Digital Signal Processing

EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

Circuits and Transforms

Shivanshu
AI21BTECH11027

7 Oct 2022

Contents

1. Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2. Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.
2. Draw the circuit using latex-tikz.
Solution: The following code yields Fig.2.2

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/Tikz%20
    Circuits/2.2.tex
```

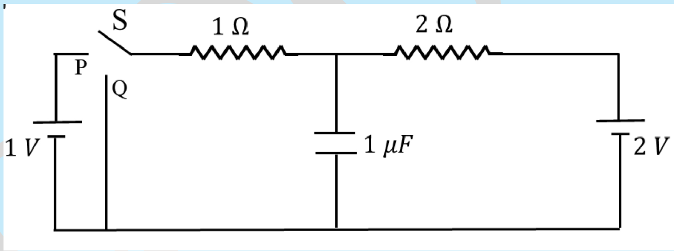

Fig. 2.1.

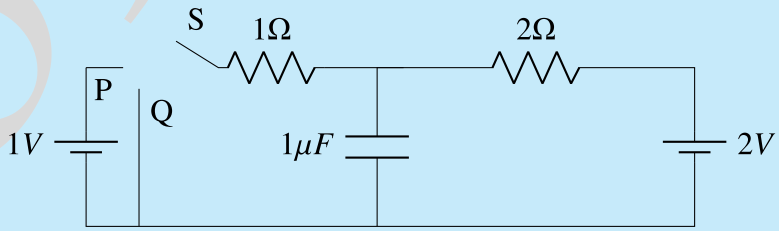

Fig. 2.2. Given Circuit

3. Find q_1.
Solution: Before switching S to Q: Calculating

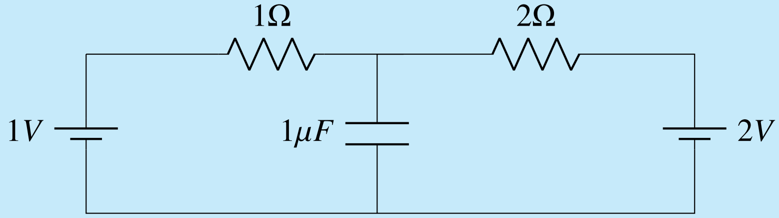

Fig. 2.3. Before switching S to Q

current,

$$1 - i - 2i - 2 = 0 \tag{2.1}$$

$$3i = -1 \Rightarrow i = \frac{-1}{3} \tag{2.2}$$

Potential Difference between capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3} \tag{2.3}$$

$$q_1 = \frac{4}{3} \cdot 1 \tag{2.4}$$

$$= \frac{4}{3}\mu C \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We know that from definition of Laplace Transform,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt U(s) \quad = \int_0^\infty u(t)e^{-st}\,dt \tag{2.6}$$

Using (1.1),

$$U(s) = \int_0^\infty u(t)e^{-st}\,dt \tag{2.7}$$

$$= \int_0^\infty e^{-st}\,dt \tag{2.8}$$

$$= -\left(0 - \frac{1}{s}\right) \tag{2.9}$$

$$= \frac{1}{s} \tag{2.10}$$

ROC is $Re(s) > 0$ since $e^{-st} < \infty$ for $t \to \infty$ The following command plots the ROC of above Laplace Transform.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/codes/2.4.py
```

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{2.11}$$

and find the ROC.
Solution: From (2.6),

$$F(s) = \int_0^\infty u(t)e^{-at}e^{-st}\,dt \tag{2.12}$$

$$= \int_0^\infty u(t)e^{-(s+a)t}\,dt \tag{2.13}$$

$$= \int_0^\infty e^{-(s+a)t}\,dt \tag{2.14}$$

$$= -\left(0 - \frac{1}{s+a}\right) \tag{2.15}$$

$$= \frac{1}{s+a} \tag{2.16}$$

ROC is

$$s + a > 0 \Rightarrow s > -a \tag{2.17}$$

The following command plots the ROC of above Laplace Transform.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/codes/2.5.py
```

6. Now consider the following resistive circuit transformed from Fig. 4.1 where

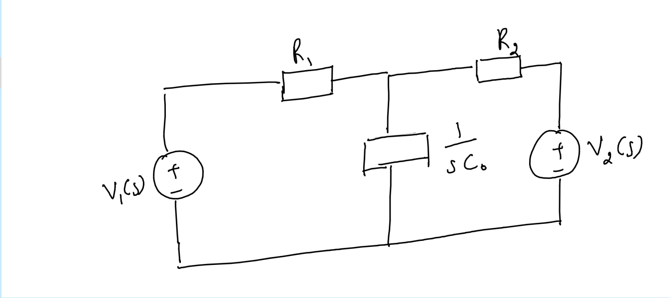

Fig. 2.4.

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.18}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.19}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:

$$R_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}\Omega \tag{2.20}$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}V \tag{2.21}$$

$$V_{C_0}(s) = V_S(s)\frac{C_0}{C_0 + R_{eff}} \tag{2.22}$$
$$= \left(\frac{4}{3s}\right)\left(\frac{\frac{1}{s}}{\frac{1}{s} + \frac{2}{3}}\right) \tag{2.23}$$
$$= \frac{6}{s(6s+9)} \tag{2.24}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Using (2.24),

$$\frac{6}{s(6s+9)} = \frac{4}{3s} - \frac{2}{9+6s} \tag{2.25}$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-\infty}} V_{C_0}(t) \tag{2.26}$$
$$\mathcal{L}^{-1}\left[V_{C_0}(s)\right] = \mathcal{L}^{-1}\left[\frac{4}{3s} - \frac{2}{9+4s}\right] \tag{2.27}$$
$$= \mathcal{L}^{-1}\left[\frac{4}{3s}\right] - \mathcal{L}^{-1}\left[\frac{2}{9+4s}\right] \tag{2.28}$$

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \tag{2.29}$$
$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}u(t) \tag{2.30}$$

Using the above equations,

$$V_{C_0}(t) = \frac{4}{3}\left(1 - e^{\frac{-3}{2}t}\right)u(t) \tag{2.31}$$

The following command plots the above equation.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/codes/2.7.py
```

8. Verify your result using ngspice.
Solution: The following command plots the ROC of above Laplace Transform.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/codes/2.8.cir
python3 2.8.py
```

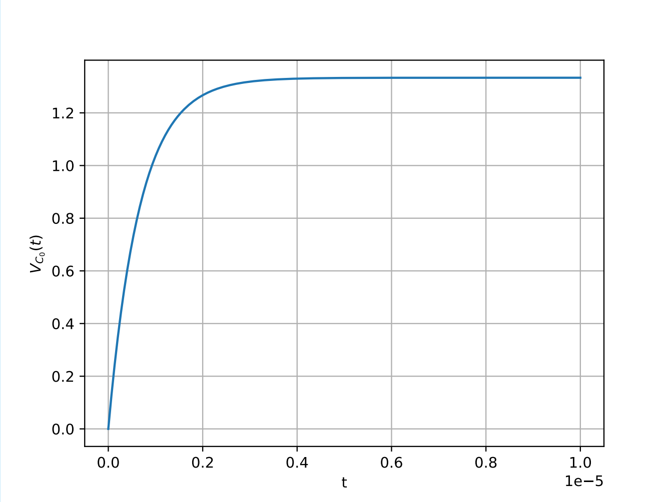

Fig. 2.5. Plot of $V_{C_0}(t)$

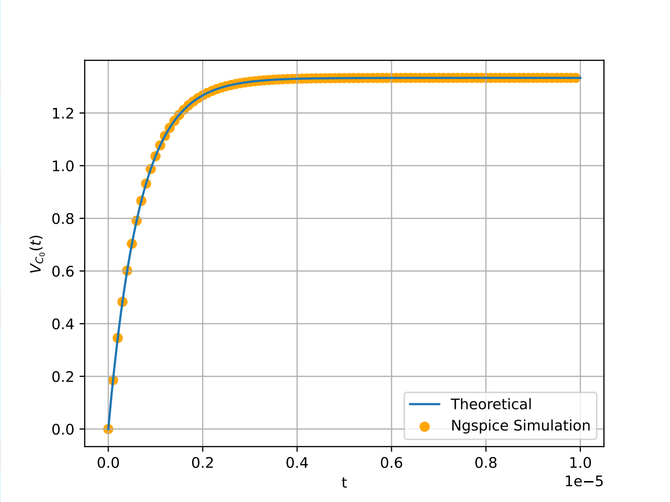

Fig. 2.6.

3. Initial Conditions

1. Find q_2 in Fig. 4.1.
Solution: At steady state, $V_{C_0} = V_{1\Omega}$

$$V_{C_0} = \frac{q_2}{C} = V_{1\Omega} = \frac{2}{1+2} = \frac{2}{3}$$
$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution: The following command plots the ROC of above Laplace Transform.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/Tikz%20
    Circuits/3.2.tex
```

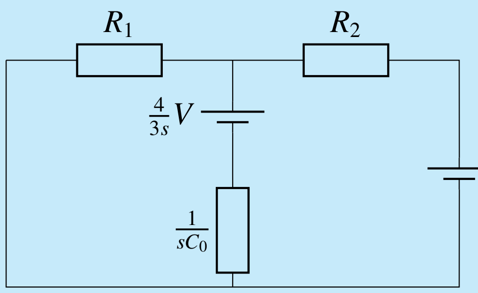

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$
Solution: Using KCL at node in Fig. 3.1

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \tag{3.1}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.2}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: From (3.2),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.3}$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.4}$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{3.5}$$

The following command plots the above equation.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/codes/3.4.py
```

5. Verify your result using ngspice.
Solution: The following command plots Fig.3.3

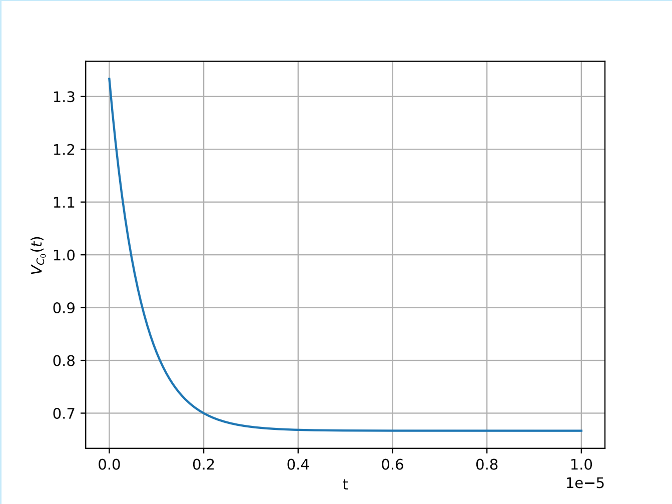

Fig. 3.2. Plot of $V_{C_0}(t)$

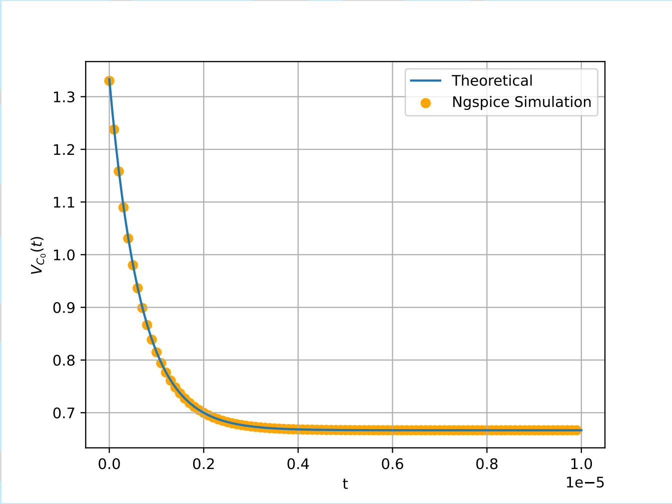

Fig. 3.3.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/Circuit/codes/3.5.cir
```

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \tag{3.6}$$

Using (3.5),

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.7}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.8}$$

4. Bilinear Transform

1. In Fig. 4.1, consider the case when S is

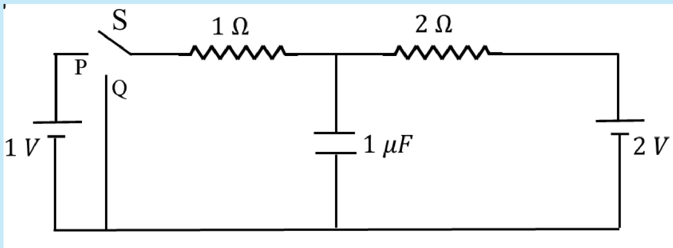

Fig. 4.1.

switched to Q right in the beginning. Formulate the differential equation. **Solution:** The rerquired differential equation is

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{4.1}$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.2}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor. **Solution:** On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - 0 = 0 \tag{4.3}$$

$$\implies V_c(s)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + sC_0V_c(s) = \frac{V_2(s)}{R_2} \tag{4.4}$$

$$\implies \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{4.5}$$

The transfer function is thus

$$H(s) = \frac{\frac{1}{R_2C_0}}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \tag{4.6}$$

On substituting the values, we get

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \tag{4.7}$$

3. Plot $H(s)$. What kind of filter is it?
Solution: The given plot can be plotted using the code given below

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Circuit/codes/4.3.py
```

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.8}$$

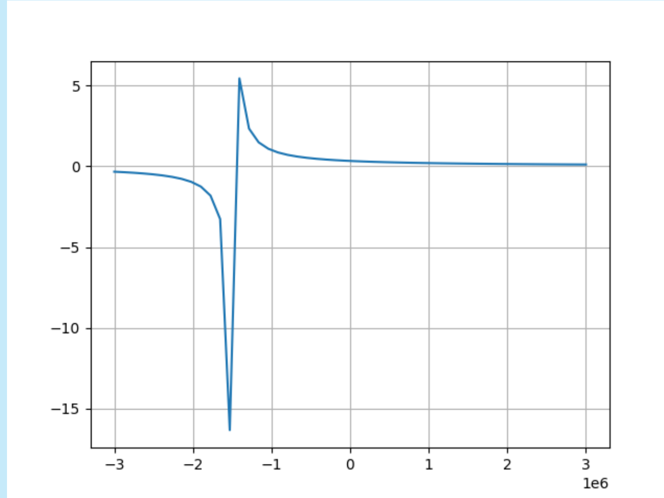

Fig. 4.2. Plot of $H(s)$

Solution:

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = 0 \tag{4.9}$$

$$\implies C_0\frac{\mathrm{d}v_c}{\mathrm{d}t} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \tag{4.10}$$

$$\implies v_c(t)|_{t=n}^{n+1} = \int_n^{n+1}\left(\frac{2u(t) - v_c(t)}{R_2C_0} - \frac{v_c(t)}{R_1C_0}\right)\mathrm{d}t \tag{4.11}$$

On doing the the above integration we will get difference equation as

$$\begin{aligned} y(n+1)&\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) \\ &= y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) \\ &\quad + \frac{1}{R_2C_0}\left(u(n) + u(n+1)\right) \end{aligned} \tag{4.12}$$

5. Find $H(z)$.
Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$
On taking the Z-transform on both sides of the difference equation

$$\begin{aligned} zY(z)&\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) \\ &= Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) \\ &\quad + \frac{1}{R_2C_0}\left(\frac{1}{1 - z^{-1}} + \frac{z}{1 - z^{-1}}\right) \end{aligned} \tag{4.13}$$

$$Y(z)\left(z+\frac{z}{2R_1C_0}+\frac{z}{2R_2C_0}-1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \quad (4.14)$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \quad (4.15)$$

$$\implies x(n) = 2u(n) \quad (4.16)$$

$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \quad (4.17)$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.18)$$

$$= \frac{\frac{1+z}{2R_2C_0}}{z+\frac{z}{2R_1C_0}+\frac{z}{2R_2C_0}-1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}} \quad (4.19)$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}-z^{-1}+\frac{z^{-1}}{2R_1C_0}+\frac{z^{-1}}{2R_2C_0}} \quad (4.20)$$

On substituting the values

$$H(z) = \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5+1+(7.5\times10^5-1)z^{-1}} \quad (4.21)$$

6. How can you obtain $H(z)$ from $H(s)$?
Solution:

$$H(s) = \frac{\frac{1}{R_2C_0}}{s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}} \quad (4.22)$$

put $s = \frac{2(1-z^{-1})}{1+z^{-1}}$ in 4.6.

$$H(z) = \frac{\frac{1+z^{-1}}{2R_2C_0}}{1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}-z^{-1}+\frac{z^{-1}}{2R_1C_0}+\frac{z^{-1}}{2R_2C_0}} \quad (4.23)$$

$$= \frac{2.5\times10^5(1+z^{-1})}{7.5\times10^5+1+(7.5\times10^5-1)z^{-1}} \quad (4.24)$$

7. find y(n).
Solution: let $a = \frac{R_1+R_2}{2C_0R_1R_2}$

$$Y(z) = \frac{1+z^{-1}}{(1-z^{-1})R_2C_0\left(1-z^{-1}+((1+z^{-1})(a))\right)} \quad (4.25)$$

$$\Rightarrow Y(z) = \frac{1+z^{-1}}{R_2C_0(1+a)(1-z^{-1})\left(1-\frac{1-a}{1+a}z^{-1}\right)} \quad (4.26)$$

$$\Rightarrow Y(z) = \frac{1}{R_2C_0(1+a)a}\left(\frac{a+1}{1-z^{-1}}-\frac{1}{1-\frac{(1-a)z^{-1}}{1+a}}\right) \quad (4.27)$$

$$\Rightarrow y(n) = \frac{1}{R_2C_0(1+a)a}\left((a+1)u(n)-\left(\frac{1-a}{1+a}\right)^n u(n)\right) \quad (4.28)$$

with $|z| > \min(1, \frac{|1-a|}{a+1})$.Above is for sampling period= $1s$. In general, for T sampling period

$$y(nT) = \frac{T}{R_2C_0(1+aT)a}\left((aT+1)u(nT)-\left(\frac{1-aT}{1+aT}\right)^{nT} u(n\right. \quad (4.29)$$

s

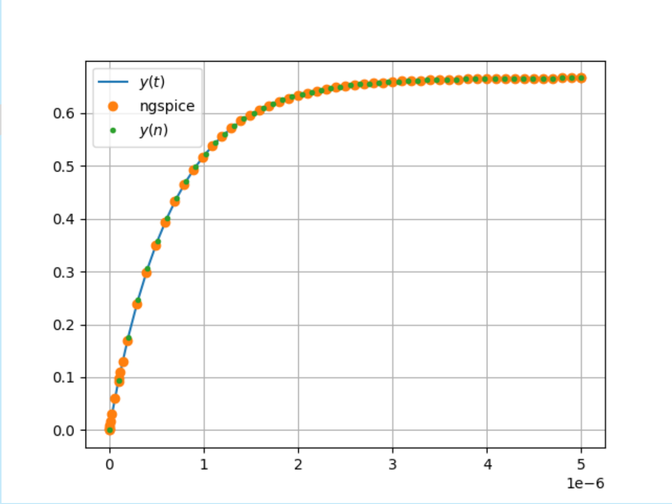

Fig. 4.3.