Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/codes/Sound_Noise.
    wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs
#Hello
#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2.

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What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: Graph of x(n) has been plotted in part 1 of Fig. 3.2.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/codes/xnyn.py
```

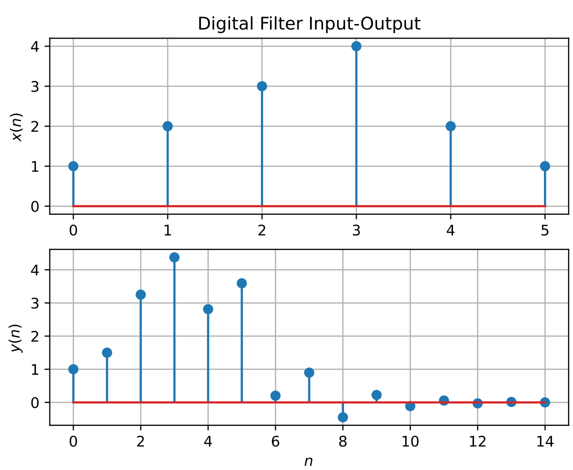

Fig. 3.2

3.3 Repeat the above exercise using a C code. **Solution:** The following code is in c and doing the same function as the above one is doing.

```
wget https://github.com/Shivanshu8211/
    EE3900/blob/master/codes/xnyn.c
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: Given that,

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.4}$$
$$= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.5}$$

So,

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1) z^{-n} \tag{4.6}$$

let $n - 1 = k$,

$$= \sum_{k=-\infty}^{\infty} x(k) z^{-(k+1)} \tag{4.7}$$
$$= z^{-1} \sum_{k=-\infty}^{\infty} x(k) z^{-k} \tag{4.8}$$
$$= z^{-1} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.9}$$
$$= z^{-1} X(z) \tag{4.10}$$

From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.11}$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.12}$$
$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.13}$$

Hence proved.

4.2 Obtain X(z) for x(n) defined in problem 3.1. **Solution:** Given

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{4.14}$$

and the Z-transform of $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{4.15}$$

$$\sum_{n=0}^{5} x(n) z^{-n} = z^{0} + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.16}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.17}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: since Z-transform is a linear operator therefore
y(z) = Y(z) and x(z) = X(z)
So,
on applying (4.13) in (3.2), we get

$$Y(z) + \frac{1}{2} z^{-1} Y(z) = X(z) + z^{-2} X(z) \tag{4.18}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2} z^{-1}} \tag{4.19}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.21}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.22}$$

Solution: The Z-transform of δn is,

$$\mathcal{Z}\{\delta n\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.23}$$

$$= \delta(0) z^{0} \tag{4.24}$$

$$= 1 \tag{4.25}$$

and the Z-transform of unit-step function $u(n)$ is,

$$U(n) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} \tag{4.26}$$

$$= 0 + \sum_{n=0}^{\infty} 1.z^{-n} \tag{4.27}$$

$$= 1 + z^{-1} + z^{-2} + \dots \tag{4.28}$$

and from (4.21),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.29}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.30}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^{n} u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - a z^{-1}} \quad |z| > |a| \tag{4.31}$$

Solution: The Z- transform will be

$$\mathcal{Z}\{a^{n} u(n)\} = \sum_{n=0}^{\infty} a^{n} z^{-n} \tag{4.32}$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^{2} + \dots. \tag{4.33}$$

Above is a infinite geometric series with first term 1 and common ratio as $\frac{a}{z}$ and it can be written as,

$$\mathcal{Z}\{a^{n} u(n)\} = \frac{1}{1 - \frac{a}{z}} \because |a| < |z| \tag{4.34}$$

Therefore,

$$a^{n} u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - a z^{-1}} \quad |z| > |a| \tag{4.35}$$

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.36}$$

Plot $\left|H\left(e^{j\omega}\right)\right|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution: Download the code for the plot **??** from the link below

```
wget https://github.com/
    Shivanshu8211/EE3900/
    blob/master/codes/dtft.py
```

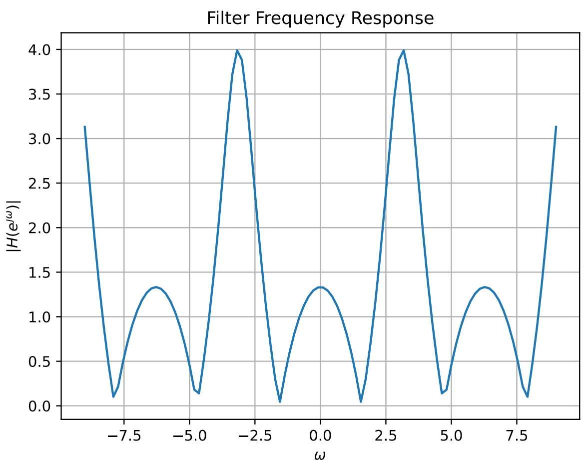

Fig. 4.6: $|H(e^{j\omega})|$

Now using (4.19), we will find $\left|H\left(e^{j\omega}\right)\right|$,

$$H\left(e^{j\omega}\right) = \frac{1+e^{-2j\omega}}{1+\frac{e^{-j\omega}}{2}} \tag{4.37}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1+e^{-2j\omega}\right|}{\left|1+\frac{e^{-j\omega}}{2}\right|} \tag{4.38}$$

$$= \frac{\left|1+e^{2j\omega}\right|}{\left|e^{2j\omega}+\frac{e^{j\omega}}{2}\right|} \tag{4.39}$$

$$= \frac{|1+\cos 2\omega + j\sin 2\omega|}{\left|e^{j\omega}+\frac{1}{2}\right|} \tag{4.40}$$

$$= \frac{\left|4\cos^2(\omega)+4j\sin(\omega)\cos(\omega)\right|}{|2e^{j\omega}+1|} \tag{4.41}$$

$$= \frac{|4\cos(\omega)|\,|\cos(\omega)+j\sin(\omega)|}{|2\cos(\omega)+1+2j\sin(\omega)|} \tag{4.42}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5+4\cos(\omega)}} \tag{4.43}$$

Since $\left|H(e^{j\omega})\right|$ is function of cosine we can say it is periodic.And from the plot **??** we can say that it is symmetric about $\omega = 0$ (even function) and it is periodic with period 2π.You can find the same from the theoritical expression $\left|H\left(e^{j\omega}\right)\right|$,

$$H(e^{j\omega}) = H(e^{j(-\omega)})\,(\text{cos is an even function}) \tag{4.44}$$

And to find period, the period of $|\cos(\omega)|$ is π and the period of $\sqrt{5+4\cos(\omega)}$ is 2π. So the period of division of both will be,

$$lcm(\pi, 2\pi) = 2\pi \tag{4.45}$$

This gives us the period of $\left|H(e^{j\omega})\right|$ as 2π

4.7 Express $h(n)$ in terms of $H\left(e^{j\omega}\right)$.
Solution: We know that

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{4.46}$$

and

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.47}$$

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.48}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.49}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.50}$$

$$= \frac{1}{2\pi}\left\{\sum_{k\neq n} h(k)\left[\frac{e^{j\omega(n-k)}}{j(n-k)}\right]_{-\pi}^{\pi} + h(n)\int_{-\pi}^{\pi} d\omega\right\} \tag{4.51}$$

$$= \frac{0+2\pi h(n)}{2\pi} \tag{4.52}$$

$$= h(n) \tag{4.53}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n<5 \tag{5.1}$$

for H(z) in (4.19).
Solution: From (4.19), we can write

$$H(z) = \frac{1+z^{-2}}{1+\frac{z^{-1}}{2}} \tag{5.2}$$

$$\begin{array}{r|lll}
 & 2z^{-1} & -4 & \\
\hline
1+z^{-1}/2 & 1 & +z^{-2} & \\
 & 2z^{-1} & +z^{-2} & \\
\hline
 & & 1 & -2z^{-1} \\
 & & -4 & -2z^{-1} \\
\hline
 & & & -5
\end{array}$$

So we can replace (4.19) as,

$$\frac{1+z^{-2}}{1+\frac{z^{-1}}{2}} = 2z^{-1} - 4 + \frac{5}{1+z^{-1}/2} \tag{5.3}$$

Now we can expand the second term of above expression as an infinite geometric series,

$$\frac{5}{1+z^{-1}/2} = 5\left(1 + \left(\frac{-1}{2z}\right) + \left(\frac{-1}{2z}\right)^2 + \dots\right) \tag{5.4}$$

where we assume $\left|\frac{1}{2z}\right| < 1$. So (5.3) will become,

$$= 2z^{-1} - 4 + 5 + \frac{-5}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + .. \tag{5.5}$$

$$= 1z^0 + \frac{-1}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + .. \tag{5.6}$$

Now to get $h(n)$ for $n < 5$ we will compare (5.6) with the below equation,

$$H(z) = \sum_{n=-\infty}^{n=\infty} h(n)z^{-n} \tag{5.7}$$

$h(n)$ will be the coefficient of z^{-n}.
Using this, from (5.6) we can write,

$$h(0) = 1 \tag{5.8}$$

$$h(1) = \frac{-1}{2} \tag{5.9}$$

$$h(2) = \frac{5}{4} \tag{5.10}$$

$$h(3) = \frac{-5}{8} \tag{5.11}$$

$$h(4) = \frac{5}{16} \tag{5.12}$$

And for $n < 0$ $h(n) = 0$.

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.13}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: The $H(z)$ can be written as,

$$H(z) = \frac{1}{1+\frac{z^{-1}}{2}} + \frac{z^{-2}}{1+\frac{z^{-1}}{2}} \tag{5.14}$$

From (4.31) we can write it as,

$$h(n) = \left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2) \tag{5.15}$$

5.3 Sketch $h(n)$. Is it bounded? Justify Theoritically.
Solution: Download the code for the plot 5.3 from the below link,

```
wget https://github.com/Charanyash/EE3900−Digital_Signal_Processing/blob/main/Assignment−1/Codes/hn.py
```

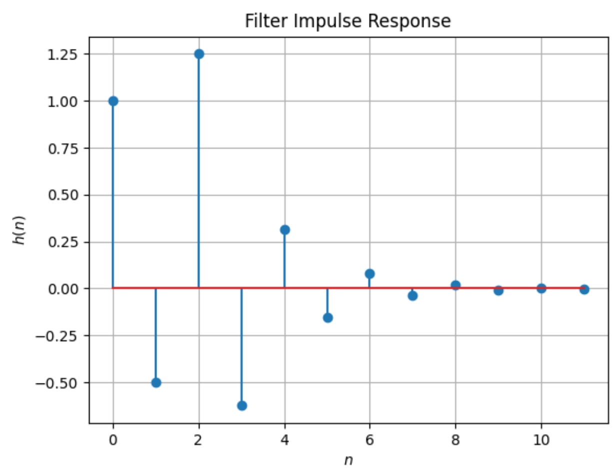

Fig. 5.3: $h(n)$ as inverse of $H(n)$

From the plot it seems like $h(n)$ is bounded and becomes smaller in magnitude as n increases. Using (5.15), we can get theoritical expression as,

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \le n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \ge 2 \end{cases} \tag{5.16}$$

A sequence $\{x_n\}$ is said to be bounded if and only if there exist a positive real number M such that,

$$|x_n| \le M, \forall n \in \mathcal{N} \tag{5.17}$$

So to say $h(n)$ is bounded we should able to find the M which satisfies (5.17).
For n ¡ 0,

$$|h(n)| \le 0 \tag{5.18}$$

For $0 \le n < 2$,

$$|h(n)| = \left|\frac{-1}{2}\right|^n \tag{5.19}$$

$$= \left(\frac{1}{2}\right)^n \le 1 \tag{5.20}$$

And for $n \ge 2$,

$$|h(n)| = \left|5\left(\frac{-1}{2}\right)\right|^n \tag{5.21}$$

$$= \left(\frac{5}{2}\right)^n \le \frac{5}{4} \tag{5.22}$$

From above three cases, we can get M as,

$$M = max\left\{0, 1, \frac{5}{4}\right\} \tag{5.23}$$

$$= \frac{5}{4} \tag{5.24}$$

Therefore, $h(n)$ is bounded using (5.17) with $M = \frac{5}{4}$ i.e.,

$$|h(n)| \le \frac{5}{4} \forall n \in \mathcal{N} \tag{5.25}$$

5.4 Convergent? Justify using the ratio test.
Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.26}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty} \left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty} \left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^n}\right| \tag{5.27}$$

$$= \lim_{n\to\infty} \left|\frac{-1}{2}\right| \tag{5.28}$$

$$= \frac{1}{2} \tag{5.29}$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.30}$$

Is the system defined by (3.2) stable for the impulse response in (5.13)?

Solution: From (5.15),

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{-1}{2}\right)^n u(n) + \left(\frac{-1}{2}\right)^{n-2} u(n-2)\right) \tag{5.31}$$

$$= 2\left(\frac{1}{1+\frac{1}{2}}\right) \tag{5.32}$$

$$= \frac{4}{3} \tag{5.33}$$

$\therefore$ the system is stable.

5.6 Verify the above result using a python code.
Solution: Download the python code from the below link

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/blob/main/
    Assignment−1/Codes/hn_stable.py
```

Then run the following command,

```
python3 hn_stable.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.34}$$

This is the definition of $h(n)$.
Solution: Download the code for the plot 5.7 from the below link,

```
wget https://github.com/Charanyash/EE3900−
    Digital_Signal_Processing/blob/main/
    Assignment−1/Codes/hndef.py
```

Note that this is same as 5.3.
For $n < 0$, $h(n) = 0$ and,

$$h(0) = \delta(0) \tag{5.35}$$

$$= 1 \tag{5.36}$$

For $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) \tag{5.37}$$

$$\implies h(1) = -\frac{1}{2}h(0) \tag{5.38}$$

$$= -\frac{1}{2} \tag{5.39}$$

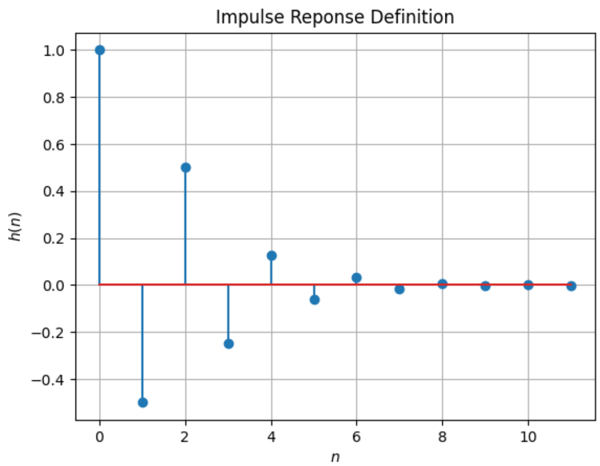

Fig. 5.7: From the definition of $h(n)$

$n = 2$,

$$h(2) + \frac{1}{2}h(1) = 0 + \delta(0) \tag{5.40}$$

$$h(2) = 1 + \frac{1}{4} \tag{5.41}$$

$$= \frac{5}{4} \tag{5.42}$$

And for $n > 2$ RHS will be 0 so,

$$h(n) = -\frac{1}{2}h(n-1) \tag{5.43}$$

Overall

$$h(n) = \begin{cases} 0 & , n < 0 \\ 1 & , n = 0 \\ -\frac{1}{2} & , n = 1 \\ \frac{5}{4} & , n = 2 \\ -\frac{1}{2}h(n-1) & , n > 2 \end{cases} \tag{5.44}$$

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.45}$$

Comment. The operation in (5.45) is known as *convolution*.

Solution: Download the code for plot 5.8 from the below link

```
wget https://github.com/Charanyash/
    EE3900−
    Digital_Signal_Processing/blob/
    main/Assignment−1/Codes/ynconv.
    py
```

Note that the plot is same that as in 3.2.

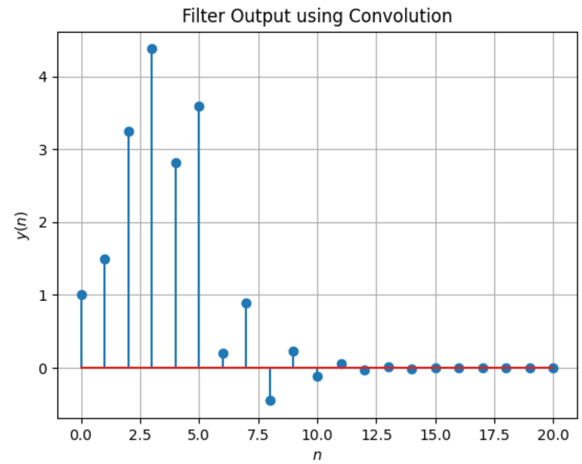

Fig. 5.8: $y(n)$ using the convolution definition

5.9 Express the above convolution using a Teoplitz matrix. **Solution:** For finding the above convolution using topleitz matrix we have to find topleitz matrix of h(n). h(n) is tending to 0 for large n. So,we take upto some n only. So,

$$h(n) = \begin{pmatrix} 1 \\ -.5 \\ . \\ . \end{pmatrix} \quad for \quad n = 0, 2...9 \tag{5.46}$$

So,topleitz matrix of $h(n)$ will be

$$top\{h(n)\} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -.5 & 1 & 0 & 0 & 0 \\ . & . & . & . & . & . \end{pmatrix} \tag{5.47}$$

and

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.48}$$

So,

$$x(n) * h(n) = top\{h(n)\}x(n) \tag{5.49}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \end{pmatrix} \tag{5.50}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.51}$$

Solution: Substitute $k := n - k$ in (5.45), we will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.52}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.53}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.54}$$

6 DFT and FFT

6.1 Compute

$$X(k) \overset{\triangle}{=} \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. **??**. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/yndft.
    py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

6.5 Wherever possible, express all the above equations as matrix equations.