Newman–Penrose Formulation of the Temporal Connection: Asymptotic Shear Control of Operational Time Transport

(Paper III of the Temporal Rasa Series)

Shivaraj S. Galagali[1]

[1]Independent Researcher, shivarajsgalagali5@gmail.com

December 10, 2025

Abstract

Paper I introduced the operational lapse $\alpha = -n_\mu U^\mu$ and showed that temporal holonomies measure the vorticity flux of a timelike reference congruence. Paper II established that clocks, light, and vorticity reconstruct the full local geometry.

In this third paper we introduce a covariant *temporal connection* derived from the vorticity of the reference congruence,

$$\mathcal{A}_\mu := \frac{1}{4\alpha}\varepsilon_{\mu\nu\rho\sigma}U^\nu\omega^{\rho\sigma} = \frac{1}{2\alpha}(\star_U\omega)_\mu, \qquad \alpha > 0,$$

and reformulate it in the Newman–Penrose / Bondi–Sachs framework. We show that its leading angular component at future null infinity is controlled entirely by the Bondi shear σ^0:

$$\boxed{\mathcal{A}_A^{(0)} = \frac{1}{4}\big(\sigma^0 m_A + \bar{\sigma}^0\bar{m}_A\big).}$$

This identifies a direct operational imprint of gravitational radiation on time-transport holonomies. Stationary spacetimes, including Kerr, satisfy the construction and yield the correct frame-dragging limit. Applications to temporal memory and asymptotic charges will be explored in future work.

Contents

Notation and Conventions

We use signature $(-,+,+,+)$. All kinematical fields $(\theta, \sigma_{\mu\nu}, \omega_{\mu\nu})$ refer to the timelike reference congruence n^μ used to define the operational lapse $\alpha = -n_\mu U^\mu$. The vorticity in this paper is always that of n^μ, not of U^μ.

A Bondi Newman–Penrose tetrad $(\ell^\mu, N^\mu, m^\mu, \bar{m}^\mu)$ satisfies

$$\ell^\mu N_\mu = -1, \qquad m^\mu \bar{m}_\mu = 1.$$

Directional derivatives are:

$$\dot{X} := U^\mu \nabla_\mu X, \qquad X' := n^\mu \nabla_\mu X.$$

In NP calculations we use the standard notation

$$D := \ell^\mu \nabla_\mu, \qquad \Delta := N^\mu \nabla_\mu.$$

1 Introduction

Paper I showed that operational synchronization is path dependent whenever the reference congruence n^μ carries vorticity. Paper II established that clocks, light, and vorticity reconstruct the full local geometry. The present paper extends the framework to null infinity and identifies the asymptotic structure of operational time transport.

We introduce a temporal connection $\mathcal{A}_\mu$ built directly from the spatial dual of the vorticity of n^μ with respect to an observer U^μ. We show that at $\mathscr{I}^+$, its leading angular component is determined by the Bondi shear. This provides a natural operational projection of gravitational radiation.

Asymptotic Behaviour

A physically natural reference congruence approaching $\mathscr{I}^+$ becomes asymptotically aligned with the outgoing null generator:

$$n^\mu = \frac{\ell^\mu}{\sqrt{-\ell \cdot N}} + O(r^{-1}), \qquad \alpha \to 1,$$

and asymptotically inertial observers satisfy

$$U^\mu = \frac{\ell^\mu + N^\mu}{\sqrt{2}} + O(r^{-1}).$$

2 Operational Temporal Connection

Paper I introduced the temporal 1-form

$$\tau = n_\mu dx^\mu.$$

Its exterior derivative is

$$d\tau = 2\nabla_{[\mu} n_{\nu]}\, dx^\mu \wedge dx^\nu = 2(\omega_{\mu\nu} - n_{[\mu} a^{(n)}_{\nu]})\, dx^\mu \wedge dx^\nu. \tag{2.1}$$

For any observer U^μ, the spatial Hodge dual of a spatial 2-form $X_{\mu\nu}$ is

$$(\star_U X)_\rho := \tfrac{1}{2} \varepsilon_{\rho\mu\nu\sigma} U^\mu X^{\nu\sigma}.$$

Then for small loops C in the rest frame of U^μ,

$$\oint_C (\star_U X)_\mu dx^\mu = \int_\Sigma X_{\mu\nu} dS^{\mu\nu}$$

whenever $X_{\mu\nu}$ is approximately constant across the surface.

We now package this into the temporal connection:

Definition 2.1 (Temporal connection)**.**

$$\boxed{\mathcal{A}_\mu = \frac{1}{4\alpha}\varepsilon_{\mu\nu\rho\sigma}U^\nu\omega^{\rho\sigma} = \frac{1}{2\alpha}(\star_U\omega)_\mu.} \tag{2.2}$$

Its spatial projection satisfies

$$h^\mu{}_\nu \mathcal{A}_\mu = \frac{1}{\alpha}(\star_U\omega)_\nu,$$

so the spatial components of $\mathcal{A}_\mu$ encode exactly the vorticity pseudo-vector that controls temporal holonomies (Paper I).

3 NP Formulation at $\mathscr{I}^+$

The Bondi shear expands as

$$\sigma = \frac{\sigma^0}{r} + O(r^{-2}), \qquad s(\sigma^0) = +2.$$

Define the sphere projection:

$$\mathcal{A}_A = \mathcal{A}_\mu\,\partial_A x^\mu, \qquad m_A = m_\mu \partial_A x^\mu.$$

We write the asymptotic expansion

$$\mathcal{A}_A = \frac{1}{r}\mathcal{A}_A^{(0)} + O(r^{-2}).$$

Theorem 3.1 (Asymptotic Temporal–Shear Correspondence)**.** *Let n^μ be asymptotically aligned with ℓ^μ and let U^μ be asymptotically inertial. Then*

$$\boxed{\mathcal{A}_A^{(0)} = \frac{1}{4}\big(\sigma^0 m_A + \bar{\sigma}^0\bar{m}_A\big).} \tag{3.1}$$

Properties.

- Under a spin rotation $m_A \to e^{i\chi}m_A$, $\mathcal{A}_A^{(0)} \to e^{i\chi}\mathcal{A}_A^{(0)}$.
- Coulombic terms contribute only to $O(r^{-2})$.
- $\mathcal{A}_A^{(0)}$ is a spin-weight $+1$ field on S^2.

4 Gauge Behaviour at $\mathscr{I}^+$

A Bondi supertranslation $u \to u + f(x^A)$ shifts

$$\sigma^0 \to \sigma^0 - \eth^2 f.$$

Thus

$$\mathcal{A}_A^{(0)} \to \mathcal{A}_A^{(0)} - \frac{1}{4}\big((\eth^2 f)m_A + (\bar{\eth}^2 f)\bar{m}_A\big).$$

5 Stationary Consistency Check: Kerr

For Kerr, the Bondi shear satisfies $\sigma^0 = 0$. Choosing $n^\mu = U^\mu_{\mathrm{ZAMO}}$ gives a temporal connection whose leading sphere component vanishes:

$$\mathcal{A}^{(0)}_A = 0.$$

This matches Theorem 3.1 and the expected frame-dragging behaviour at subleading order.

6 Conclusion

We introduced a covariant temporal connection built from the vorticity of a timelike reference congruence and derived its asymptotic form at $\mathscr{I}^+$. Our main result is:

$$\mathcal{A}^{(0)}_A = \frac{1}{4}\big(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A\big),$$

identifying a direct operational imprint of gravitational radiation on time-transport holonomies.

A NP Derivation of Theorem 3.1

At large radius,

$$\ell^\mu = \partial_r + O(r^{-1}), \qquad m^\mu = \frac{1}{\sqrt{2}\,r}\hat{m}^A \partial_A + O(r^{-2}).$$

The shear expands as

$$\sigma = \frac{\sigma^0}{r} + O(r^{-2}).$$

NP torsion identities yield

$$\nabla_{[\mu}\ell_{\nu]} = -\sigma m_{[\mu} m_{\nu]} - \bar{\sigma}\bar{m}_{[\mu}\bar{m}_{\nu]} + O(r^{-2}).$$

Since n^μ is asymptotically aligned with ℓ^μ, the vorticity components satisfy

$$\omega_{\mu\nu} = -\frac{1}{r}\left(\sigma^0 m_{[\mu} m_{\nu]} + \bar{\sigma}^0 \bar{m}_{[\mu}\bar{m}_{\nu]}\right) + O(r^{-2}).$$

Contracting with $U^\mu = (\ell^\mu + N^\mu)/\sqrt{2} + O(r^{-1})$ yields

$$\omega_{\nu A} U^\nu = -\frac{1}{2r}\big(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A\big) + O(r^{-2}).$$

Using $\mathcal{A}_A = \frac{1}{2\alpha}(\star_U \omega)_A$ and $\alpha \to 1$,

$$\mathcal{A}_A = \frac{1}{4r}\big(\sigma^0 m_A + \bar{\sigma}^0 \bar{m}_A\big) + O(r^{-2}),$$

establishing Theorem 3.1.

References

[1] R. M. Wald, *General Relativity*, University of Chicago Press (1984).

[2] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, Freeman (1973).

[3] E. T. Newman and R. Penrose, "An Approach to Gravitational Radiation by a Method of Spin Coefficients," J. Math. Phys. **3**, 566 (1962).

[4] H. Bondi, M. van der Burg, A. Metzner, Proc. Roy. Soc. A **269**, 21 (1962).

[5] R. K. Sachs, Proc. Roy. Soc. A **270**, 103 (1962).

[6] R. Penrose and W. Rindler, *Spinors and Space-Time*, Vols. 1–2, Cambridge University Press.

[7] R. Geroch, "Asymptotic Structure of Space-Time," in *Asymptotic Structure of Space-Time*, Plenum (1977).

[8] A. Strominger, *Lectures on the Infrared Structure of Gravity*, Princeton University Press (2023).

[9] J. Winicour, "Characteristic Evolution and Matching," Living Rev. Relativ. **15**, 2 (2012).

[10] Shivaraj S. Galagali, "Operational Time Geometry in General Relativity," (2025). doi:10.5281/zenodo.17813825.

[11] Shivaraj S. Galagali, "Operational Reconstruction of Local Spacetime Geometry," (2025). doi:10.5281/zenodo.17833292.

[12] M. L. Ruggiero and A. Tartaglia, "Gravitomagnetic Effects," Int. J. Mod. Phys. D **13**, 905 (2004).

[13] B. Mashhoon, "Gravitoelectromagnetism: A Brief Review," Class. Quant. Grav. **25**, 085014 (2008).

[14] Y. B. Zeldovich and A. G. Polnarev, "Radiation of Gravitational Waves by a Cluster of Relativistic Particles," Sov. Astron. **18**, 17 (1974).

[15] D. Christodoulou, "Nonlinear Nature of Gravitation and Gravitational-Wave Experiments," Phys. Rev. Lett. **67**, 1486 (1991).

[16] K. S. Thorne, "Gravitational-Wave Bursts with Memory," Phys. Rev. D **45**, 520 (1992).

[17] W. Kinnersley, "Type D Vacuum Metrics," J. Math. Phys. **10**, 1195 (1969).

[18] S. A. Teukolsky, "Rotating Black Holes: Separable Wave Equations for Gravitational and Electromagnetic Perturbations," Phys. Rev. Lett. **29**, 1114 (1972).