materials are online. Download the lecture course notebook (in `.pdf`) from:

`http://malroy.econ.ox.ac.uk/ccw/Games.shtml`

This contains the problem sets for the classes in week 8 of this term and week 1 of next term, a full reading list, and other information. There are also links to this site from the department's website.

There are some books that I recommend: **Osborne**, M.J. (2004) *An Introduction to Game Theory*, OUP; **Gibbons**, R. (1992) *A Primer in Game Theory*, Harvester-Wheatsheaf; **Osborne**, M.J. and **Rubinstein**, A. (1994) *A Course in Game Theory*, MIT Press; **Binmore**, K.G. (2007) *Playing for Real*, OUP; **Kuhn**, H.W. (1997) *Classics in Game Theory*, Princeton; amongst many others.

If you have questions about the course, please either email me at:

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Or come and see me in my office in the department (room 2133) anytime on Mondays.

Strategic-Form Games

Definition 1. A strategic-form (or normal-form) game consists of three objects:

1. *Players.* A set of agents who play the game, $N = \{1, \ldots, n\}$, with typical element $i \in N$.
2. *Strategies.* For each $i \in N$ there is a nonempty set of strategies S_i with typical element $s_i \in S_i$.
3. *Payoffs.* A payoff function $u_i : S \mapsto \mathcal{R}$ assigned to each player i, where $S = \times_{i \in N} S_i$.

Anything with these three features can be written as a strategic-form game:

$$\mathcal{G} = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle.$$

Notation. $s \in S = \times_{i \in N} S_i$ is called a "strategy profile".

Note. For today's lecture it would suffice to use a preference relation $\succeq_i$ over strategy profiles for each player i, rather than moving directly to a function u_i; however, tomorrow's lecture will require vNM utility functions.

The Prisoners' Dilemma

"Two suspects are arrested for a crime, and interviewed separately. If they both keep quiet (they *cooperate* with each other) they go to prison for a year. If one suspect supplies incriminating evidence (*defects*) then that one is freed, and the other one is imprisoned for nine years. If *both* defect then they are imprisoned for six years. Their preferences are solely contingent on any jail term they individually serve."

Players. The players are the two suspects $N = \{1, 2\}$.

Strategies. The strategy set for player 1 is $S_1 = \{C, D\}$, and for player 2 is $S_2 = \{C, D\}$.

Payoffs. Represent the payoffs in the strategic-form matrix:

	C	D
C	-1 -1	0 -9
D	-9 0	-6 -6

Dominance

Of course, game theory does not only provide a language within which to describe interactions.

What would be expected to *happen* when a given game $\mathcal{G}$ is actually played?

Definition 2. Strategy $s_i \in S_i$ *strictly dominates* strategy $s_i' \neq s_i \in S_i$ for player $i \in N$ if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i} = \times_{j \neq i} S_j$. (Replace $>$ with $\geq$ for *weakly*.)

Definition 3. Strategy $s_i' \in S_i$ is *strictly dominated* if there is an $s_i \in S_i$ that strictly dominates it.

The idea might be that a player would never play a strictly dominated strategy, since they can always do better by choosing a different strategy, no matter what their opponents play.

Definition 4. $s_i \in S_i$ is *strictly dominant* for $i \in N$ if it strictly dominates all $s_i' \neq s_i \in S_i$.

If a player has a strictly dominant strategy, it is always better than their other strategies, no matter the strategies chosen by the other players. Surely they would play it?

Notation. $s_{-i} = \{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n\} \in S_{-i}$ is a list of strategies for all players except i.

Dominant-Strategy Equilibrium

The Prisoners' Dilemma provides a simple example of a game with a strictly dominant strategy:

$$
\begin{array}{cccc}
 & C & & D \\
C & \boxed{\begin{matrix} & -1 \\ -1 & \end{matrix}} & \Rightarrow & \boxed{\begin{matrix} & 0 \\ -9 & \end{matrix}} \\
 & \Downarrow & & \Downarrow \\
D & \boxed{\begin{matrix} & -9 \\ 0 & \end{matrix}} & \Rightarrow & \boxed{\begin{matrix} & -6 \\ -6 & \end{matrix}}
\end{array}
$$

D is a strictly dominant strategy for each player. C is a strictly dominated strategy for each player.

$\{D, D\}$ is a *dominant-strategy equilibrium*.

There are no real strategic issues in the one-shot Prisoners' Dilemma. As long as players are rational (they choose the best available action given their preferences), each will play D. Formally:

Definition 5. $s^* \in S$ is a *dominant-strategy equilibrium* if $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$ for all players $i \in N$ and for all strategy profiles $(s_i, s_{-i}) = s \in S$.

Iterated Deletion of Strictly Dominated Strategies

It is easier to illustrate this concept than it is to define it, so consider the following game:

	L	M	R
T	$4 \quad 3$	$2 \quad 7$	$0 \quad 4$
B	$5 \quad 5$	$5 \quad -1$	$-4 \quad -2$

For the column player, M is strictly better than R, leaving:

	L	M
T	$4 \quad 3$	$2 \quad 7$
B	$5 \quad 5$	$5 \quad -1$

For the row player, B is clearly better than T:

	L	M
B	$5 \quad 5$	$5 \quad -1$

$\longrightarrow$

	L
B	$5 \quad 5$

L beats M, leaving $\{B, L\}$, which *survives iterated deletion of strictly dominated strategies*.

Formally Deleting Strictly Dominated Strategies

If column player was rational, won't play R. If row player was rational *and* knew that column player was rational, won't play T. If column player was rational, knew that row player was rational, *and* knew that row player knew that column player was rational, won't play M.

Will they really play $\{B, L\}$?

Definition 6. $A \subseteq S$ *survives iterated deletion of strictly dominated strategies* if $A = \times_{i \in N} A_i$ and there is a collection $\{\{A_i^t\}_{i \in N}\}_{t=0}^{T}$ of sets such that, for all $i \in N$:

1. $A_i^0 = S_i$ and $A_i^T = A_i$.
2. $A_i^{t+1} \subseteq A_i^t$ for all $t \in \{0, \dots, T-1\}$.
3. For all $t \in \{0, \dots, T-1\}$, every strategy of player i, $a_i \in A_i^t \setminus A_i^{t+1}$ is strictly dominated in the game $\langle N, \{A_j^t\}_{j \in N}, \{u_j^t\}_{j \in N} \rangle$, where u_j^t for all $j \in N$ is the function u_j restricted to $\times_{i \in N} A_i^t$.
4. No $a_i \in A_i^T$ is strictly dominated in the game $\langle N, \{A_j^T\}_{j \in N}, \{u_j^T\}_{j \in N} \rangle$.

More on Deleting Strictly Dominated Strategies

Notice that deleting strictly dominated strategies need not result in a unique strategy profile.

	L	M	R
T	$4 \quad 3$	$6 \quad 7$	$0 \quad 4$
B	$5 \quad 5$	$5 \quad -1$	$-4 \quad -2$

For the column player, M is strictly better than R, so $A_C^0 = \{L, M, R\}$ and $A_C^1 = \{L, M\}$:

	L	M
T	$4 \quad 3$	$6 \quad 7$
B	$5 \quad 5$	$5 \quad -1$

No further deletion of strictly dominated strategies is possible: $A = \{T, B\} \times \{L, M\}$. Clearly many games will have no strictly dominated strategies at all. The iterated deletion of strictly dominated strategies may not help say what will happen in a given game $\mathcal{G}$ at all…

On the other hand, A is non-empty, and does not depend upon the strategy-deletion order…

Nash Equilibrium

In the previous example, outcomes like $\{T, L\}$ are not eliminated. Should they be?

- Column player would play M if row player played T!
- Row player would play B if column player played L!

In other words, after the game, both players would "regret" playing their strategies.

Definition 7a. A *Nash equilibrium* is a strategy profile $s^* \in S$ such that for each $i \in N$,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \forall s_i \in S_i.$$

At s^*, no i regrets playing s_i^*. Given all the other players' actions, i couldn't have done better.

So: find what each player would do for each strategy combination of their opponents. If no player has an incentive to deviate from their part in a particular strategy profile, then it is Nash…

Battle of the Sexes

“Two M.Phil. students need to meet up to discuss their love for economics. They can meet in either the pub or the cafe. One likes coffee, and prefers the cafe. The other, however, is a big fan of beer—and prefers the pub. They would both rather meet (wherever it may be) than miss each other.”

Players. The players are the first student, (row) and the second (column).

Strategies. Row chooses $x \in \{\text{Cafe,Pub}\}$, and column chooses $y \in \{\text{Cafe,Pub}\}$.

Payoffs. Represent the payoffs in the strategic-form matrix:

	Cafe	Pub
Cafe	3 4	1 1
Pub	0 0	4 3

- Neither strategy is strictly (or weakly) dominated.
- Deletion of dominated strategies does not rule out anything.

Finding Nash Equilibria

The arrows represent what each player would do given the choice of their opponent.

	Cafe		Pub
Cafe	4, 3	$\Leftarrow$	1, 1
	$\Uparrow$		$\Downarrow$
Pub	0, 0	$\Rightarrow$	3, 4

Both {Cafe,Cafe} and {Pub,Pub} are Nash equilibria. Neither player has an incentive to deviate from their strategy given the strategy of their opponent. Alternatively, underline the payoff associated with each player's *best response*. Consider the game on slide 10:

	L	M	R
T	$4, 3$	$\underline{6}, \underline{7}$	$\underline{0}, 4$
B	$\underline{5}, \underline{5}$	$5, -1$	$-4, -2$

Best-Response Functions

Nash equilibria are strategy profiles where "every payoff is underlined". This suggests an alternative (equivalent) definition for Nash equilibrium involving best responses.

Definition 8. The *best-response function* for player $i \in N$ is a set-valued function B_i such that:

$$B_i(s_{-i}) = \left\{ s_i \in S_i \mid u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}),\ \ \forall s_i' \in S_i \right\} .$$

So that $B_i(s_{-i}) \subseteq S_i$ "tells" player i what to do when the other players play s_{-i}. Hence:

Definition 7b. $s^* \in S$ is a Nash equilibrium if and only if $s_i^* \in B_i(s_{-i}^*)$ for all $i \in N$.

In words: a Nash equilibrium is a strategy profile of mutual best responses. Each player picks a best response to the combination of strategies the other players pick.

The Stag Hunt

"Two hunters simultaneously choose to hunt for rabbits, or to hunt for a stag. Successfully capturing a stag requires coordination, but there will be lots of meat. Anyone can catch a rabbit, but there will be less meat, especially when both are hunting rabbits (all that noise...)"

Players. Hunters 1 and 2 (row and column respectively).

Strategies. Each player can choose either Rabbit (R) or Stag (S).

Payoffs. Represent the payoffs in the strategic-form matrix:

	R	S
R	$\underline{3}$, $\underline{3}$	4, 0
S	0, 4	$\underline{5}$, $\underline{5}$

- There are Nash equilibria at $\{R, R\}$ and $\{S, S\}$. Which one gets played?
- The latter is Pareto optimal...but is the former, somehow, "safer"?

Hawk-Dove

This is the classic biological game, where two players may either *fight* over a resource (Hawk) or *yield* (Dove). A Hawk beats a Dove, gaining the resource, of value v, with 0 for the Dove. Two Doves split the payoff v equally. Two Hawks have equal chance of winning the fight. The loser pays a cost c, where $v < c$.

Represent this as a symmetric strategic-form game, and remember $v - c < 0$:

	Hawk	Dove
Hawk	$\frac{v-c}{2}$, $\frac{v-c}{2}$	v, 0
Dove	0, v	$\frac{v}{2}$, $\frac{v}{2}$

$$\xrightarrow[(v = 4 \text{ and } c = 6)]{}$$

	Hawk	Dove
Hawk	-1, -1	$\underline{4}$, $\underline{0}$
Dove	$\underline{0}$, $\underline{4}$	2, 2

There are two *asymmetric* Nash equilibria at {Hawk,Dove} and {Dove,Hawk}.

Matching Pennies

"Two players (row and column) each have a coin. They must simultaneously choose whether to put their coins down heads-up (H) or tails-up (T). If the coins match, row gives column their coin, if they don't match, column gives row their coin."

Players. As usual, row player and column player.

Strategies. Row chooses from $\{H, T\}$, and column from $\{H, T\}$.

Payoffs. Represent the payoffs in the strategic-form matrix:

	H	T
H	-1, $\underline{1}$	$\underline{1}$, -1
T	$\underline{1}$, -1	-1, $\underline{1}$

- Both H and T can be *best responses*, depending on the other player's strategy.
- There is no pure-strategy Nash equilibrium to this game. There is a mixed equilibrium.

Rock-Scissors-Paper

What would rational players do in the following game?

	R	S	P
R	0, 0	$\underline{1}$, -1	-1, $\underline{1}$
S	-1, $\underline{1}$	0, 0	$\underline{1}$, -1
P	$\underline{1}$, -1	-1, $\underline{1}$	0, 0

"Would row ever play R? Yes, if row thought column was playing S. Is this a rational belief? Yes, if row believes column believes row will play P. Is this a rational belief? Yes, if row believes column believes row believes column will play R. Is this a rational belief? Yes if…"

Eventually this process will return to "…believes row will play R". R is said to be *rationalisable*.

A formal definition is slightly more cumbersome…

Rationalisability

Here is a reasonably sharp definition—see textbooks for others.

Definition 9. Strategy $s_i \in S_i$ is *rationalisable* in the game $\mathcal{G} = \langle N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ if for all $j \in N$, there is a set $R_j \subseteq S_j$ such that (i) $s_i \in R_i$ and (ii) every action $s_j \in R_j$ is a best response to a belief $\mu_j(s_j)$ of player j whose support is a subset of $R_{-j} = \times_{k\neq j} R_k$.

This is “circular”. A strategy is rationalisable if it is a best response to a combination of opponents' strategies that are all rationalisable. However, the formal definition does make sense!

- D is rationalisable in the Prisoners' Dilemma.
- Cafe and Pub are rationalisable in the Battle of the Sexes.
- R and S are rationalisable in the Stag Hunt.
- Hawk and Dove are rationalisable in the Hawk-Dove game on slide 16.
- H and T are rationalisable in Matching Pennies.
- R, S, and P are rationalisable in Rock-Scissors-Paper.

Nash, Rationalisability, and Dominance

Recall the game from slide 8 (redrawn below). Is M rationalisable?

	L	M	R
T	4 3	2 7	0 4
B	5 5	5 -1	-4 -2

No. For column to play M, column must believe row will play T. For this belief to be rational, column must believe that row believes that column will play R. For *this* belief to be rational there must be a belief for column to which R is a best response. There is not—R is strictly dominated.

In fact, only B is rationalisable for row and L rationalisable for column. Which suggests…

Let $R = \times_{i \in N} R_i$, where R_i is the set of **rationalisable strategies** for $i \in N$. Z is the set of **Nash equilibria**, and A is the set that survives **iterated deletion of strictly dominated strategies**. Then

$$Z \subseteq R \subseteq A \subseteq S.$$

Note. Under the current definition of rationalisability $R = A$ if the mixed extension of the game is used.