Operational Reconstruction of Local Spacetime Geometry in General Relativity: Clocks, Light, and Congruence Kinematics as Complete Observables

(Paper II of the Temporal Rasa Series)

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Abstract

Paper I introduced the operational lapse $\alpha = -n_\mu U^\mu$ and a temporal connection encoding the non-integrability of observer time. Here we show that, for any smooth physical observer congruence U^μ, the directly measurable operational data set

$$\{\theta,\ \sigma_{\mu\nu},\ \omega_{\mu\nu},\ a_\mu;\ \rho, p, q_\mu, \pi_{\mu\nu}\}$$

is sufficient to reconstruct the full local spacetime geometry $(g_{\mu\nu}, T_{\mu\nu})$ up to diffeomorphisms on a simply-connected convex normal neighborhood.

The reconstruction proceeds in four steps: (i) symmetric kinematics from photon frequency transport, (ii) vorticity from infinitesimal temporal holonomies, (iii) curvature from the Einstein–Bianchi system, and (iv) metric reconstruction via Cartan structure equations.

This establishes the operational foundation for temporal gauge structure and infrared temporal geometry developed in later papers.

Contents

1 Observer Kinematics and Operational Quantities

We take signature $(-+++)$ and use a physical observer congruence U^μ with $U_\mu U^\mu = -1$. The spatial projector is

$$h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu. \tag{1.1}$$

The spatial covariant derivative acting on any tensor $X_{\nu\cdots}$ is

$$D_\mu X_{\nu\cdots} := h_\mu{}^\rho h_\nu{}^\sigma \cdots \nabla_\rho X_{\sigma\cdots}, \tag{1.2}$$

with all free indices projected into the observer rest space.

The 1+3 decomposition of $\nabla_\mu U_\nu$ is

$$\nabla_\mu U_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} + \tfrac{1}{3}\theta\, h_{\mu\nu} - U_\mu a_\nu, \tag{1.3}$$

where $a_\mu = U^\rho \nabla_\rho U_\mu$ is locally measurable (by an accelerometer).

We retain the reference congruence n^μ from Paper I and the operational lapse

$$\alpha = -n_\mu U^\mu > 0. \tag{1.4}$$

Operationally,

$$\alpha = \frac{d\tau_n}{d\tau_U},$$

the Lorentz factor of U^μ relative to n^μ.

The temporal one-form is $\tau = n_\mu dx^\mu$. The temporal connection $\mathcal{A}_\mu$ is defined by

$$\mathcal{A}_\mu dx^\mu = \alpha^{-1}\tau.$$

2 Photon Frequency Transport and Symmetric Kinematics

Let k^μ be an affinely parametrized null geodesic and $\nu = -k_\mu U^\mu$ the measured photon frequency.

Paper I derived a general exact transport identity for ν/α in terms of $(\theta, \sigma_{\mu\nu}, \omega_{\mu\nu}, a_\mu^{(n)}, v^\mu)$ for an arbitrary pair of congruences U^μ and n^μ. When the reference congruence is chosen to coincide with the physical observer,

$$n^\mu = U^\mu, \qquad \alpha = 1, \qquad v^\mu = 0, \tag{2.1}$$

this decomposition becomes trivial and the general law reduces consistently to the standard observer-based frequency transport:

$$\frac{d\nu}{d\lambda} = -k^\mu k^\nu \left(\sigma_{\mu\nu} + \tfrac{1}{3}\theta\, h_{\mu\nu}\right). \tag{2.2}$$

Because $k^\mu k^\nu$ is symmetric, vorticity does not contribute.

At an event p, write $k^\mu = \omega(U^\mu + N^\mu)$ with N^μ spatial and $N_\mu N^\mu = 1$. Define the directional transport coefficient

$$\Delta_p(N) = -\sigma_{\mu\nu} N^\mu N^\nu - \tfrac{1}{3}\theta. \tag{2.3}$$

Proposition 2.1. *Full-sky directional measurements of $\Delta_p(N)$ uniquely determine θ and $\sigma_{\mu\nu}$ at p.*

Proof. Expand $\Delta_p(N)$ in spherical harmonics on the observer's sky. The $\ell = 0$ mode yields θ, the $\ell = 2$ modes yield the five independent components of $\sigma_{\mu\nu}$; higher harmonics vanish by construction. □

3 Temporal Holonomy and Vorticity

Temporal non-integrability is encoded in $\tau = n_\mu dx^\mu$:

$$d\tau = 2\nabla_{[\mu} n_{\nu]} dx^\mu \wedge dx^\nu = 2(\omega_{\mu\nu} - n_{[\mu} a^{(n)}_{\nu]}) dx^\mu \wedge dx^\nu. \tag{3.1}$$

The curvature of the temporal connection $\mathcal{A}_\mu = \alpha^{-1} n_\mu$ is

$$F_{\mu\nu} = 2\nabla_{[\mu} \mathcal{A}_{\nu]}.$$

For infinitesimal spatial 2-surfaces orthogonal to U^μ, the *vorticity* term dominates the holonomy, and one effectively measures $\omega_{\mu\nu}$.

For a small spatial loop C enclosing a 2-surface with area bivector $S^{\mu\nu}$ in the rest space of U^μ, we have

$$\oint_C \mathcal{A}_\mu dx^\mu = \omega_{\mu\nu} S^{\mu\nu} + O(\epsilon^3). \tag{3.2}$$

Here $S^{\mu\nu}$ is spatial, $S^{\mu\nu} U_\nu = 0$.

Proposition 3.1. *Holonomies over three independent infinitesimal spatial loops determine $\omega_{\mu\nu}$ uniquely.*

Proof. Choose three independent spatial 2-planes in the observer rest space. The three resulting holonomies give three independent linear combinations of the components of $\omega_{\mu\nu}$, which can be inverted to obtain $\omega_{\mu\nu}$. □

4 Operational Dataset

At each event, the measured dataset is

$$\mathcal{O}_{\mathrm{op}}[U] = \{\theta, \sigma_{\mu\nu}, \omega_{\mu\nu}, a_\mu;\ \rho, p, q_\mu, \pi_{\mu\nu}\}.$$

Matter variables can be defined operationally by local stress-energy measurements in the rest frame of U^μ.

5 Einstein–Bianchi Closure and Curvature

5.1 Ricci Tensor

Einstein's equation gives

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right), \tag{5.1}$$

so once $(\rho, p, q_\mu, \pi_{\mu\nu})$ are known, the Ricci tensor is determined.

5.2 Weyl Tensor

Define the electric and magnetic parts of the Weyl tensor with respect to U^{μ}:

$$E_{\mu\nu} = C_{\mu\alpha\nu\beta}U^{\alpha}U^{\beta}, \qquad B_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\alpha\beta}C^{\alpha\beta}{}_{\nu\gamma}U^{\gamma}. \tag{5.2}$$

The spatial Bianchi constraints take the form

$$D^{\nu}\sigma_{\mu\nu} - \tfrac{2}{3}D_{\mu}\theta + 4\pi q_{\mu} = -E_{\mu\nu}U^{\nu}, \tag{5.3}$$
$$D^{\nu}\omega_{\mu\nu} = B_{\mu\nu}U^{\nu}. \tag{5.4}$$

These constraints determine the longitudinal components $E_{\mu\nu}U^{\nu}$ and $B_{\mu\nu}U^{\nu}$. Together with symmetry, trace-free conditions, and the evolution equations, this fixes the full tensors $E_{\mu\nu}$ and $B_{\mu\nu}$.

The evolution equations can be written schematically as

$$\dot{E}_{\langle\mu\nu\rangle} = -\theta E_{\mu\nu} + \mathrm{curl}B_{\mu\nu} + \mathcal{S}^{(E)}_{\mu\nu}, \tag{5.5}$$
$$\dot{B}_{\langle\mu\nu\rangle} = -\theta B_{\mu\nu} - \mathrm{curl}E_{\mu\nu} + \mathcal{S}^{(B)}_{\mu\nu}, \tag{5.6}$$

where the source terms $\mathcal{S}^{(E)}_{\mu\nu}$, $\mathcal{S}^{(B)}_{\mu\nu}$ depend algebraically on $(\theta, \sigma_{\mu\nu}, \omega_{\mu\nu}, a_{\mu})$ and the matter fields.

5.3 Full Riemann Tensor

The Riemann tensor decomposes as

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu} - \tfrac{1}{3}R\, g_{\mu[\rho}g_{\sigma]\nu}. \tag{5.7}$$

Thus $R_{\mu\nu\rho\sigma}$ is fully determined once $(E_{\mu\nu}, B_{\mu\nu})$ and $R_{\mu\nu}$ are known.

6 Metric Reconstruction via Cartan Integration

Choose an orthonormal frame $e^{a}{}_{\mu}$ at a point in a simply-connected convex normal neighborhood $\mathcal{U}$. The torsion-free, metric-compatible Cartan equations are

$$de^{a} + \omega^{a}{}_{b} \wedge e^{b} = 0, \tag{6.1}$$
$$d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b} = R^{a}{}_{b}, \tag{6.2}$$

where $R^{a}{}_{b}$ is the curvature 2-form corresponding to $R^{\rho}{}_{\sigma\mu\nu}$.

Given the curvature on $\mathcal{U}$, these equations uniquely determine $(e^{a}{}_{\mu}, \omega^{a}{}_{b})$ up to a global Lorentz transformation. The metric is then reconstructed as

$$g_{\mu\nu} = \eta_{ab}e^{a}{}_{\mu}e^{b}{}_{\nu}. \tag{6.3}$$

The remaining freedom is diffeomorphism invariance.

7 Reconstruction Theorem

Theorem 7.1. *Let $\mathcal{U}$ be a simply-connected convex normal neighborhood. If the operational data $\mathcal{O}_{\mathrm{op}}[U]$ satisfy the Einstein–Bianchi system, then there exists a unique spacetime geometry $(g_{\mu\nu}, T_{\mu\nu})$ on $\mathcal{U}$, up to diffeomorphisms, that reproduces all operational measurements.*

Sketch. Sections 1.3–2.3 reconstruct θ and $\sigma_{\mu\nu}$ from photon transport. Section 3 reconstructs $\omega_{\mu\nu}$ from temporal holonomy. Matter variables fix $R_{\mu\nu}$ via Einstein's equation. The Bianchi constraints (5.3)–(5.4) and evolution equations determine $E_{\mu\nu}$ and $B_{\mu\nu}$, hence the full curvature. Cartan's equations then integrate the orthonormal frame and metric uniquely modulo diffeomorphisms. □

8 Examples

Plane gravitational wave. For an observer congruence adapted to a plane gravitational wave, shear measurements determine $\dot{h}_{+,\times}$ via the standard relation between $\sigma_{\mu\nu}$ and the transverse-traceless metric perturbation. The Bianchi evolution then yields curvature components $\propto \ddot{h}_{+,\times}$, and Cartan integration reconstructs the local TT waveform up to gauge.

Weak static potential. For static observers in a weak-field metric, $a_i = \partial_i \Phi$ directly measures the Newtonian potential's gradient. Einstein's equation reduces to Poisson's equation $\nabla^2 \Phi = 4\pi\rho$. Integrating recovers Φ and hence the metric to the relevant order.

Rotating congruence. In flat spacetime, a rigidly rotating congruence has $\omega_{\mu\nu} \neq 0$ but vanishing curvature. Temporal holonomies detect non-zero vorticity even though $R_{\mu\nu\rho\sigma} = 0$, distinguishing rotation from gravity.

9 Discussion

The key result is that all kinematical components of $\nabla_\mu U_\nu$ (shear, expansion, vorticity, and acceleration) are operationally measurable: the symmetric part from photon frequency transport and the antisymmetric part from temporal holonomy. Together with matter variables, the Einstein–Bianchi system determines the curvature, and Cartan integrability reconstructs the metric.

Although n^μ enters intermediate constructions such as α and the temporal connection, the reconstructed geometric quantities are independent of this choice. The temporal connection introduced in Paper I provides an operational route to vorticity and will play a central role in the temporal gauge structure developed in subsequent papers.

References

[1] S. S. Galagali, "Operational Time Geometry in General Relativity," (2025).

[2] G. F. R. Ellis, "Dynamics of pressure-free matter in general relativity," J. Math. Phys. **8**, 1171 (1967).

[3] J. Ehlers, "Contributions to the Relativistic Mechanics of Continuous Media," Gen. Rel. Grav. **25**, 1225 (1993) (English translation of the 1961 original).

[4] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press (1973).

[5] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology*, Cambridge University Press (2012).

[6] E. Cartan, "Sur les variétés à connexion affine," Ann. Éc. Norm. Sup. **39**, 325 (1922).

[7] R. W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*, Springer (1997).

[8] T. Frankel, *The Geometry of Physics*, Cambridge University Press (2011).

[9] R. Maartens and B. A. Bassett, "Gravito-electromagnetism," Class. Quant. Grav. **15**, 705 (1998).

[10] G. F. R. Ellis and H. van Elst, "Cosmological Models," in *Theoretical and Observational Cosmology*, Springer (1999).

[11] H. Friedrich, "Hyperbolic reductions for Einstein's equations," Class. Quant. Grav. **13**, 1451 (1996).

[12] J. L. Synge, *Relativity: The General Theory*, North-Holland (1960).

[13] V. A. Brumberg, *Essential Relativistic Celestial Mechanics*, Adam Hilger (1991).

[14] S. M. Kopeikin, M. Efroimsky and G. Kaplan, *Relativistic Celestial Mechanics of the Solar System*, Wiley (2011).

[15] N. Ashby, "Relativity in the Global Positioning System," Living Rev. Relativ. **6**, 1 (2003).

[16] P. Delva and J. Lodewyck, "Atomic clocks: new prospects in metrology and geodesy," Acta Futura **10**, 67 (2017).

[17] P. Wolf et al., "Testing Relativity Using Space Clocks," Gen. Rel. Grav. **48**, 10 (2016).

[18] A. Ramos and C. F. Sopuerta, "Gravitational-wave measurements of shear and expansion," Phys. Rev. D **100**, 104016 (2019).

[19] C. F. Sopuerta, "Kinematical quantities in general relativity and their measurement," Class. Quant. Grav. **21**, 771 (2004).

[20] A. Raychaudhuri, "Relativistic Cosmology I," Phys. Rev. **98**, 1123 (1955).

[21] S. Kar and S. SenGupta, "The Raychaudhuri equations: A brief review," Pramana **69**, 49 (2007).

[22] R. M. Wald, *General Relativity*, University of Chicago Press (1984).

[23] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Freeman (1973).

[24] E. Poisson, *A Relativist's Toolkit*, Cambridge University Press (2004).

[25] Y. Choquet–Bruhat, *General Relativity and the Einstein Equations*, Oxford University Press (2009).