# Temporal Soft Charges and Memory in Asymptotically Flat Gravity: Operational Infrared Structure from Clock Holonomy

## (Paper IV of the Temporal Rasa Series)

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**Abstract**

Papers I–III developed an operational description of relativistic time based on a timelike reference congruence $n^\mu$, a physical observer $U^\mu$, and the measurable scalar $\alpha = -n_\mu U^\mu$. Non-integrability of clock transport is encoded in the vorticity $\omega_{\mu\nu}$ of $n^\mu$.

Here we analyse the infrared structure of this non-integrability at future null infinity $\mathscr{I}^+$. Using the covariant temporal connection

$$\mathcal{A}_\mu = \frac{1}{2\alpha}(\star_U \omega)_\mu, \qquad (\star_U \omega)_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}U^\nu \omega^{\rho\sigma},$$

we show that its radiative mode at $\mathscr{I}^+$ is fixed completely by the Bondi shear $\sigma^0$:

$$\boxed{\mathcal{A}_A^{(0)} = \frac{1}{4}\left(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A\right).}$$

This is a spin-weight $+1$ projection of gravitational radiation and contains no new degrees of freedom.

We define the associated soft symmetry, Wald–Zoupas charge, and the resulting "temporal holonomy memory". Only the electric-parity component of $\sigma^0$ contributes to the permanent effect. These structures establish the IR completion of the operational time geometry developed in Papers I–III.

## Contents

# 1 Introduction

Papers I–III introduced an operational time geometry based on:

- a reference congruence $n^\mu$,
- a physical observer $U^\mu$,
- the observable lapse $\alpha = -n_\mu U^\mu$,
- and the vorticity two-form $\omega_{\mu\nu}$ of $n^\mu$.

Clock transport is path dependent whenever $\omega_{\mu\nu} \neq 0$. The resulting temporal connection,

$$\mathcal{A}_\mu = \frac{1}{2\alpha}(\star_U \omega)_\mu,$$

encodes this non-integrability.

In asymptotically flat spacetimes, the IR limit of this structure projects onto the Bondi radiative data. Our goals:

1. Identify the radiative mode $\mathcal{A}_A^{(0)}$ at $\mathscr{I}^+$.
2. Show it is determined entirely by the shear $\sigma^0$.
3. Define the associated soft symmetry and charge.
4. Derive a "temporal memory" in clock holonomy.

No new fields are introduced; all IR data descend from the gravitational shear.

# 2 Temporal Connection and Holonomy

Let $n^\mu$ be a unit timelike congruence and $U^\mu$ a physical observer. The vorticity of $n^\mu$ is

$$\omega_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma \nabla_{[\rho} n_{\sigma]}, \qquad h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu.$$

The spatial Hodge dual relative to $U^\mu$ is

$$(\star_U \omega)_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} U^\nu \omega^{\rho\sigma}.$$

**Definition 2.1 (Temporal connection)**

$$\boxed{\mathcal{A}_\mu = \frac{1}{2\alpha}(\star_U \omega)_\mu.}$$

For any small loop $C$ in the rest frame of $U^\mu$,

$$\oint_C \mathcal{A}_\mu dx^\mu = \int_\Sigma \omega_{\mu\nu} dS^{\mu\nu},$$

the vorticity flux first identified in Paper I.

## 3 Asymptotic Structure at $\mathscr{I}^+$

Bondi coordinates $(u, r, x^A)$ yield

$$\sigma = \frac{\sigma^0(u, x^A)}{r} + O(r^{-2}), \qquad m_A = m_\mu \partial_A x^\mu.$$

A physically natural asymptotic observer satisfies

$$U^\mu = \frac{\ell^\mu + N^\mu}{\sqrt{2}} + O(r^{-1}), \qquad \alpha = 1 + O(r^{-1}),$$

and $n^\mu$ aligns with $\ell^\mu$.

Projecting $\mathcal{A}_\mu$ to the sphere:

$$\mathcal{A}_A = \mathcal{A}_\mu \partial_A x^\mu = \frac{1}{r}\mathcal{A}_A^{(0)} + O(r^{-2}).$$

**Theorem 3.1 (Temporal–shear correspondence)**

$$\boxed{\mathcal{A}_A^{(0)} = \frac{1}{4}(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A).}$$

This is a spin-weight +1 field determined by shear alone.

**Properties.**

- No Coulombic ($\ell = 0, 1$) data contribute.
- No new state-dependent function enters; IR data is inherited from $\sigma^0$.
- Under $m_A \to e^{i\chi} m_A$, the field transforms with spin +1.

## 4 Electric and Magnetic Parity

Decompose

$$\sigma^0 = \sigma_E^0 + i\sigma_B^0.$$

In a real dyad basis $\{e_A^{(1)}, e_A^{(2)}\}$ defined through $m_A = \frac{1}{\sqrt{2}}(e_A^{(1)} + ie_A^{(2)})$, the radiative mode becomes:

$$\mathcal{A}_A^{(0)} = \frac{1}{2}\big[\sigma_E^0\, e_A^{(1)} - \sigma_B^0\, e_A^{(2)}\big].$$

Thus:

$$\boxed{\mathcal{A}_A^E \propto \sigma_E^0, \qquad \mathcal{A}_A^B \propto \sigma_B^0.}$$

Only $\sigma_E^0$ contributes to the permanent memory.

## 5 Soft Symmetry and Charge

At $\mathscr{I}^+$, define the operational symmetry

$$\delta_\Lambda \mathcal{A}_A^{(0)} = 0,$$

(no radiative gauge term appears at leading order), and the charge:

$$\boxed{Q_\Lambda = \frac{1}{32\pi} \int_{S^2} \Lambda\, D^A \big(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A\big) d\Omega.}$$

Only $\ell \geq 2$ modes contribute. The algebra is abelian:

$$\{Q_\Lambda, Q_{\Lambda'}\} = 0, \qquad \{Q_\Lambda, Q_\xi^{\mathrm{BMS}}\} = 0.$$

## 6 Temporal Memory

Since $N = \partial_u \sigma^0$,

$$\partial_u \mathcal{A}_A^{(0)} = \frac{1}{4}(N m_A + \bar{N} \bar{m}_A).$$

Integrating:

$$\Delta \mathcal{A}_A^{(0)} = \frac{1}{4}(\Delta\sigma^0 m_A + \Delta\bar{\sigma}^0 \bar{m}_A).$$

The change in a clock holonomy around a loop $C$ is:

$$\boxed{\Delta \ln \mathcal{H}[C] = \oint_C \Delta \mathcal{A}_A^{(0)} dx^A.}$$

If $\Delta \mathcal{A}_A^{(0)} = \partial_A f$ (pure gauge), the integral vanishes. The permanent temporal memory arises only from the electric parity of $\sigma^0$, exactly paralleling E-mode displacement memory.

## 7 Observer-Class Invariance

If $\tilde{U}^\mu$ is any asymptotically inertial observer:

$$\tilde{U}^\mu = U^\mu + O(r^{-1}), \qquad U_\mu \tilde{U}^\mu = -1 + O(r^{-1}),$$

then

$$\tilde{\mathcal{A}}_A^{(0)} = \mathcal{A}_A^{(0)}.$$

Thus the temporal IR mode is invariant under the standard Bondi class of asymptotic inertial observers.

## 8 Conclusion

The operational temporal connection of Papers I–III acquires a clean infrared limit at future null infinity. The radiative mode

$$\mathcal{A}_A^{(0)} = \frac{1}{4}(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A)$$

encodes a spin-weight +1 projection of gravitational radiation, defines a consistent abelian soft charge, and generates a permanent temporal memory proportional to the E-mode shear.

The temporal IR sector contains no new dynamical degrees of freedom: it is a physically operational projection of the Bondi shear. This prepares the ground for the temporal information channel and soft entanglement developed in Papers V–VIII.

## A Derivation of the Asymptotic Formula

The Bondi–Sachs NP tetrad satisfies

$$\ell^\mu N_\mu = -1, \qquad m^\mu \bar{m}_\mu = 1.$$

At large $r$:

$$\ell^\mu = \partial_r + O(r^{-1}), \qquad m^\mu = \frac{1}{\sqrt{2}r}\hat{m}^A \partial_A + O(r^{-2}),$$

$$\sigma = \frac{\sigma^0}{r} + O(r^{-2}).$$

NP torsion identities yield

$$\nabla_{[\mu}\ell_{\nu]} = -\sigma\, m_{[\mu} m_{\nu]} - \bar{\sigma}\, \bar{m}_{[\mu}\bar{m}_{\nu]} + O(r^{-2}).$$

Since $n^\mu \sim \ell^\mu$,

$$\omega_{\mu\nu} = -\frac{1}{r}\left(\sigma^0 m_{[\mu} m_{\nu]} + \bar{\sigma}^0 \bar{m}_{[\mu}\bar{m}_{\nu]}\right) + O(r^{-2}).$$

Contract with $U^\nu = (\ell^\nu + N^\nu)/\sqrt{2}$:

$$\omega_{\nu A} U^\nu = \frac{1}{2r}(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A) + O(r^{-2}).$$

Since $\alpha = 1 + O(r^{-1})$,

$$\mathcal{A}_A = \frac{1}{2\alpha}\omega_{\nu A} U^\nu = \frac{1}{4r}(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A) + O(r^{-2}),$$

giving the stated result.

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