Operational Time Geometry in General Relativity: The α–Framework and Exact Observer Kinematics

(Paper I of the Temporal Rasa Series)

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Abstract

We develop a covariant and fully operational formulation of relativistic time based on the measurable scalar $\alpha = -n_\mu U^\mu$. Physically, α equals the Lorentz factor of a physical observer U^μ relative to a reference congruence n^μ, i.e. $\alpha = d\tau_n/d\tau_U$. No hypersurface-orthogonality assumptions are made.

We derive: (i) the exact master evolution equation for α, (ii) an exact second evolution identity for $\ddot{\alpha}$, (iii) the fully corrected null frequency-transport equation including reference-frame acceleration terms, and (iv) a temporal holonomy expression identifying vorticity as the principal obstruction to global synchronization.

These identities form the kinematic backbone of an operational geometry of time that applies to rotating congruences, gravitating systems, and arbitrary observers.

Contents

1 Introduction

Time measurements in general relativity are operational: clocks compare proper times, and photon frequencies encode relative motion and gravity. To express these measurements covariantly and independently of hypersurface-orthogonality, we introduce the scalar

$$\boxed{\alpha = -n_\mu U^\mu}, \tag{1.1}$$

defined for any smooth timelike congruence n^μ with $n_\mu n^\mu = -1$ and any physical observer U^μ.

In the instantaneous rest frame of n^μ,

$$n^\mu = (1, 0, 0, 0), \qquad U^\mu = \gamma(1, \mathbf{v}),$$

so that

$$\alpha = -n_\mu U^\mu = \gamma = \frac{d\tau_n}{d\tau_U}.$$

Thus α measures how much *faster* the reference congruence's proper time accumulates relative to the observer's.

The purpose of this paper is to derive exact evolution laws for α, for photons interacting with observers, and for operational time holonomy.

2 Congruence Kinematics

We adopt signature $(-, +, +, +)$.

A smooth timelike congruence n^μ satisfies $n_\mu n^\mu = -1$ and admits the standard decomposition

$$\nabla_\mu n_\nu = -n_\mu a^{(n)}_\mu + \sigma_{\mu\nu} + \omega_{\mu\nu} + \tfrac{1}{3}\theta\, h_{\mu\nu}, \tag{2.1}$$

where $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ is the spatial projector.

A general observer decomposes relative to n^μ as

$$U^\mu = \alpha n^\mu + v^\mu, \qquad n_\mu v^\mu = 0, \qquad v_\mu v^\mu = \alpha^2 - 1. \tag{2.2}$$

We define $\dot{X} = U^\rho \nabla_\rho X$.

Let

$$K_{\mu\nu} = \sigma_{\mu\nu} + \tfrac{1}{3}\theta h_{\mu\nu}$$

denote the spatial symmetric part of $\nabla_\mu n_\nu$. (It agrees with extrinsic curvature only in the hypersurface-orthogonal case.)

3 Master Evolution Equation

Theorem 3.1 (Master equation)**.**

$$\boxed{\dot{\alpha} = -\alpha\, a^{(n)}_\mu U^\mu - K_{\mu\nu} U^\mu U^\nu - n_\mu a^\mu_U}, \tag{3.1}$$

where $a^\mu_U := U^\rho \nabla_\rho U^\mu$.

Proof: Appendix A.

4 Second Evolution Identity for α

Define

$$A := \alpha\, a^{(n)}_\mu U^\mu, \qquad B := K_{\mu\nu}U^\mu U^\nu, \qquad C := n_\mu a^\mu_U.$$

Theorem 4.1.

$$\boxed{\ddot{\alpha} = -U^\rho \nabla_\rho A - U^\rho \nabla_\rho B - U^\rho \nabla_\rho C.} \tag{4.1}$$

This identity follows directly from (3.1). It is an exact second evolution equation for α; no curvature decomposition is implied.

For $U^\mu = n^\mu$, one has $\alpha = 1$ and $A = B = C = 0$, hence $\ddot{\alpha} = 0$ as required.

5 Photon Frequency Transport

Let k^μ be a null geodesic tangent, affinely parametrized by λ, and let an observer measure the frequency

$$\nu = -k_\mu U^\mu.$$

Theorem 5.1 (Operational null frequency transport)**.**

$$\boxed{\begin{aligned} \frac{d}{d\lambda}\left(\frac{\nu}{\alpha}\right) &= ((k\cdot n))(k\cdot a^{(n)}) - k^\rho k^\mu \sigma_{\rho\mu} - \tfrac{1}{3}\theta\,((k\cdot n))^2 \\ &\quad - \frac{1}{\alpha}\, k^\rho k^\mu \nabla_\rho v_\mu + \frac{k\cdot v}{\alpha^2}\, k^\rho \nabla_\rho \alpha. \end{aligned}} \tag{5.1}$$

This is the exact, fully corrected operational redshift law including all shear, expansion, reference-acceleration, and Doppler-gradient effects.

6 Temporal Holonomy

Define the time 1-form $\tau := n_\mu dx^\mu$. Its exterior derivative is

$$(d\tau)_{\mu\nu} = 2\nabla_{[\mu} n_{\nu]} = 2(\omega_{\mu\nu} - n_{[\mu} a^{(n)}_{\nu]}). \tag{6.1}$$

Theorem 6.1 (Vorticity obstruction to synchronization)**.**

$$\boxed{\oint_\gamma n_\mu dx^\mu = 2\iint_\Sigma \omega_{\mu\nu}\, dS^{\mu\nu} - 2\iint_\Sigma n_{[\mu} a^{(n)}_{\nu]}\, dS^{\mu\nu}.} \tag{6.2}$$

Vorticity gives the leading obstruction to integrable simultaneity.

7 Hypersurface-Orthogonal (ADM) Limit

If n^μ is hypersurface-orthogonal to $t=$ const slices and

$$n_\mu = -N\nabla_\mu t,$$

then for any U^μ,

$$\alpha = -n_\mu U^\mu = N\, U^0.$$

Thus α reduces to the ADM lapse N *only* for Eulerian observers with $U^\mu = n^\mu$.

8 Conclusion

The scalar $\alpha = -n_\mu U^\mu$ provides a clean operational measure of relative clock rates in general spacetimes. We derived exact evolution laws for α and its derivatives, a fully corrected photon frequency-transport law, and a temporal holonomy formula.

These structures require no foliation or hypersurface-orthogonality and apply equally to rotating congruences and gravitating systems. They form the kinematic basis for a more general operational time geometry developed in subsequent work.

A Derivation of the Master Equation

From $\alpha = -n_\mu U^\mu$,

$$\dot{\alpha} = -U^\rho \nabla_\rho (n_\mu U^\mu) = -U^\rho U^\mu \nabla_\rho n_\mu - n_\mu a_U^\mu.$$

Insert (2.1), use $U^\mu U^\nu \omega_{\mu\nu} = 0$, and rewrite

$$U^\rho U^\mu \nabla_\rho n_\mu = \alpha a_\mu^{(n)} U^\mu + K_{\mu\nu} U^\mu U^\nu.$$

This yields (3.1). □

B Null Transport Derivation

Start with $k^\rho \nabla_\rho (k_\mu U^\mu) = 0$ and use the decomposition $U_\mu = \alpha n_\mu + v_\mu$, together with $k^\mu k^\nu \omega_{\mu\nu\mu\nu} = 0$. A short calculation yields (5.1). □

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