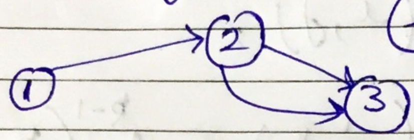
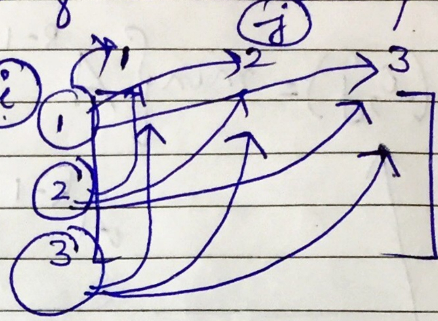
All-Pairs Shortest Paths problem

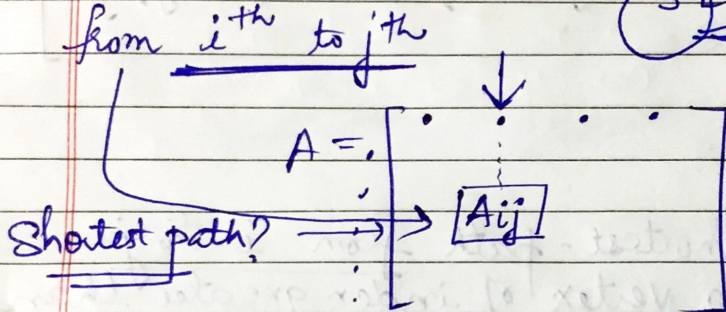
(using Dynamic Programming)

Floyd-Warshal Algorithm

⇒ to determine a matrix A such that $A(i, j)$ is the length of a shortest path from i to j.







∴ ⇒ **All-pairs shortest paths**

Dynamic Programming :→ (i.e. involving principle of optimality)

⇒ involves sequence of decisions !

$$\boxed{A^{k}(i,j) = \min\left\{A^{k-1}(i,j),\ A^{k-1}(i,k) + A^{k-1}(k,j)\right\} \quad (\text{for } k \geq 1)}$$

All-pairs shortest paths

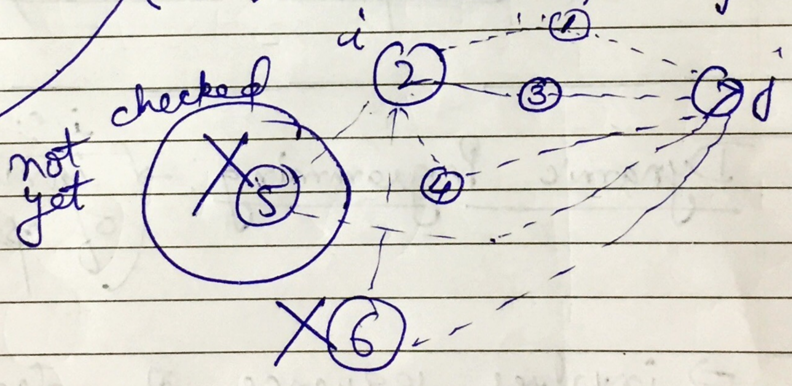
(using dynamic programming) → Floyd – Warshal Algorithm

$$\boxed{A^{k}(i,j) = \min\left\{A^{k-1}(i,j),\ A^{k-1}(i,k) + A^{k-1}(k,j)\right\}}$$

The length of a shortest-path from i to j going through no vertex of index greater than k.

(e.g. $A^{4}(2,7)$ → shortest path from

4 से बड़ी Value अभी नहीं Check हुई

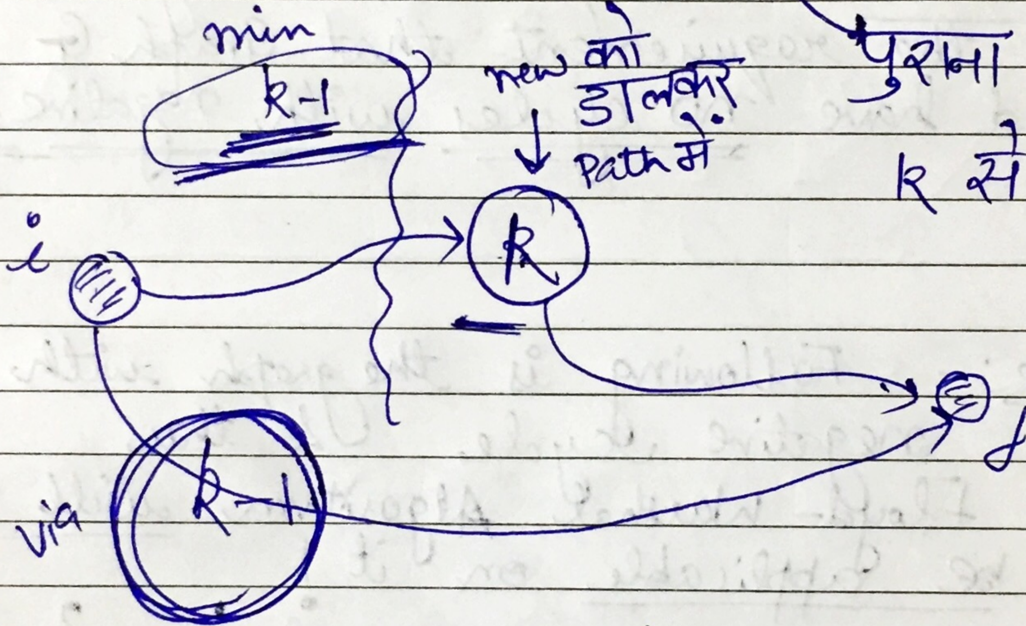


$$A^{k}(i,j) = \min\{A^{k-1}(i,j),\ A^{k-1}(i,k) + A^{k-1}(k,j)\}$$

पुराना Decision

पुराना Decision
i से k तक

पुराना Decision
k से j तक



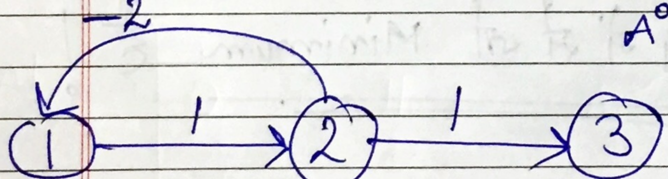
इन में से जो Minimum है।

$$\boxed{A^k(i,j) = \min\left\{ A^{k-1}(i,j),\ A^{k-1}(i,k) + A^{k-1}(k,j) \right\}}$$

Is not true for graphs with cycles of negative length.

<u>Important note</u>:- Floyd-Warshal Algorithm has the requirement that Graph G should have <u>no cycles</u> <u>with negative length</u>.

<u>for example</u>:- Following is the graph with negative cycle & thus Floyd-Warshal Algorithm <u>will not be applicable</u> on it!



$$A^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & \infty \\ -2 & 0 & 1 \\ \infty & \infty & 0 \end{bmatrix} \end{array}$$

<u>not going through any vertex</u>

$$A^0(i,j) = \underline{cost(i,j)}$$

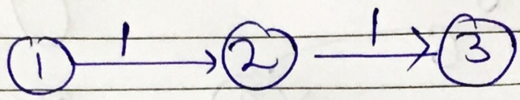
only considers <u>direct edge</u>

via vertex 2

$A^2(1,3)$

from to

If no negative cycle was there!



$$A^2(1,3) = \min\{A^1(1,3),\ A^1(1,2) + A^1(2,3)\}$$

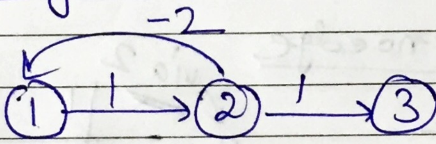
via 2

$$= \min\{\infty,\ 1+1\}$$

$$= 2$$

where $A^1(1,3) = \min\{A^0(1,3)\} = \infty$

But if cycle is there :—



→ $A^2(1,3) = \min\{A^1(1,3),\ A^1(1,2) + A^1(2,3)\}$

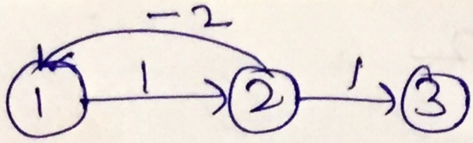
→ $A^1(1,2) = \min\{A^0(1,2)\} = ? = -\infty$

⇒ $A^1(2,3) = \min\{A^0(2,3)\} = 1$

$\boxed{1,2,1,2,1,2,1,2,1,2}\ \boxed{\ldots}\ \boxed{1,2}$

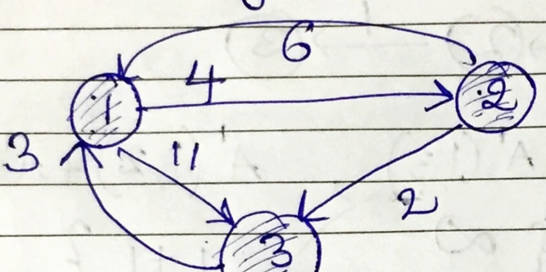
$\therefore$ $A^2(1,3)$ is $-\infty$

the length of path (can be made arbitrarily small) e.g.



If we allow graph G to contain a cycle of negative length, then the shortest path between any two vertices on this cycle has length $-\infty$.

Floyd-Warshal Algorithm



$O(V^3)$ or $O(n^3)$

$n \rightarrow$ vertices

$A^0 =$

	1	2	3
1	0	4	11
2	6	0	2
3	3	∞	0

for no cycle (or loop)

for noedge

via 1

$A^1 =$

	1	2	3
1	0	4	11
2	6	0	2
3	3	7	0

via 2

$A^2 =$

	1	2	3
1	0	4	6
2	6	0	2
3	3	7	0

via 3

$A^3 =$

	1	2	3
1	0	4	6
2	5	0	2
3	3	7	0