Assignment-2

EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

Shivanshu
AI21BTECH11027

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1. Difference Equation

1.1 **[Question 2.4 from Discrete-Time Signal Processing by Alan V. Oppenheim] :** Consider the linear constant-coefficient difference equation:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1] \tag{1.1}$$

Determine $y[n]$ for $n \geq 0$ when $x(n) = \delta[n]$ and $y(n) = 0, n < 0$.

Solution: By applying Z-transform,

$$\mathcal{Z}\{y[n]\} - \frac{3}{4}\mathcal{Z}\{y[n-1]\} + \frac{1}{8}\mathcal{Z}\{y[n-2]\} = 2\mathcal{Z}\{x[n-1]\} \tag{1.2}$$

As,

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{1.3}$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = 2z^{-1}X(z) \tag{1.4}$$

As,

$$x(n) = \delta[n] \tag{1.5}$$

$$\mathcal{Z}\{x(n)\} = \mathcal{Z}\{\delta(n)\} = 1 \tag{1.6}$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = 2z^{-1} \tag{1.7}$$

$$Y(z)(1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}) = 2z^{-1} \tag{1.8}$$

$$Y(z) = \frac{2z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \tag{1.9}$$

$$= \frac{-8}{4z-1} + \frac{8}{2z-1} \tag{1.10}$$

$$= \frac{4z^{-1}}{1 - \frac{1}{2}z^{-1}} - \frac{2z^{-1}}{1 - \frac{1}{4}z^{-1}} \tag{1.11}$$

Case-1: When ROC is $|z| > \frac{1}{2}$, the signal x(n) is causal. Both terms in (1.11) are causal. As,

$$a^{n-1}u(n-1) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{z^{-1}}{1 - az^{-1}} \quad |z| > |a| \tag{1.12}$$

$$y(n) = 4\left(\frac{1}{2}\right)^{n-1} u(n-1) - 2\left(\frac{1}{4}\right)^{n-1} u(n-1) \tag{1.13}$$

$$y(n) = \left(\frac{1}{2}\right)^{n-3} u(n-1) - \left(\frac{1}{4}\right)^{n-\frac{3}{2}} u(n-1) \tag{1.14}$$

Case-2: When ROC is $\frac{1}{4} < |z| < \frac{1}{2}$, ROC is a ring. The signal is two-sided. Thus the pole $p_1 = \frac{1}{2}$ provides anti-causal part and $p_2 = \frac{1}{4}$ provides causal part.

$$-a^{n-1}u(-n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{z^{-1}}{1 - az^{-1}} \quad |z| < |a| \tag{1.15}$$

$$y(n) = -4\left(\frac{1}{2}\right)^{n-1} u(-n) - 2\left(\frac{1}{4}\right)^{n-1} u(n-1) \tag{1.16}$$

$$= \left(\frac{1}{2}\right)^{n-3} u(-n) - \left(\frac{1}{4}\right)^{n-\frac{3}{2}} u(n-1) \tag{1.17}$$

Case-3: When ROC is $|z| < \frac{1}{4}$, the signal x(n) is anti-causal. Both terms in (1.11) are anti-causal.
Using (1.12),

$$y(n) = -4\left(\frac{1}{2}\right)^{n-1} u(-n) + 2\left(\frac{1}{4}\right)^{n-1} u(-n) \tag{1.18}$$

$$y(n) = \left(\frac{1}{2}\right)^{n-3} u(-n) + \left(\frac{1}{4}\right)^{n-\frac{3}{2}} u(-n) \tag{1.19}$$