Finding disjoint Paths in Graphs

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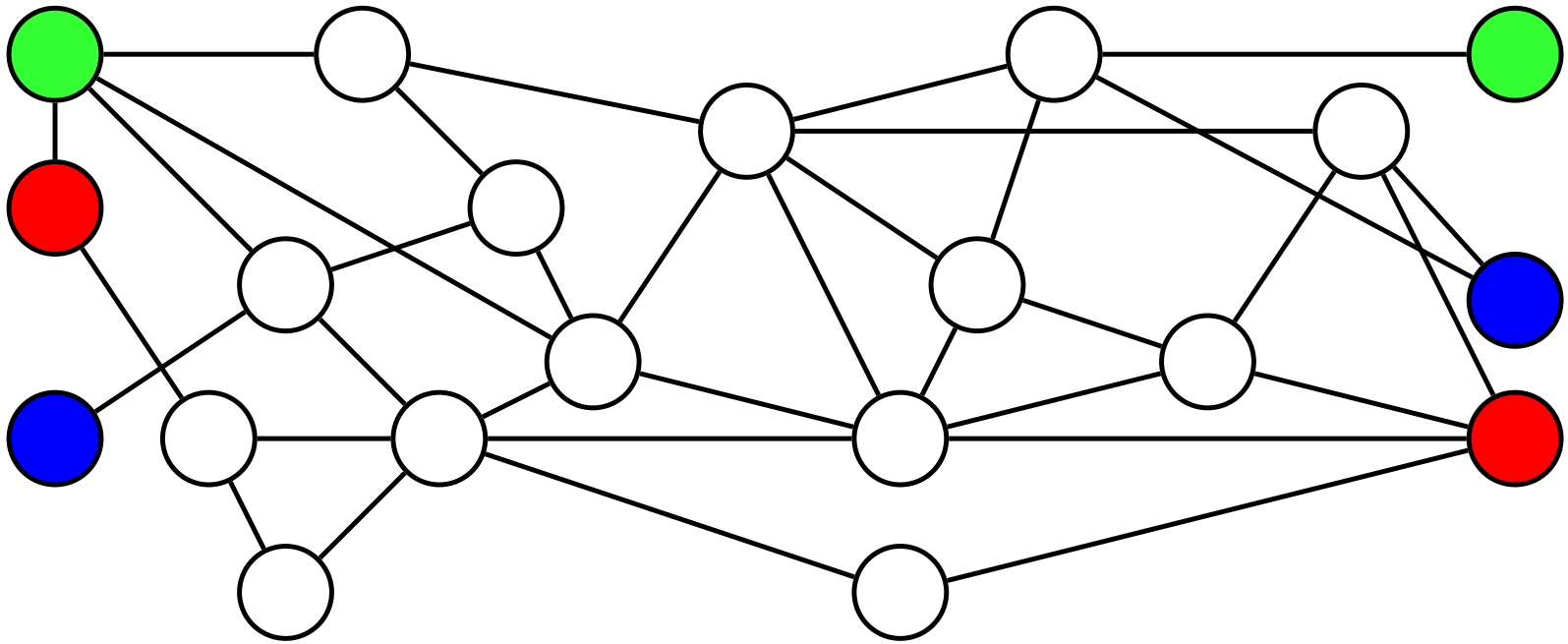
Finding Disjoint Paths

Given: A graph with sources and targets.
Problem: Connect sources and targets by disjoint paths.



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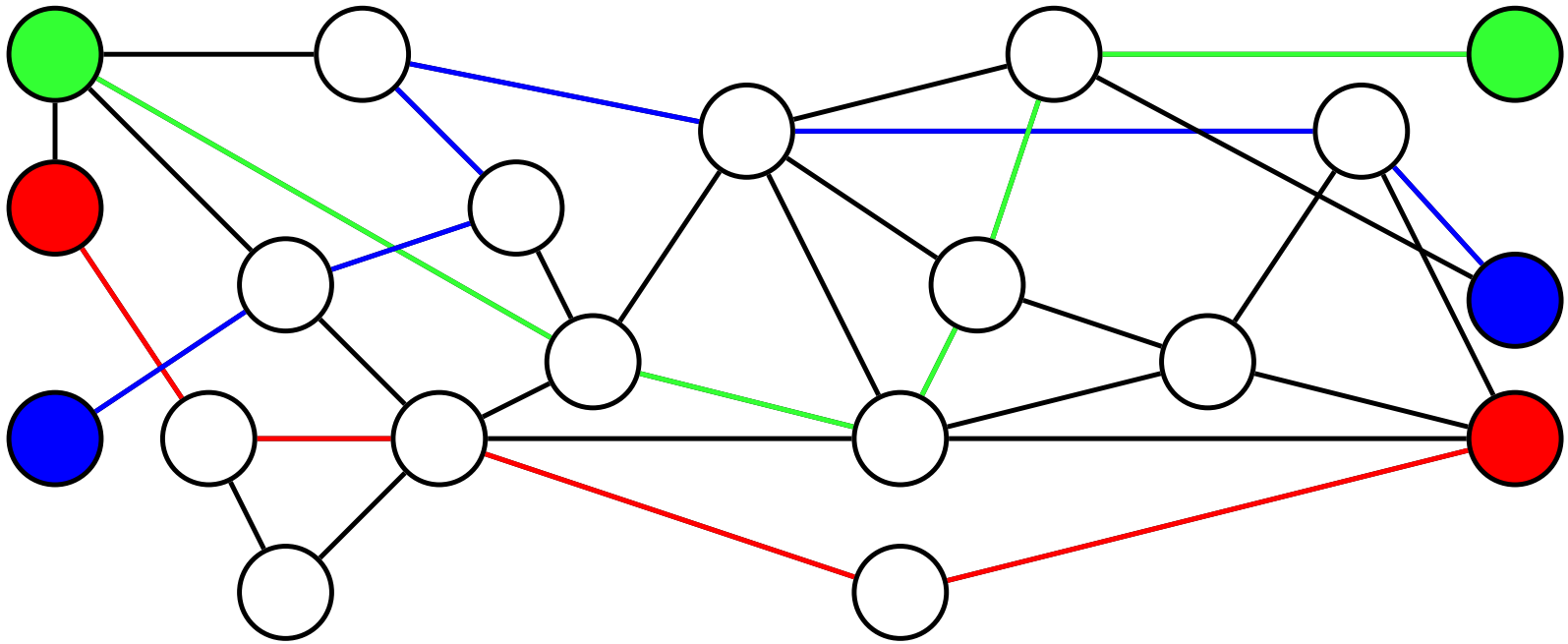
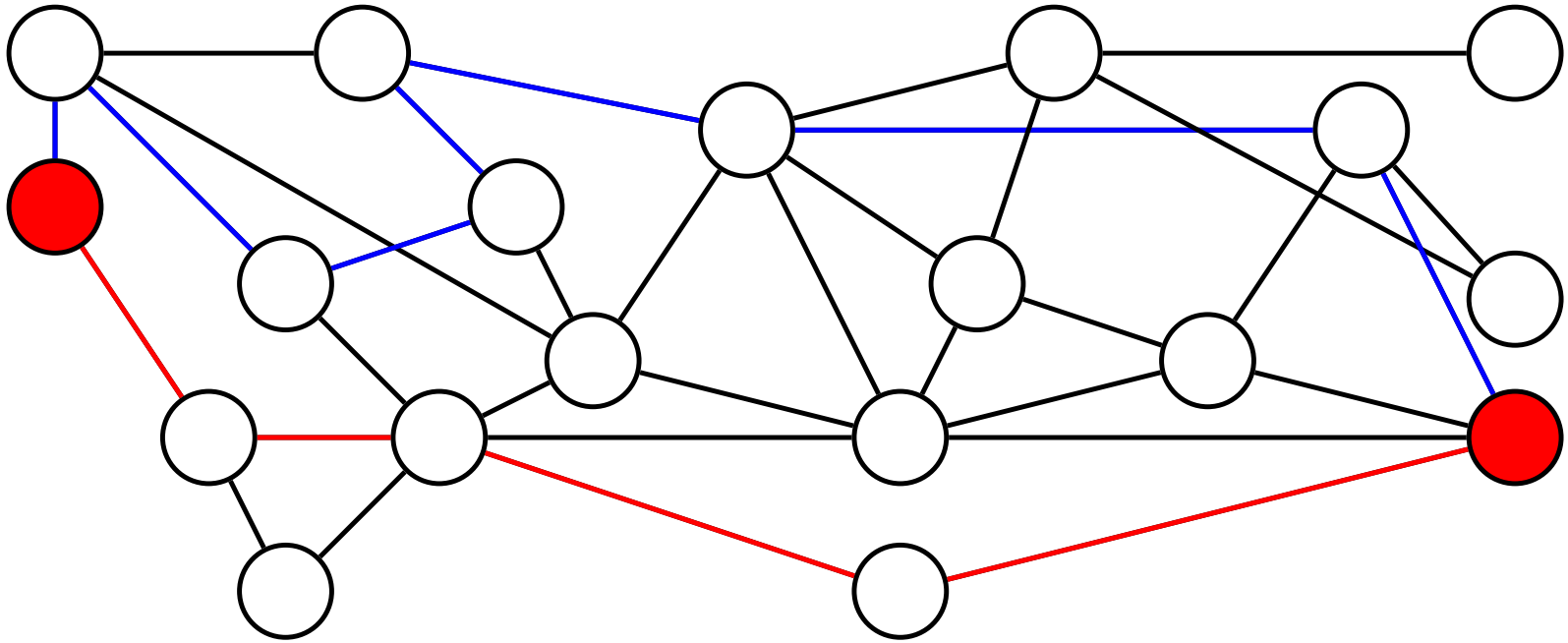


Several version

- single/multiple sources and targets.



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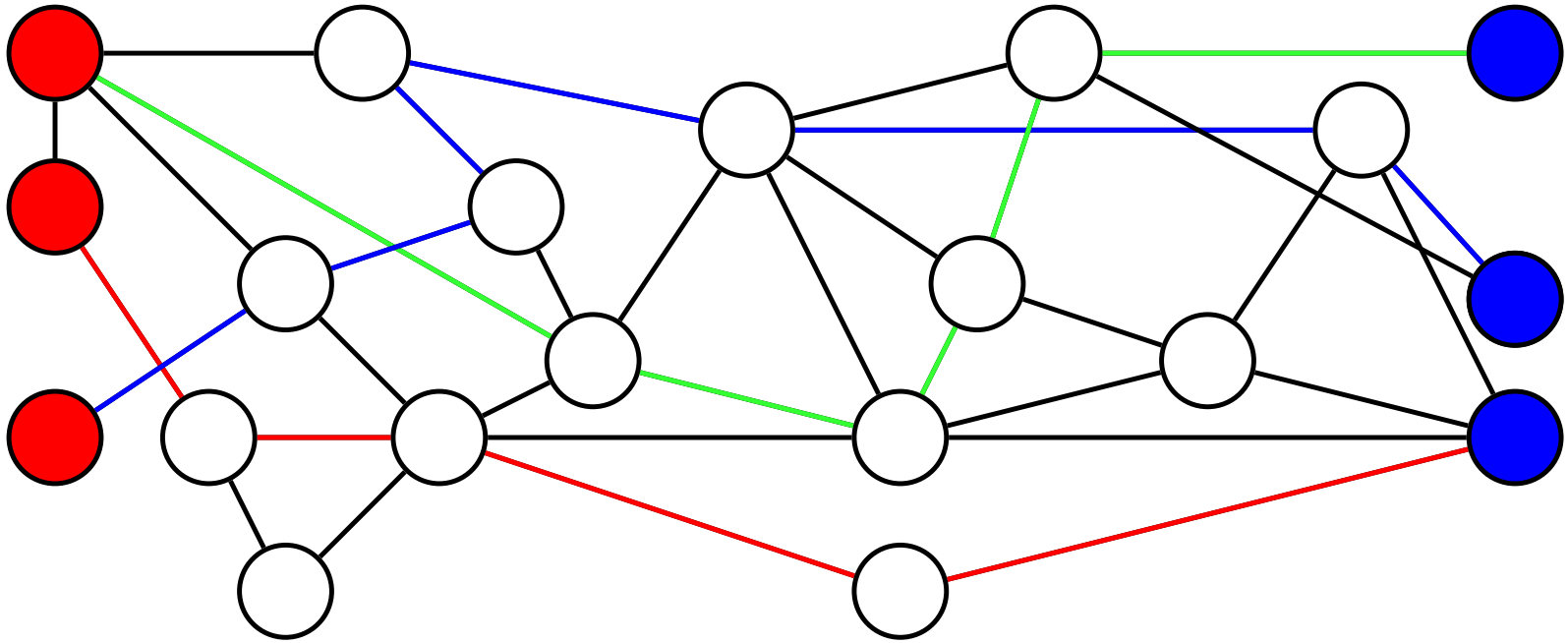

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- single/multiple sources and targets.
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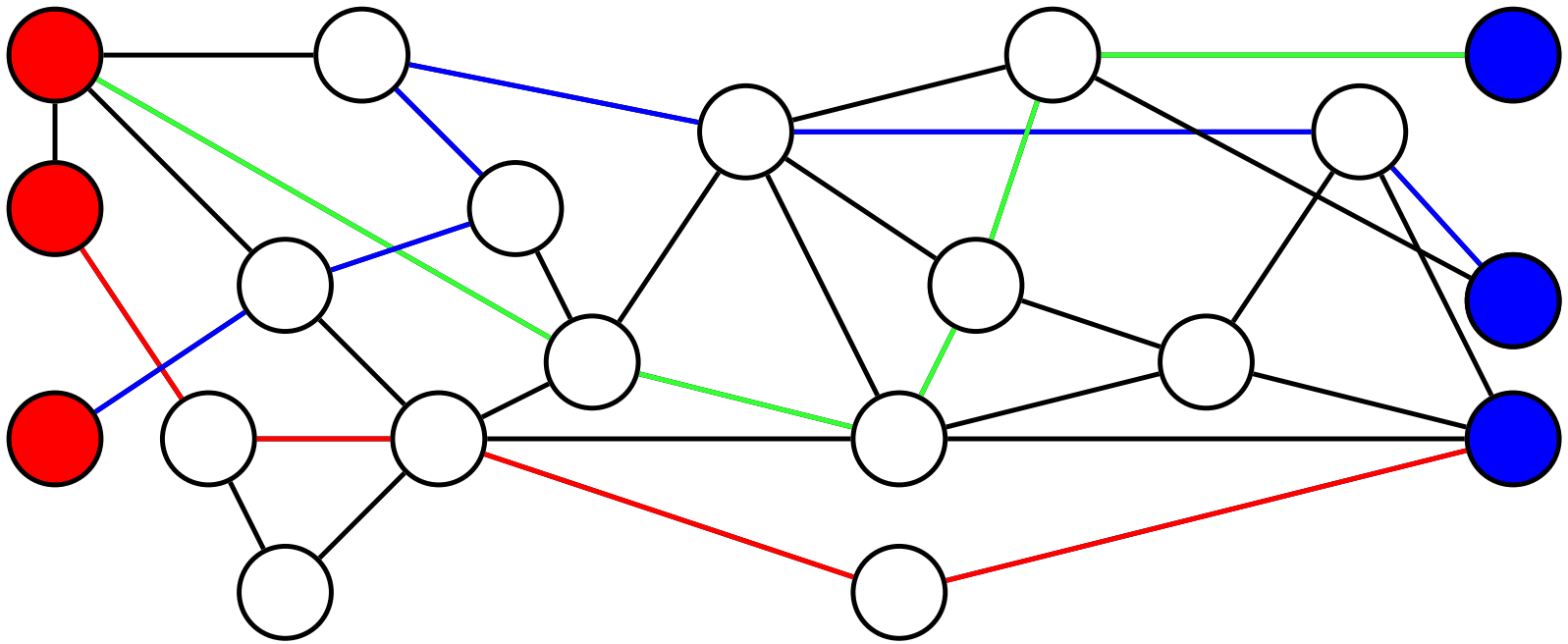

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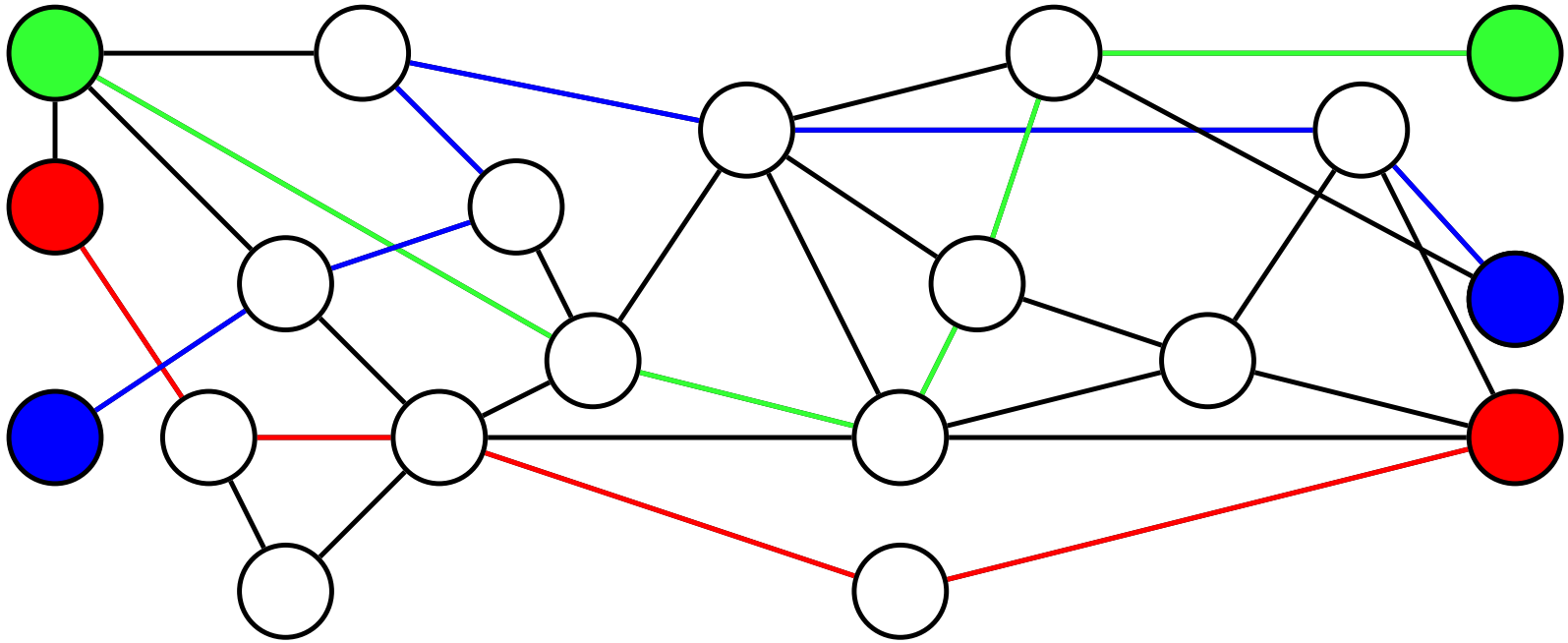

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- single/multiple sources and targets.
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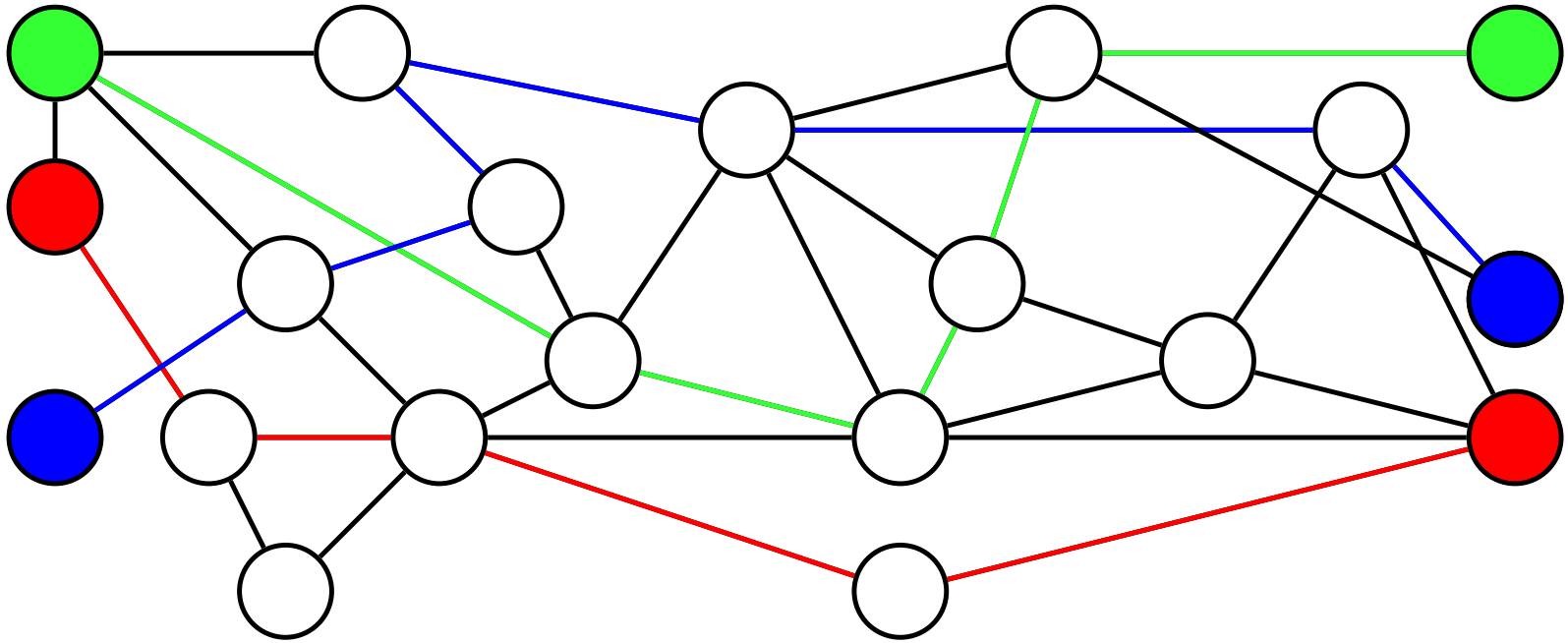
- vertex-/edge-disjoint paths.



The problem

Several version

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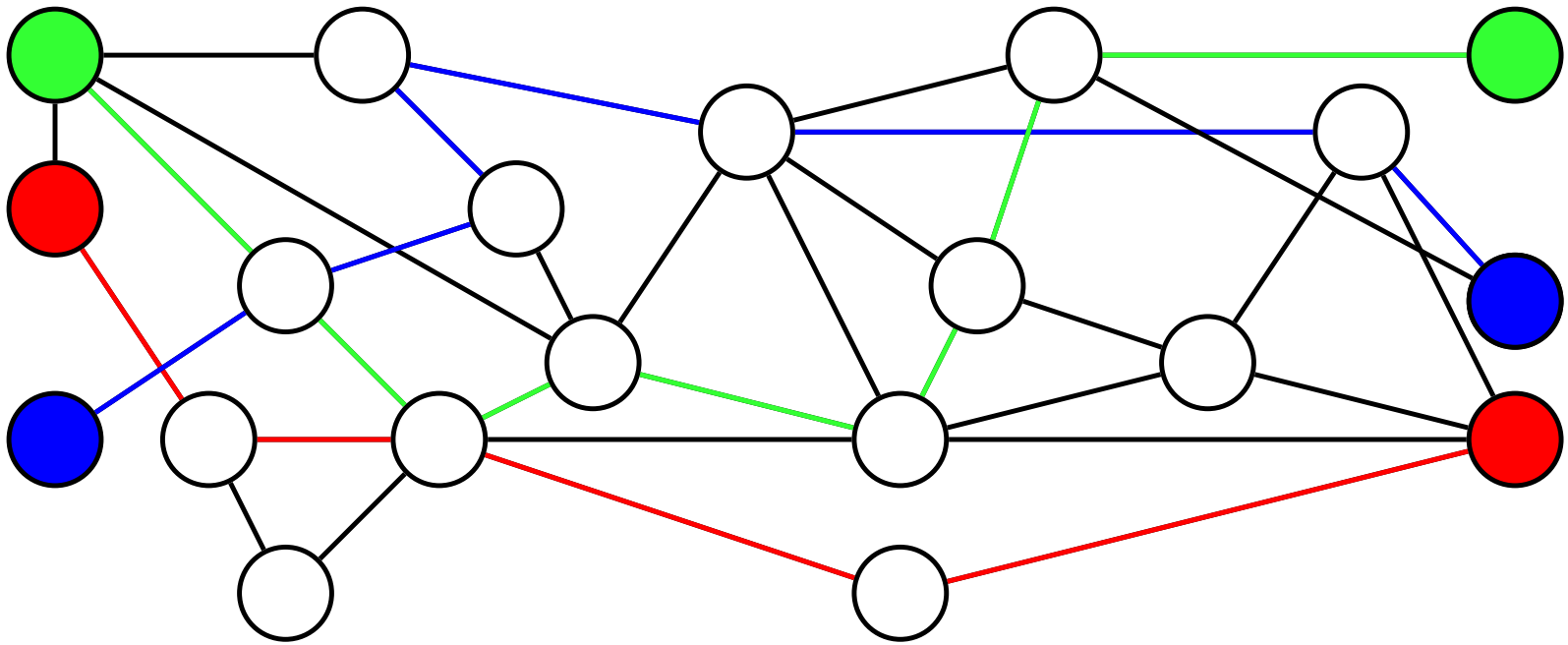

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- single/multiple sources and targets.
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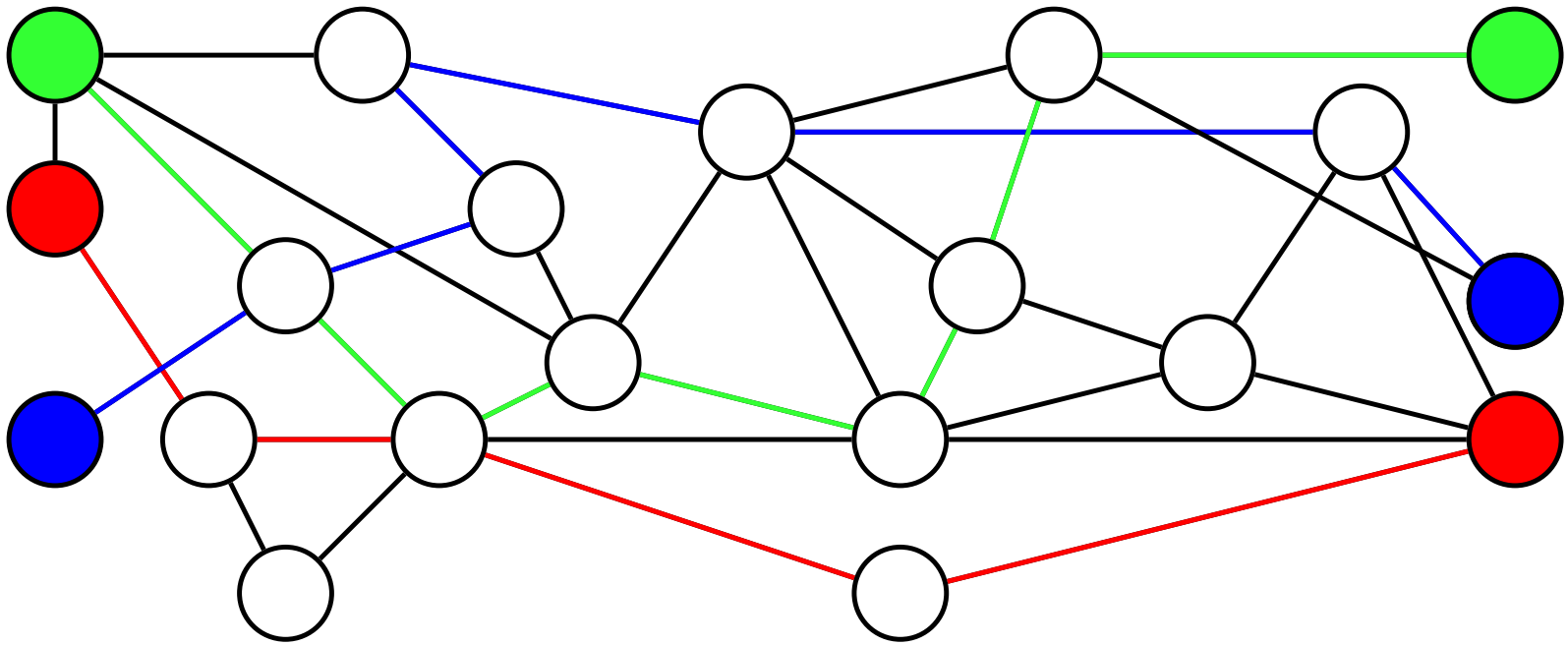
- vertex-/edge-disjoint paths.
- in undirected/directed graphs



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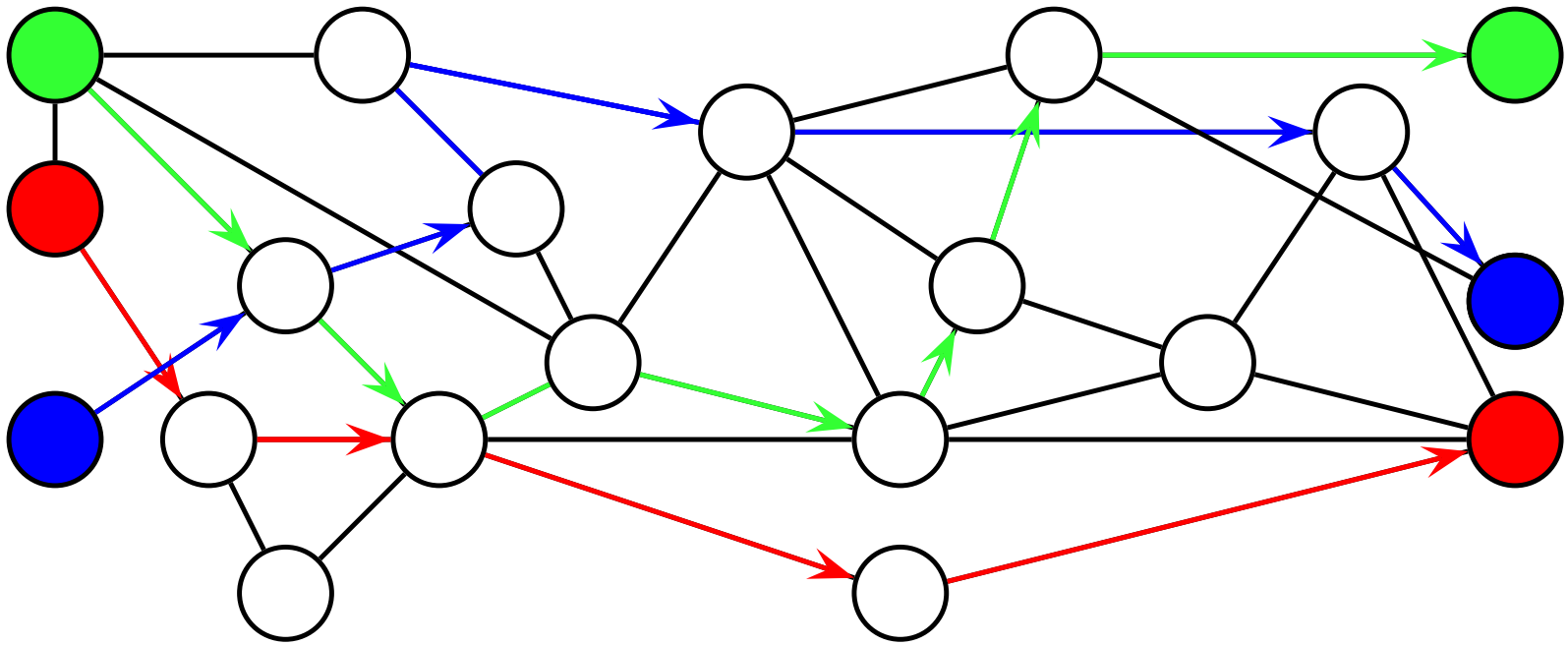
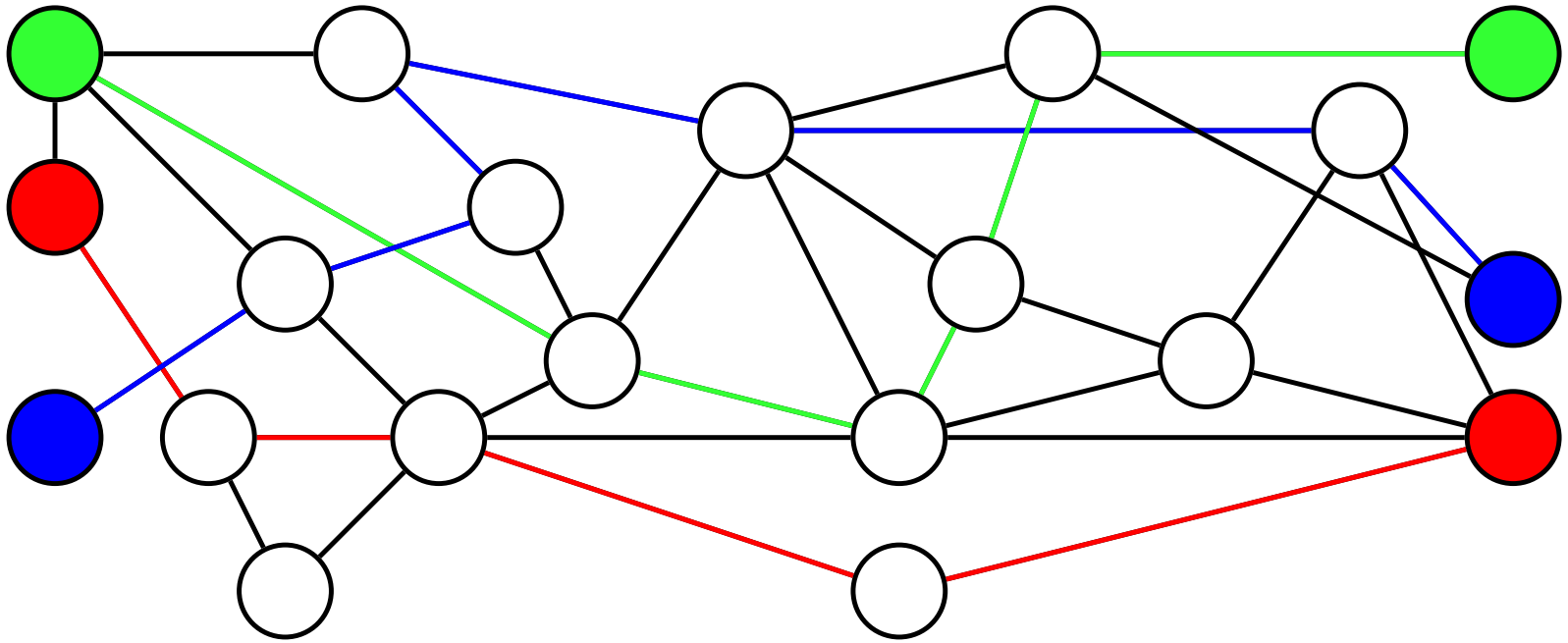


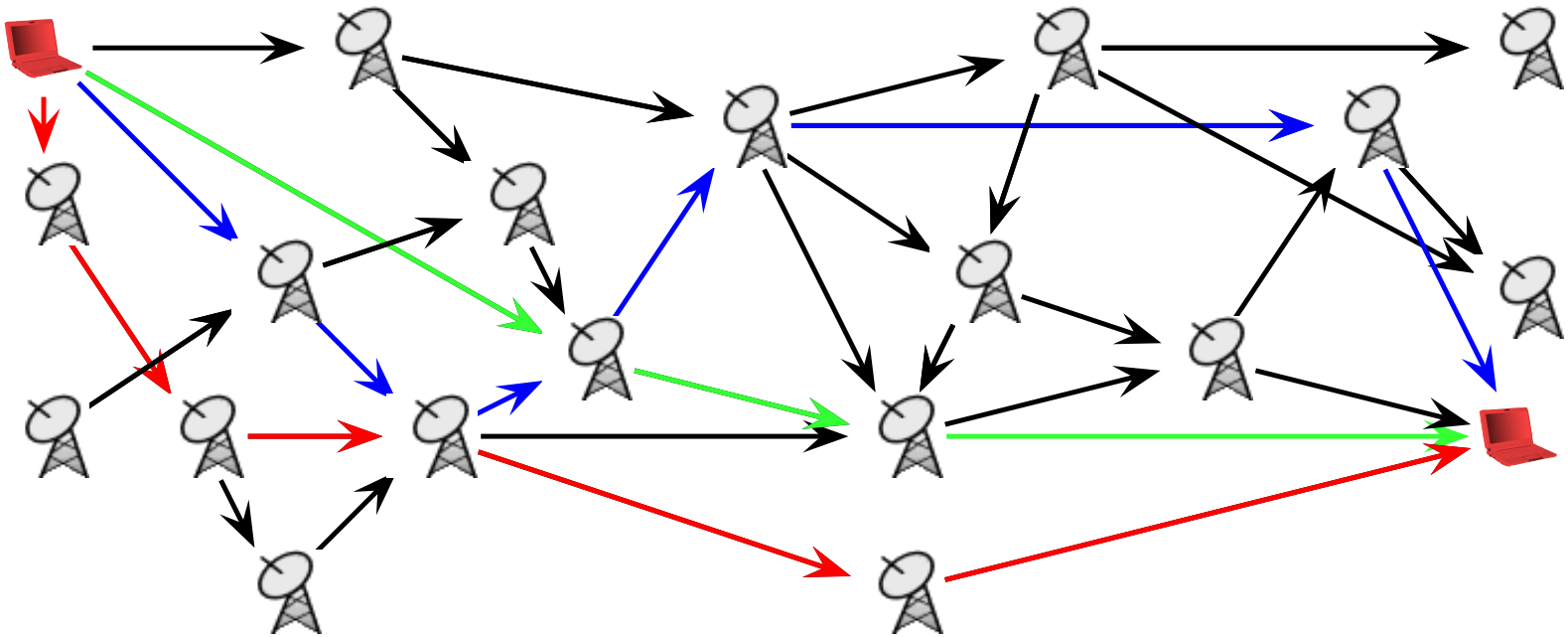
Applications

- routing in computer/traffic networks
- reliability of networks
- VLSI design



A first problem

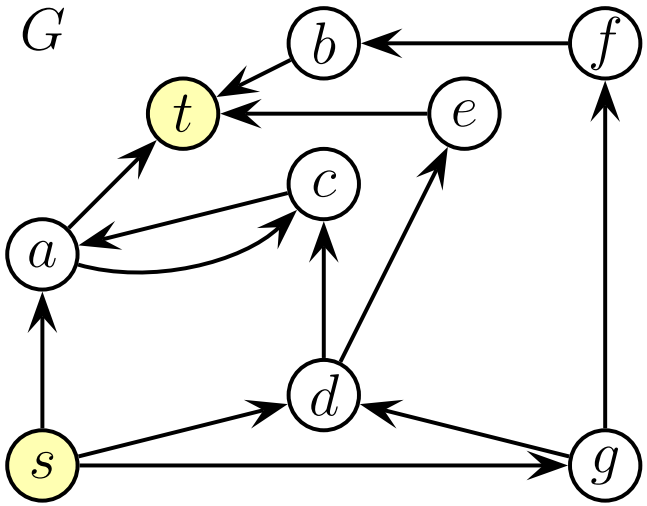
Given: vertices s and t in a directed graphs, $k \in \mathbb{N}$
Output: k edge-disjoint paths from s to t, if they exist.

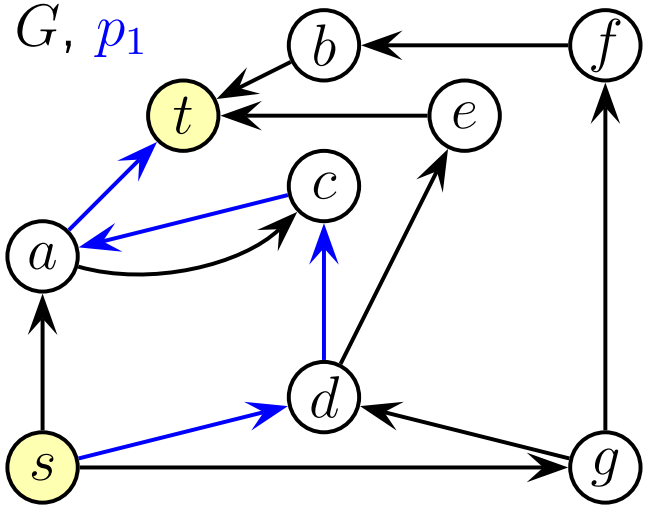


A first approach

Repeat:
Return a path p.
Delete p from G
Until there is no path in G

The approach does not work





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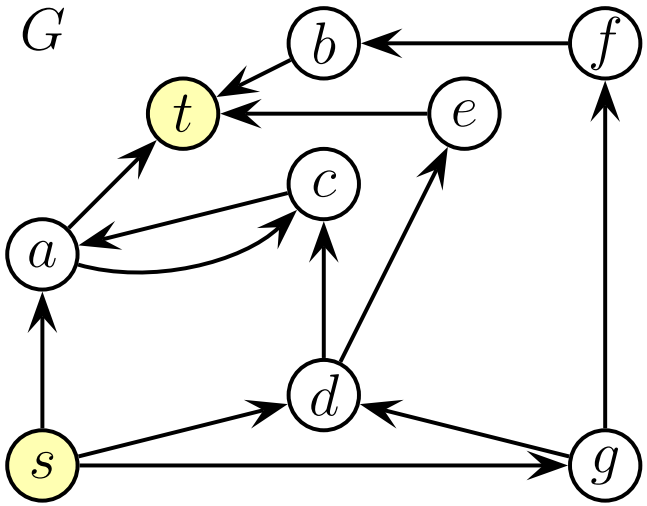

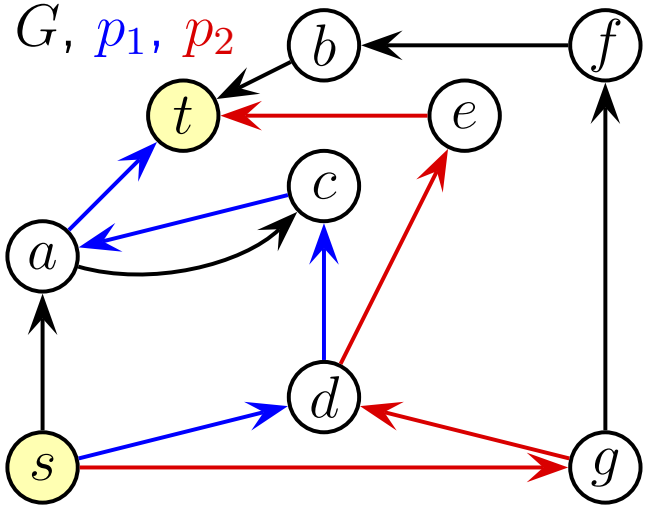

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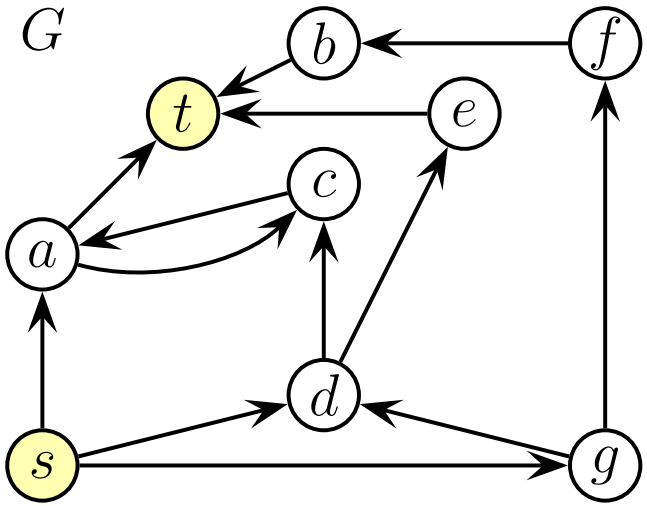
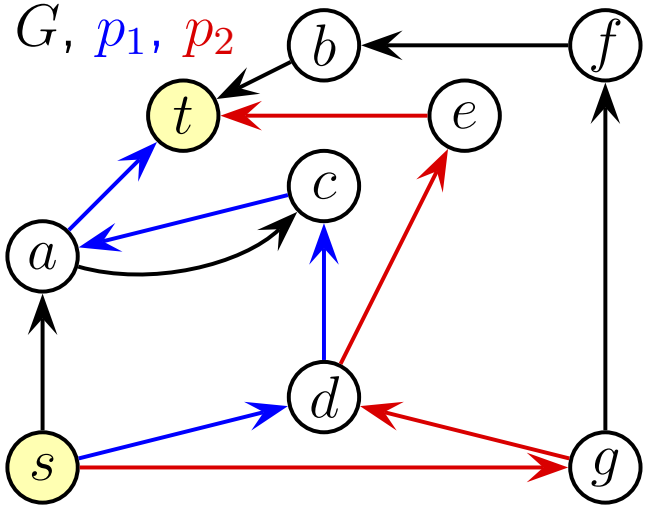
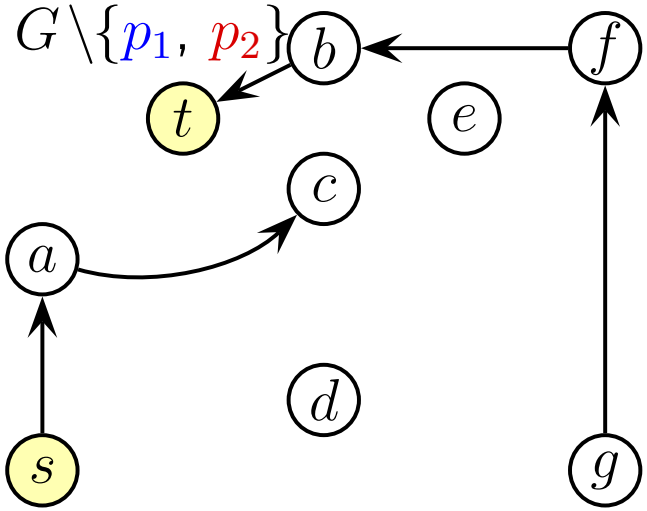
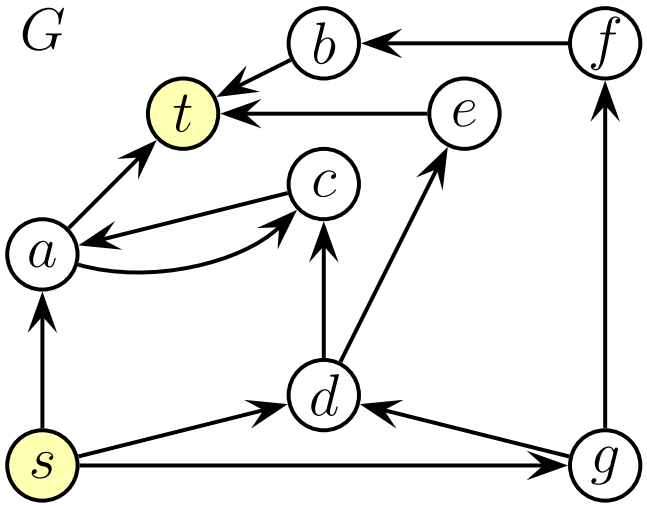


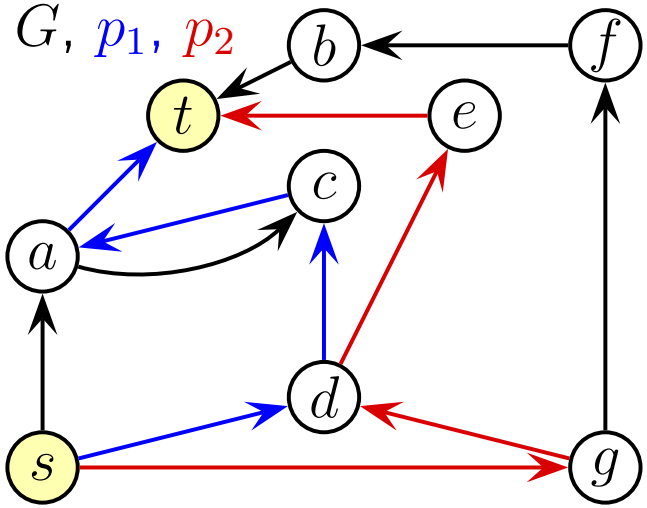


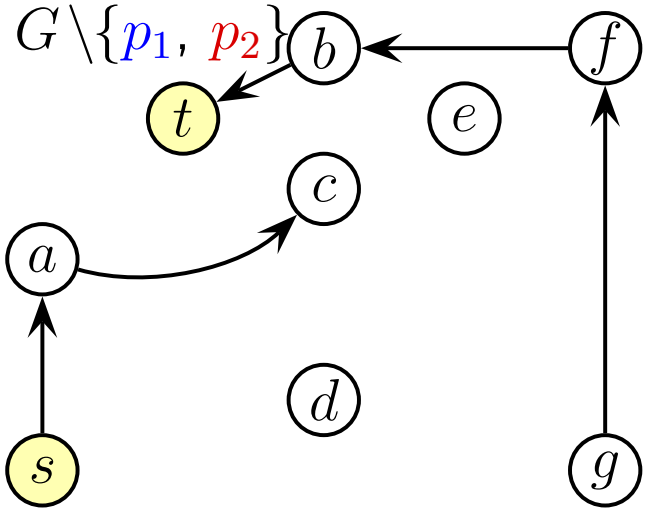


In $G \setminus \{p_1, p_2\}$ there is no path from s to t.

The approach does not work



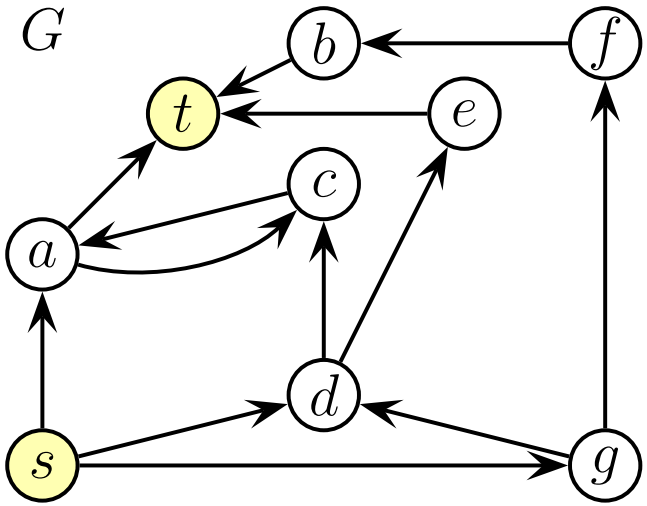


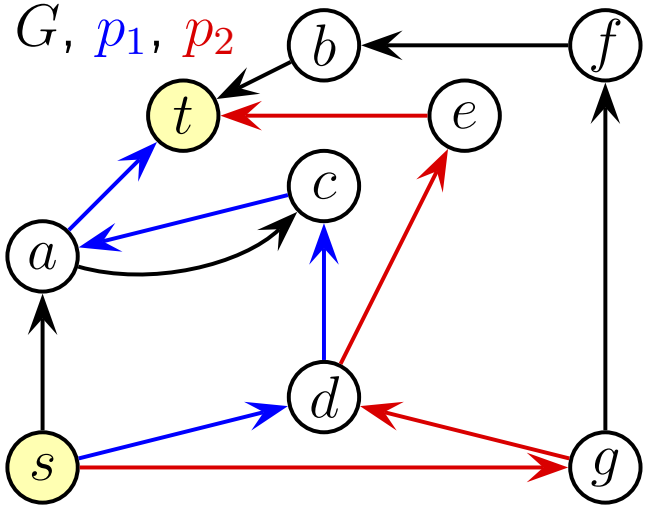


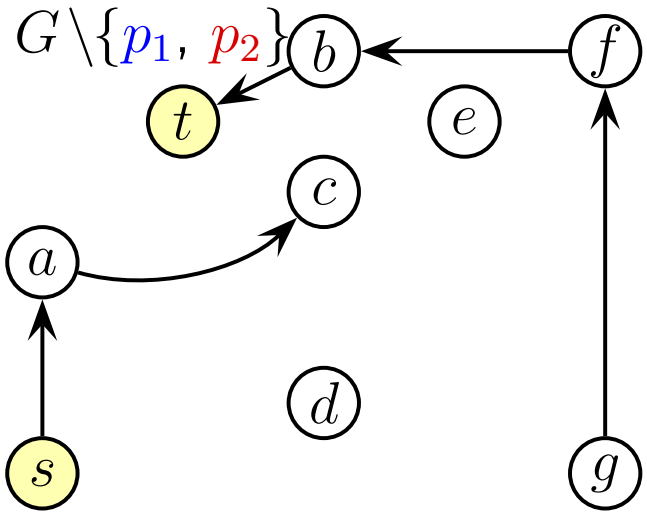
In $G \setminus \{p_1, p_2\}$ there is no path from s to t.

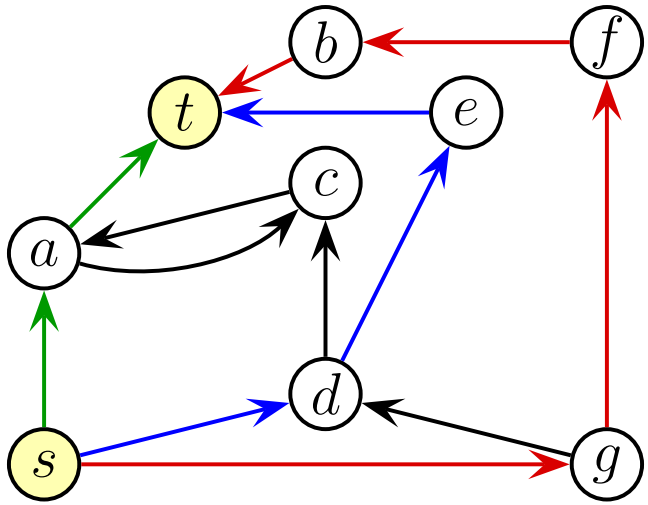
However, in G there are three edge-disjoint paths $p_i : s \to t$:

The approach does not work



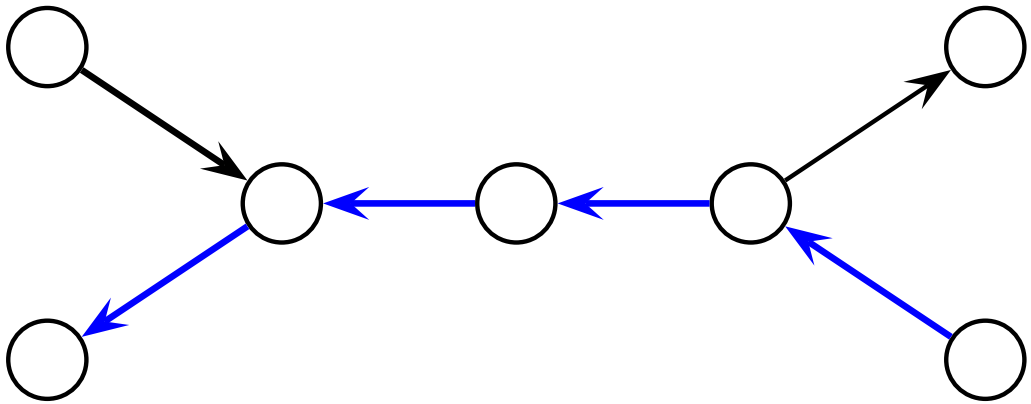






Conclusion

Problem: We make wrong decisions.



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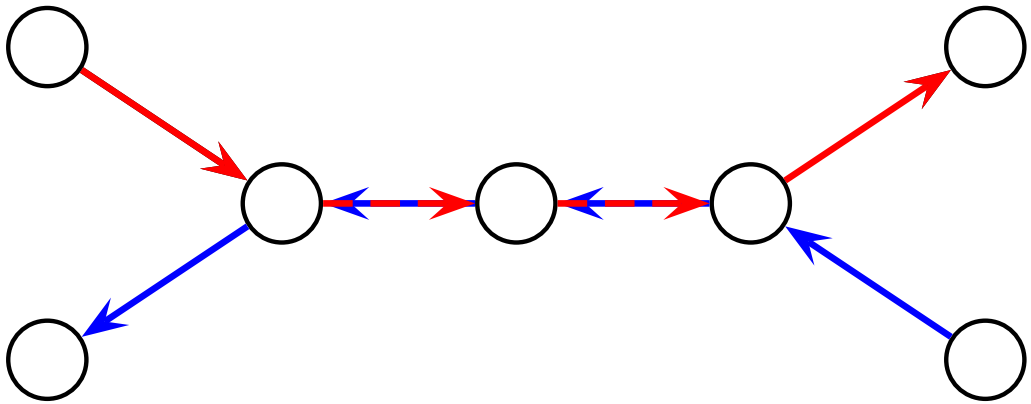
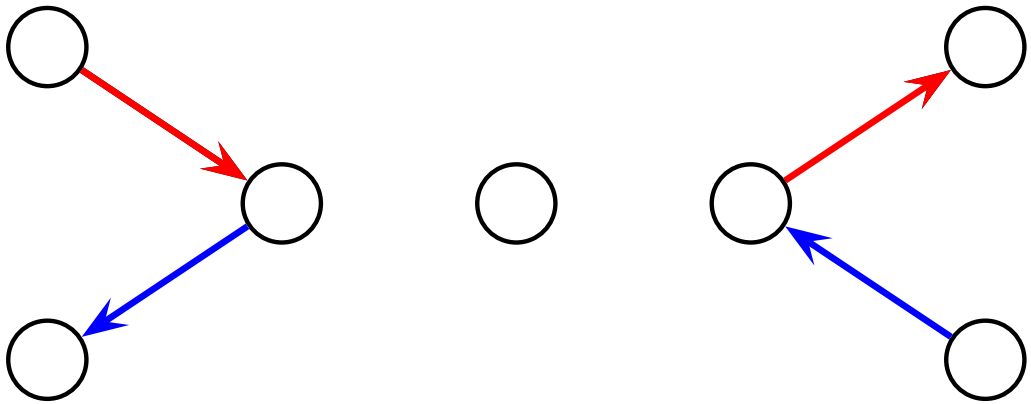
Solution: Edges of already constructed paths may be used in reverse direction.



Conclusion

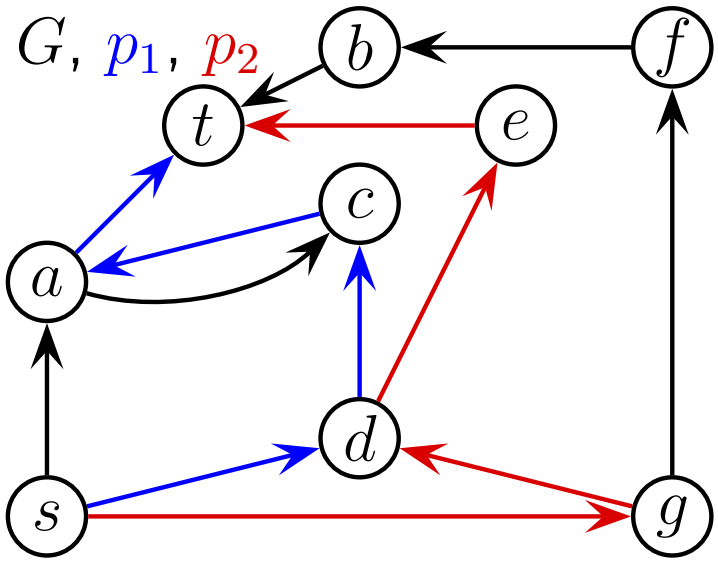
Problem: We make wrong decisions.
Solution: Edges of already constructed paths may be used in reverse direction. Afterward delete doubly visited edges.

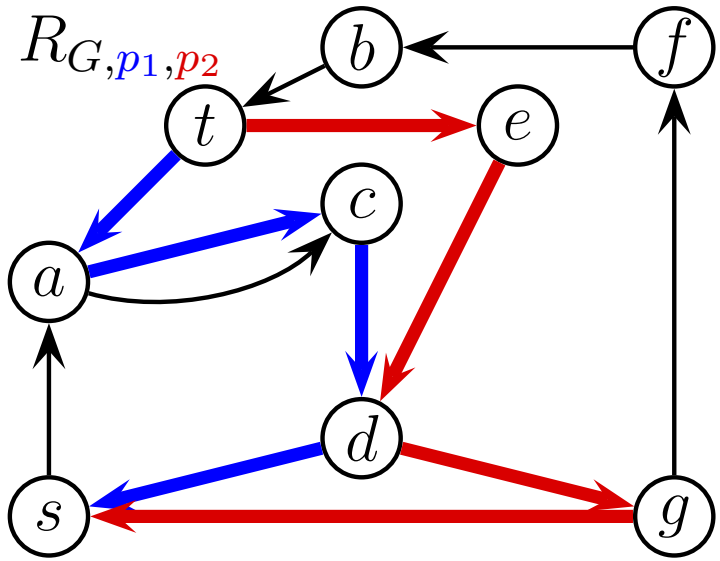


residual graph $G_{p_1,\ldots,p_k}$

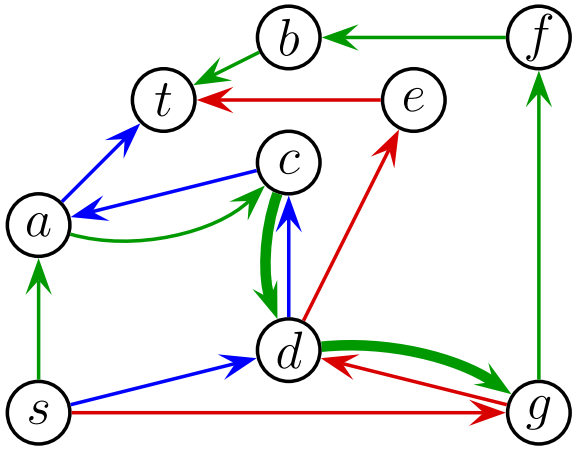
Given: A graph G, edge-disjoint paths $p_1, \ldots, p_k$

$G_{p_1,\ldots,p_k}$**:** Graph obtained from G be reversing the edges of $p_1, \ldots, p_k$:





Deletion of edges from a path in the residual graph



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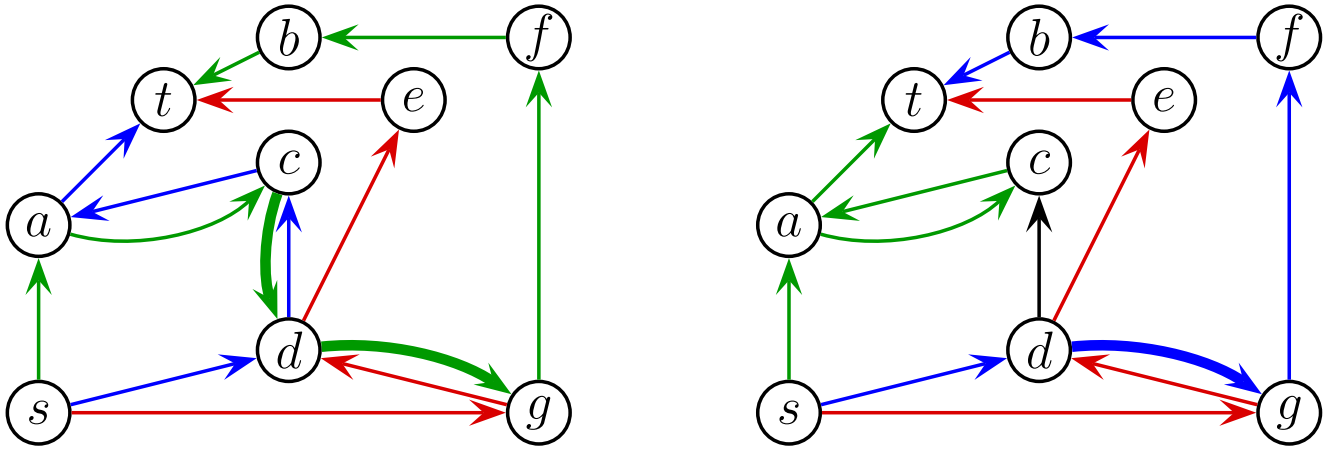

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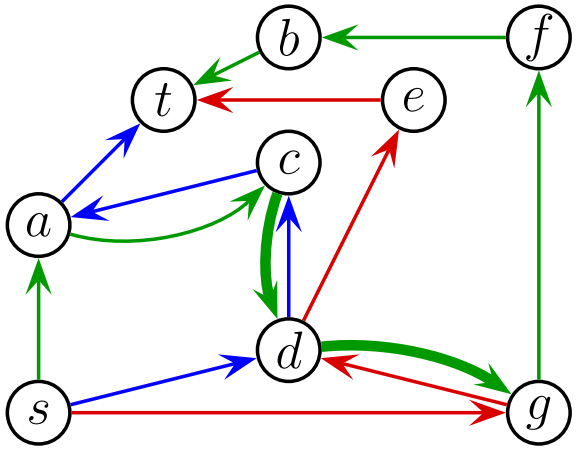

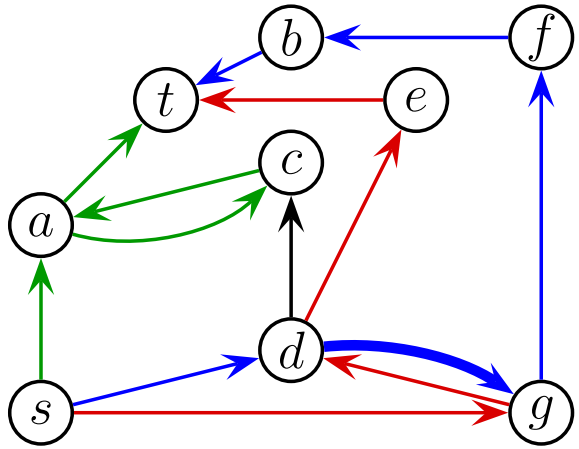

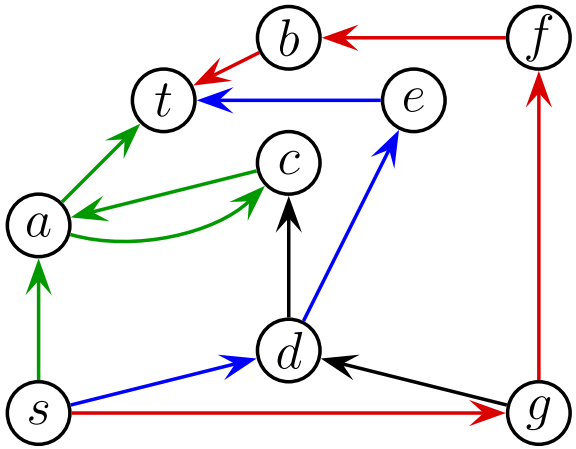

The algorithm of Ford and Fulkerson

(1) **For** $i = 1$ **to** k:
(2) Construct $R_{G,p_1,\ldots,p_{i-1}}$ $(= G$ if $i = 1)$. $O(m+n)$
(3) Find a path $p : s \to t$ in $R_{G,p_1,\ldots,p_{i-1}}$. $O(m+n)$
(4) Delete all backward edges from p and the corresponding forward edges from p. $O(m+n)$
(5) Rename the resulting paths $p_1, \ldots, p_i$. $O(m+n)$
(6) **End For**

Running time: $O(k(m+n))$.

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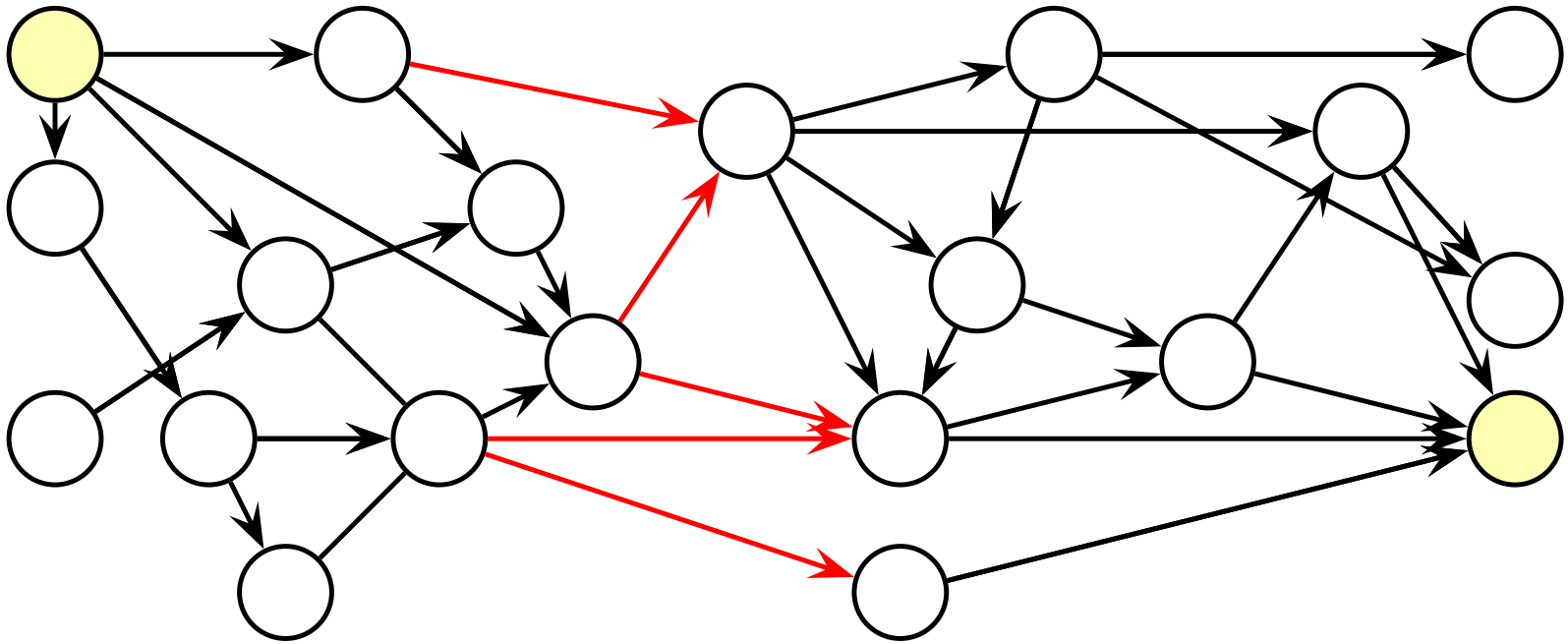
Correctness

We have to show: If there is no path $p : s \to t$ in $R_{p_1,\ldots,p_k}$, then G has at most k edge-disjoint paths from s to t.

s,t-edge cut

An s,t-edge cut is set S of edges with s,t being not connected in $G \setminus S$.



Lemma 1

For an s-t edge-cut C, there are at most $|C|$ edge-disjoint paths from s to t.

Proof

Every paths from s to t must visit an edge of C.

Lemma 2

If there is no path $p : s \to t$ in $R_{p_1,\ldots,p_k}$, then $G = (V, E)$ has at most k edge-disjoint paths from s to t.

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If there is no path $p : s \to t$ in $R_{p_1,\ldots,p_k}$, then $G = (V, E)$ has at most k edge-disjoint paths from s to t.

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- Assume there is no path $p : s \to t$ in $R_{p_1,\ldots,p_k}$.
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$\Rightarrow$ $F = S \times (V \setminus S)$ is an s,t-edge cut in G.

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$\Rightarrow$ Every path from s to t uses an edge in F.

- No path $p : s \to t$ uses an edge in $(V \setminus S) \times S$ (otherwise there is an backward edge in $S \times (V \setminus S)$ in $R_{p_1,\ldots,p_k}$).

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$\Rightarrow$ $|F| = k$.

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If there is no path $p : s \to t$ in $R_{p_1,\ldots,p_k}$, then $G = (V, E)$ has at most k edge-disjoint paths from s to t.

Proof

- Assume there is no path $p : s \to t$ in $R_{p_1,\ldots,p_k}$.
- Let S be the vertices reachable from s in $R_{p_1,\ldots,p_k}$.

$\Rightarrow$ $F = S \times (V \setminus S)$ is an s,t-edge cut in G.

$\Rightarrow$ Every path from s to t uses an edge in F.

- No path $p : s \to t$ uses an edge in $(V \setminus S) \times S$ (otherwise there is an backward edge in $S \times (V \setminus S)$ in $R_{p_1,\ldots,p_k}$).
- All edges in F are visited by a path in $p_1, \ldots, p_k$.

$\Rightarrow$ $|F| = k$.

$\Rightarrow$ There are at most $k = |F|$ paths (Lemma 1).

Observation 1

Lemma 1 and the line $|F| = k$ in the last lemma also imply the following theorem.

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Menger's Theorem

There are k-edge disjoint paths between two vertices if and only if there are no s, t-edge cut of size $k - 1$.

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Lemma 1 and the line $|F| = k$ in the last lemma also imply the following theorem.

Menger's Theorem

There are k-edge disjoint paths between two vertices if and only if there are no s, t-edge cut of size $k - 1$.

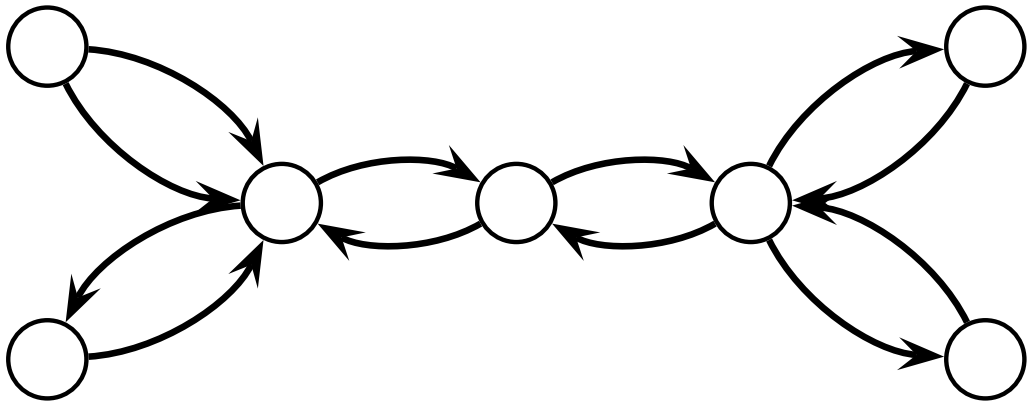
Menger's Theorem (vertex version)

There are k-vertex disjoint paths between two nonadjacent vertices if and only if there are no s, t-vertex cut of size $k - 1$.

Other versions

If we want to find edge-disjoint paths in an undirected graph

- we replace an edge $\{u, v\}$ by directed edges $(u, v), (v, u)$.



Other versions

If we want to find edge-disjoint paths in an undirected graph

- we replace an edge $\{u, v\}$ by directed edges $(u, v), (v, u)$.
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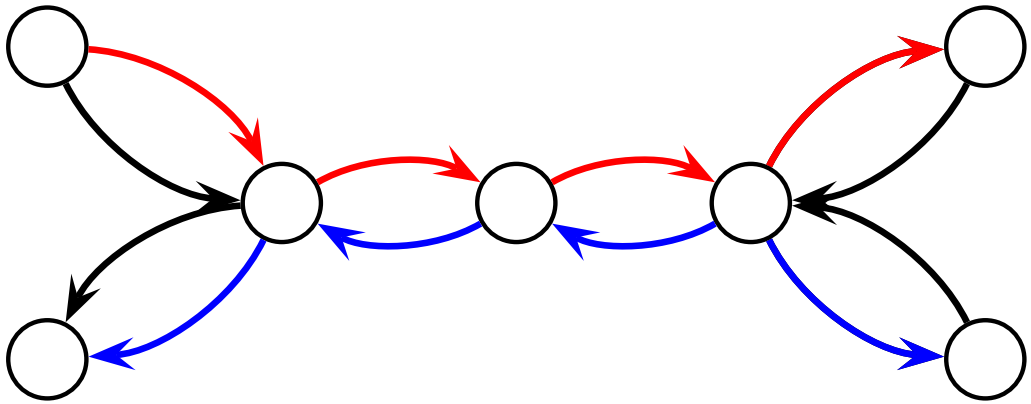
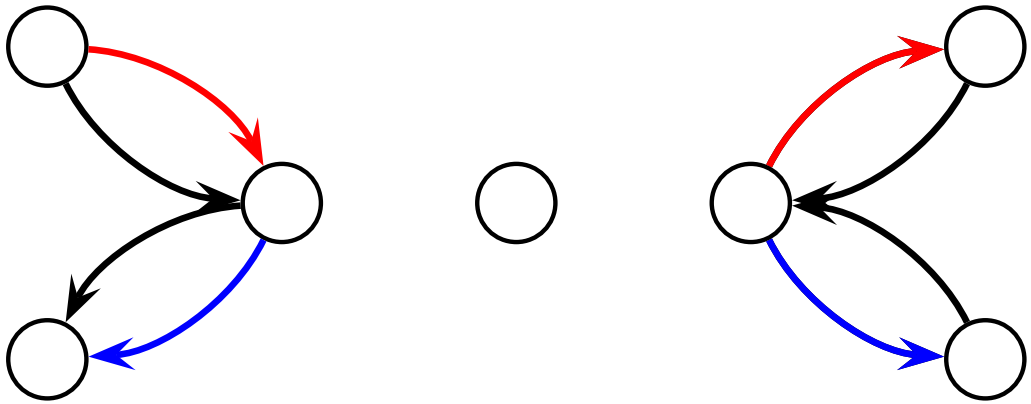


Other versions

If we want to find edge-disjoint paths in an undirected graph

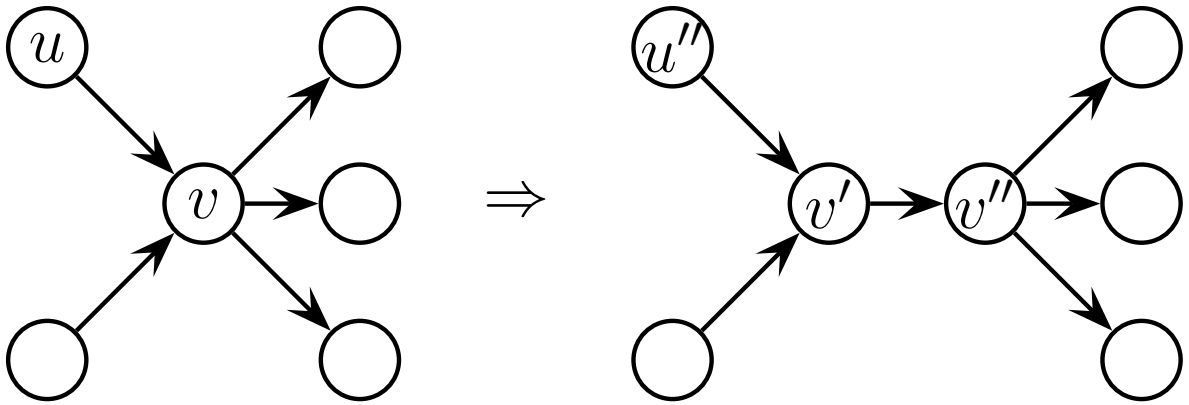
- we replace an edge $\{u, v\}$ by directed edges $(u, v), (v, u)$.
- A solution might use the same edge in different directions.
- Doubly visited edges can be removed as backward edges and their counterparts.



Other versions

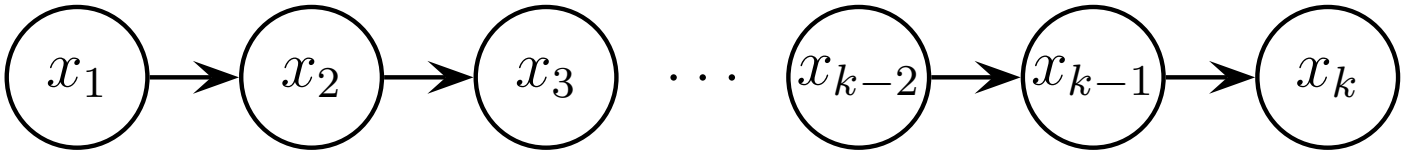
For finding vertex-disjoint paths in a directed graph G, let G' be obtained from G by replacing

- each vertex by vertices v' and v'' and an edge (v, v').
- each edge (u, v) by an edge (u', v').

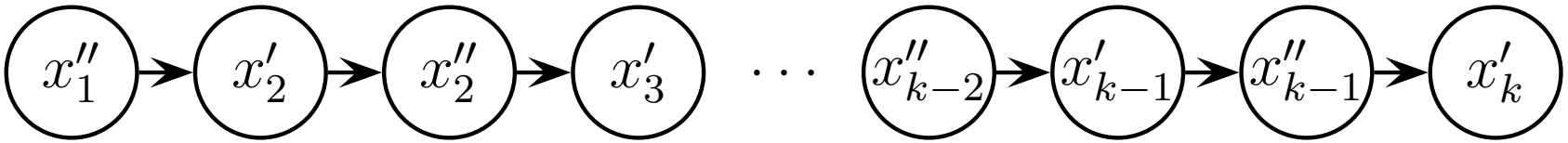


Observation 1:

For every path



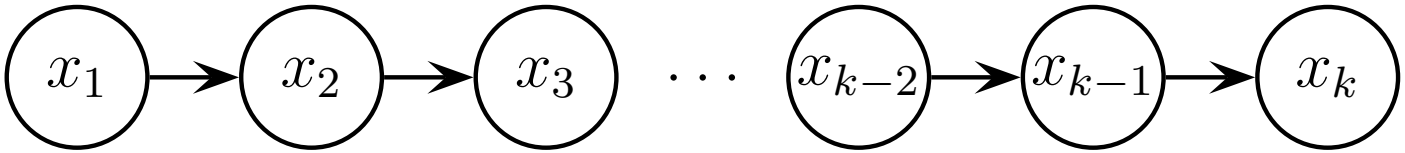
in G, there is a path in G'



and vice versa.

Observation 1:

For every path



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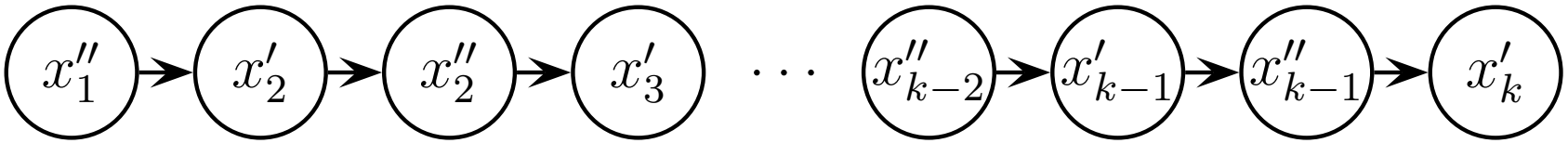

and vice versa.

Observation 2:

There are k internally vertex-disjoint paths $s \to t$ in G
$\Leftrightarrow$ There are k edge-disjoint path $s'' \to t'$ in G'.

Other versions

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- we replace an vertex $\{u, v\}$ by directed edges $(u, v), (v, u)$.

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Theorem

Given two vertices s and t in an undirected or directed graph, k edge or internally vertex-disjoint paths from s to t can be computed in $O(k(m+n))$ time.

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Given two vertices s and t in an undirected or directed graph G, a maximum number edge or internally vertex-disjoint paths from s to t can be computed in $O(n(m+n))$ time.

Proof

There are at most $n-1$ such paths since $\deg(s) \leq n-1$.

Question?

Can a maximum number of disjoint paths from s and t be computed faster.

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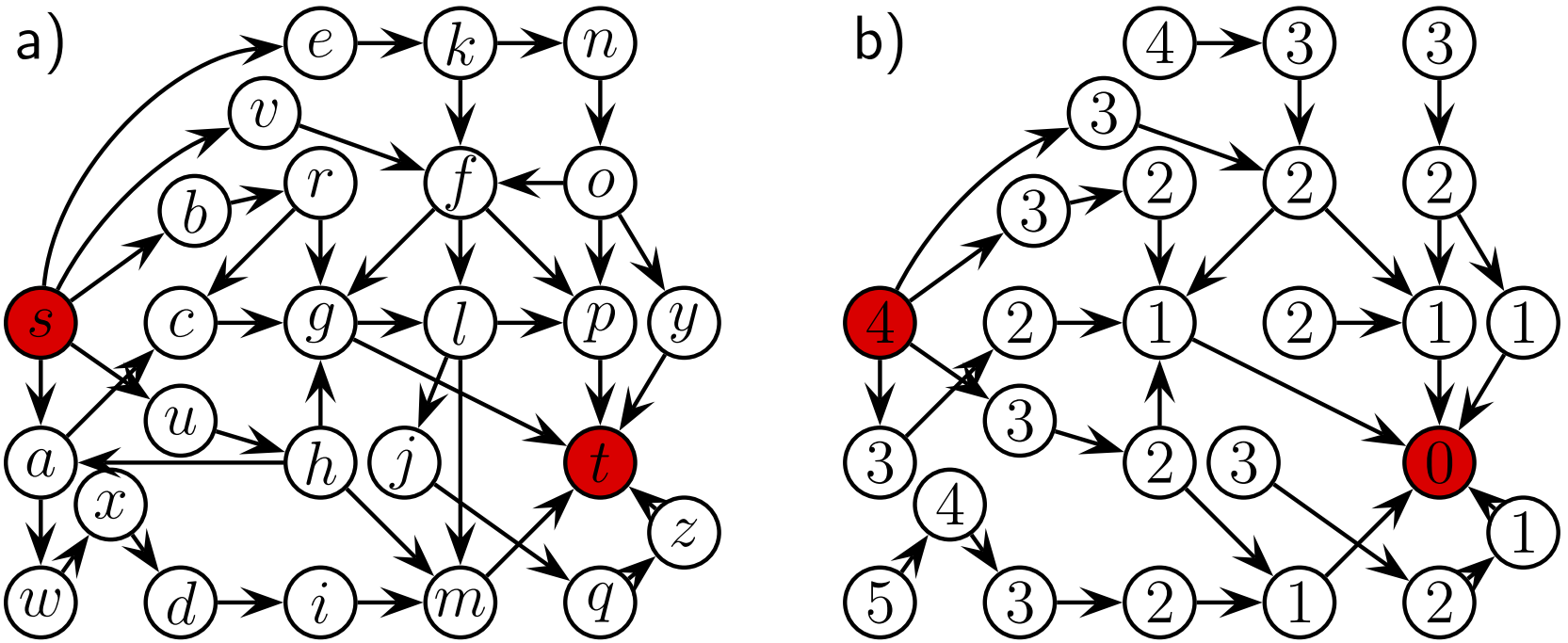
Definition

Given: Vertices s and t in a directed G.
$L(G, s, t)$**:** The graph obtained from G after

(1) removing all edge (v,w) with $dist_G(v, t) \neq dist_G(w, t) + 1$
(2) removing all vertices not reachable from s.
(3) removing all vertices from which t is not reachable.

Construction of $L(G, s, t)$



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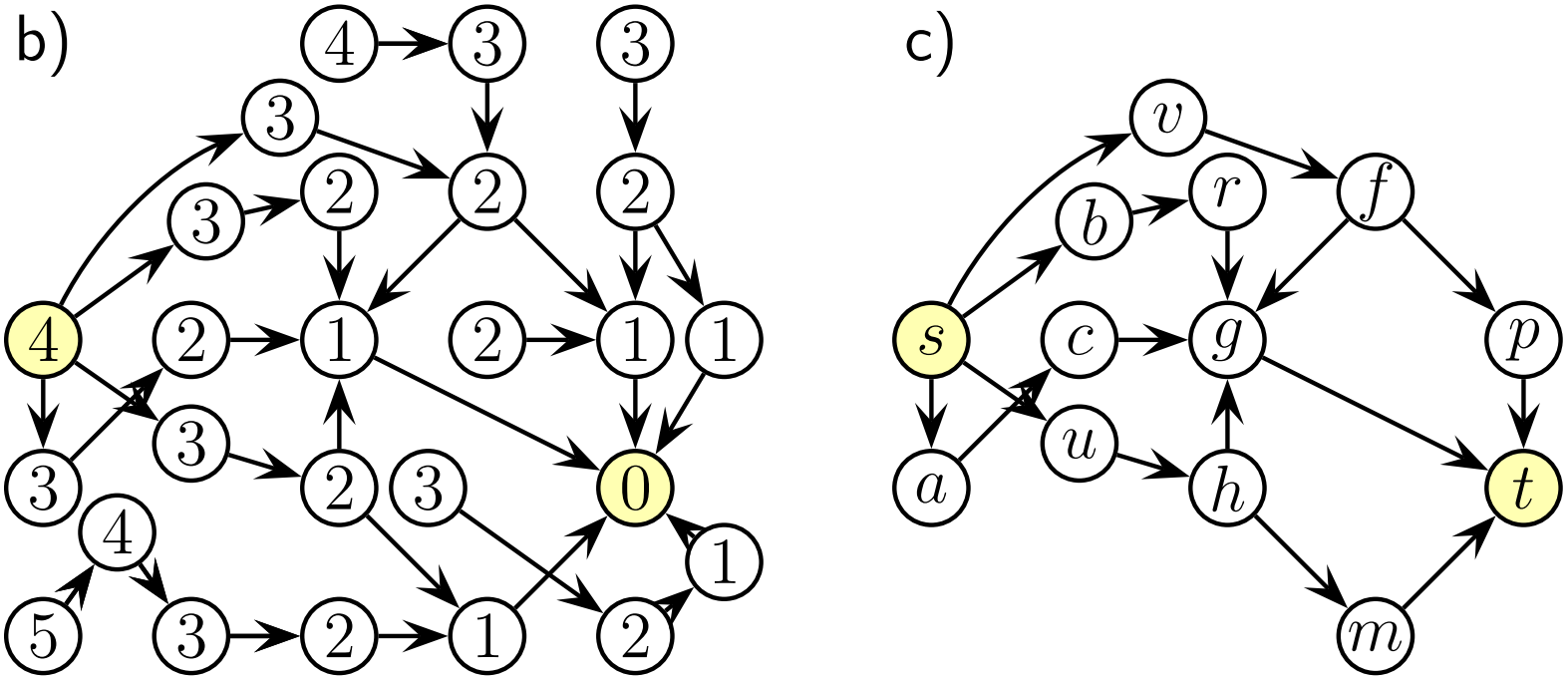


Advantages of a level graph

- we can construct a path p in reverse direction in $O(|p|)$ instead of $O(m+n)$ time:

  Starting with $p : t \to t$ allways add a vertex with a distance one larger than the current start point.
- Time for computing the set T of edges of $L(G, s, t)$ from which t is not reachable in $L(G \setminus p, s, t)$: $O(|T|)$

  (During the construction of p, remove the edges of p from $L(G, s, t)$, after removing an edge (u, v) update $\mathrm{outdeg}(u)$, and for each vertex with $\mathrm{outdeg}(u) = 0$ remove all edges ending in u.)
- Time for computing the set S of edges of $L(G, s, t)$ not reachable from S in $L(G \setminus p, s, t)$: $O(|S|)$.

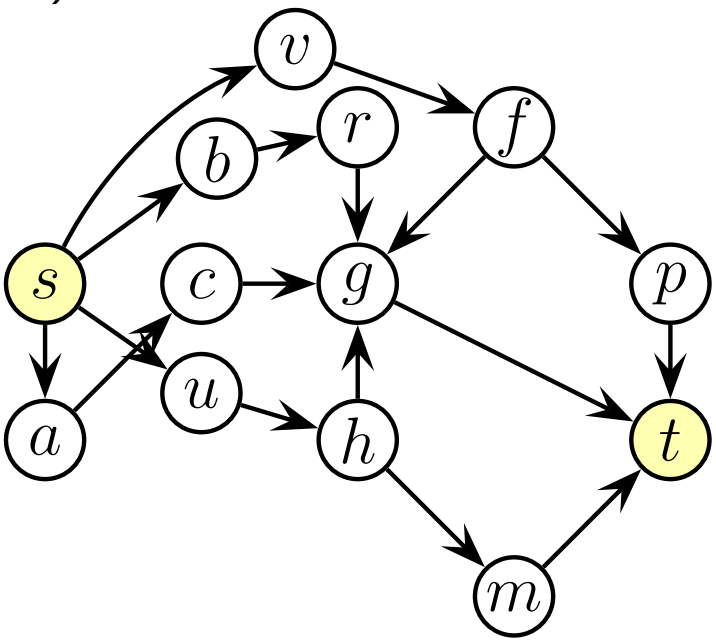
Advantages of a level graph

- $L(G \setminus p, s, t)$ can be constructed in a time linear in the edges of $L(G, s, t) - L(G \setminus p, s, t)$.

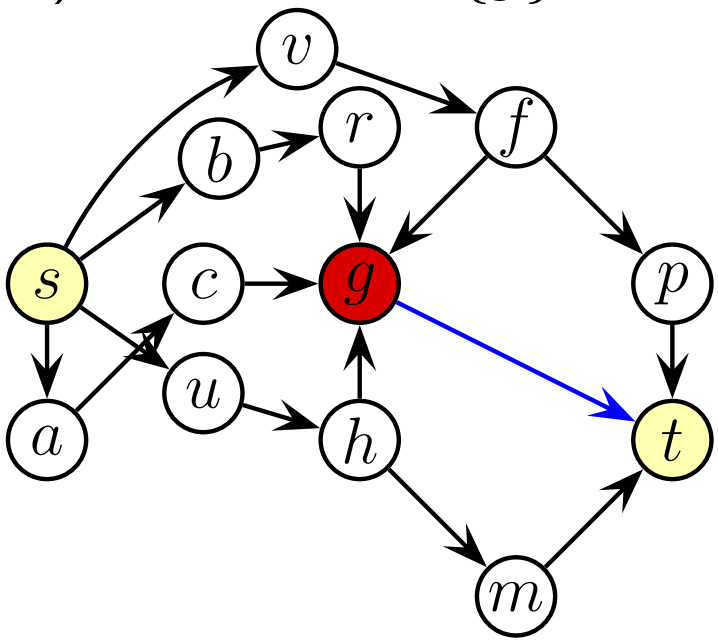
$\Rightarrow$ Applying the algorithm recursively on $L(G \setminus p, s, t)$ a maximal number of edge-disjoint paths in $L(G, s, t)$ can be computed in $O(m + n)$ time.

Update of the level graph

a) $T = \emptyset$



b) $T = T \cup \{g\}$



Update of the level graph

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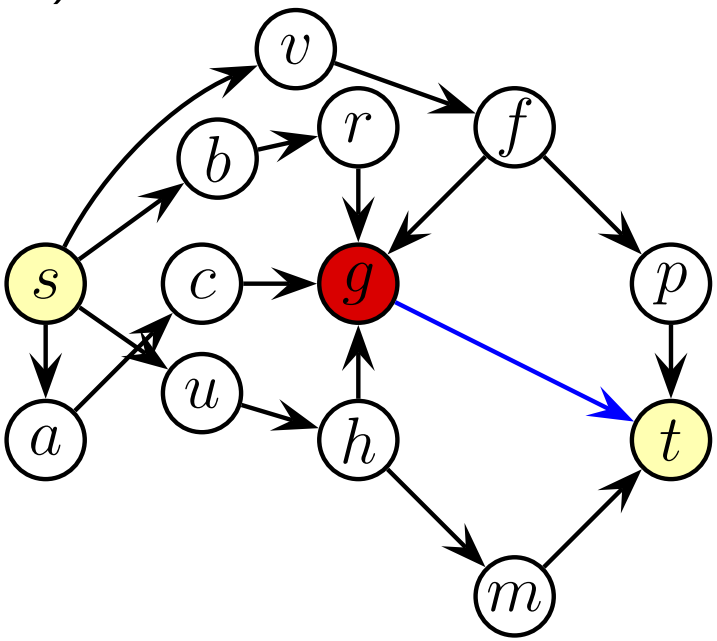
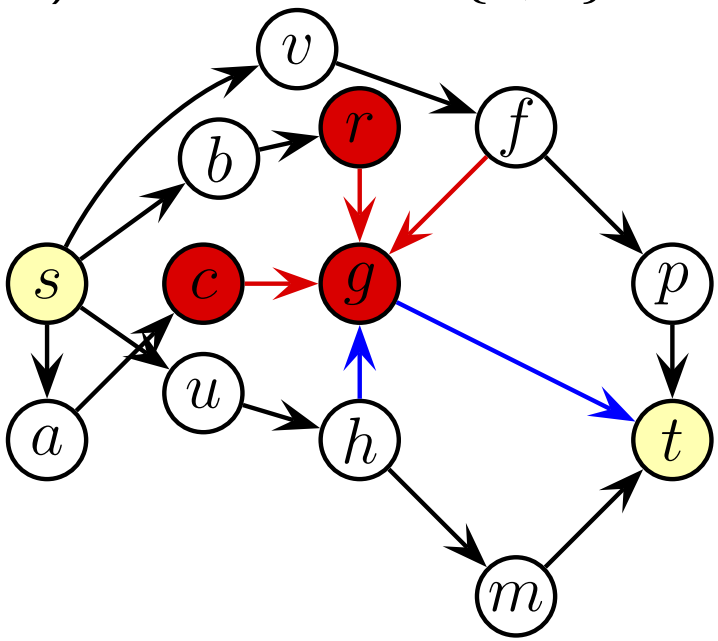


c) $T = T \cup \{c, r\}$



Update of the level graph

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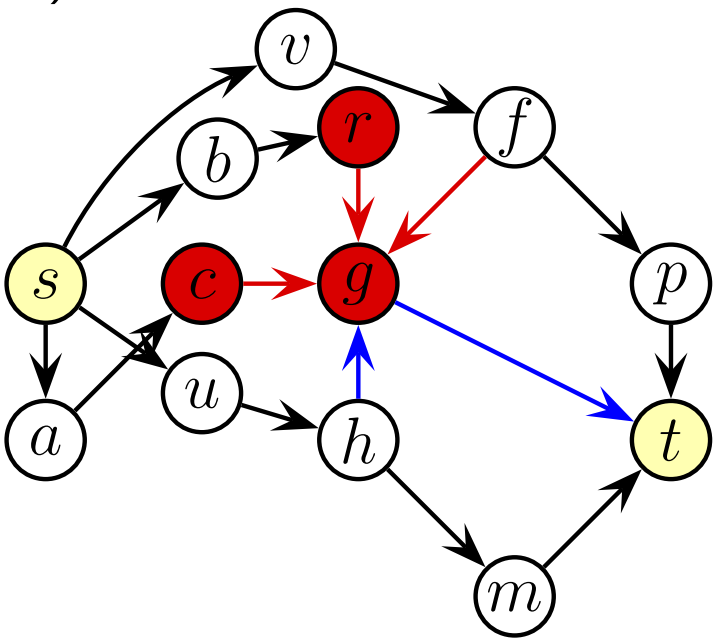
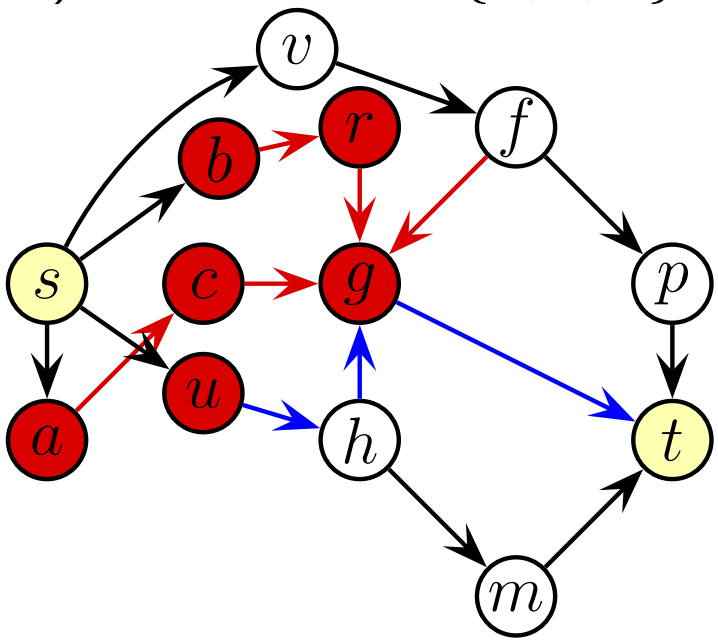
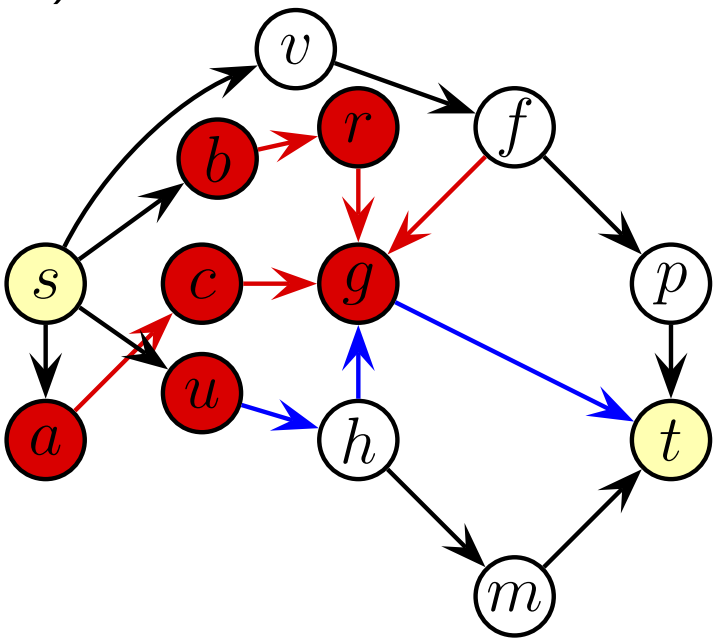


d) $T = T \cup \{a, b, u\}$

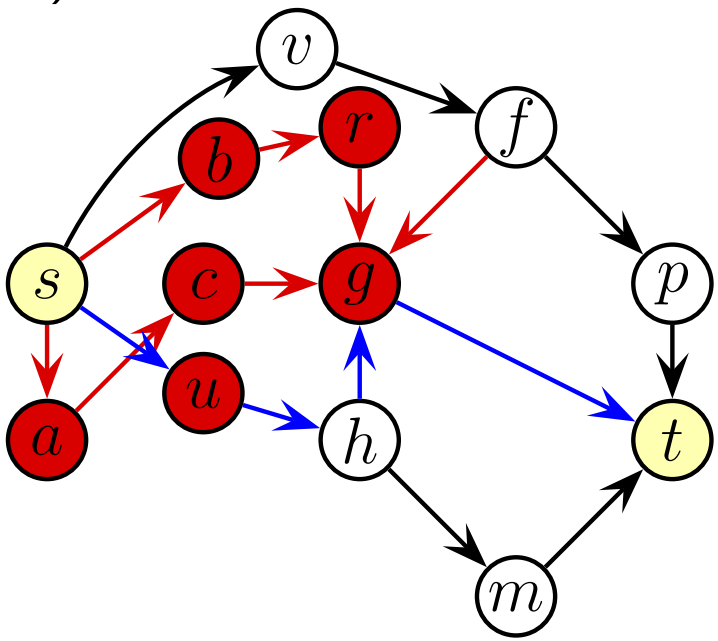


Update of the level graph

d)



e) $T = T \cup \emptyset$



Update of the level graph

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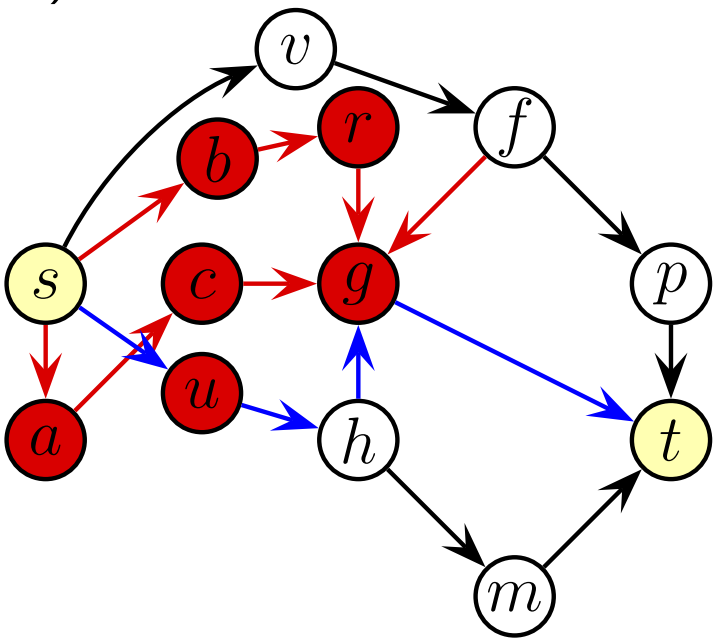
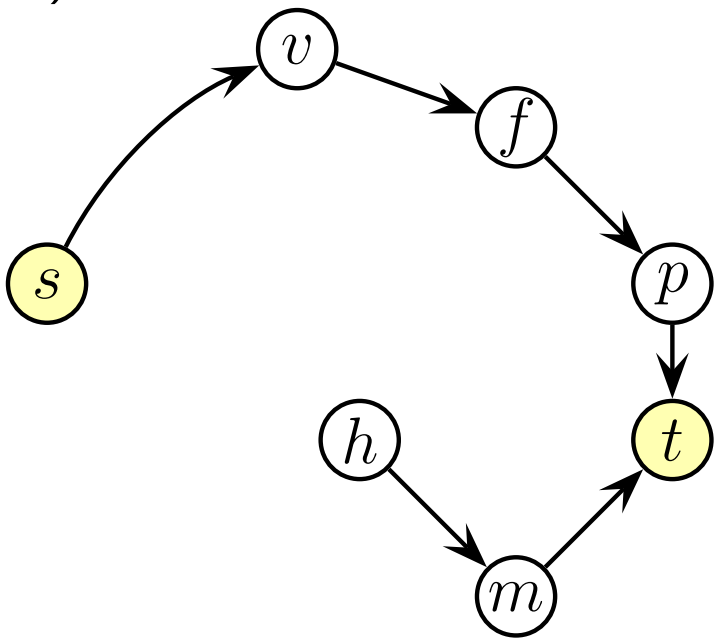


f) $T = T \cup \emptyset$



The algorithm of Dinitz

(1) $R := G$;
(2) **while** t is reachable from s in R
(3) Construct $G' = L(R, s, t)$, $j = 0$
(4) **while** t is reachable from s in G'
(5) $j = j + 1$.
(6) Construct a path $q_j : s \to t$ in $G'$and update G' as the level graph without q_j
(7) **end while**
(8) Replace $p_1, \ldots, p_i, q_1, \ldots, q_j$ by edge-disjoint paths $p'_1, \ldots, p'_{i+j}$ in G.
(9) Rename $p'_1, \ldots, p'_{i+j}$ in $p_1, \ldots, p_{i+j}$.
(10) $i := i + j$
(11) $R := R_{G,p_1,\ldots,p_i}$
(12) **end while**

Analyzing runnig time

- The outer loop divides the algorithms into rounds.
- We will show the distance between s and t in the residual graph R increases after each round.

$\Rightarrow$ After $\sqrt{m}$ rounds the distance is $O(\sqrt{m})$.

$\Rightarrow$ There are at most $O(\sqrt{m})$ edge-disjoint paths in R.

$\Rightarrow$ There are at most $O(\sqrt{m})$ additional paths in G.

$\Rightarrow$ The algorithm stops after $O(\sqrt{m})$ additional rounds.

$\Rightarrow$ The total running time is $O(\sqrt{m}(m+n))$.

Lemma

the distance between s and t in R increases after each round.

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Proof

- Each path $p : s \to t$ in R has $\mathrm{dist}_R(s,t)$ edges.
- R has no edges (v,w) with $dist_R(v,t) > dist_R(w,t) + 1$.

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- Each path $p : s \to t$ in R has $\mathrm{dist}_R(s,t)$ edges.
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- $\mathrm{dist}_R(v,t) = \mathrm{dist}_R(w,t) + 1$ for all edges (v,w) on p.

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$\Rightarrow$ The residual graph R' of the next round has also no edges (v,w) with $\text{dist}_R(v,t) > dist_R(w,t) + 1$.

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- Paths exclusively existing of such edges were removed in the previous round.

$\Rightarrow$ Each path in p' has length $> \text{dist}_R(s,t)$.

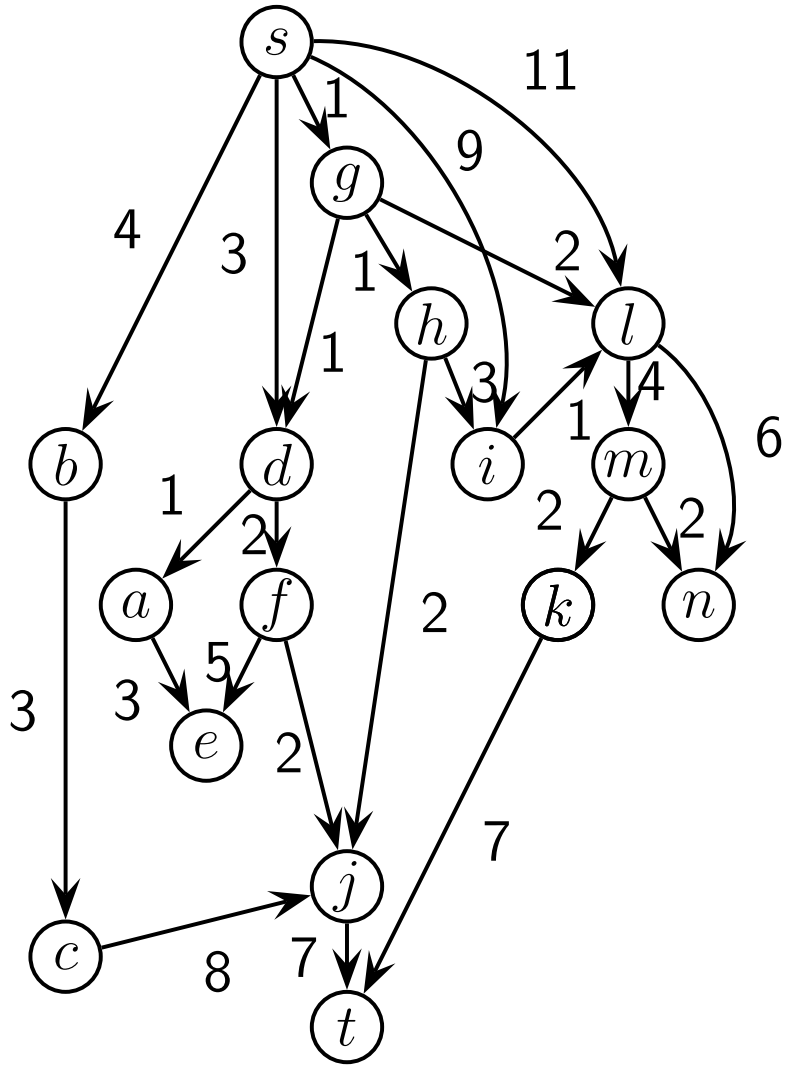
A new problem

Given: vertices s and t in a directed weighted graph, $k \in \mathbb{N}$
Output: k edge-disjoint paths from s to t, such that the sum of the edge weights is minimal.

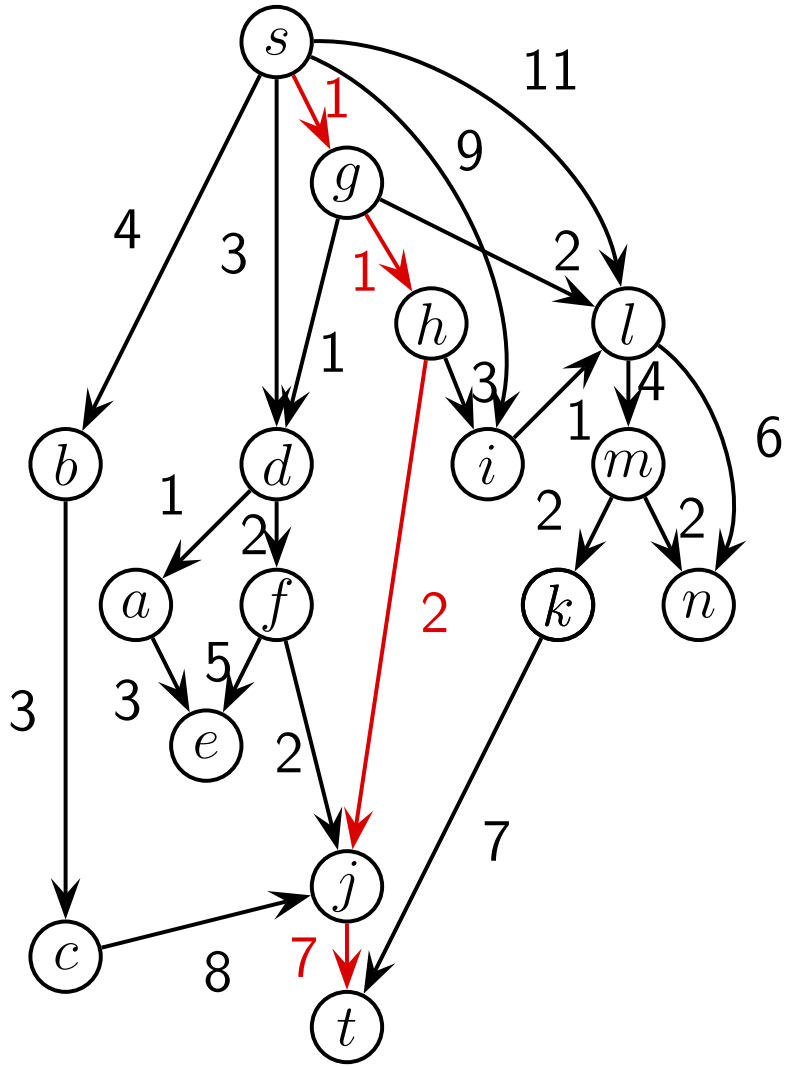
A simple approach

- Let $g(v, w)$ be the weight of (v, w).
- Assume we have constructed edge-disjoint paths $p_1, \ldots, p_i$ of shortest total length.
- Search for a shortest path in $R_{p_1,\ldots,p_k}$, where we assign weight $-g(v, w)$ to a backward edge (w, v).

Example:



Example:



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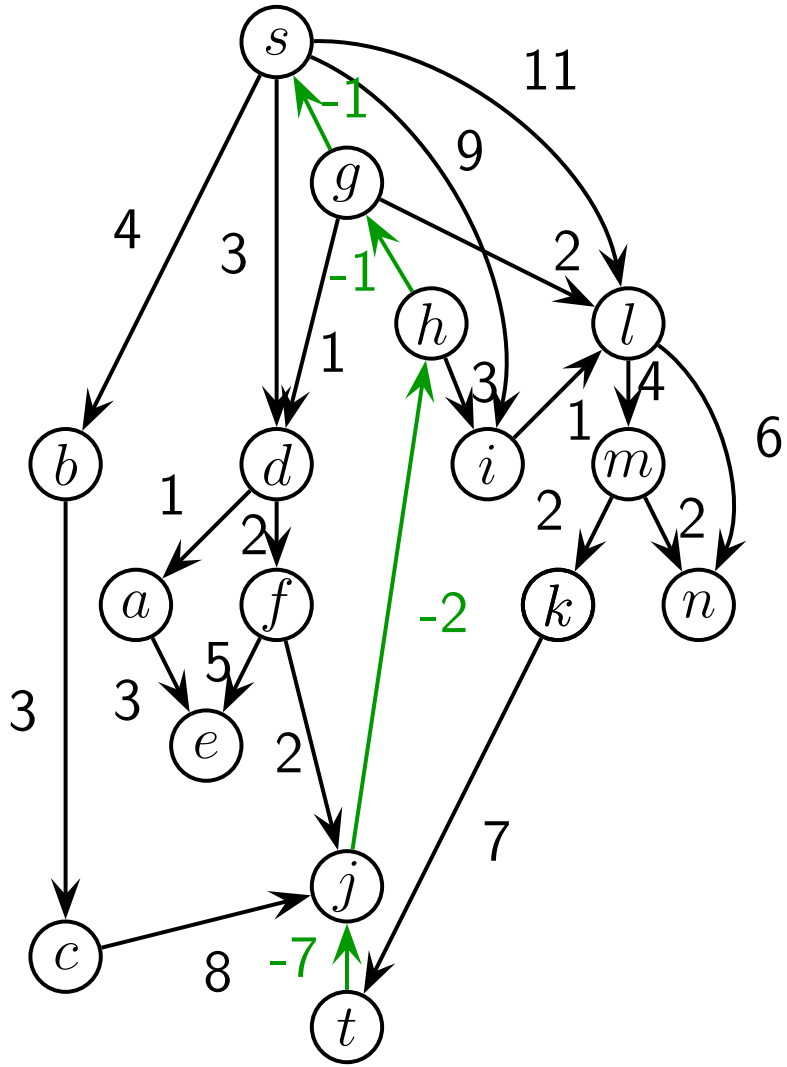

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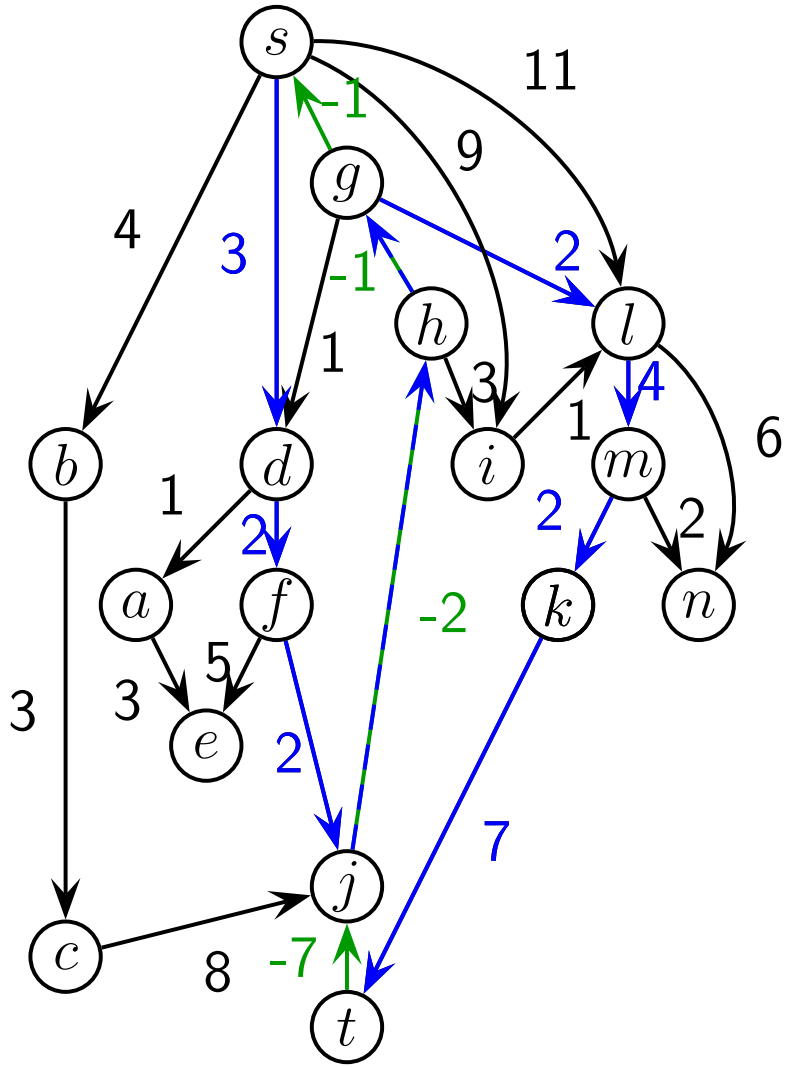

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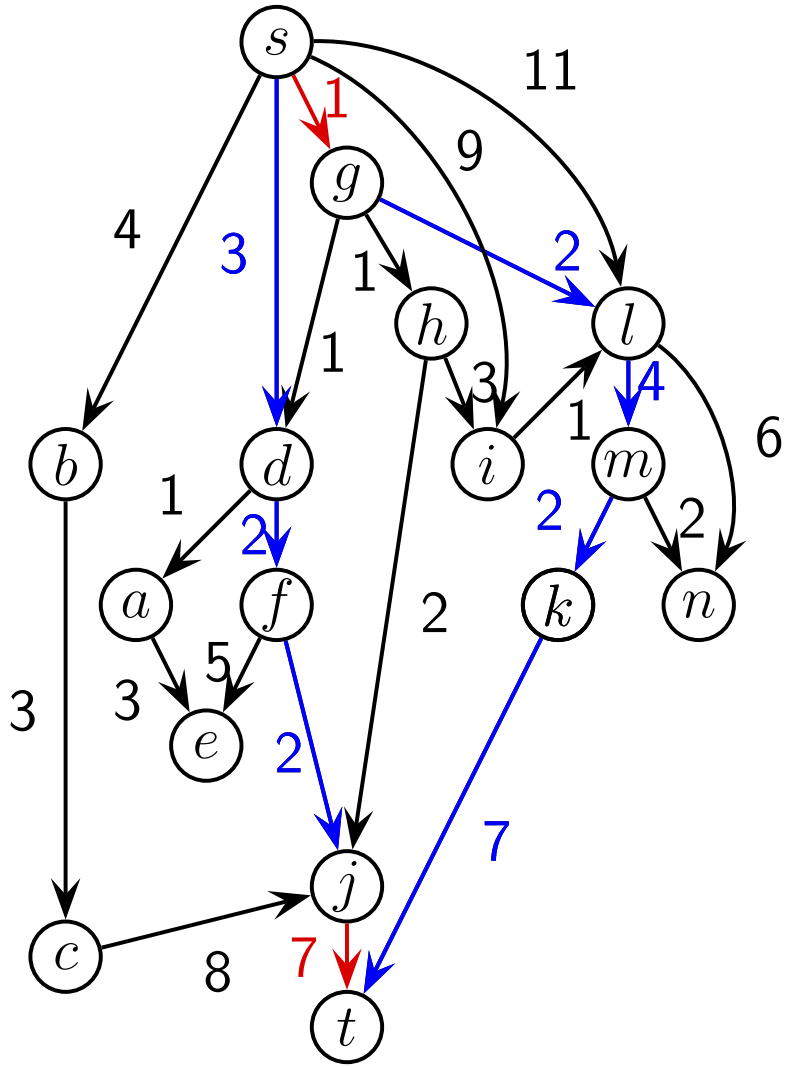
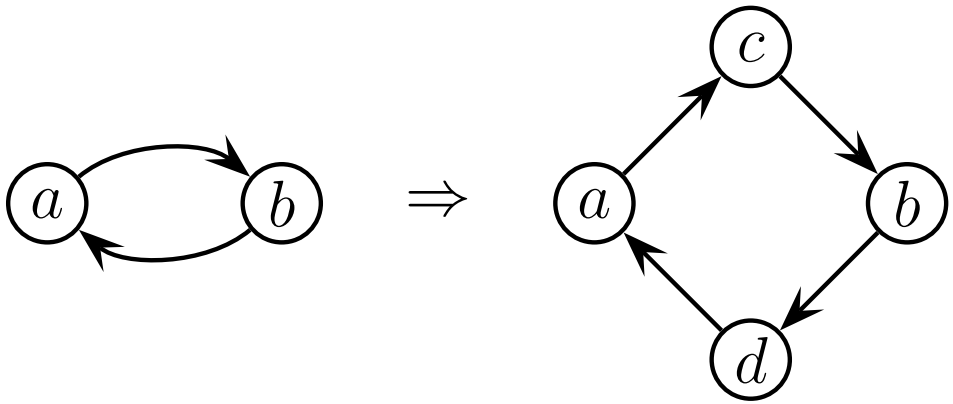


Simplification

We assume that the graph is antisymmetric, i.e.,

$$(v, w) \in E \Rightarrow (w, v) \notin E.$$

Otherwise the following replacement is possible.



Simplification (positive edge weights)

- We replace $g(v,w)$ by $g^*(v,w) := g(v,w) + \text{dist}_g(s,v) - \text{dist}_g(s,w)$.
- For a path $p = ((u_i,v_i))_{1\le i\le k}$ from a vertex u to a vertex v, we have

$$\begin{aligned}&\textstyle\sum_i g^*((u_i,v_i))\\ =&\textstyle\sum_i g((u_i,v_i)) + \sum_{1\le i\le k}\text{dist}_g(s,u_i) - \sum_{1\le i\le k}\text{dist}_g(s,v_i)\\ =&\text{dist}_g(s,u) - \text{dist}_g(s,v) + \textstyle\sum_i g((u_i,v_i)).\end{aligned}$$

$\Rightarrow$ A shortest path $p : u \to v$ with respect to g is also a shortest path with respect to g^* and vice versa.

$\Rightarrow$ We only have to compute shortest with respect to g^*.

Shortest disjoint paths

Advantages of the new edge weights

- All edge weights are positive.
- Edges on a shortest path p have weight 0.
- Edges in R_p have weight 0.

Correctness of the simple approach

The correctness of the simple approach follows from the following lemma.

Lemma

- Let $p_1, \ldots, p_k$ simple edge-disjoint paths from s to t of shortest total length d.
- Let p be a shortest path from s to t in G.
- Then there are edge-disjoint paths $q_1 \ldots, q_{k-1}$ from s to t in R_p of total length $\leq d$.

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- Then there are edge-disjoint paths $q_1 \ldots, q_k$ from s to t in R_p of total length $\leq d$.

Lemma

- Let $p_1, \ldots, p_k$ simple edge-disjoint paths from s to t of shortest total length d.
- Let p be a shortest path from s to t in G.
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I_u should mean

- $q_1, \ldots, q_k$ are edge-disjoint paths on $(V_G, E_G \cup E_{R_{G,p}})$ of length $\leq d$ except that edges of $p[u,t]$ are used ≤ 2 times.
- The backward edges of $q_1, \ldots, q_k$ are part of $p[s,u]$.
- $q_k[s,u] = p[s,u]$.

I_s holds for $q_i = p_i$ for $i < k$, $q_k = p$. We want that I_t holds.

Proof (Sketch)

- Asumme that I_u holds for a vertex u on p.
- Then we show that it also holds for a vertex after u on p.
- After a finite number of steps I_t holds.

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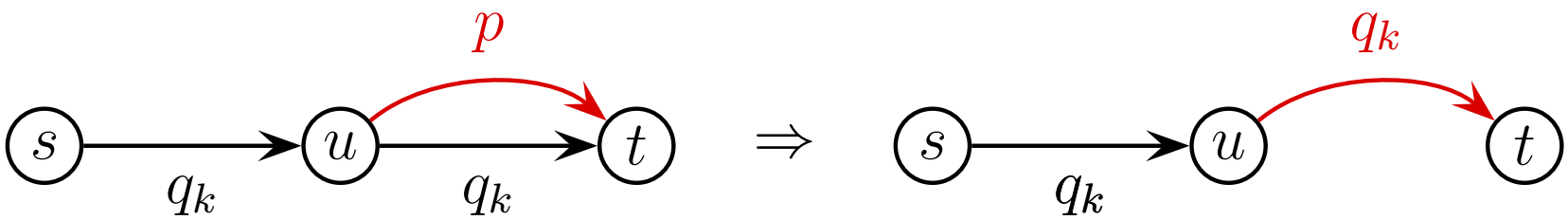
- Asumme that I_u holds for a vertex u on p.
- Let (v, w) be the next edge on $p[u, t]$ that is part of $q_1, \ldots, q_k$.

Case 1 ((v, w) does not exist)

- Replace $q_k[u, t]$ by $p[u, t]$.



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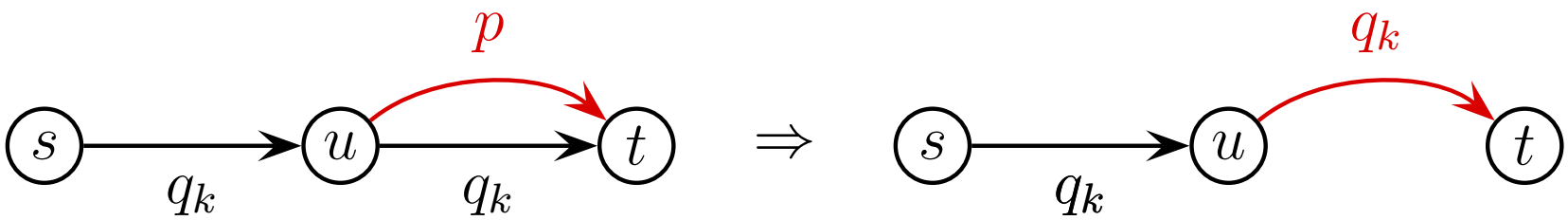

Case 1 ((v, w) does not exist)

- Replace $q_k[u, t]$ by $p[u, t]$.
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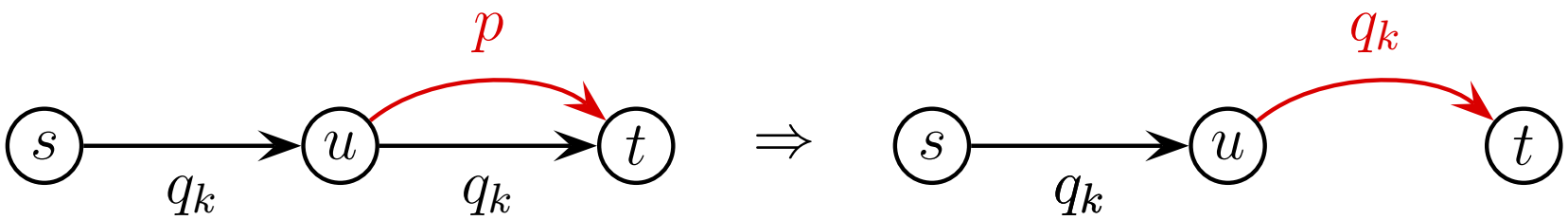
- Assumption: $q_k[u, t]$ was shorter than $p[u, t]$



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- Replace $q_k[u, t]$ by $p[u, t]$.
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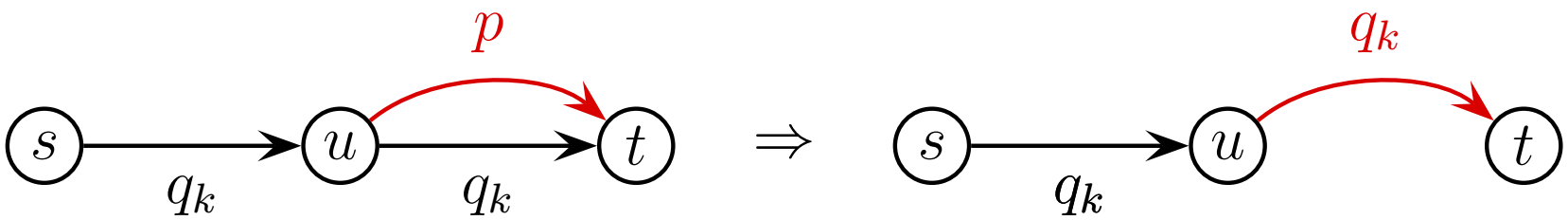
$\Rightarrow$ q_k uses a backward edge.



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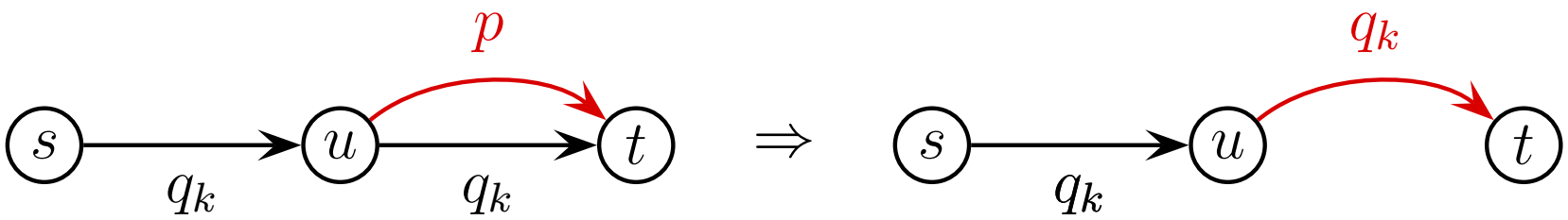

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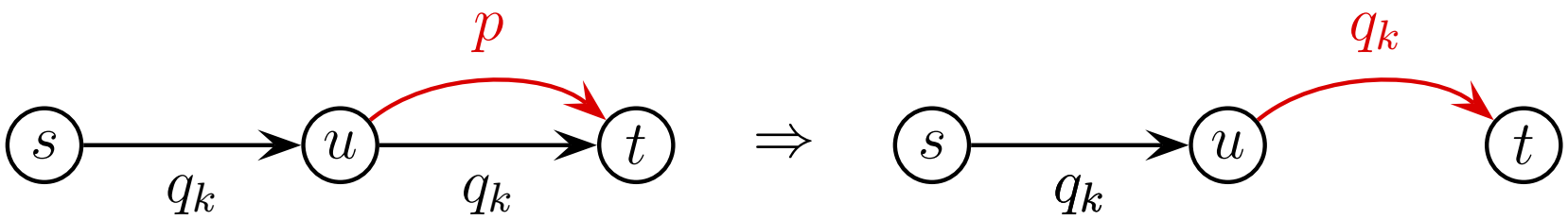

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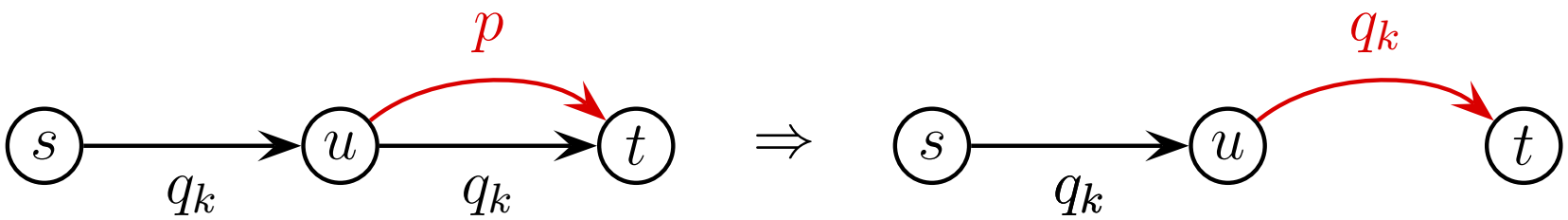
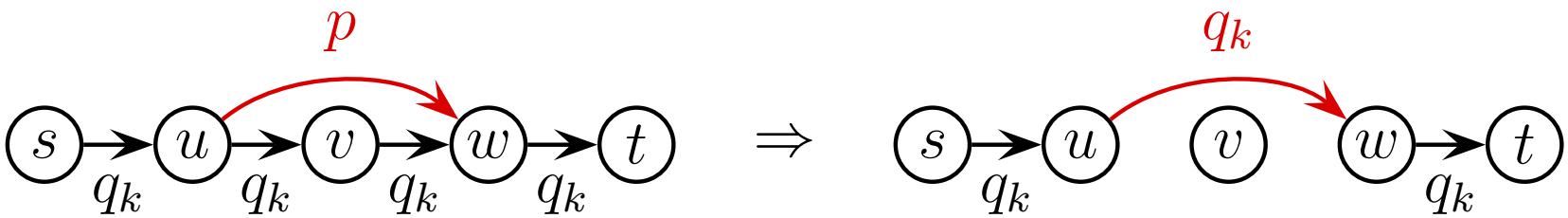
$\Rightarrow$ I_t holds after the replacement.



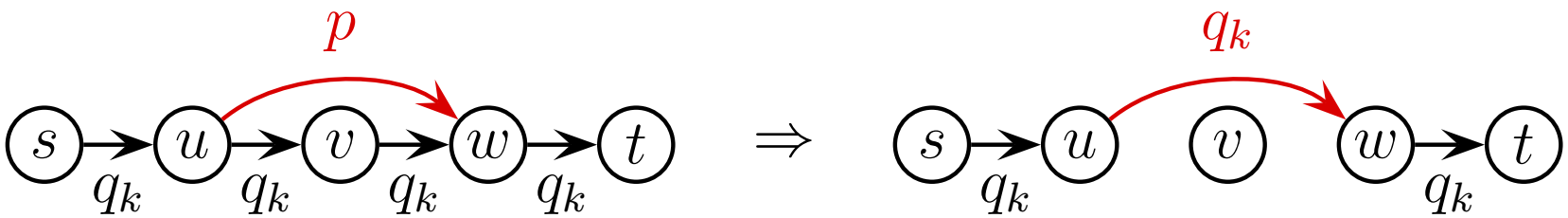
Case 2 ((v, w) appears on q_k)

- Replace $q_k[u, w]$ to $p[u, w]$



Case 2 ((v, w) appears on q_k)

- Replace $q_k[u, w]$ to $p[u, w]$
- Shorten the resulting paths if it is not simple.



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- Replace $q_k[u, w]$ to $p[u, w]$
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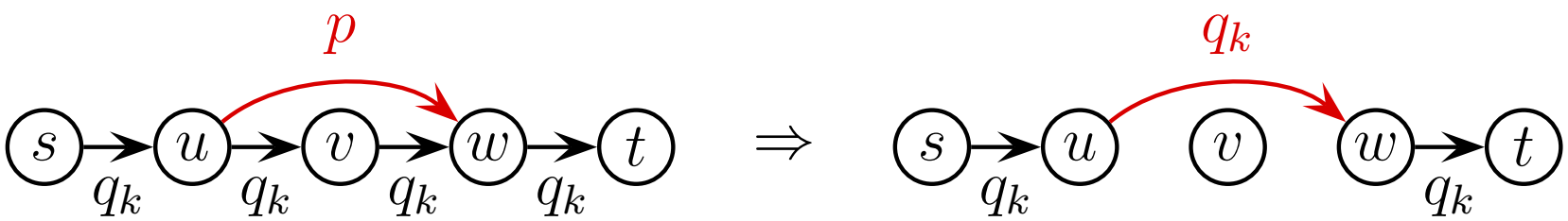

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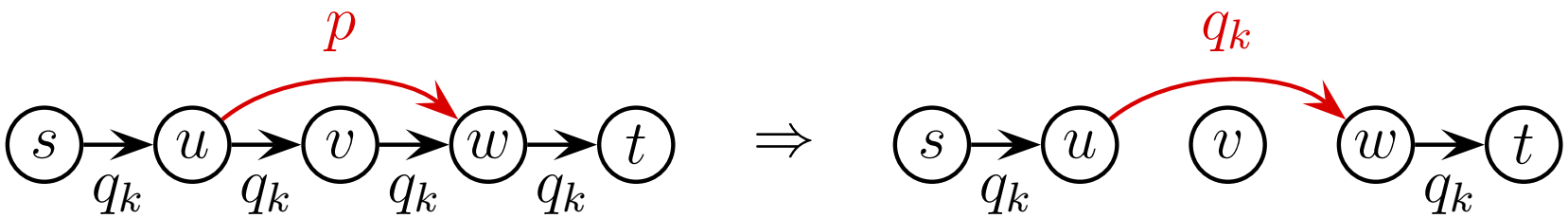
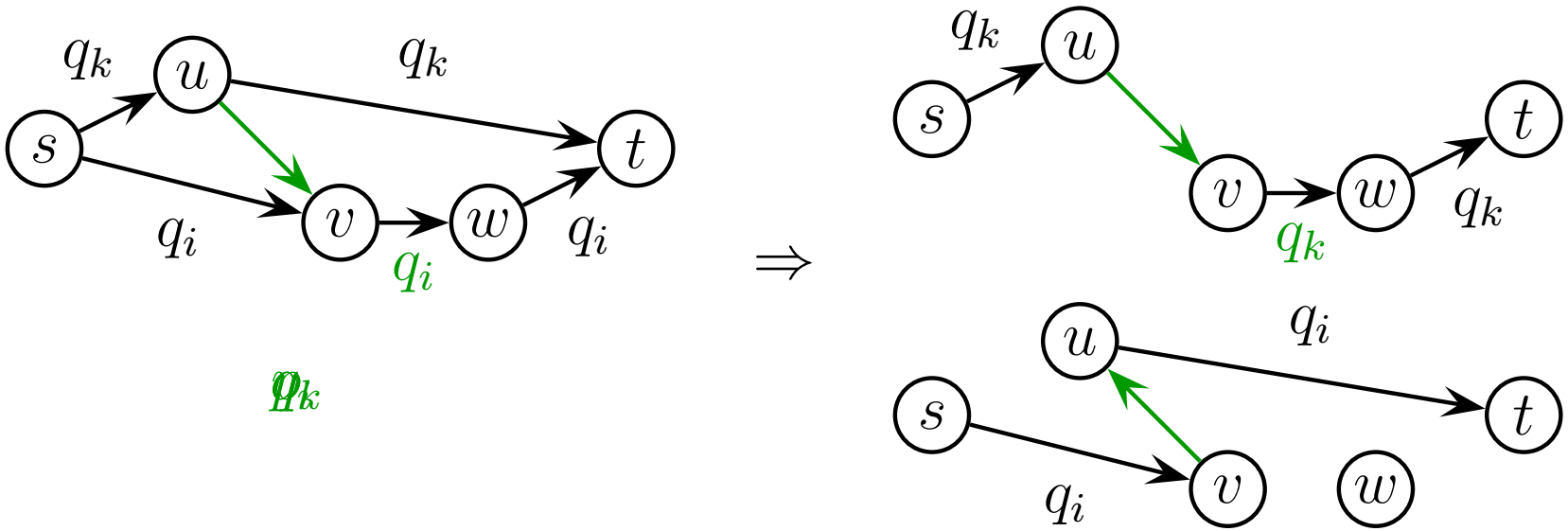
$\Rightarrow$ $I_{w'}$ holds for a vertex after u on p.



Case 3 ((v, w) appears on p_i for $i < k - 1$)

- Replace q_k by $q_k[s, u] \circ p[u, v] \circ q_i[v, t]$ and q_i by $q_i[s, v] \circ p[v, u] \circ q_k[u, t]$.



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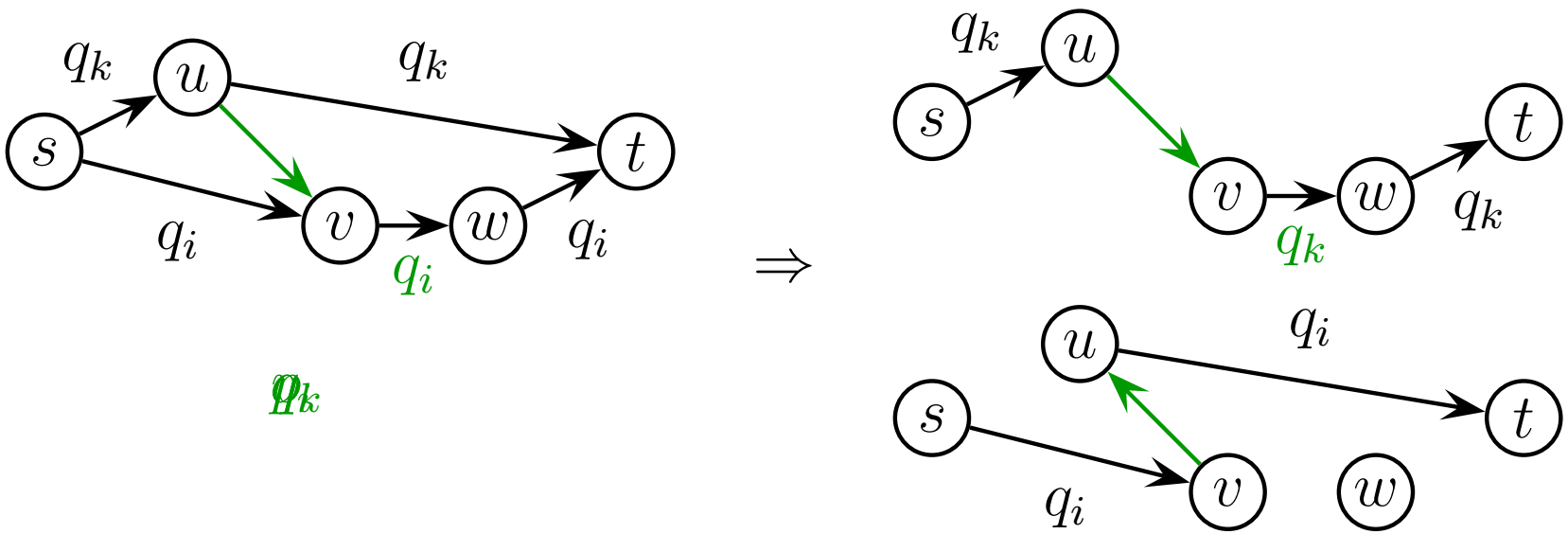

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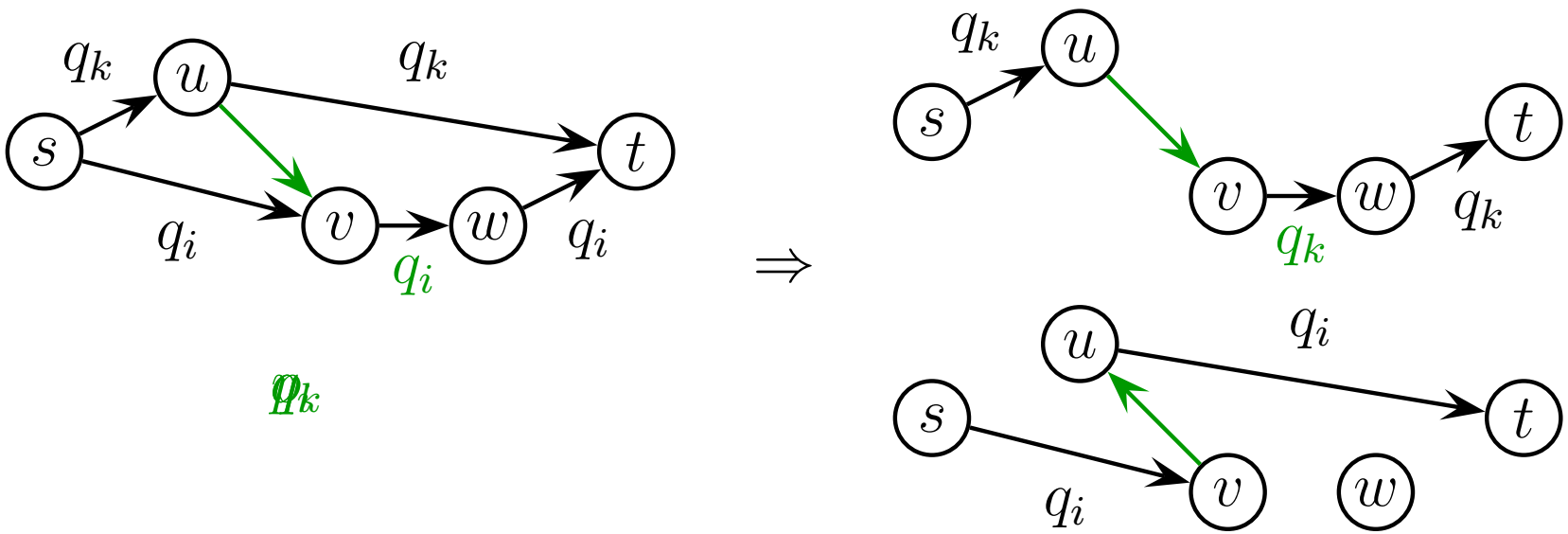
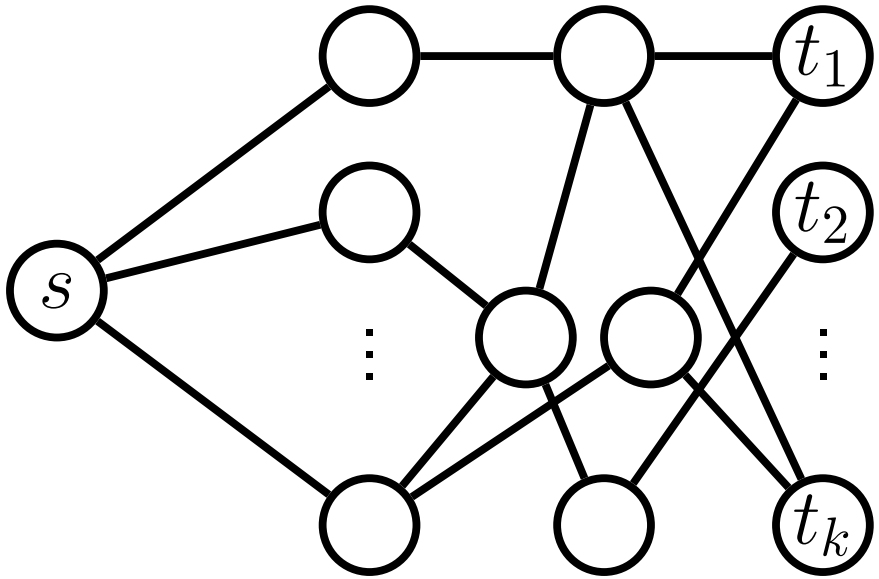


Multiple sources and targets

single source/mutiple target

Given: Vertices $s, t_1, \ldots, t_k$ in a graph.
Output: Disjoint paths $p_1 : s \to t_1, \ldots, p_k : s \to t_k$.
Solution: Reduction to single source/single target version.



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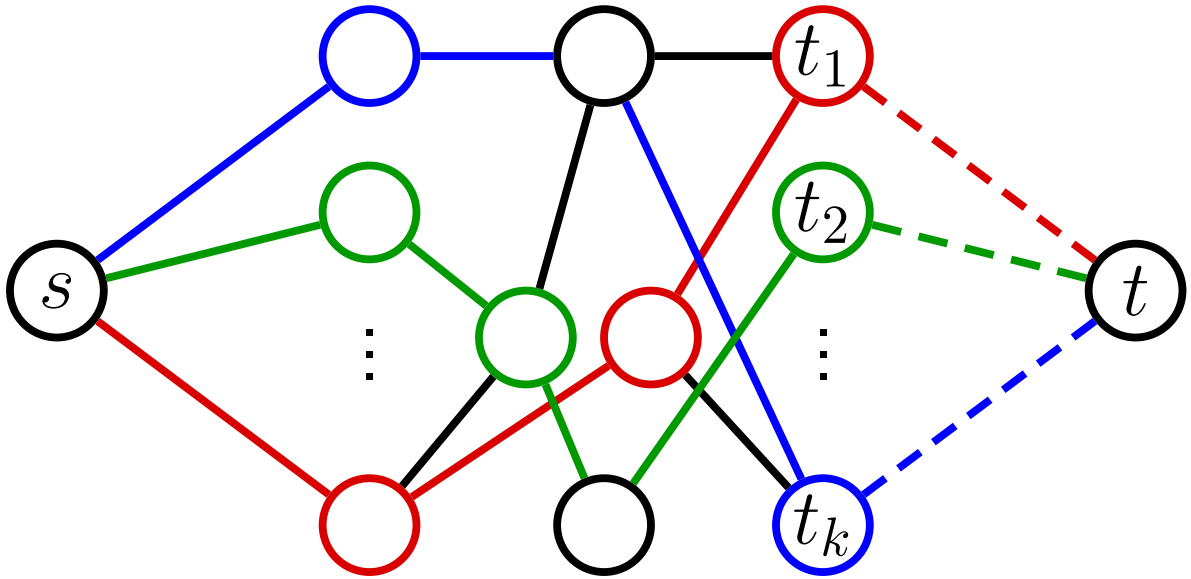

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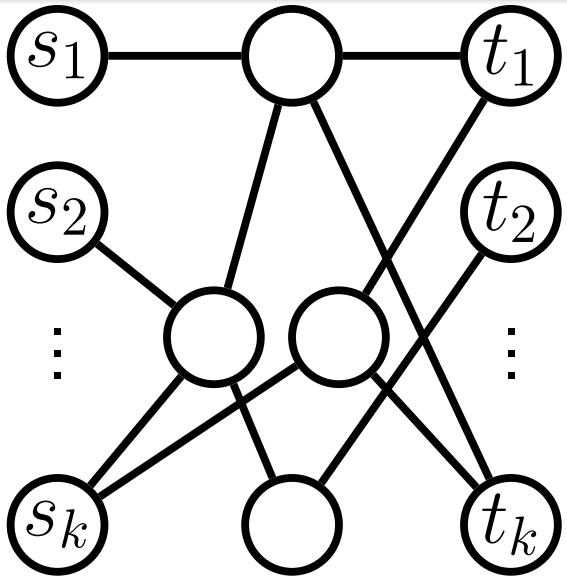

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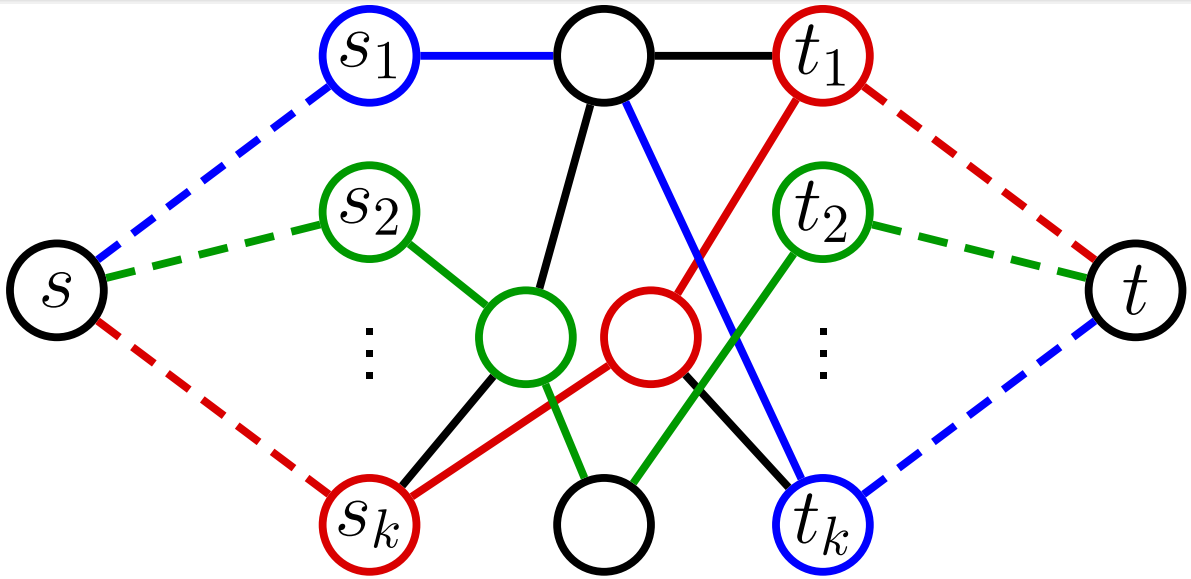

The k-VDPP

The (k-)disjoint path-problem (k-)VDPP

Given: Vertex pairs $(s_1, t_1), \ldots, (s_k, t_k)$ in a graph.
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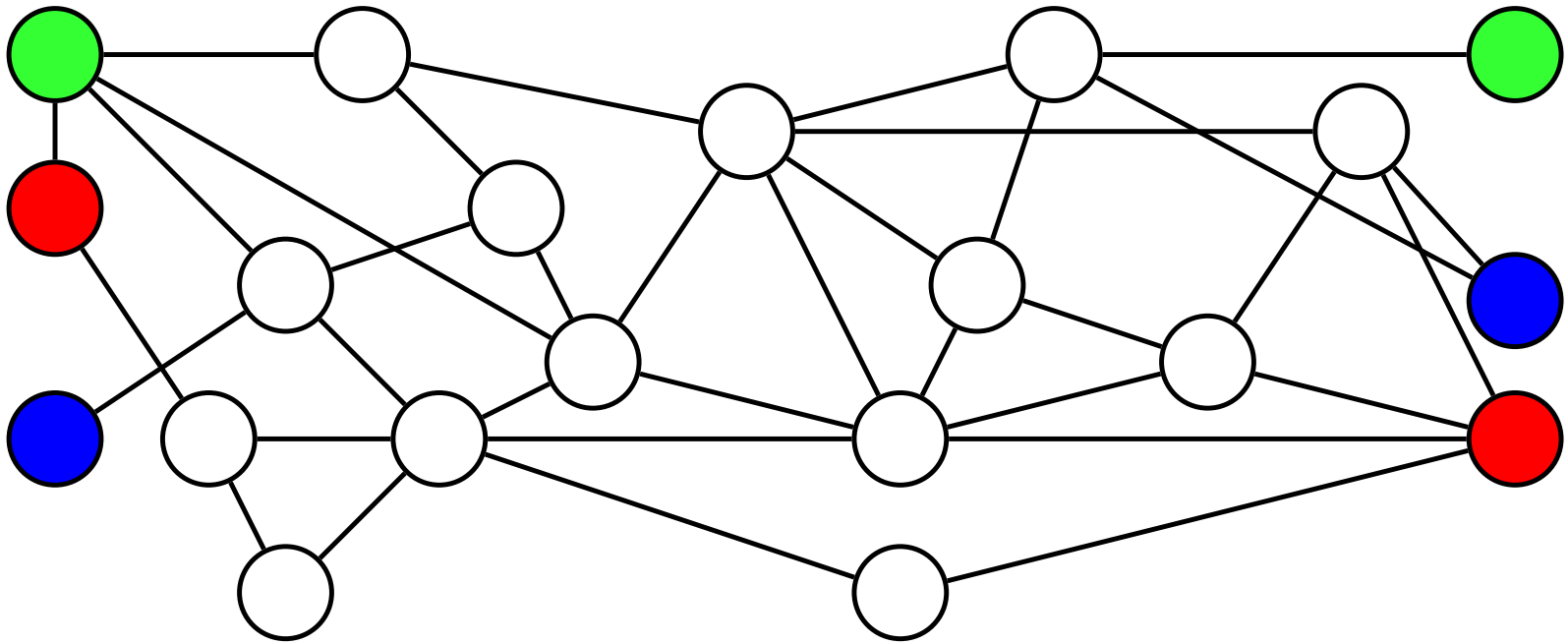

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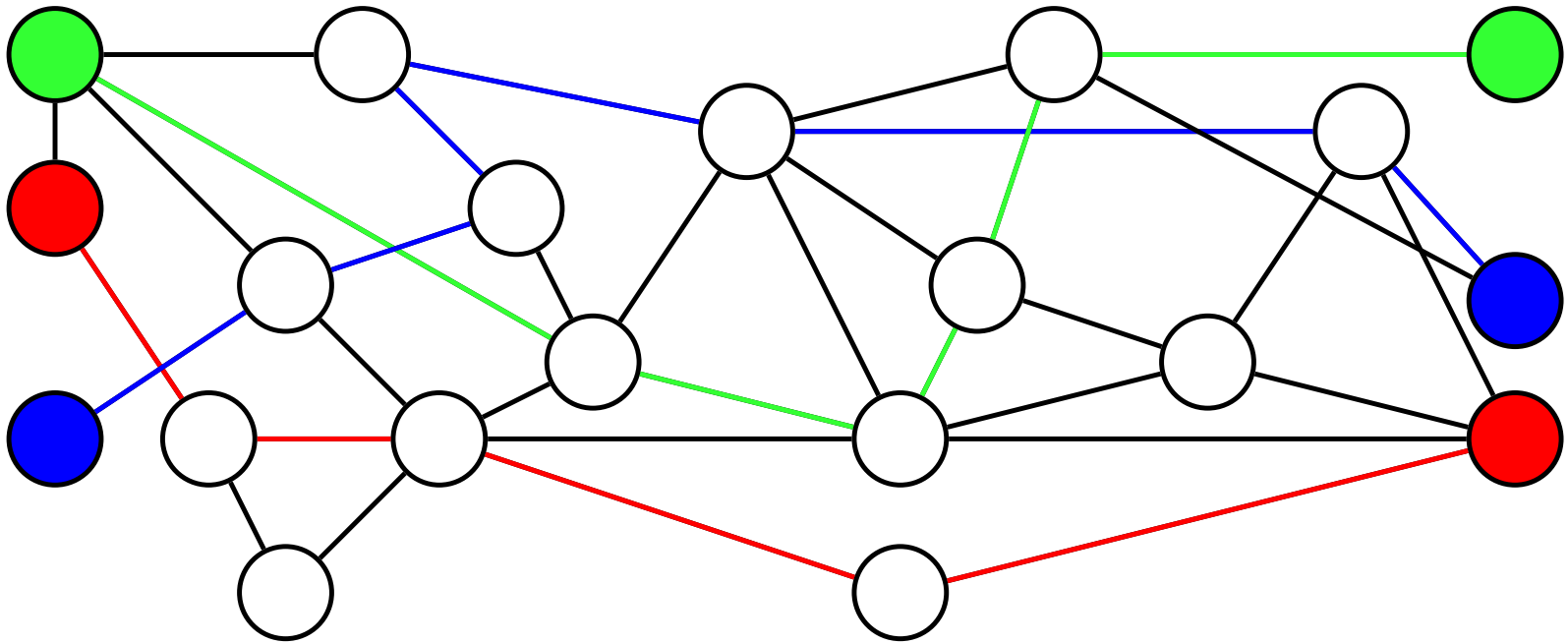
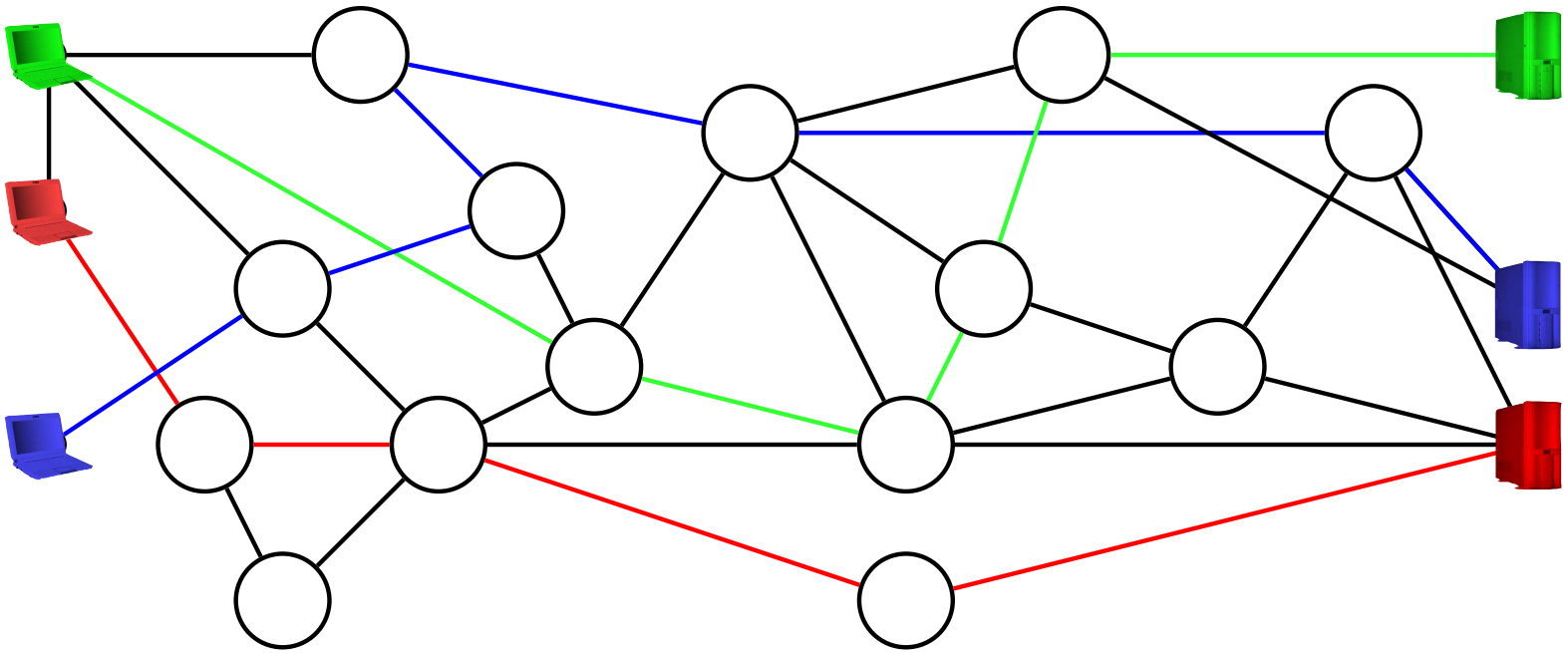
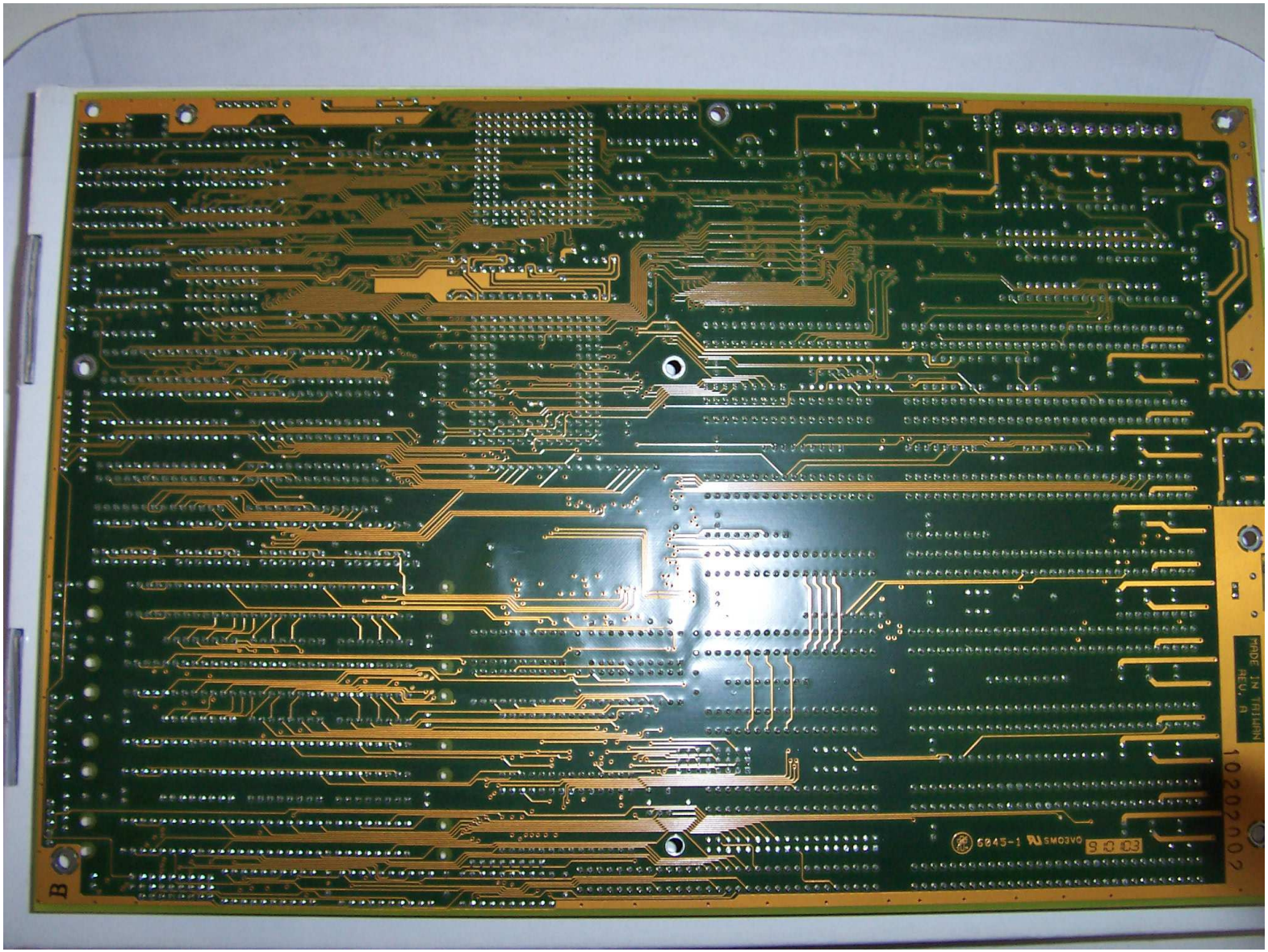


Applications: Simple Computer Netwerks



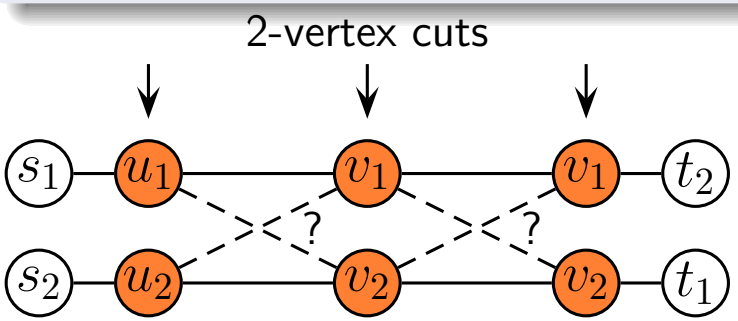
Applications: Computer Hardware



Solving the 2-VDPP (Shiloach's algorithm)

Step 1: Remove all small vertex-cuts (Sketch)

- Reduce the problem to one connected component



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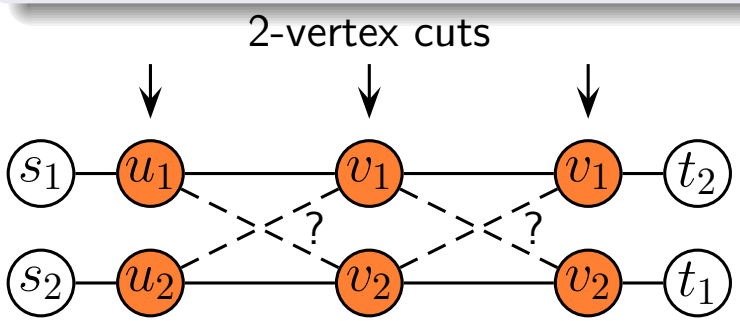

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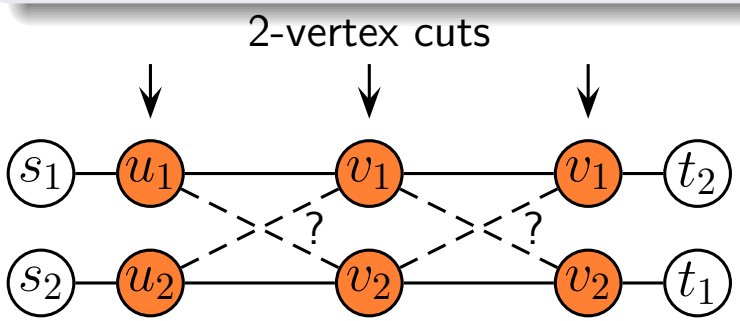

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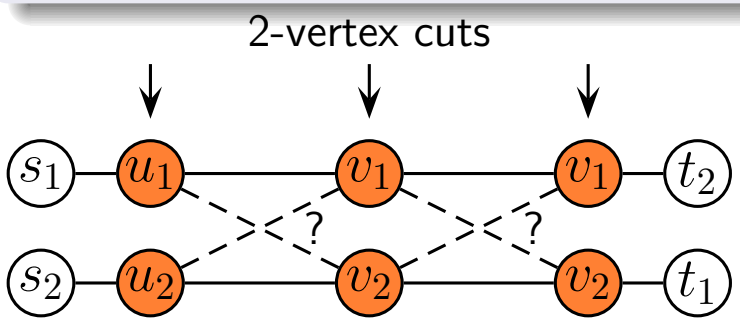

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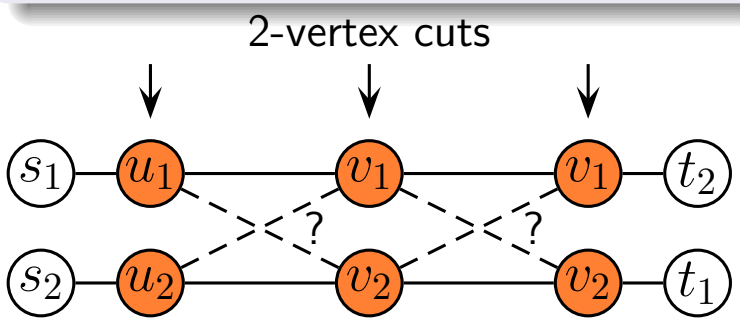

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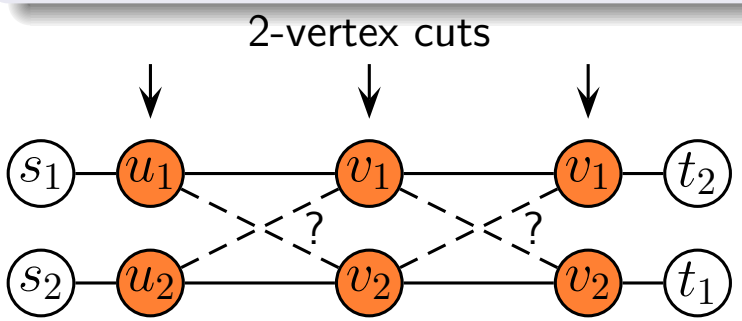

Solving the 2-VDPP (Shiloach's algorithm)

Step 2: Test if G is planar

A graph is *planar* if it can be drawn in the plane without any crossing edges.

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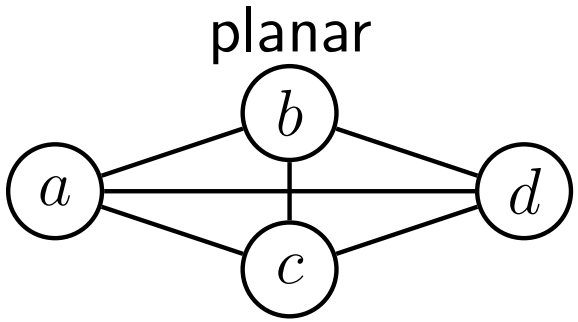

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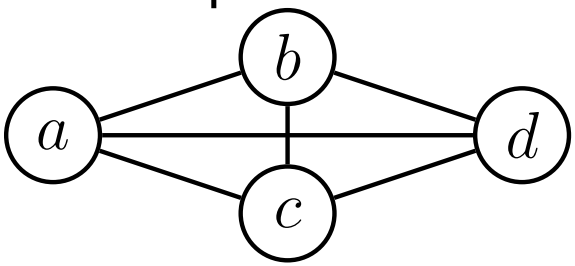
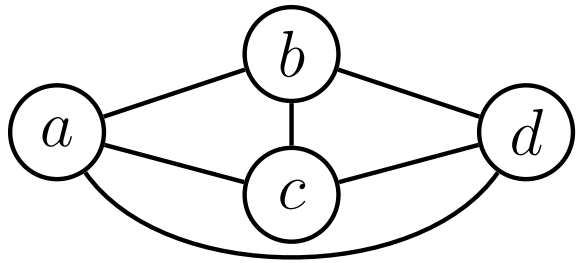


since it can be also drawn as

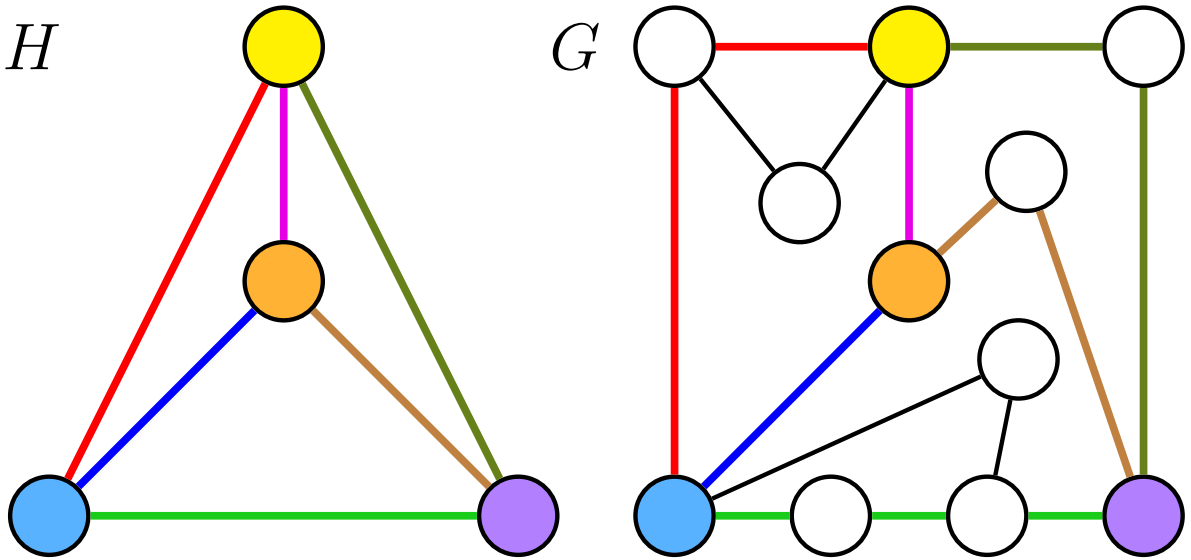


Solving the 2-VDPP (Shiloach's algorithm)

A *subdivision* H

of a graph $G = (V, E)$ is a graph (V', E') for which there are injective mappings $\varphi_1 : V' \to V$ $\varphi_2 : E' \to \mathcal{P}$ such that

- $\mathcal{P}$ is a set of internally vertex-disjoint paths in G.
- $\varphi_2((u, v)) : \varphi_1(u) \to \varphi_1(v)$.

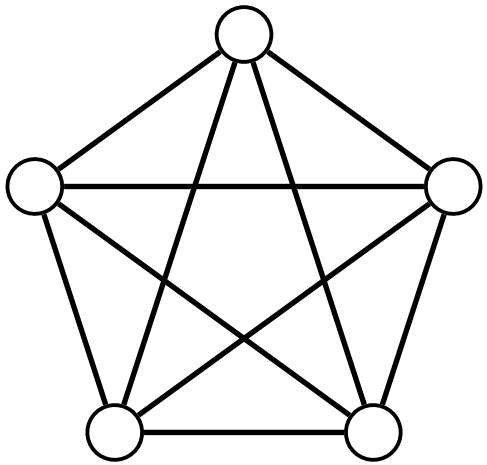


Solving the 2-VDPP (Shiloach's algorithm)

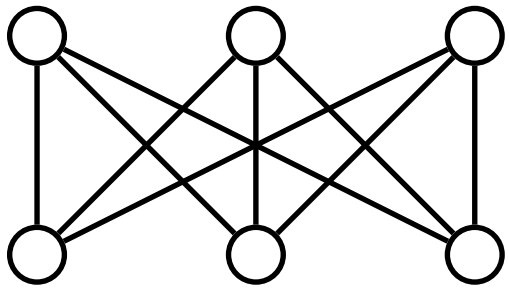
Theorem of Kuratowski

A graph is planar if does not contain a subgraph being a subdivion of the K_5 or $K_{3,3}$.

K_5



$K_{3,3}$



Solving the 2-VDPP (Shiloach's algorithm)

Step 2: Test if G is planar

Use one of the well known $O(m+n)$-time algorithms to compute a subgraph of G being a subdivion of a K_5 of $K_{3,3}$ if it exists.

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Step 3: If G is planar

Solve the problem as shown on the following slides.

Solving the 2-VDPP on a planar triconnected graph

Case 1

s_1 and t_1 are on the boundary of a common face F and each of the two boundary paths contains one of s_2 and t_2

Solving the 2-VDPP on a planar triconnected graph

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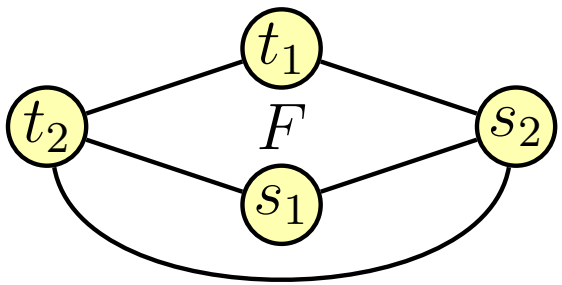
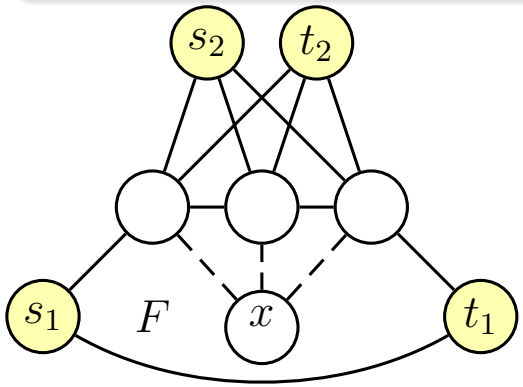
- Output that the instance is not solvable.



Solving the 2-VDPP on a planar triconnected graph

Case 2

s_1 and t_1 are on the boundary of a common face F and one of the two boundary paths p does contain neither s_2 nor t_2.

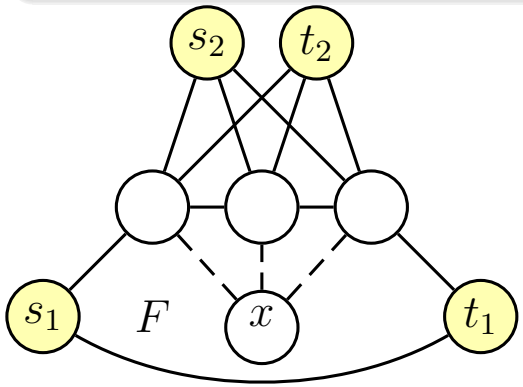


Solving the 2-VDPP on a planar triconnected graph

Case 2

s_1 and t_1 are on the boundary of a common face F and one of the two boundary paths p does contain neither s_2 nor t_2.

- Let q_1, q_2, q_3 disjoint paths from s_2 to t_2.
- Return p, q with $q \in \{q_1, q_2, q_3\}$ not visiting a vertex of p.
- Such a path must exist: otherwise there is a subdivision of a $K_{3,3}$ after adding a new vertex x into F:



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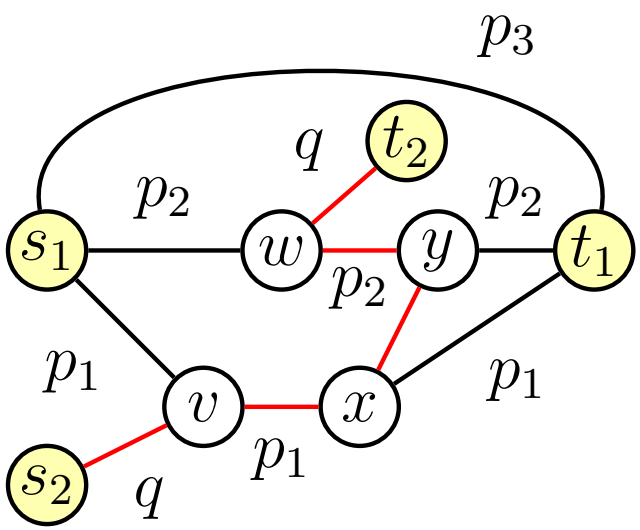

Solving the 2-VDPP on a planar triconnected graph

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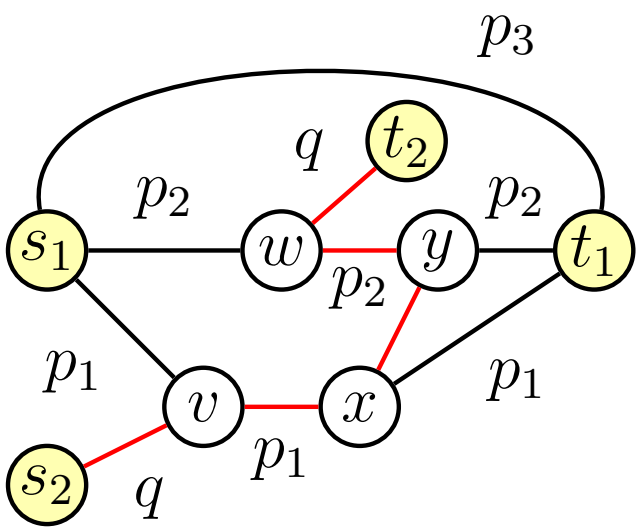
s_1 and t_1 are not on the boundary of a common face F.

Solution

One can always find a solution.



Solving the 2-VDPP (Shiloach's algorithm)

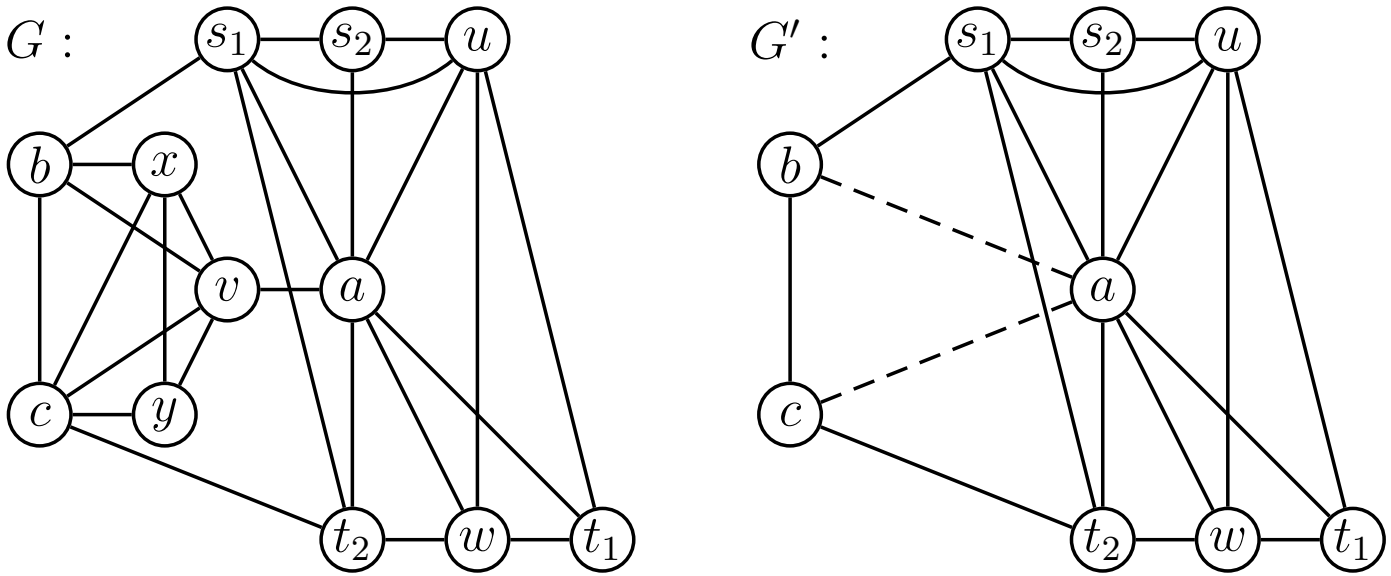
Step 4: If a subdivion of a K_5 is found

- If $\{s_1, s_2, t_1, t_2\}$ is separated by a 3-vertex cut S from at least one vertex v of the K_5:

  -Remove the connected component containing v.

  -Insert edges between all vertex pairs in S.

  -Solve the 2-VDPP on the new instance.



Solving the 2-VDPP (Shiloach's algorithm)

Step 5: If a subdivion of a K_5 is found

- If $\{s_1, s_2, t_1, t_2\}$ are not separated by a 3-vertex cut S from at least one vertex v of the K_5:

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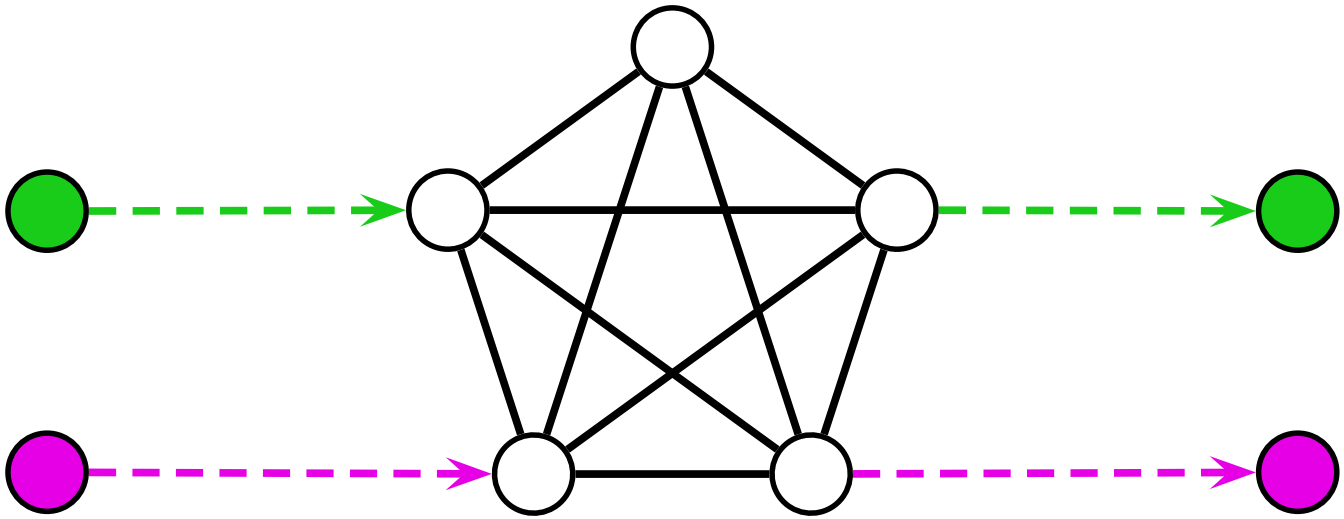

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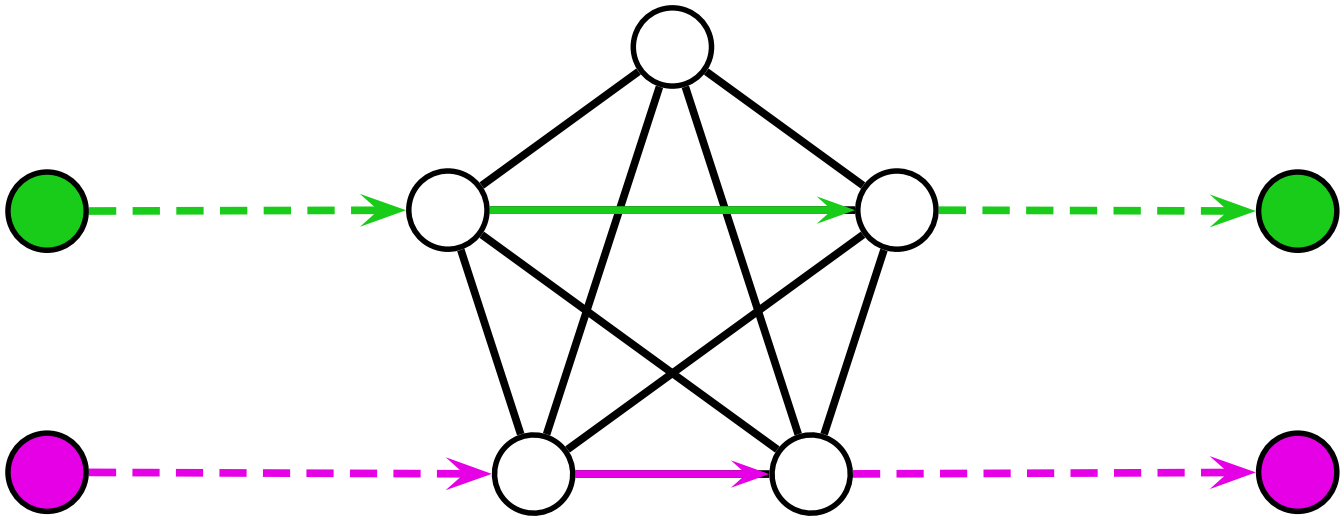

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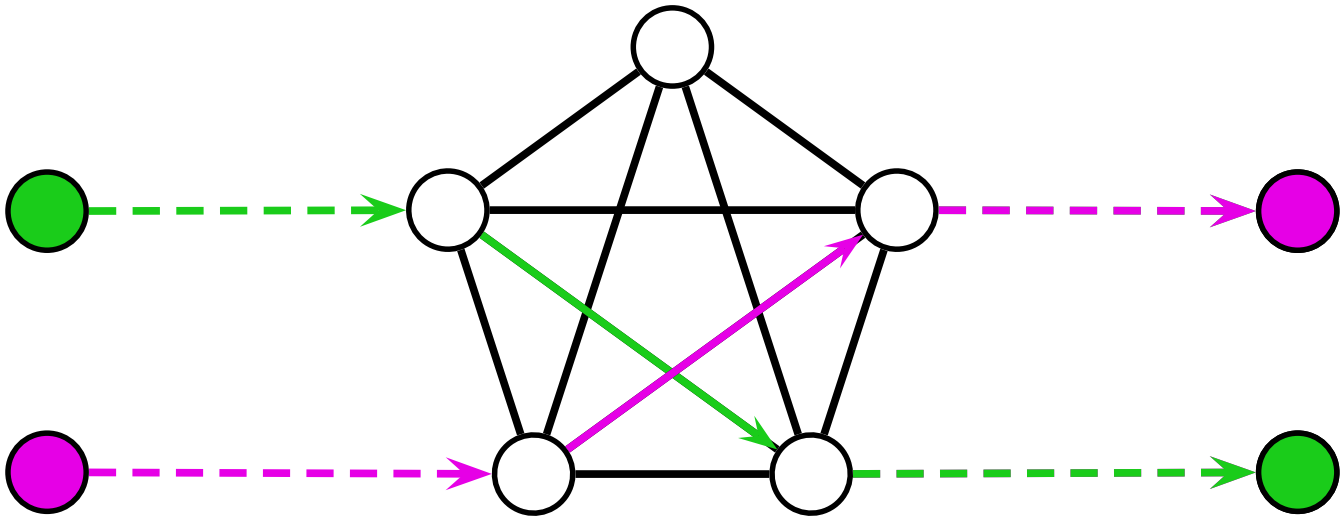

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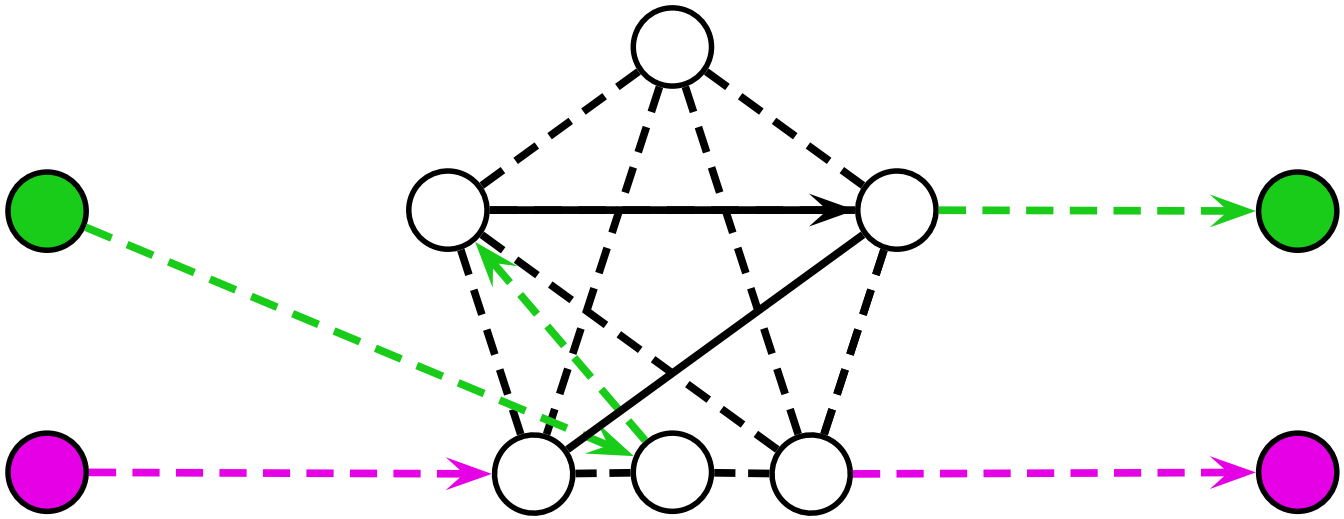

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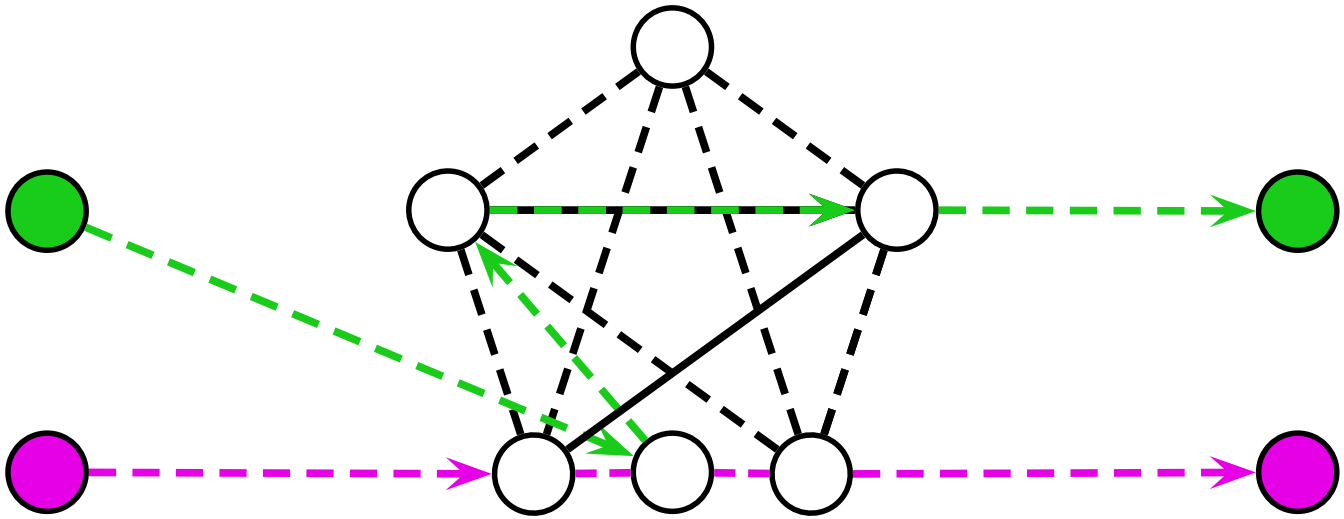

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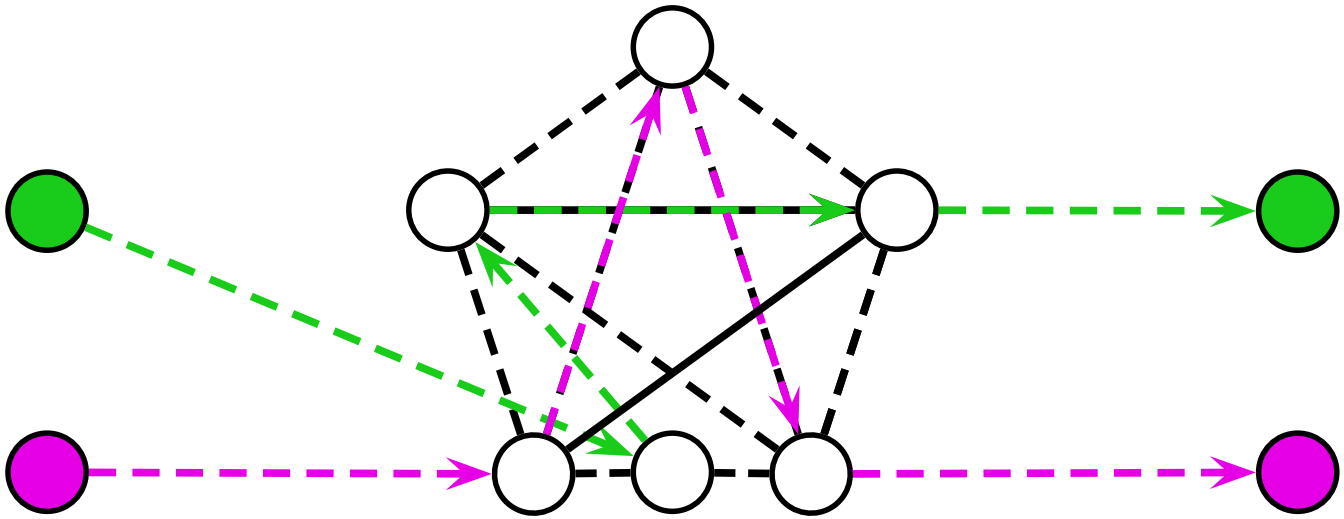

Solving the 2-VDPP (Shiloach's algorithm)

Step 6: If a subdivion of a $K_{3,3}$ is found

- Similar to the case of a K_5.

Known results

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The k-DPP on undirected graphs can be solved in polynomial time for all fixed $k \in \mathbb{N}$.

Fortune, Hopcroft and Wyllie (1980)

The k-DPP on directed graphs is $\mathcal{NP}$-hard for all $k \geq 2$.

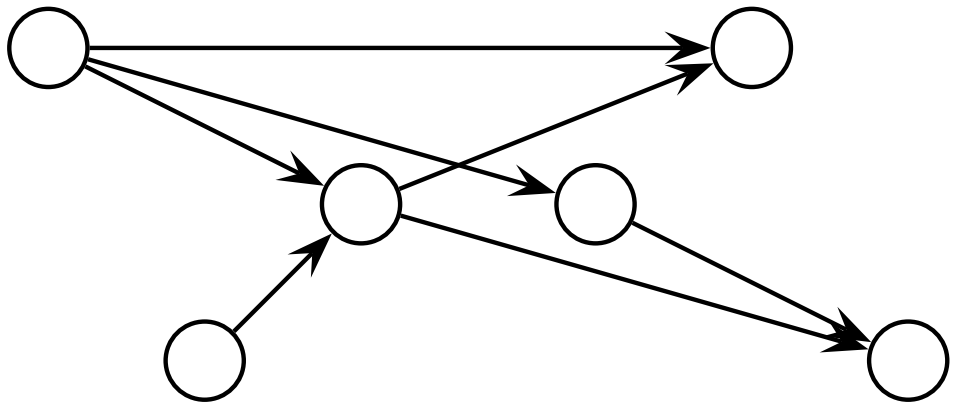
The k-DPP on directed acyclic graphs

A directed acyclic graph (dag)

is a directed graph $G(V, E)$ such that there is a mapping $\tau : V \to \{1, \ldots, |V|\}$ with

$$\tau(v) \leq \tau(w) \text{ for all } (v, w) \in E.$$

τ is called a topological numbering of G.



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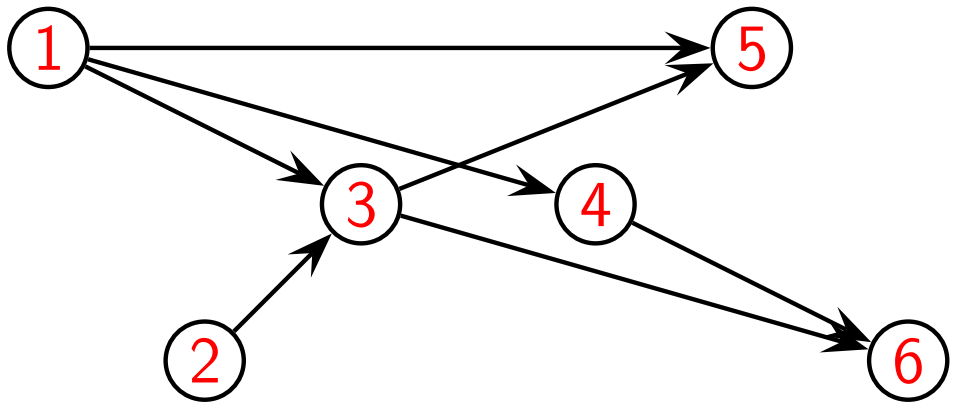

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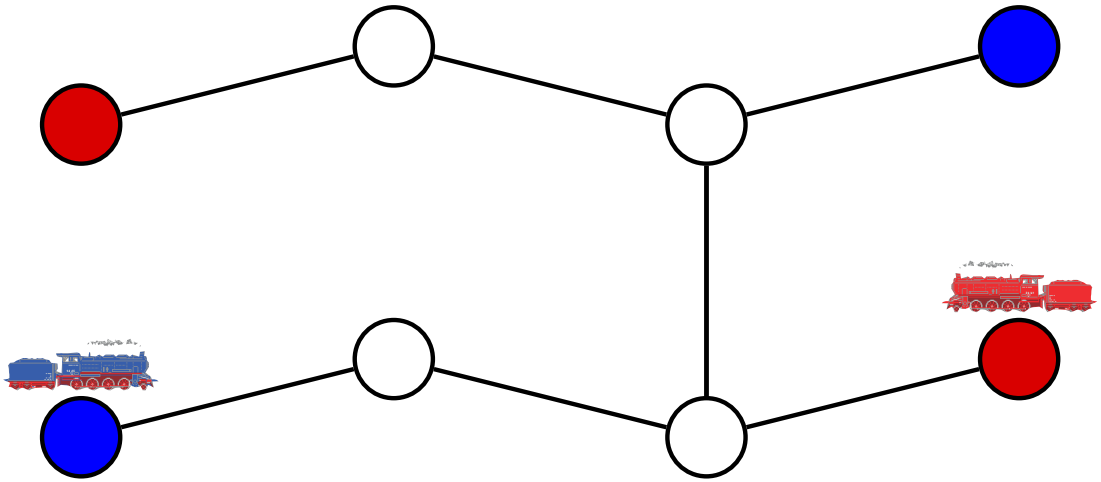
motivation:



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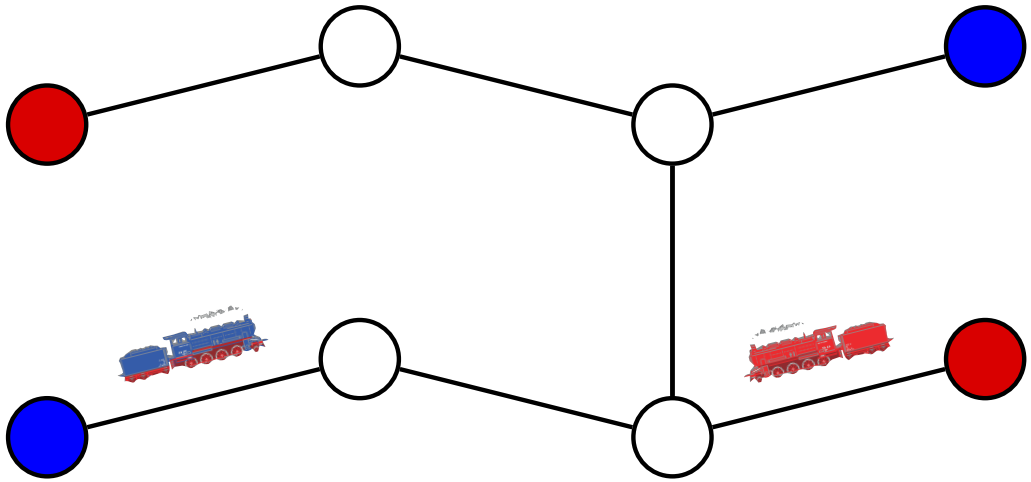

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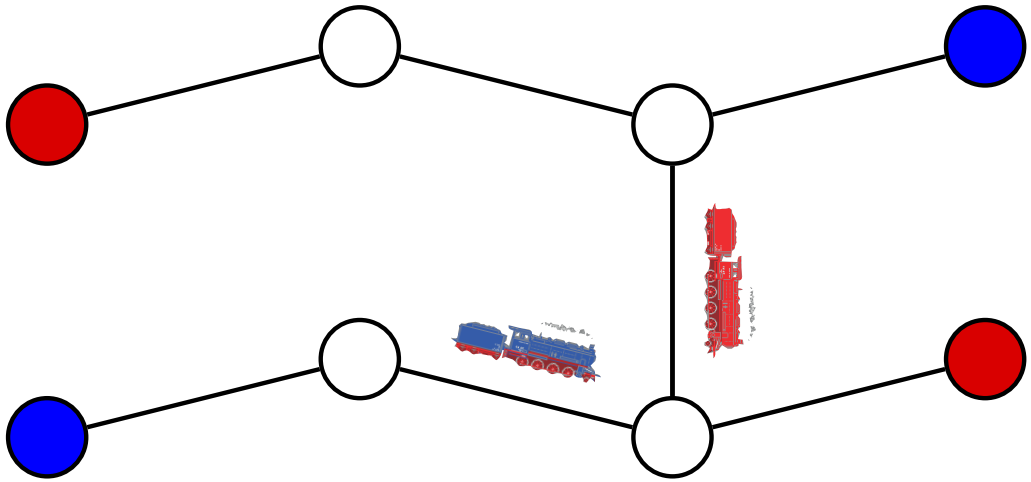

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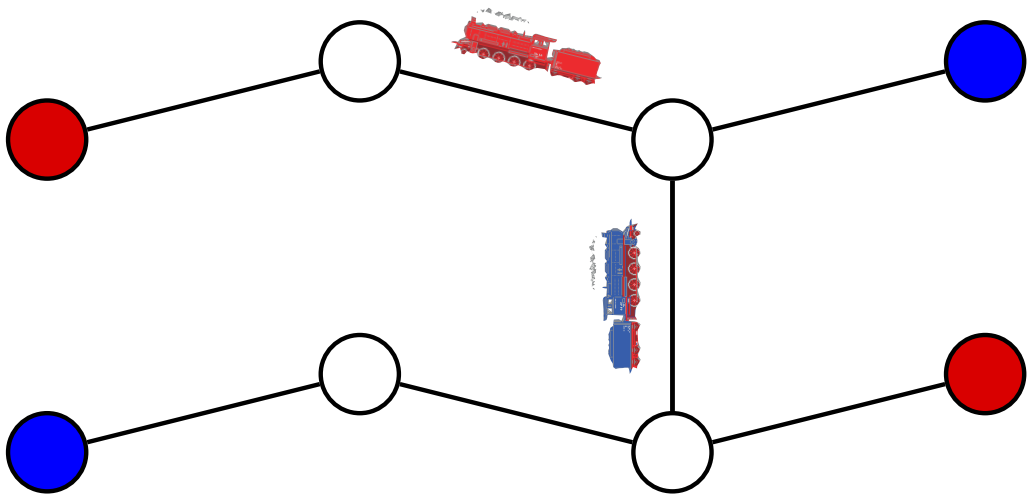

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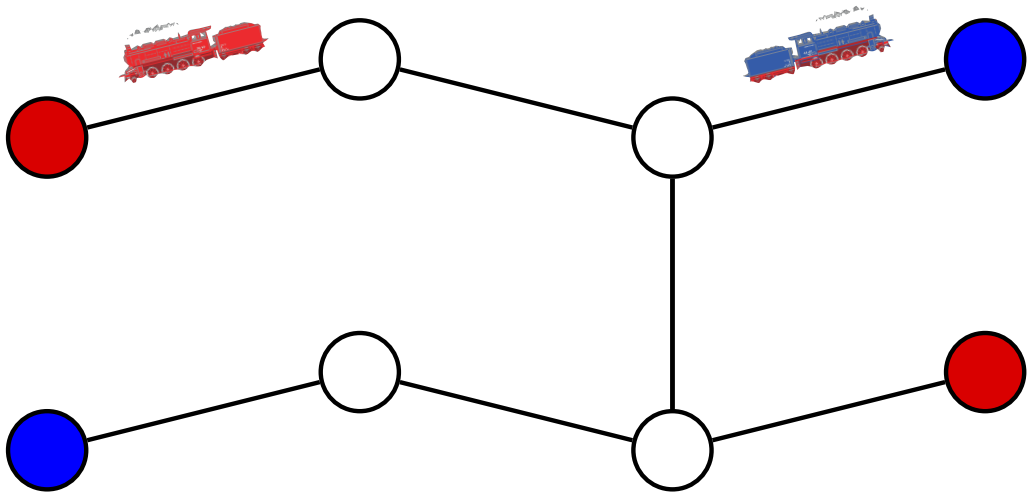

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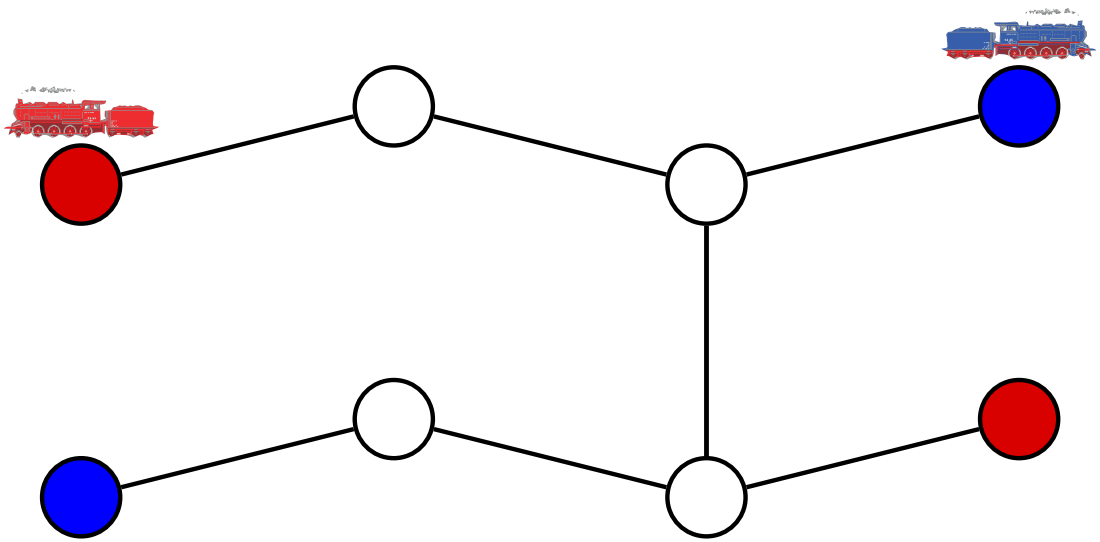

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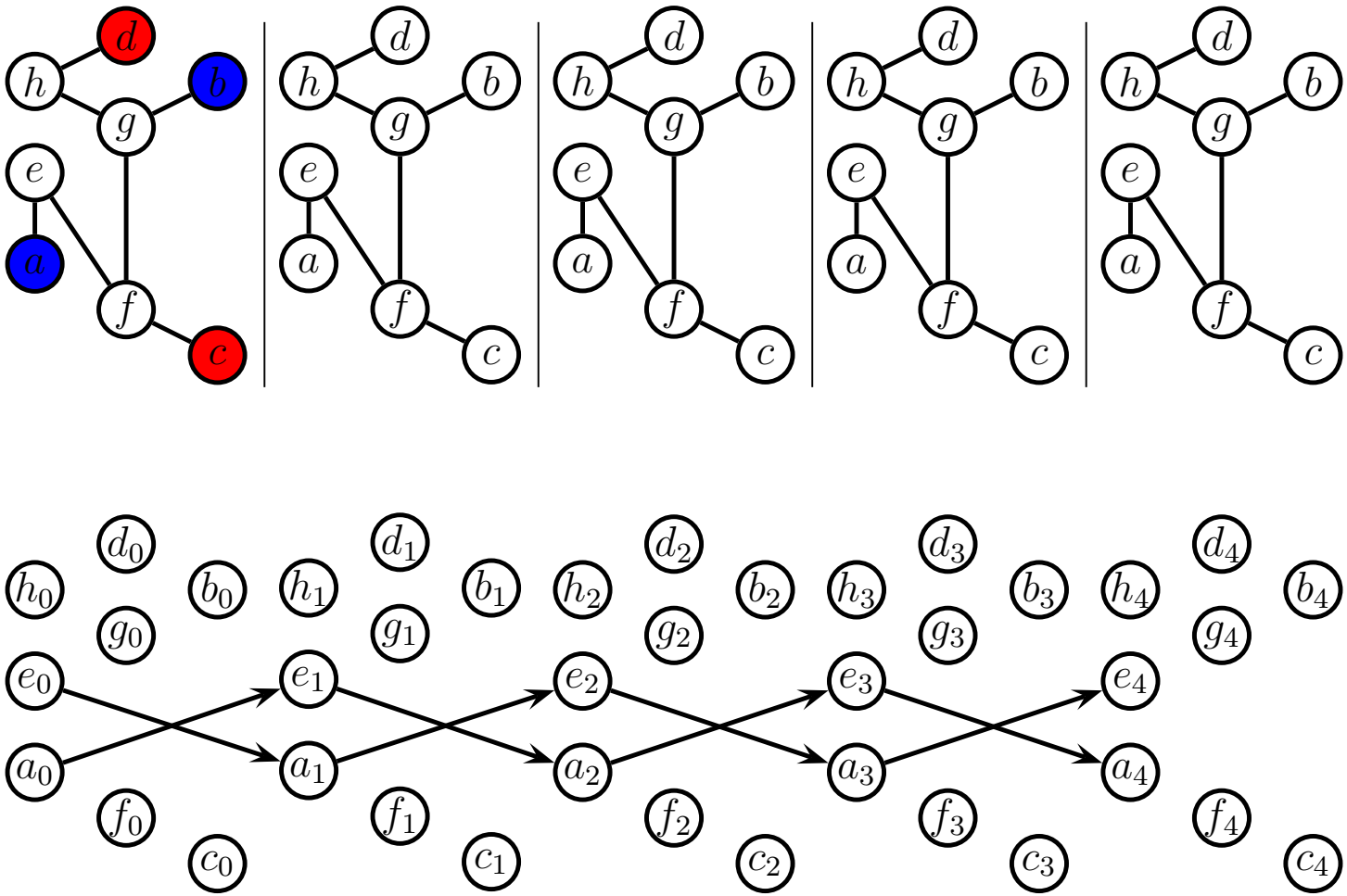

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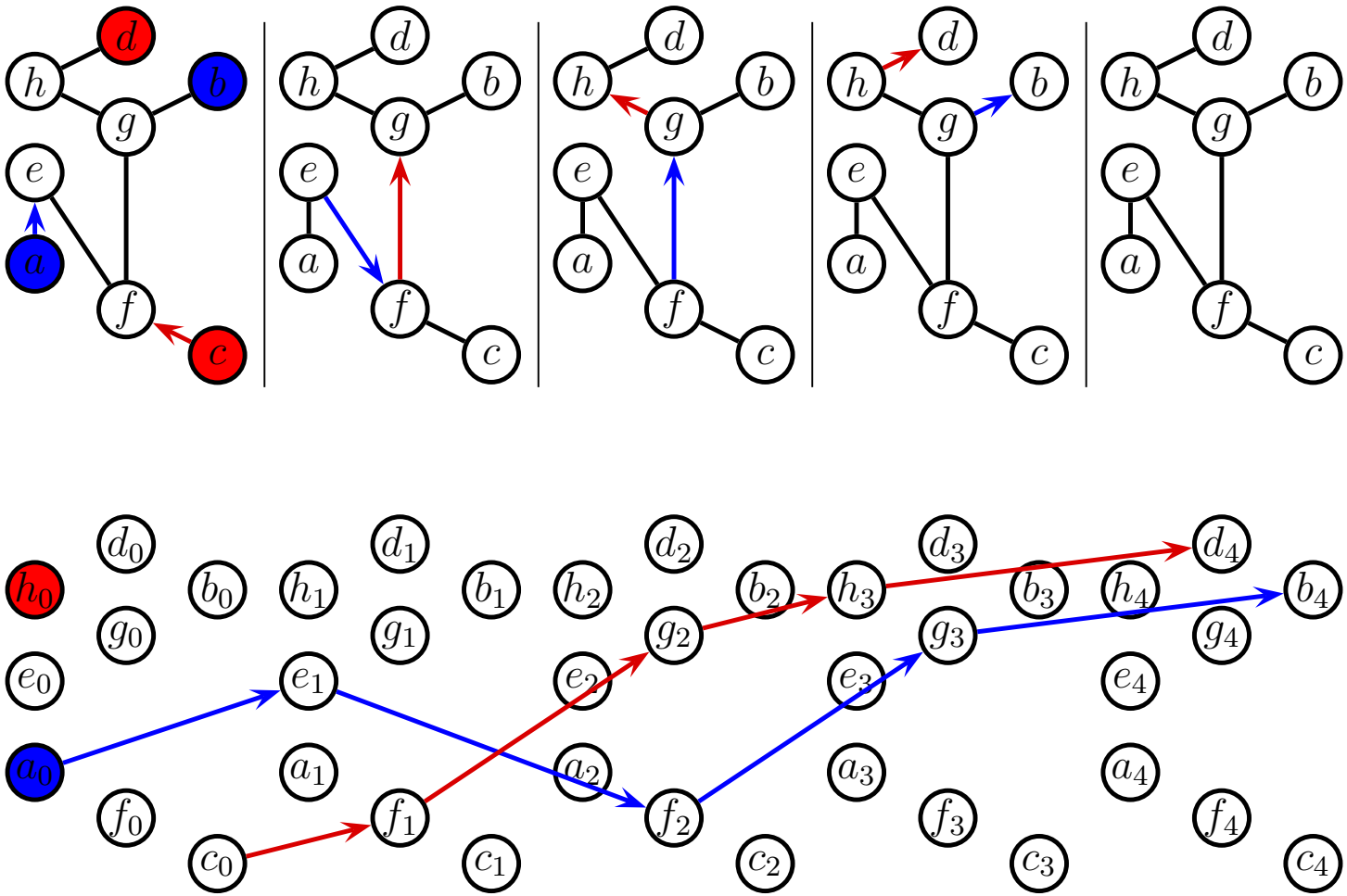

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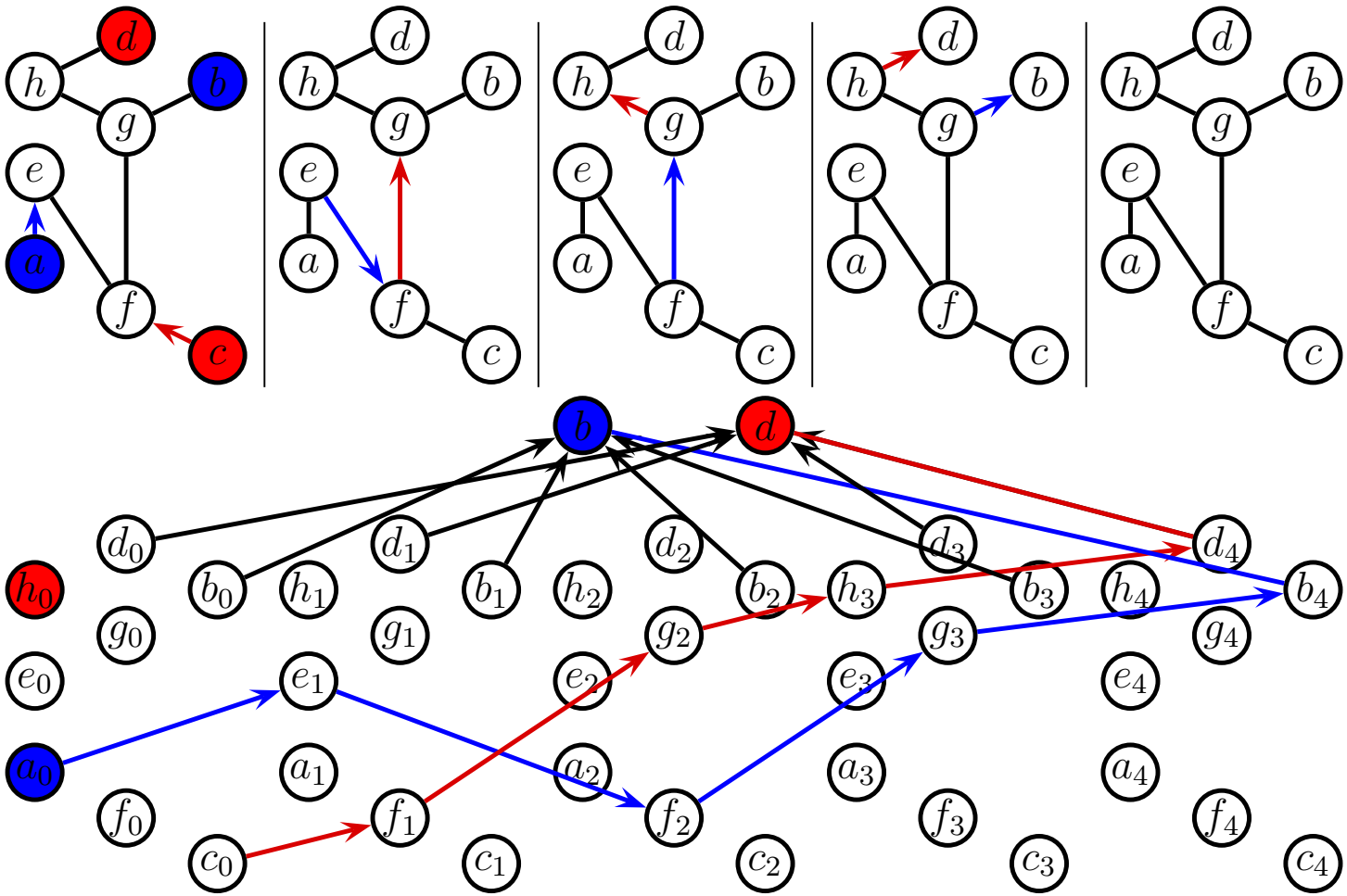

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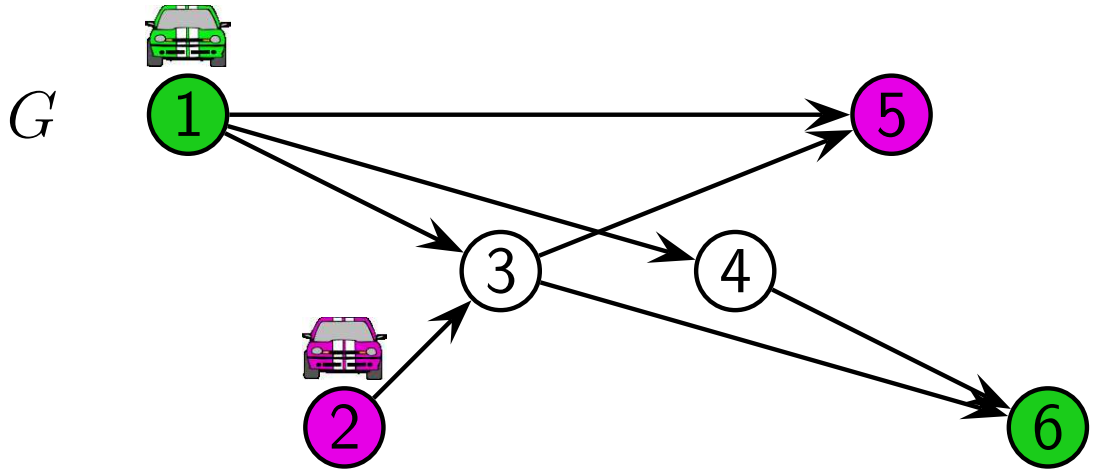

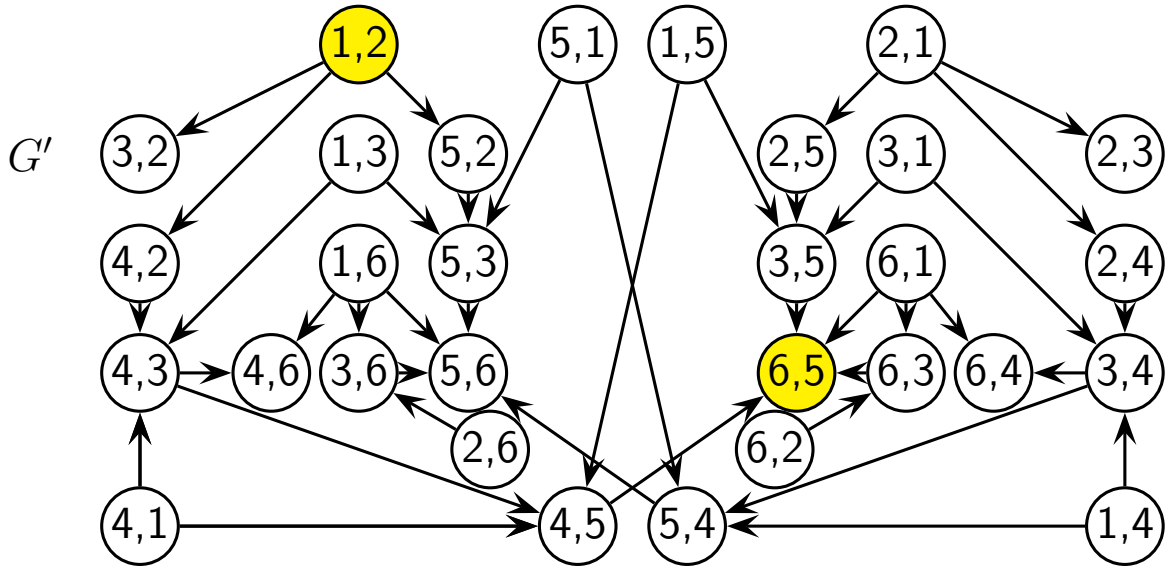

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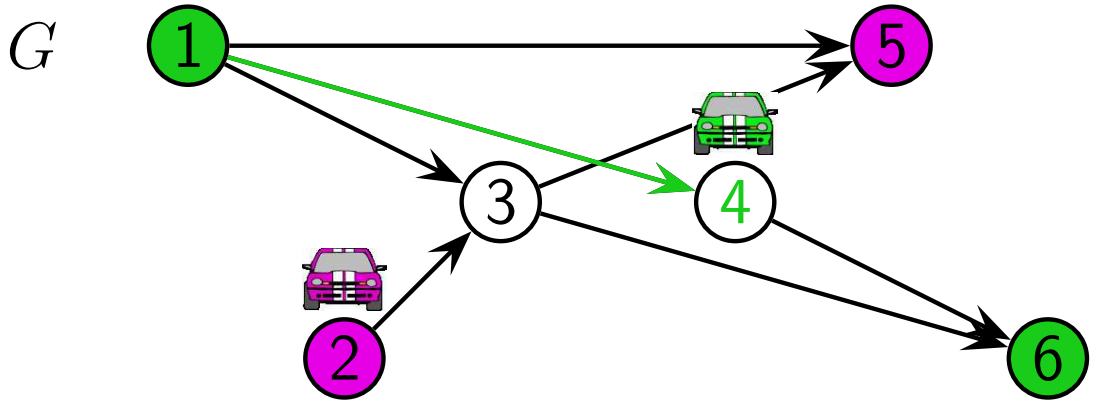

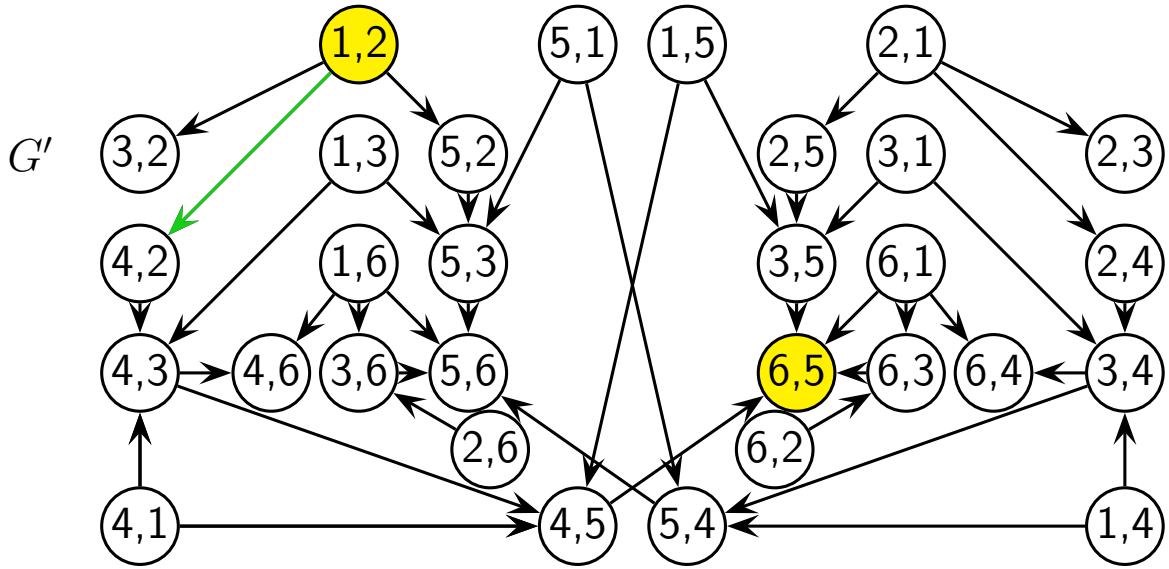

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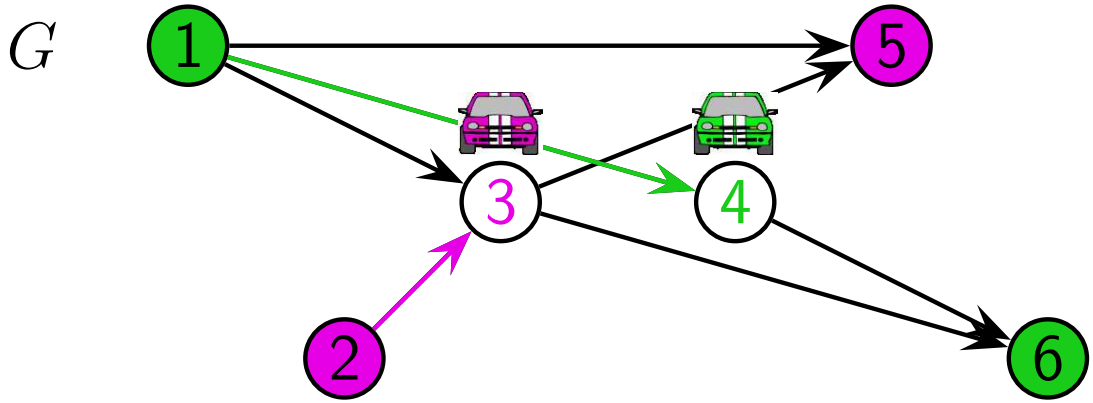

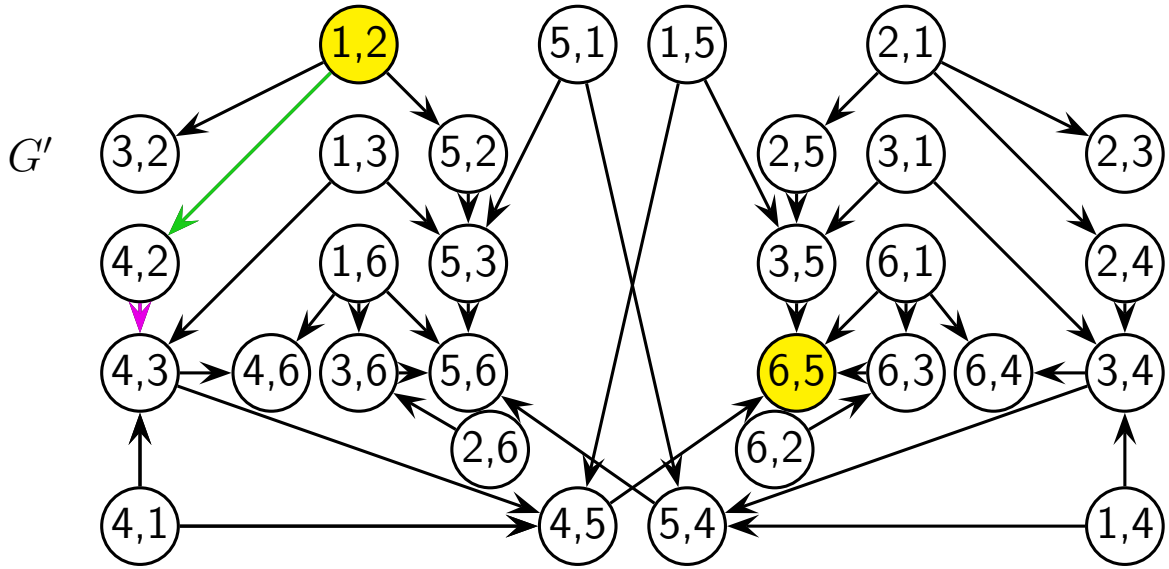

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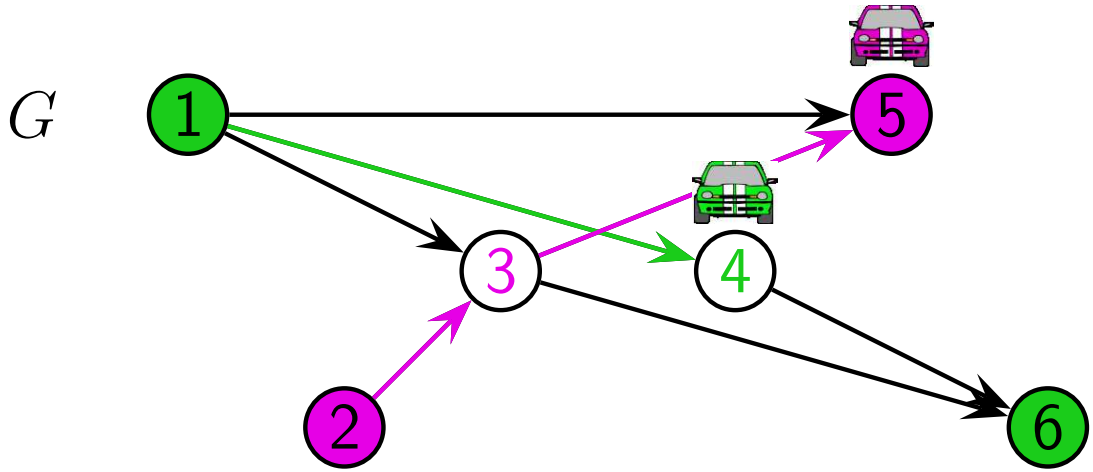

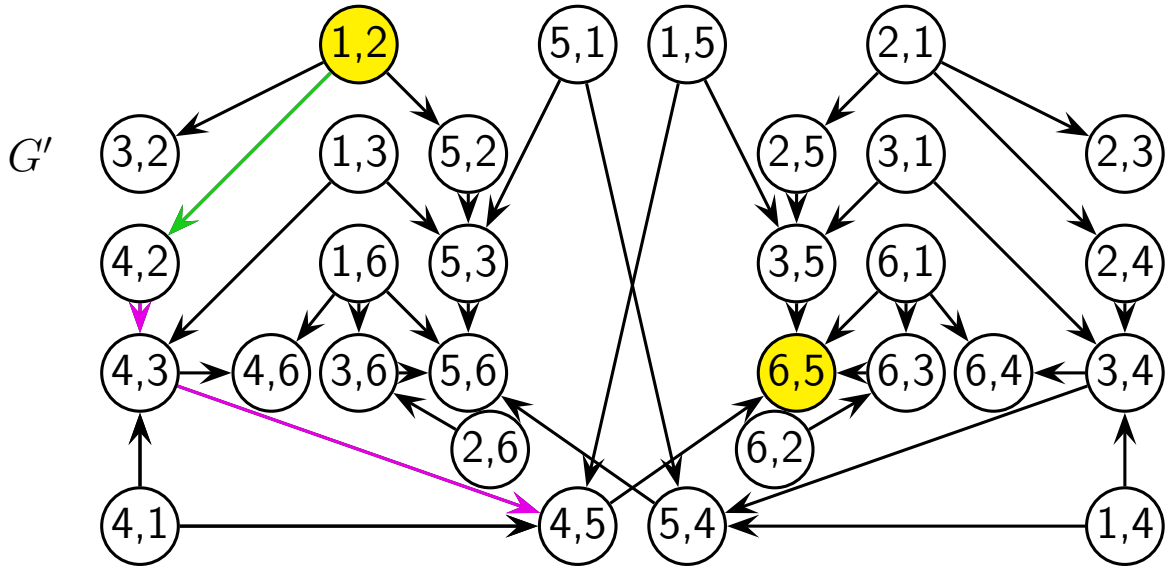

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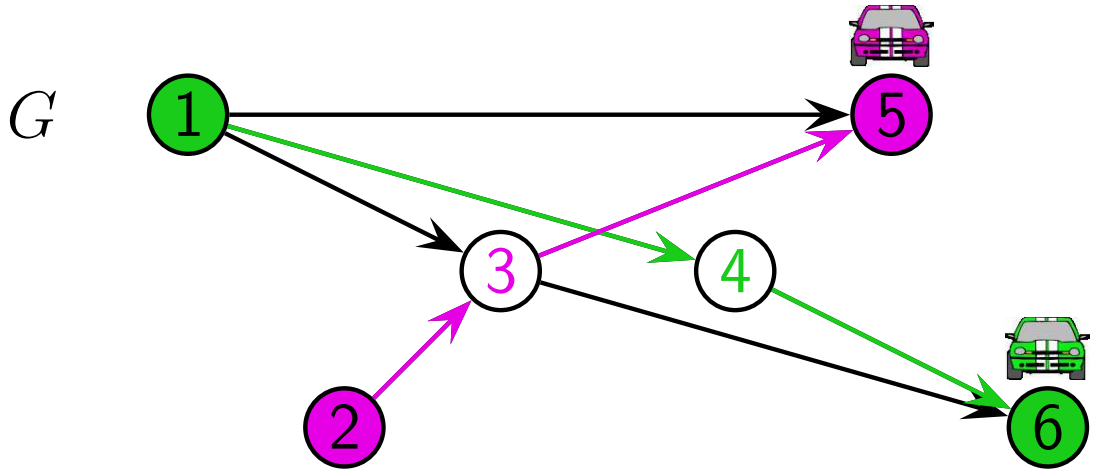

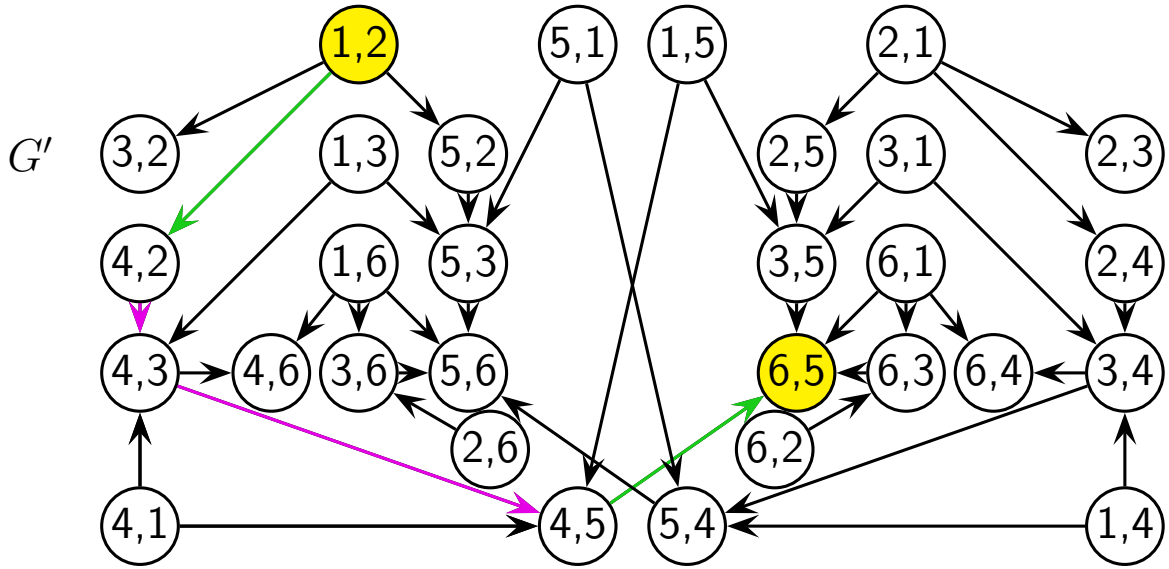

The 2-VDPP: the algorithm of Lucchesi and Giglio

Observation

- There are k vertex-disjoint paths $p_i : s_i \to t_i$ of total length d in G

  if and only if there is a path
  $p : (s_1, \ldots, s_k) \to (t_1, \ldots, t_k)$ of length d in G'.

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Disjoint paths of shortest total length solving the k-VDPP can be computed in $O(mn^{k-1})$ time.

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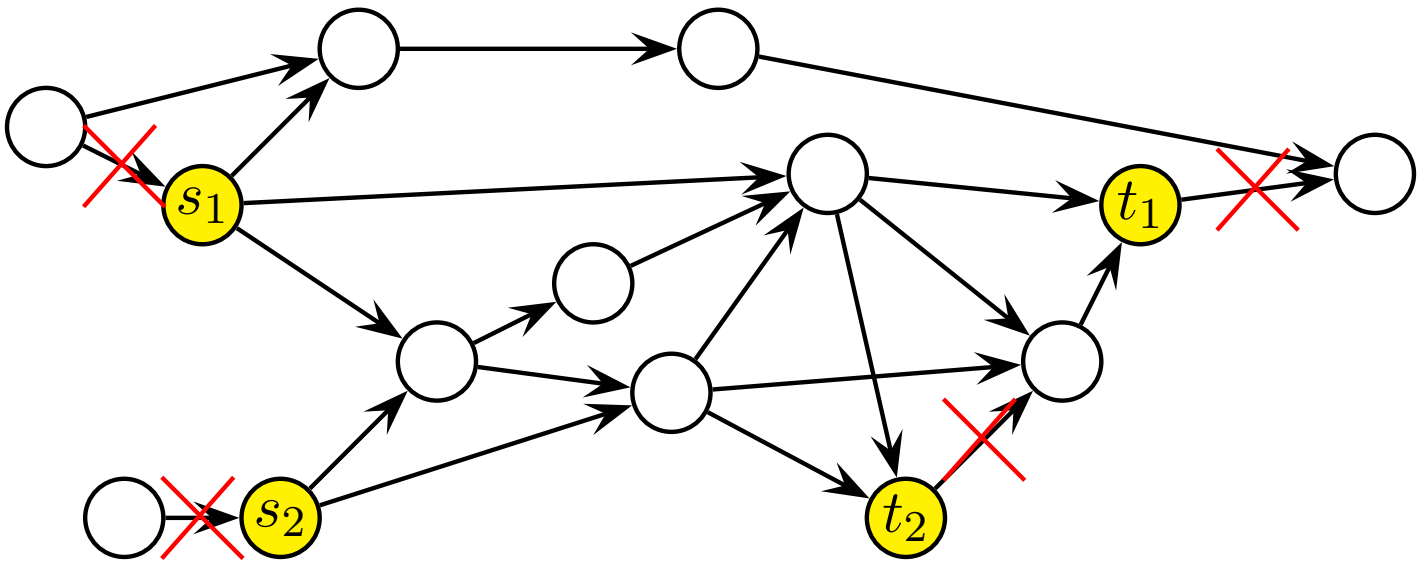
Remark

: The result can be extended to weighted graphs with a running time linear in the computation of a shortest weighted

The algorithm of Lucchesi and Giglio

1. Delete all ingoing edges of s_i and outgoing edges of t_i.



The algorithm of Lucchesi and Giglio

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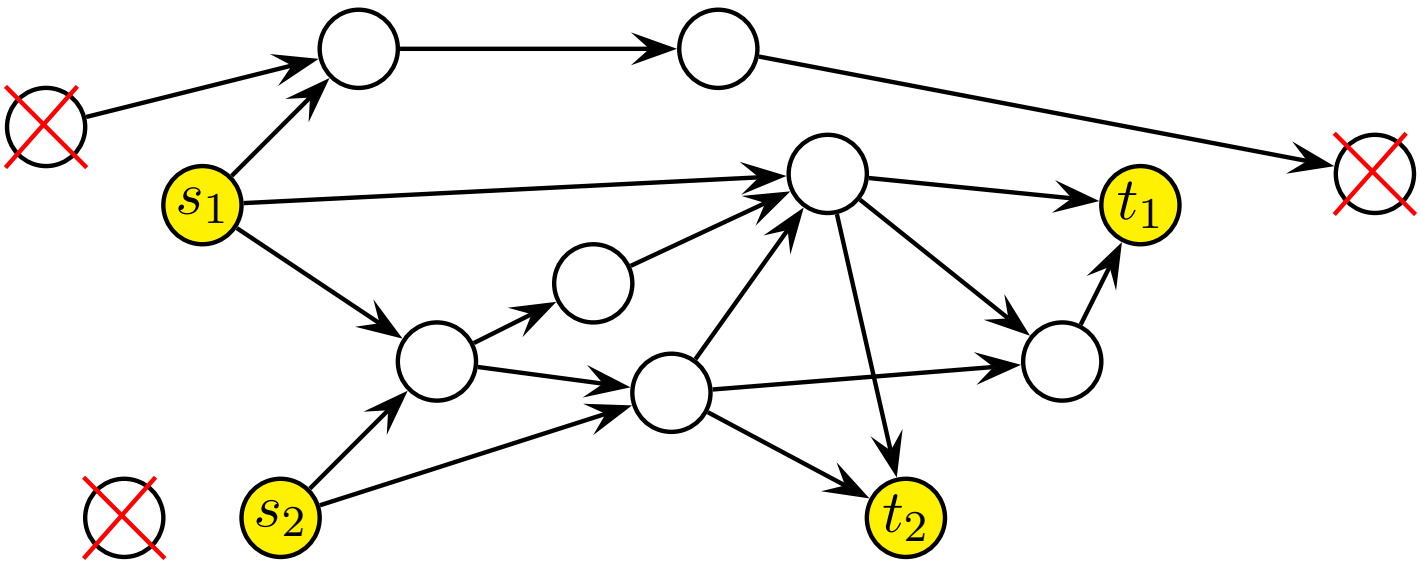
2. Delete all vertices $v \notin \{s_1, s_2\}$ with $\text{indeg}(v) = 0$ and all $v \notin \{t_1, t_2\}$ with $\text{outdeg}(v) = 0$.



The algorithm of Lucchesi and Giglio

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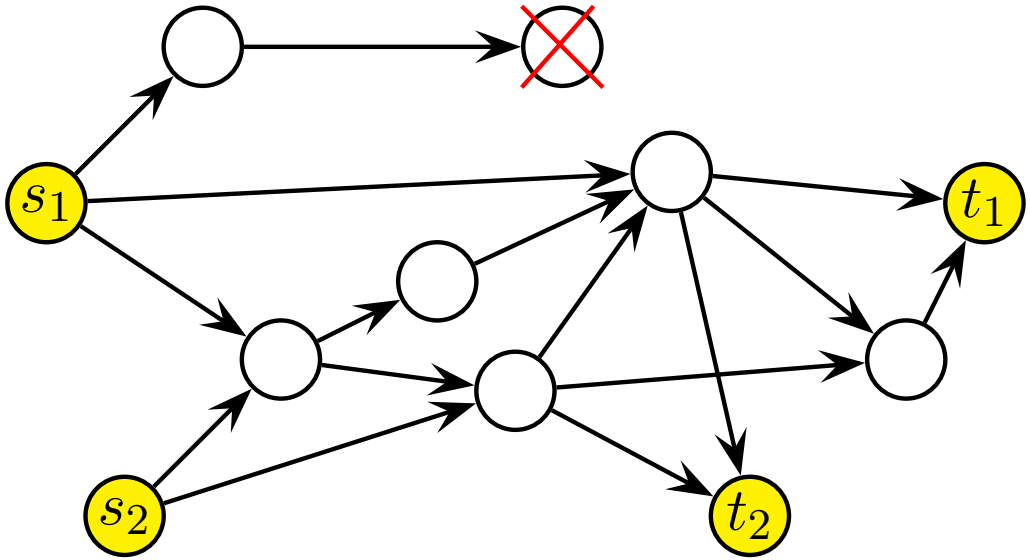

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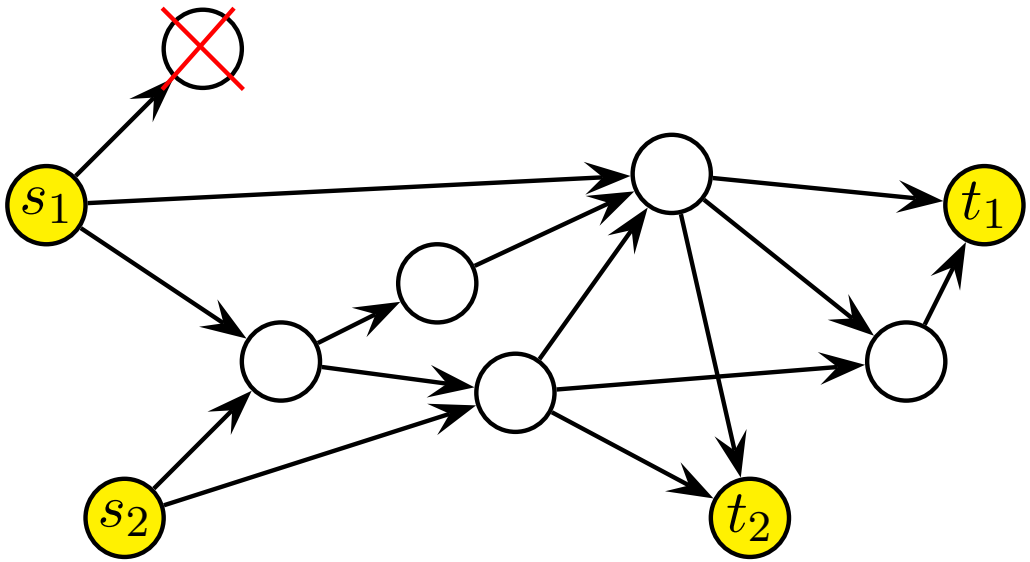
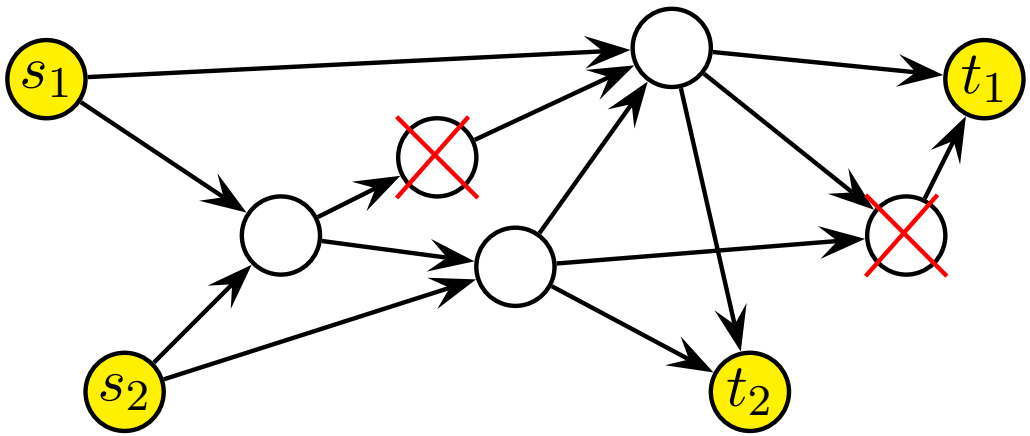


The 2-VDPP: the algorithm of Lucchesi and Giglio

The algorithm of Lucchesi and Giglio

1. Delete all ingoing edges of s_i and outgoing edges of t_i.
2. Delete all vertices $v \notin \{s_1, s_2\}$ with $\text{indeg}(v) = 0$ and all $v \notin \{t_1, t_2\}$ with $\text{outdeg}(v) = 0$.
3. Delete delete all vertices $v \notin \{t_1, t_2\}$ with $\text{indeg}(v) = 1$ or $\text{outdeg}(v) = 1$ by edge contractions.



The 2-VDPP: the algorithm of Lucchesi and Giglio

The algorithm of Lucchesi and Giglio

1. Delete all ingoing edges of s_i and outgoing edges of t_i.
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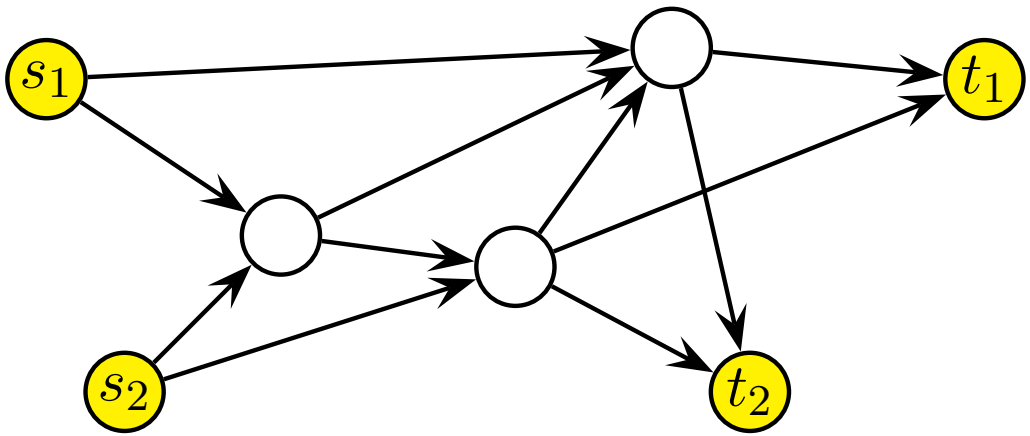
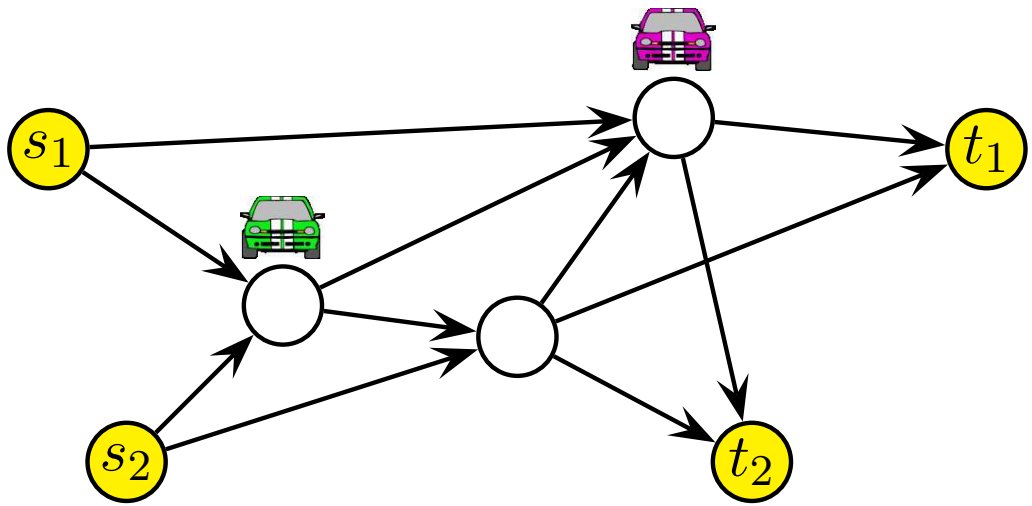


The algorithm of Lucchesi and Giglio

Oberseravation:

For every pair of different vertices x, y there are
- two disjoint paths from $\{x, y\}$ to $\{t_1, t_2\}$,



The algorithm of Lucchesi and Giglio

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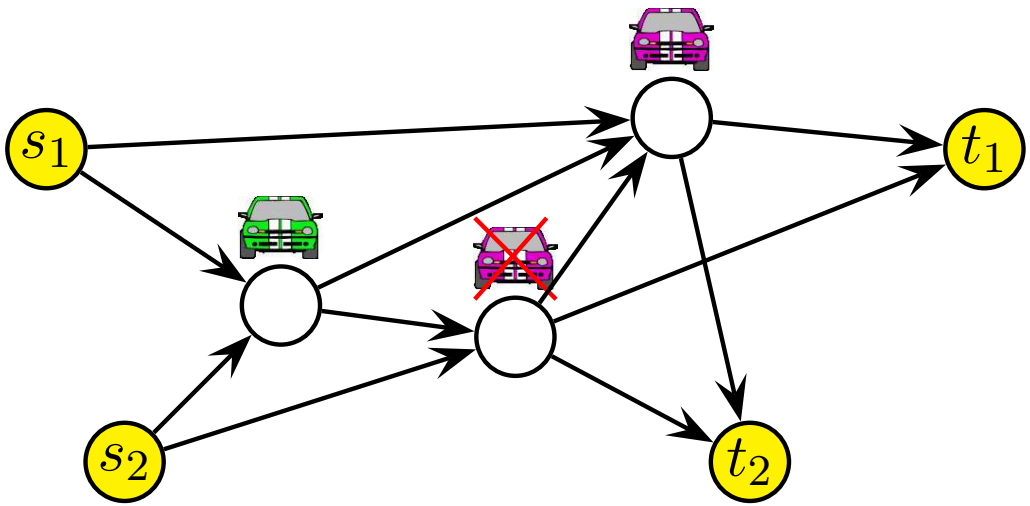
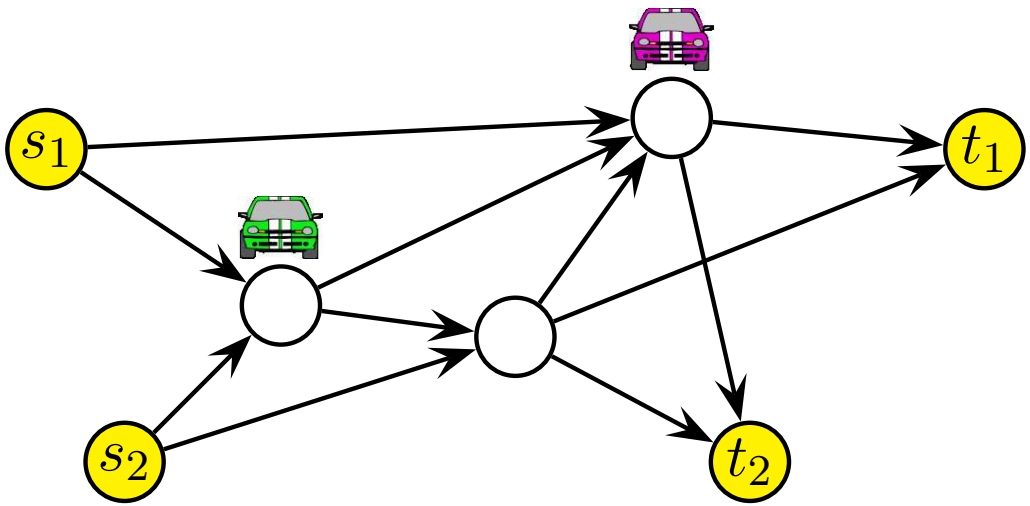


The algorithm of Lucchesi and Giglio

Oberseravation:

For every pair of different vertices x, y there are
- two disjoint paths from $\{x, y\}$ to $\{t_1, t_2\}$,
- two disjoint paths from $\{s_1, s_2\}$ to $\{x, y\}$.



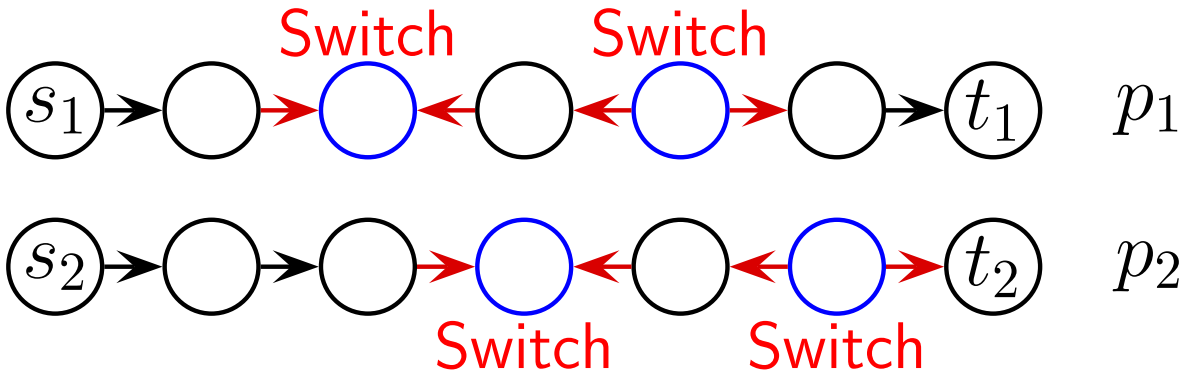
The algorithm of Lucchesi and Giglio

4. Construct disjoint paths $p_1 : s_1 \to t_1$ and $p_2 : s_2 \to t_2$ ignoring edge direction.

The 2-VDPP: the algorithm of Lucchesi and Giglio

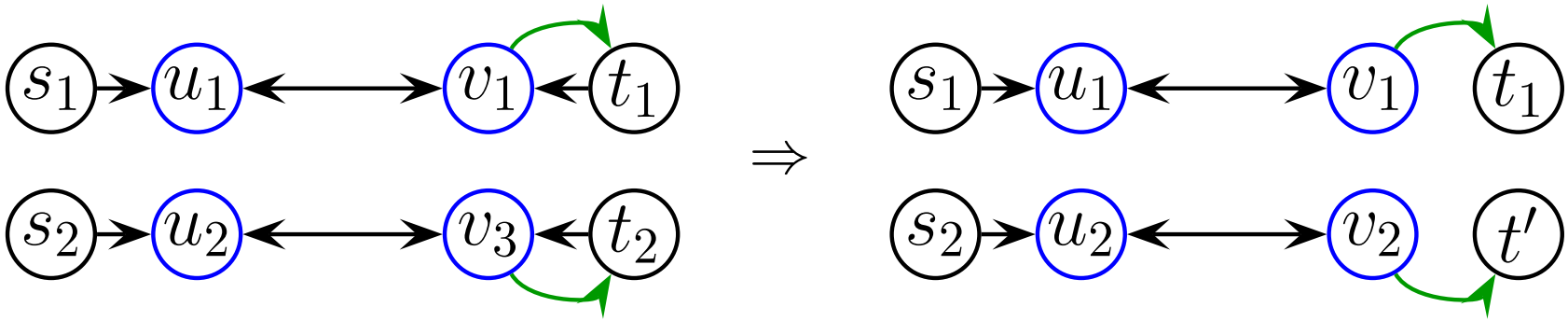
Definition

A vertex with two incoming or two outgoing edges is called a switch.



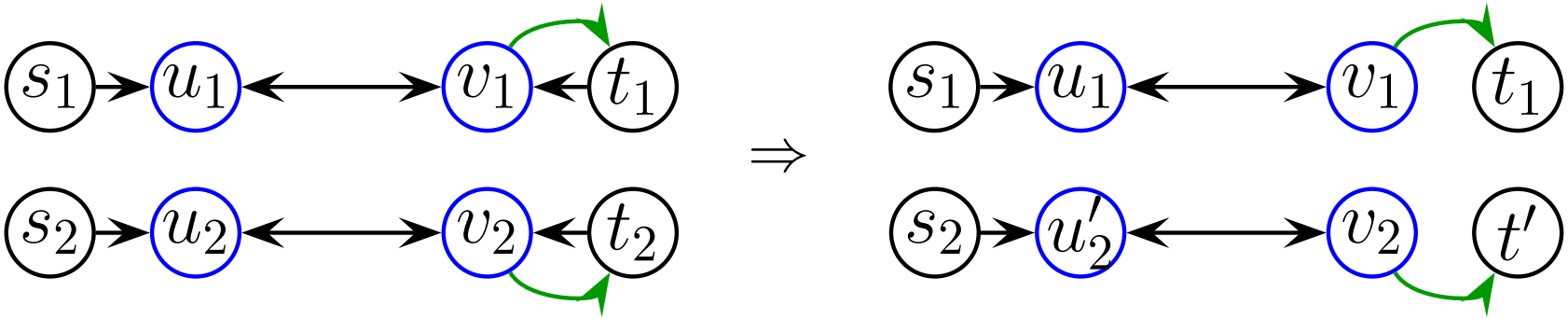
The algorithm of Lucchesi and Giglio

5. Remove the switches:
   - u_i: switch with the smallest topological number on p_i.
   - v_i: switch with the smallest topological number on p_i.
   - Construct disjoint paths in G connecting $\{s_1, s_2\}$ with $\{u_1, u_2\}$
   - Construct disjoint paths in G connecting $\{v_1, v_2\}$ with $\{t_1, t_2\}$
   - Use these paths to remove the switch as shown on the slides.

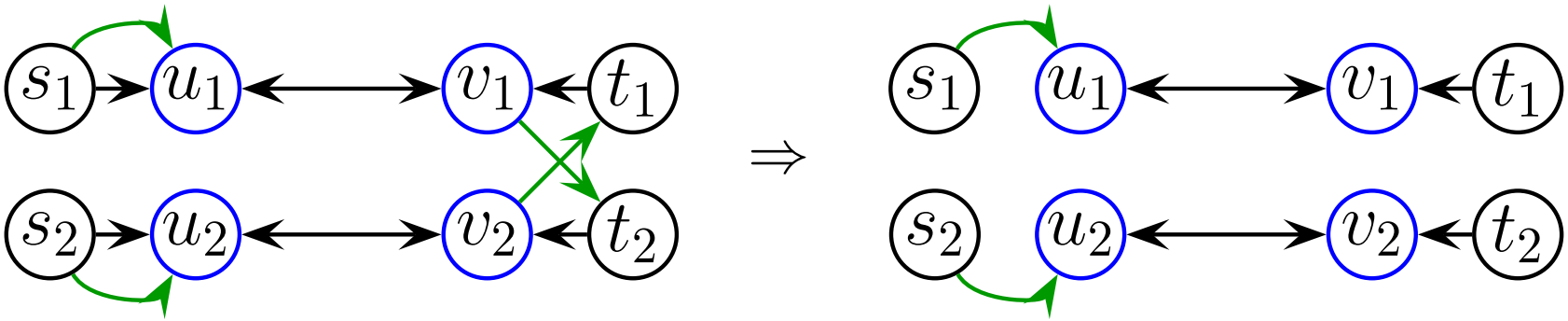


A switch with the smallest topological number on

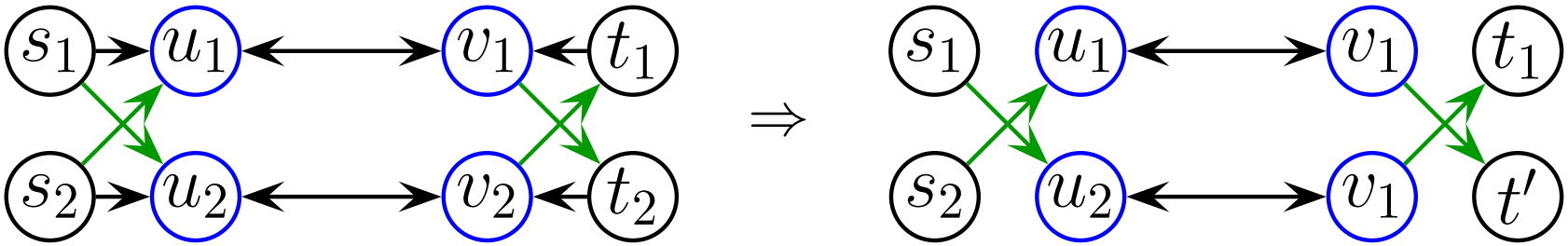
Fall 1



Fall 2



Fall 3



Conclusion

Observation

- The running time is dominated by the removal of the switches.
- We have to remove at most n switches.
- For each them four paths have to be constructed in $O(m+n)$ time.

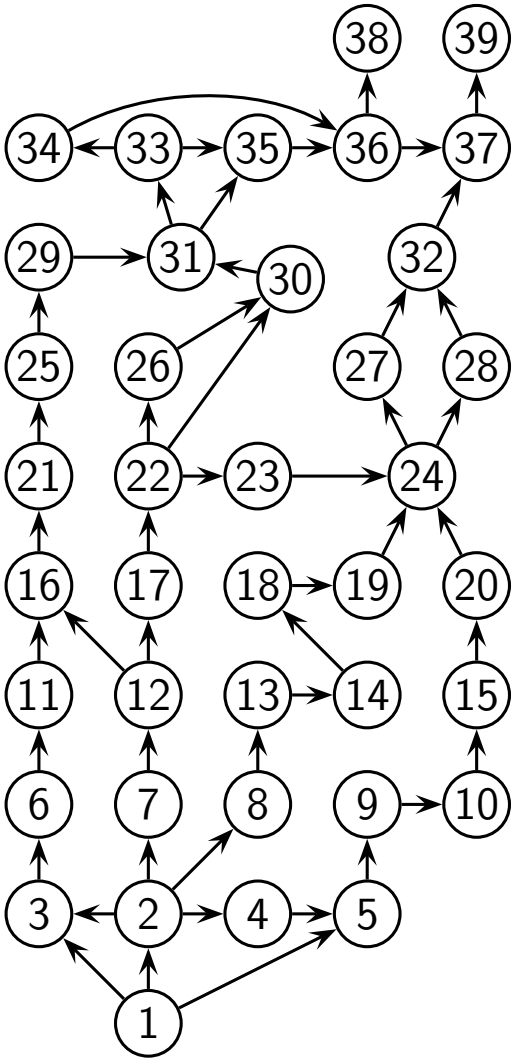
$\Rightarrow$ Lemma: The whole running time is $O(n(m+n))$.

The 2-VDPP: A new Approach

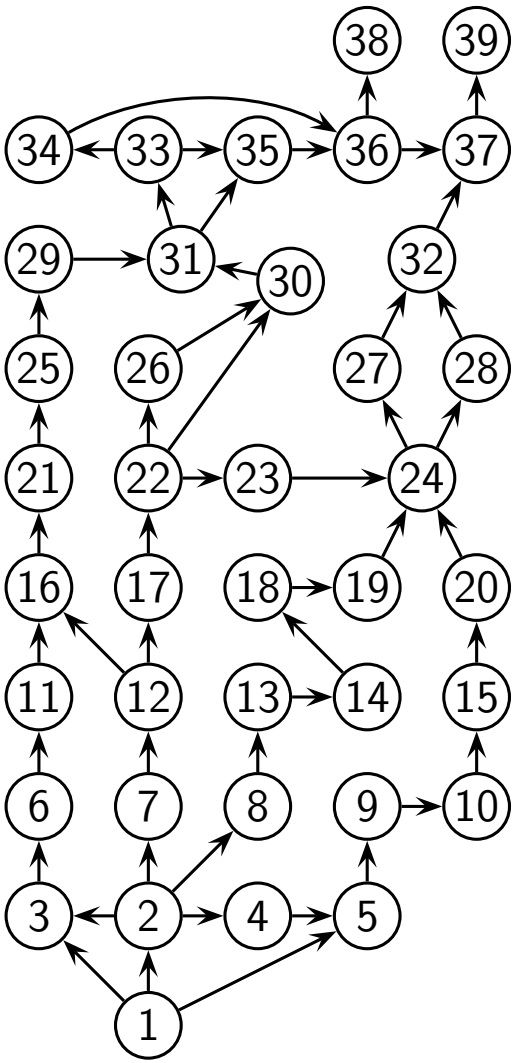
Improving the running time

- We use a data structure that allows us:
- To compute for fixed s_1, s_2 and any pair of vertices t_1, t_2:
- paths p_1, p_2 connecting $\{s_1, s_2\}$ and $\{t_1, t_2\}$ arbitrarily.
- to predict in $O(1)$ time which pairs a connected.
- Delay the paths replacements until the very end.

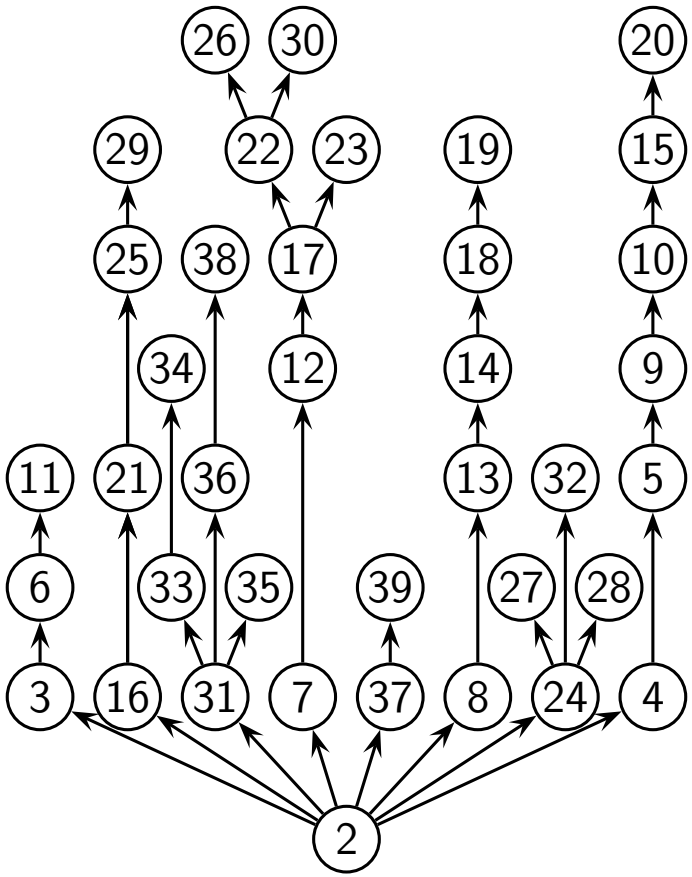
The 2-VDPP: A new Approach



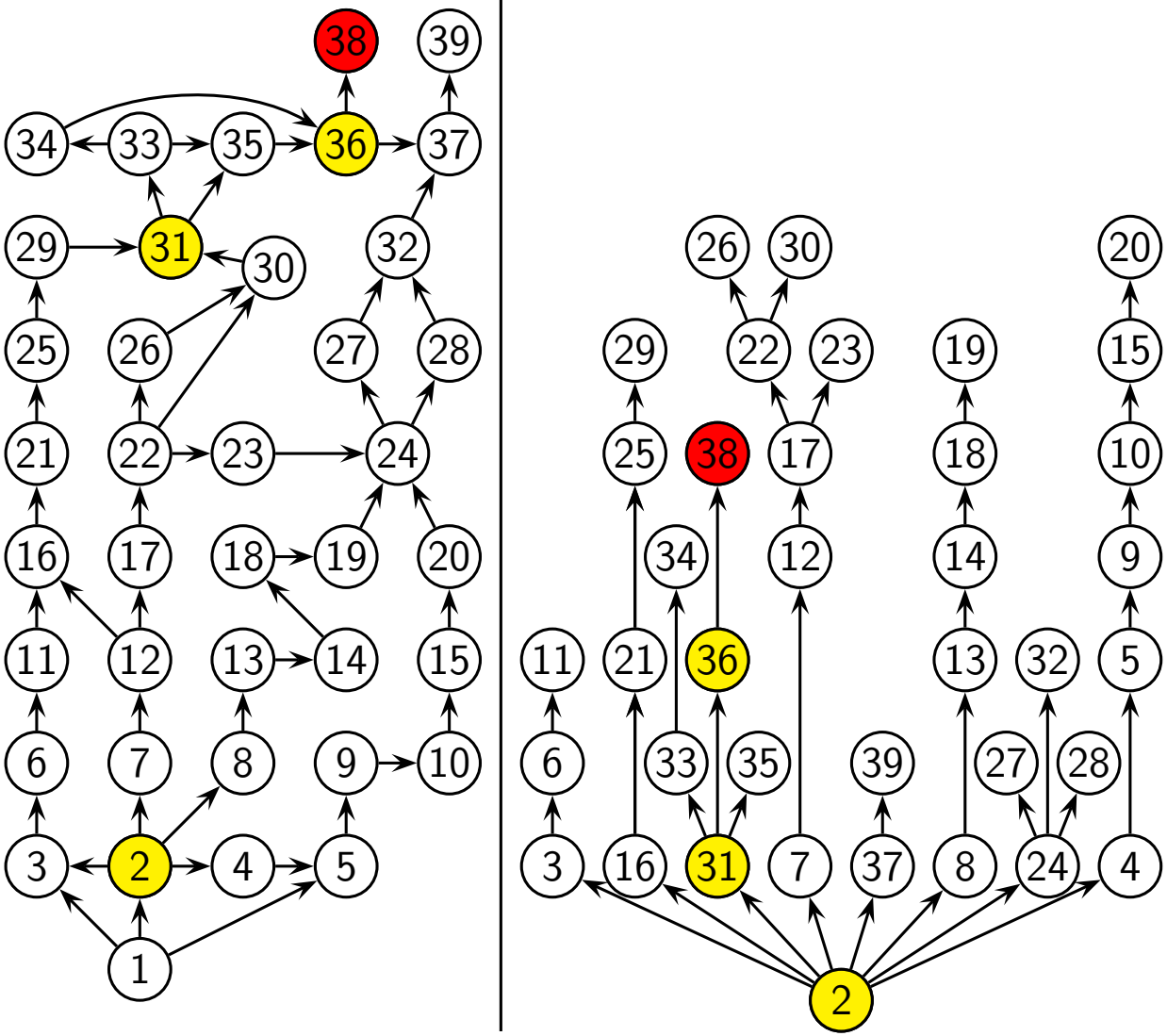
The 2-VDPP: A new Approach



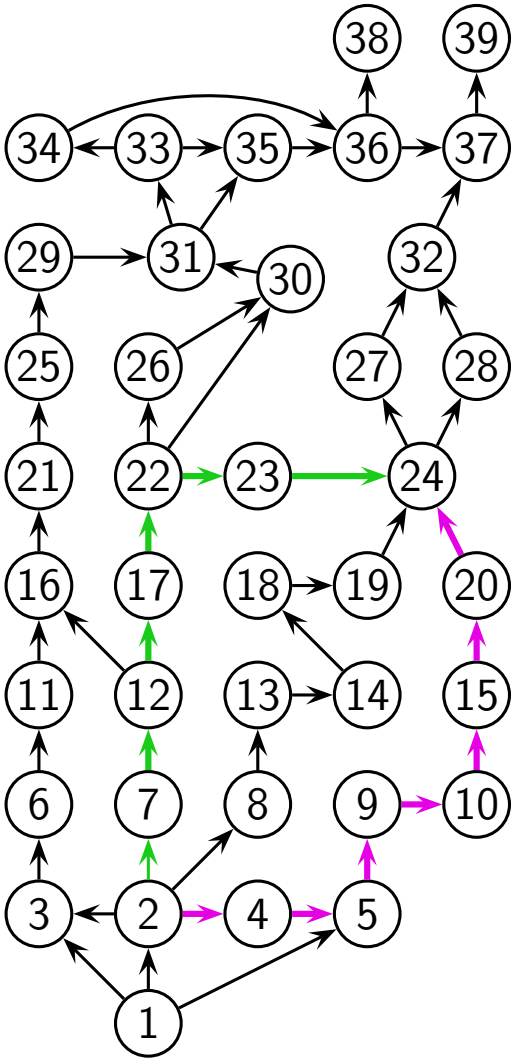
Dominator tree



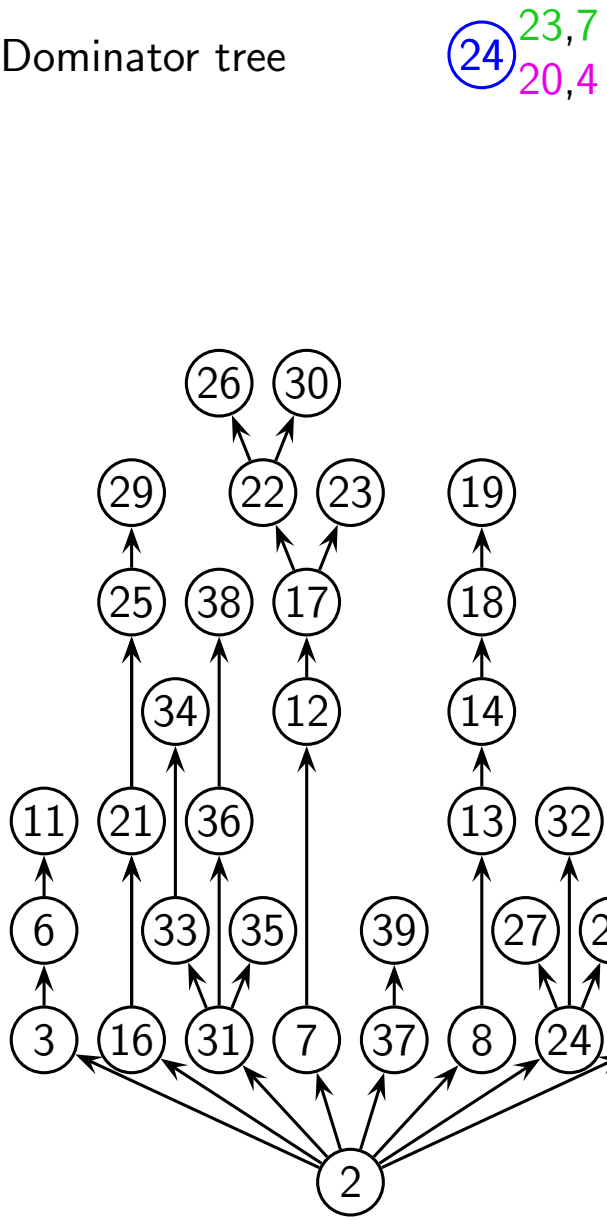
The 2-VDPP: A new Approach



The 2-VDPP: A new Approach



Dominator tree



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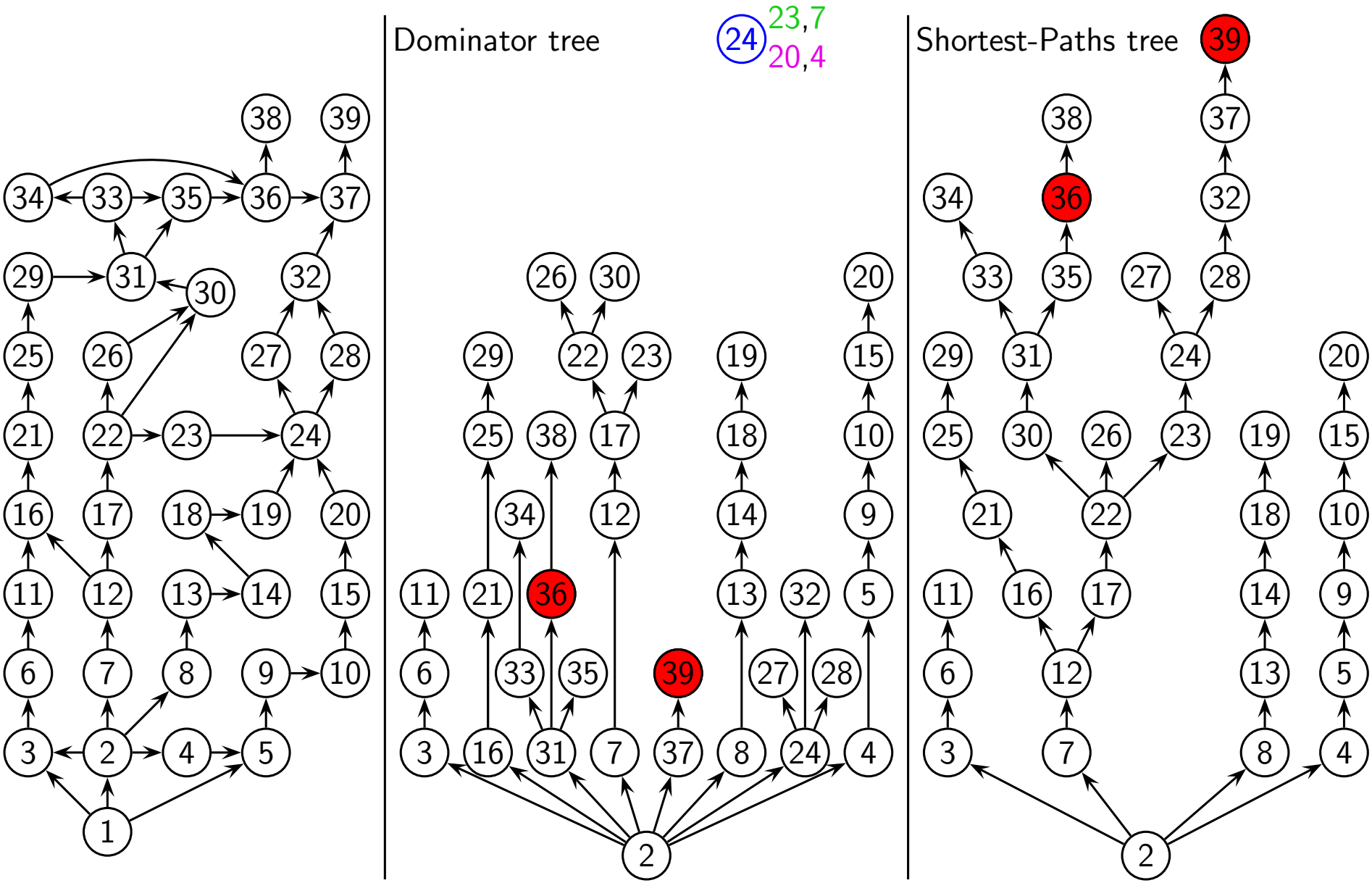

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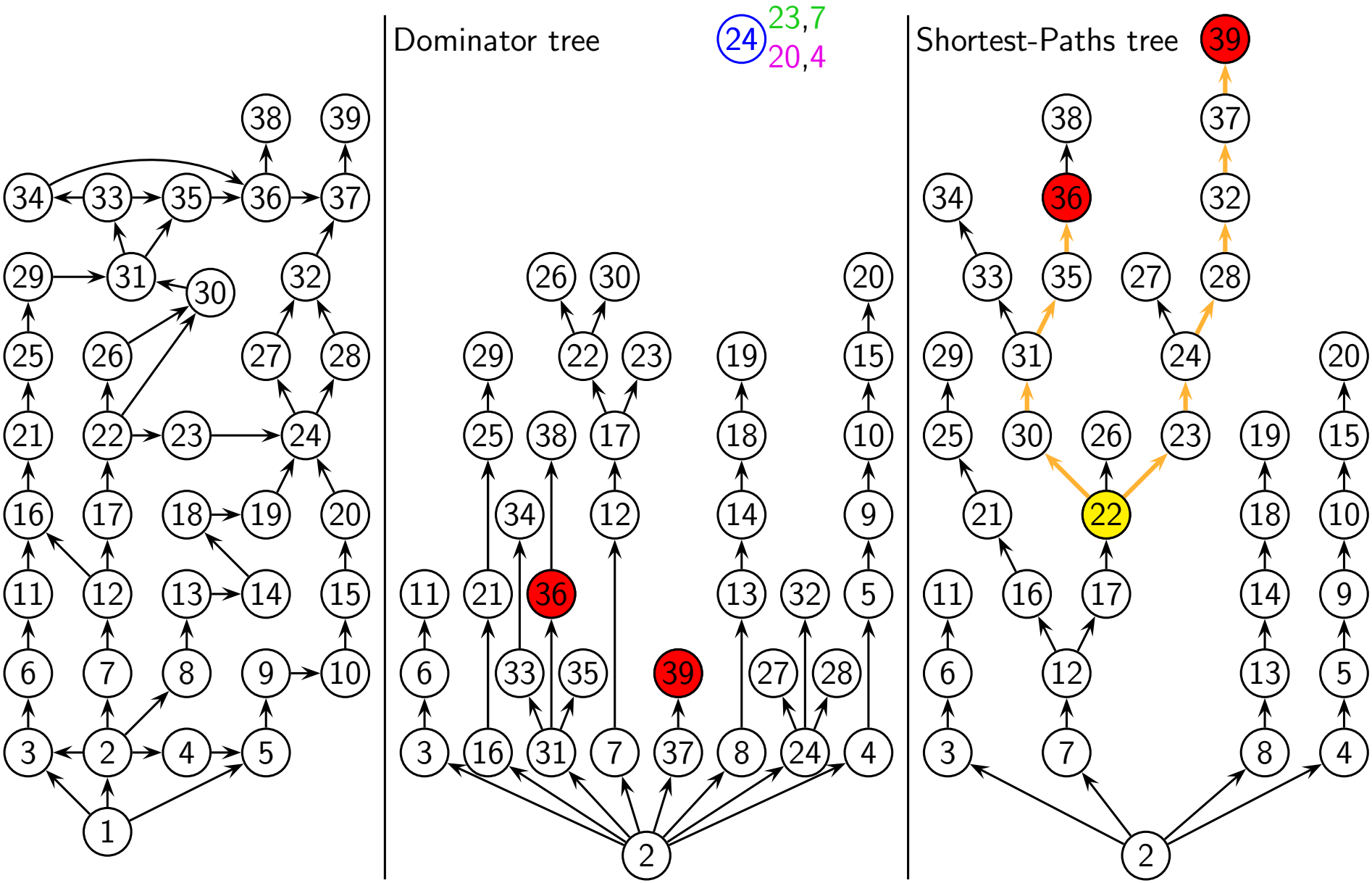

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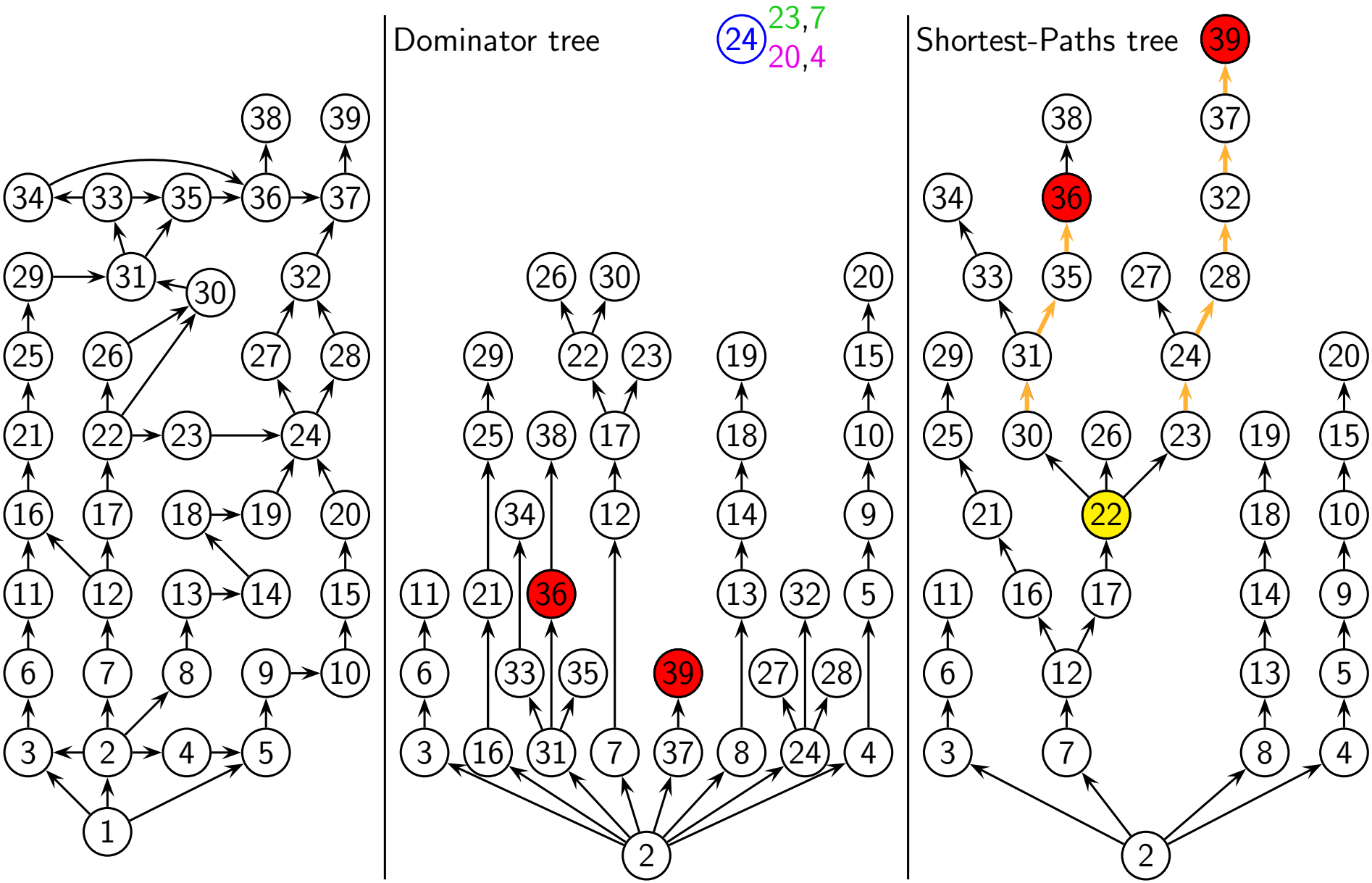

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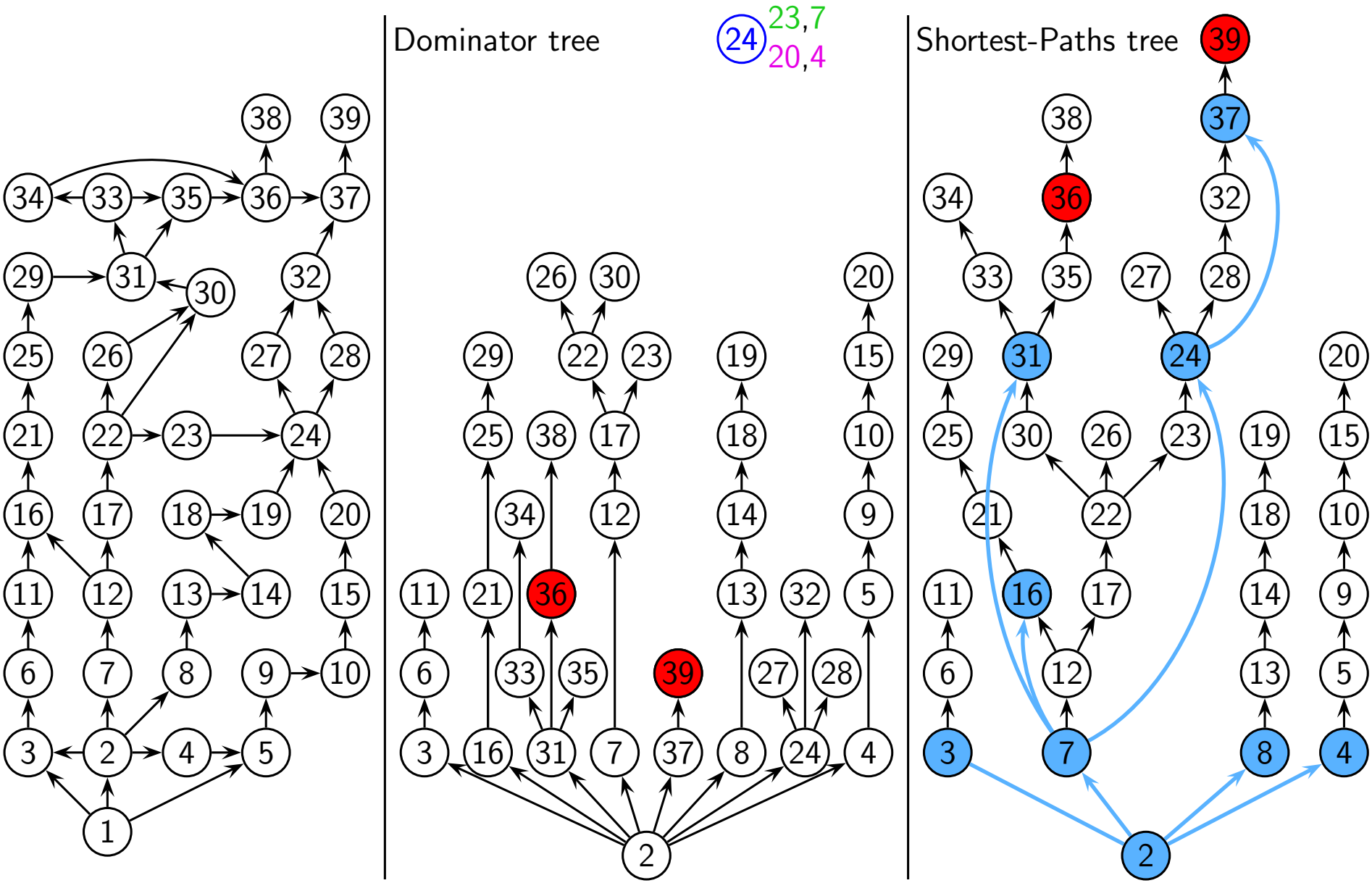

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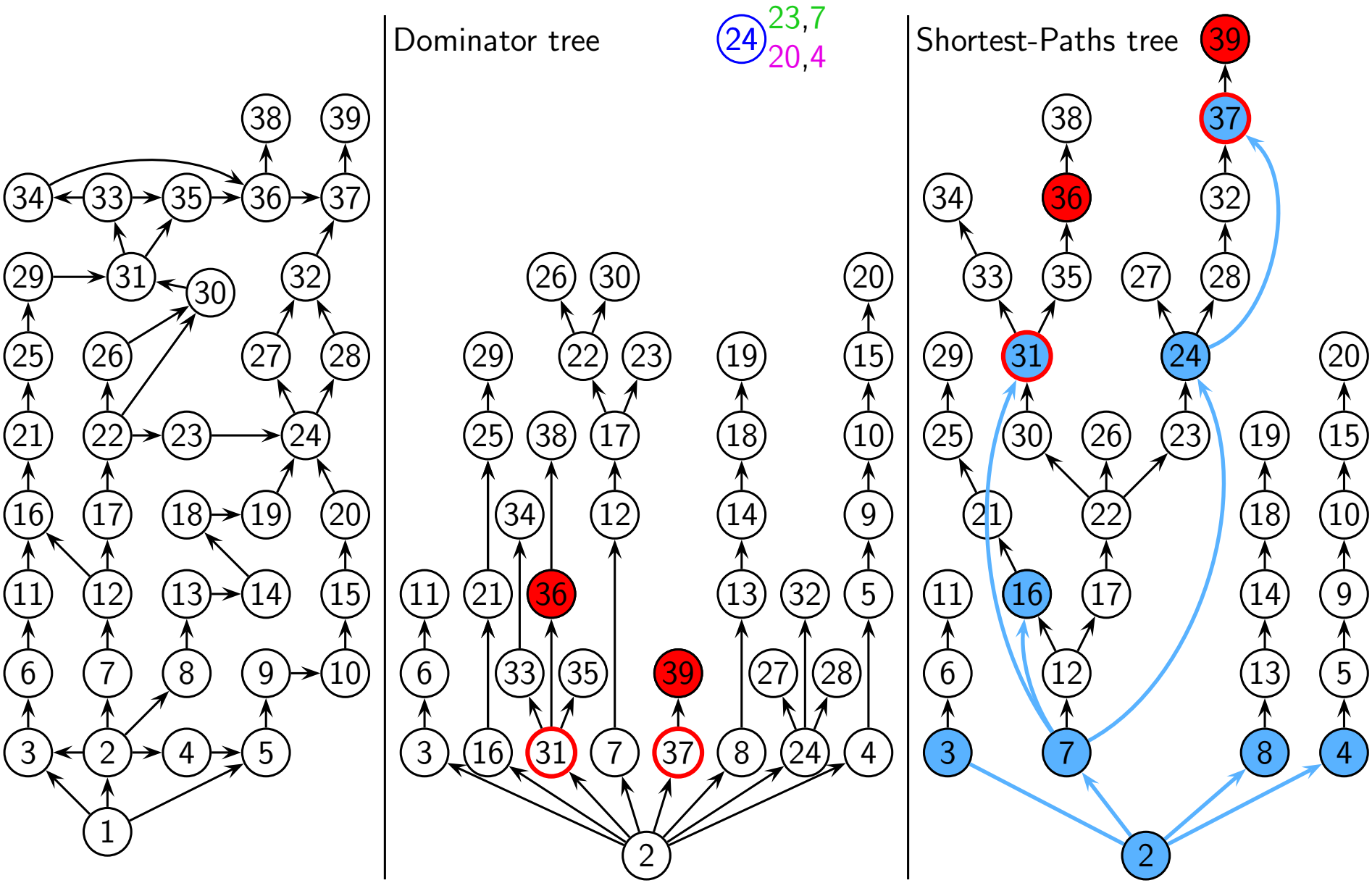
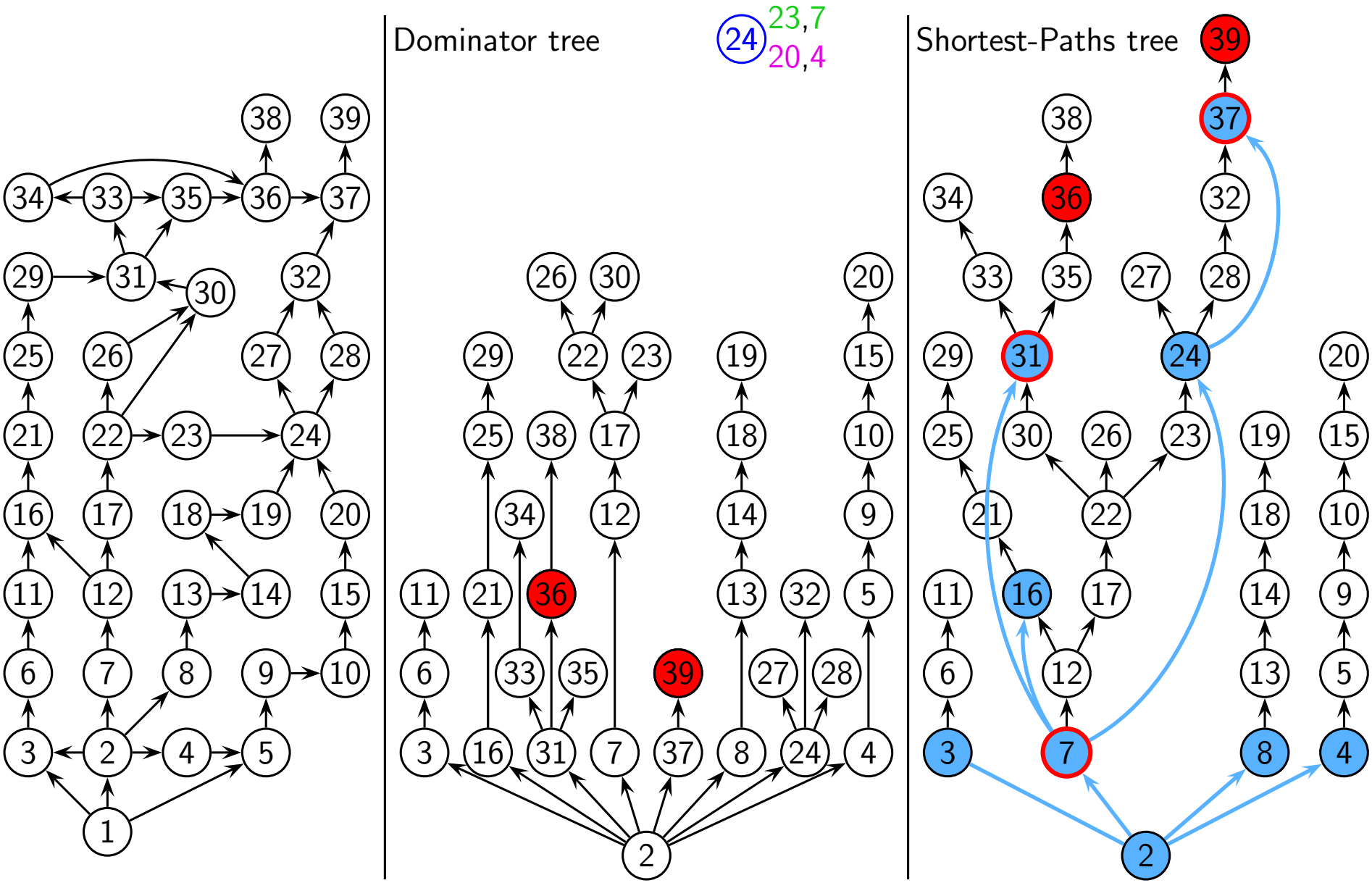


The 2-VDPP: A new Approach



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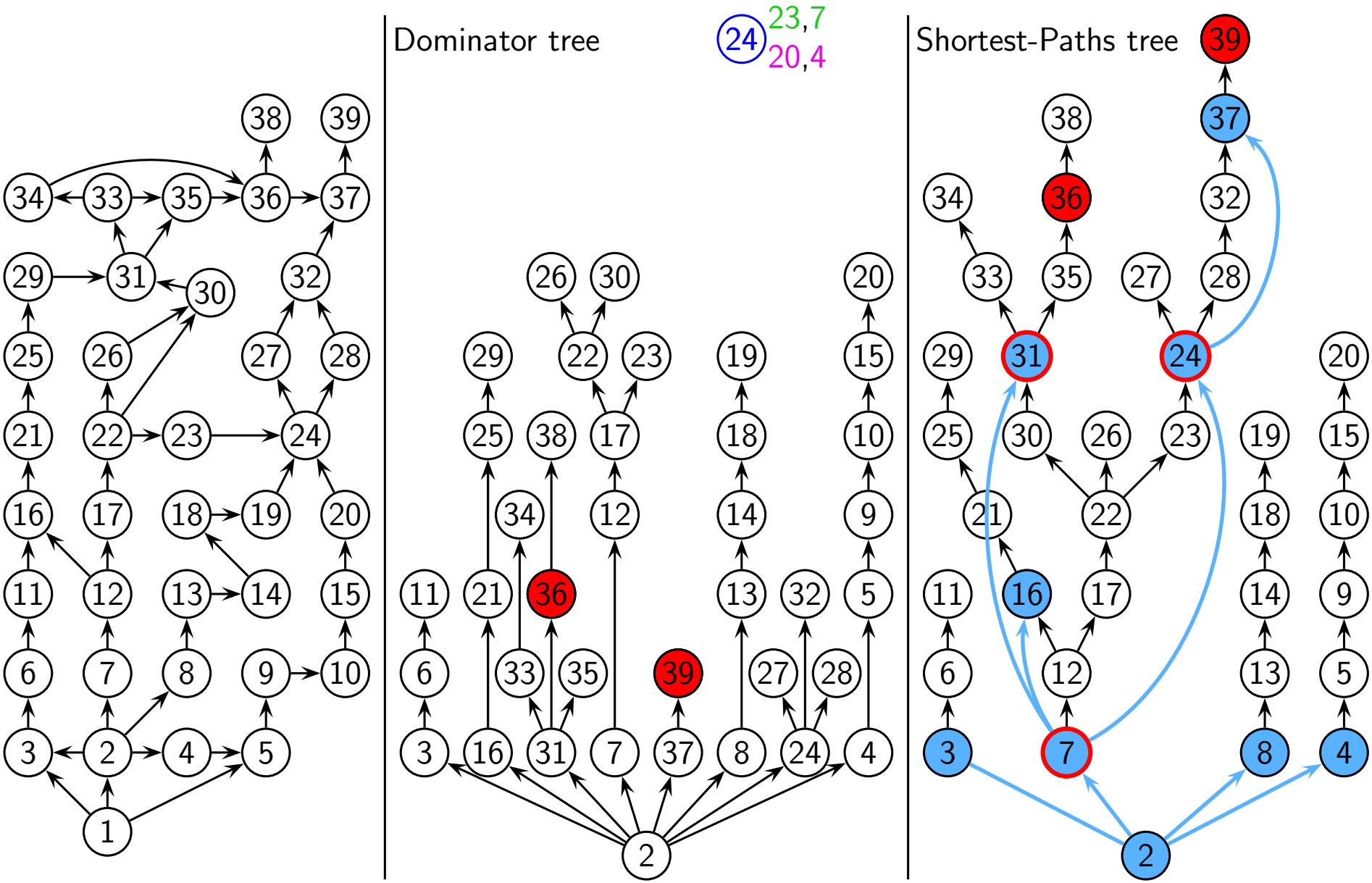

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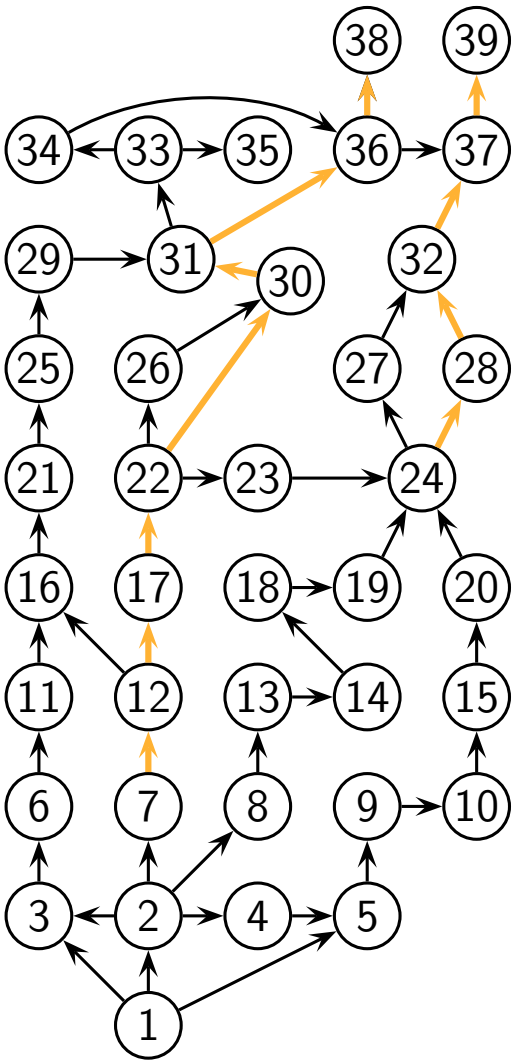
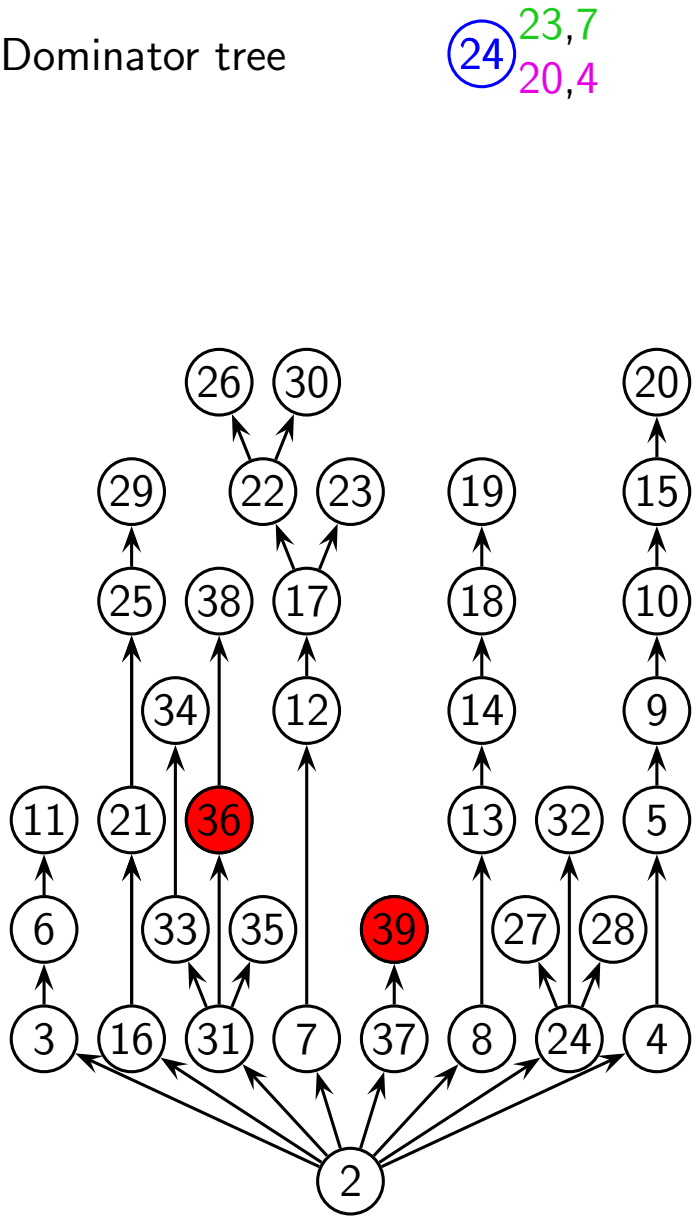
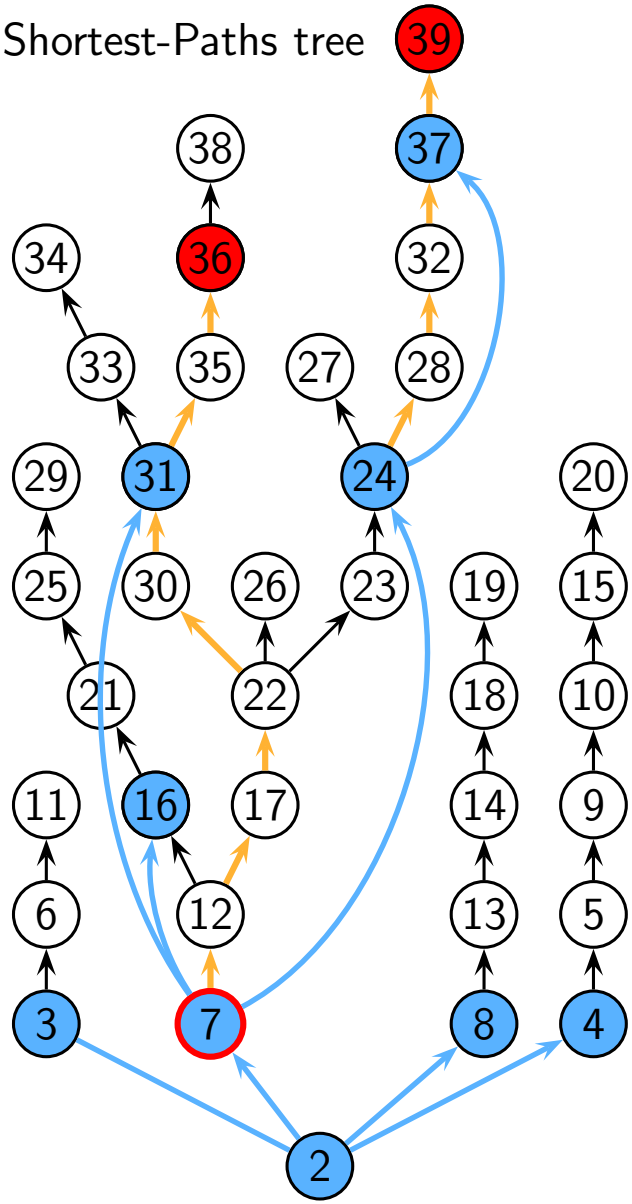


Dominator tree



Shortest-Paths tree



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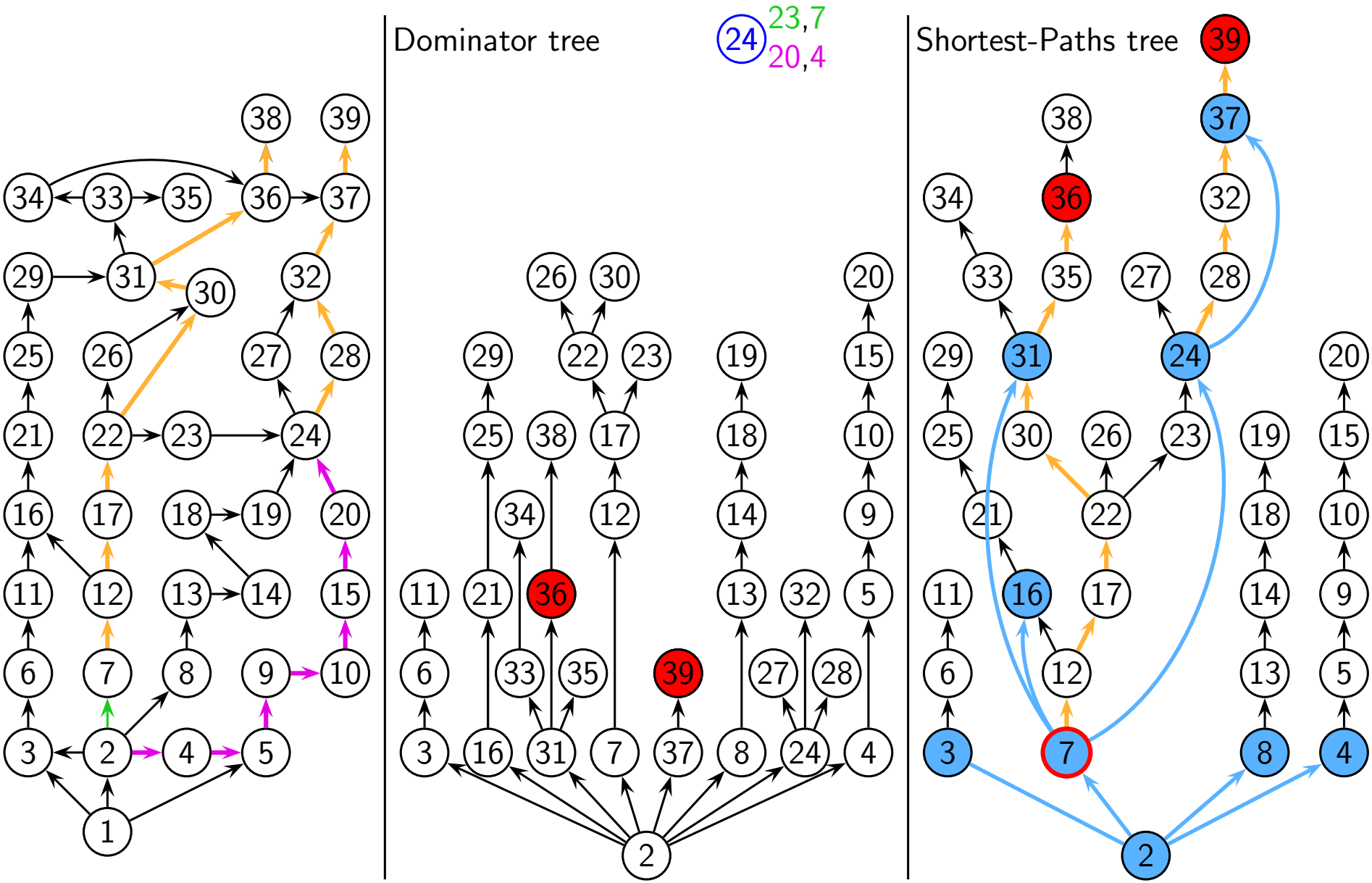
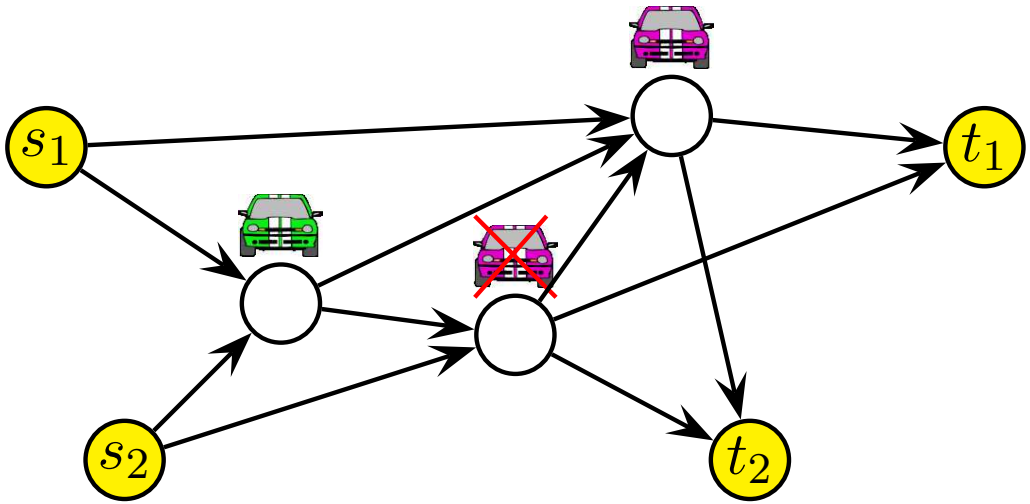


Thank You!



Thank You!