Bio- and neuro-imaging methods: Computed Tomography

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Reference material : Principles of computerized tomographic imaging by Avinash C.Kak and Malcolm Slaney, IEEE press, 1988.

Imaging a medium based on energy absorbance measurements

Parametrisation of straight lines in the plane

Let $\ell_\theta(s)$ be a straight line in the Euclidean plane, determined through its support vector $\vec{s}$ of length $\|\vec{s}\| = s$ and having angle θ with the canonical vector $\vec{e}_x$, i.e., $\langle \vec{e}_s, \vec{e}_x \rangle = \cos(\theta)$ and $\langle \vec{e}_s, \vec{e}_y \rangle = \sin(\theta)$ where $\vec{e}_s = \frac{\vec{s}}{\|\vec{s}\|} = \cos(\theta)\,\vec{e}_x + \sin(\theta)\,\vec{e}_y$. A vector $\vec{u}$ determines a point (with respect to the origin) on the line if it satisfies $\langle \vec{u}, \vec{e}_s \rangle = s$. In coordinates:

$$u_x s_x + u_y s_y = s_x^2 + s_y^2 = s^2$$

We may express our points in a new coordinate system $(\vec{e}_s, \vec{e}_z)$, where $\vec{e}_z = R_{\pi/2}(\vec{e}_s)$, a rotation of $\vec{e}_s$ over an angle of $\frac{\pi}{2}$. Hence, in the canonical coordinate system, we have $\vec{e}_z = \cos\left(\theta + \frac{\pi}{2}\right)\vec{e}_x + \sin\left(\theta + \frac{\pi}{2}\right)\vec{e}_y = -\sin(\theta)\,\vec{e}_x + \cos(\theta)\,\vec{e}_y$. Any point on the straight line can now be given in the new coordinate system as $\vec{u} = \langle \vec{u}, \vec{e}_s \rangle \vec{e}_s + \langle \vec{u}, \vec{e}_z \rangle \vec{e}_z = s\vec{e}_s + z\vec{e}_z$, where s is a constant (by construction of our straight line), and $z \in \mathbb{R}$. We thus have (see also figure 1)

$$\ell_\theta(s) = \left\{ s\vec{\mathrm{e}}_s + z\vec{\mathrm{e}}_z \middle| z \in \mathbb{R}, \vec{\mathrm{e}}_s = \cos\theta\vec{\mathrm{e}}_x + \sin\theta\vec{\mathrm{e}}_y, \vec{\mathrm{e}}_z = -\sin\theta\vec{\mathrm{e}}_x + \cos\theta\vec{\mathrm{e}}_y \right\} \ .$$

Absorbance of ray energy when traversing a medium

We will here derive the de Beer–Lambert law of absorbance. Thereto we will take an infinitesimal part δz of the ray with respect to $\vec{e}_z$ (z-coordinates on the ray). At both sides of the infinitesimal part of the medium, we have a relation between the incoming and outgoing intensity (energy) that is determined by the local attenuation coefficient $\mu(s, z) > 0$ as follows

$$I(s\vec{e}_s + (z + \delta z)\,\vec{e}_z) - I(s\vec{e}_z + z\vec{e}_z) = -\mu(s\vec{e}_z + z\vec{e}_z)\,\delta z\, I(s\vec{e}_z + z\vec{e}_z) \ . \tag{1}$$

The law implies that the difference of intensity is proportional to the length of the homogeneous medium that is traversed

In the limit $\delta z \to 0$, equation (1) becomes

$$\frac{\partial}{\partial z} I(s\vec{e}_z + z\vec{e}_z) = -\mu(s\vec{e}_z + z\vec{e}_z)\, I(s\vec{e}_z + z\vec{e}_z) \iff I(s\vec{e}_z + z\vec{e}_z) = \exp\left(-\int^{z} \mu(s\vec{e}_z + v\vec{e}_z)\,\mathrm{d}v\right) \ .$$

Let the initial intensity at the source (emitter) be given by $\exp\left(-\int^{z_0} \mu(s\vec{e}_z + v\vec{e}_z)\,\mathrm{d}v\right) = I(s\vec{e}_z + z_0\vec{e}_z)$, we then obtain

$$\frac{I(s\vec{e}_z + z\vec{e}_z)}{I(s\vec{e}_z + z_0\vec{e}_z)} = \exp\left(-\int_{z_0}^{z} \mu(s\vec{e}_z + v\vec{e}_z)\,\mathrm{d}v\right) \ .$$

From here on we will only work with the log-absorbance values, i.e.,

$$\log\left(\frac{I(s\vec{e}_z + z_0\vec{e}_z)}{I(s\vec{e}_z + z\vec{e}_z)}\right) = \int_0^z \mu(s\vec{e}_z + v\vec{e}_z)\,\mathrm{d}v > 0 \ . \tag{2}$$

The inverse problem

The goal is to inverse a set of equations of the type (2), so we can get from the measurements $I(s\vec{e}_z + z\vec{e}_z)$ and the initial conditions $I(s\vec{e}_z + z_0\vec{e}_z)$ for a set of (s, θ)-values — that determine the coordinate systems $(\vec{e}_s, \vec{e}_z)$ — back to an estimate of the absorbance field μ for any position vector $\vec{u} = x\vec{e}_x + y\vec{e}_y$ of interest. Indeed, biological tissues have typical absorbance values and our aim is to reconstruct the inner structure of the body solely by inversion of the equations obtained, e.g., by X-ray (absorbance) imaging.

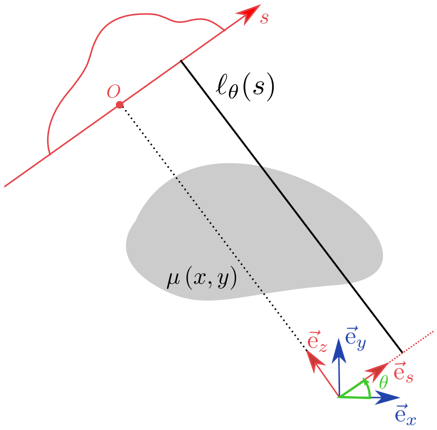


Figure 1: Construction of the line integrals for the absorbance imaging with the Radon transform.

Radon transform

The Radon transform[1] consists of an imaging technique by line integrals, also called projection operators. Getting back to the parametric ray equation, we may express the line integrals as

$$p_\theta(s) = \int_{\ell_\theta(s)} \mu(x,y)\,\mathrm{d}z = \iint_{\mathbb{R}^2} \mu(x,y)\,\delta(x\cos\theta + y\sin\theta - s)\,\mathrm{d}x\mathrm{d}y$$

where the Dirac delta follows from the fact that vectors in the plane that are on the line satisfy $\langle \vec{u}, \vec{e}_s \rangle = s$ where $\vec{u} = x\vec{e}_x + y\vec{e}_y$, see figure 1.

When varying s and θ, we are imaging the plane. Suppose for now that we have a single Dirac distribution $\delta(x - x_0, y - y_0)$ that attenuates in a non-attenuating background. Then we have the following projections

$$p_\theta(s) = \begin{cases} 1 & \text{if } x_0\cos\theta + y_0\sin\theta = s \\ 0 & \text{otherwise} \end{cases}$$

In the (θ, s)-plane this gives ones in the positions

$$(\theta, x_0\cos\theta + y_0\sin\theta) = \left(\theta, \underbrace{\sqrt{x_0^2 + y_0^2}}_{=A(x_0,y_0)} \cos\left(\theta - \underbrace{\operatorname{sign}(y_0)\arccos\left(\frac{x_0}{\sqrt{x_0^2+y_0^2}}\right)}_{=\varphi(x_0,y_0)}\right)\right)$$

and zeros elsewhere, whence the name *sinogram*. Any measurement set covering the (θ, s) can thus be seen as a superposition of these kind of sinusoidal functions.

Exercise 1 (Understanding the Radon transform)**.** Using a standard Radon transform, show that a Dirac-delta function (single pixel absorbance) in an image indeed results in a sinusoidal function in the (θ, s)-plane. Show that the location of the Dirac-delta indeed influences the amplitude and the phase as predicted with the above formula. How does the imaging behave when the spread of the distribution becomes larger. To that purpose compute the (θ, s)-plane image of (1) a disc of varying diameter, (2) a gaussian with varying full width at half maximum. How does translation of an object in the plane influence the (θ, s)-plane image ? Can you predict how image rotation influences the (θ, s)-plane image ? Set up a numerical experiment to confirm your intuition.

Python tip: for Radon transforms and inverse Radon transforms, it is recommended to use the package `scikit-image`. See Radon transform in scikit-image for documentation.
Matlab tip: call help on the function `radon`.

Image reconstruction

The goal is, given a set of imaged points $p_\theta(s)$, to reconstruct the absorbance pattern, and hence the (absorbance) geometry of the underlying medium. To do so, we will first recall the Fourier slice theorem, and then show how to reconstruct the medium from projections by a clever use of the Fourier transform.

[1] In image processing, this is also called the Hough transform, which is not only determined on simple lines, but on more generic parametric curves. Johann Radon has devised the technique in 1917 together with its inversion.

The Fourier slice theorem

First, recall the two-dimensional Fourier transform of the absorbance μ

$$\mathscr{F}\left[\mu\right]\left(\nu_x,\nu_y\right)=\iint_{\mathbb{R}^2}\mu\left(x,y\right)\mathrm{e}^{-\imath 2\pi\left(x\nu_x+y\nu_y\right)}\mathrm{d}x\mathrm{d}y$$

and look at how the Fourier transform along s of a projection along z relates to this double integral:

$$\begin{aligned}\mathscr{F}\left[p_\theta\right]\left(\nu_s\right)&=\int_{\mathbb{R}}p_\theta\left(s\right)\mathrm{e}^{-\imath 2\pi s\nu_s}\mathrm{d}s\\&=\int_{\mathbb{R}}\int\int_{\mathbb{R}^2}\mu\left(x,y\right)\delta\left(x\cos\theta+y\sin\theta-s\right)\mathrm{e}^{-\imath 2\pi s\nu_s}\mathrm{d}x\mathrm{d}y\mathrm{d}s\\&=\int\int_{\mathbb{R}^2}\mu\left(x,y\right)\mathrm{e}^{-\imath 2\pi\left(x\nu_s\cos\theta+y\nu_s\sin\theta\right)}\mathrm{d}x\mathrm{d}y\\&=\mathscr{F}\left[\mu\right]\left(\nu_s\cos\theta,\nu_s\sin\theta\right)\end{aligned}$$

which is a straight line through the origin in the Fourier plane.

Theorem 1 (*The Fourier slice theorem*)**.** *The Fourier transform of a projection is a projection (slice) of the Fourier transform.*

Inverse Fourier transform

Collecting data over different angles now allows us to invert the Fourier transform, since we are occupying the whole Fourier plane. However, the Fourier slices are obtained in polar coordinates, hence the inverse Fourier transform passes through a change of coordinates

$$\begin{cases}\nu_x=\nu_s\cos\theta\\\nu_y=\nu_s\sin\theta\end{cases}\implies\mathrm{d}\nu_x\mathrm{d}\nu_y=J^{-1}\left(\nu_s,\theta\right)\mathrm{d}\nu_s\mathrm{d}\theta=\nu_s\mathrm{d}\nu_s\mathrm{d}\theta$$

from which

$$\mu\left(x,y\right)=\int_0^{2\pi}\int_0^{+\infty}\mathscr{F}\left[\mu\right]\left(\nu_s\cos\theta,\nu_s\sin\theta\right)\mathrm{e}^{\imath 2\pi\left(x\nu_s\cos\theta+y\nu_s\sin\theta\right)}\nu_s\mathrm{d}\nu_s\mathrm{d}\theta\ .$$

But, we have $p_{\theta+\pi}\left(s\right)=p_\theta\left(-s\right)$ and thus $\mathscr{F}\left[p_{\theta+\pi}\right]\left(\nu_s\right)=\mathscr{F}\left[p_\theta\right]\left(-\nu_s\right)$ which means we may integrate only over half a circle

$$\begin{aligned}\mu\left(x,y\right)&=\int_0^{\pi}\int_0^{+\infty}\mathscr{F}\left[\mu\right]\left(\nu_s\cos\theta,\nu_s\sin\theta\right)\nu_s\mathrm{e}^{\imath 2\pi\left(x\nu_s\cos\theta+y\nu_s\sin\theta\right)}\mathrm{d}\nu_s\mathrm{d}\theta\\&\quad+\int_0^{\pi}\int_0^{+\infty}\mathscr{F}\left[\mu\right]\left(-\nu_s\cos\theta,-\nu_s\sin\theta\right)\nu_s\mathrm{e}^{-\imath 2\pi\left(x\nu_s\cos\theta+y\nu_s\sin\theta\right)}\mathrm{d}\nu_s\mathrm{d}\theta\\&=\int_0^{\pi}\int_{-\infty}^{+\infty}\mathscr{F}\left[\mu\right]\left(\nu_s\cos\theta,\nu_s\sin\theta\right)\left|\nu_s\right|\mathrm{e}^{\imath 2\pi\left(x\nu_s\cos\theta+y\nu_s\sin\theta\right)}\mathrm{d}\nu_s\mathrm{d}\theta\end{aligned}\tag{3}$$

Consider now the inner integral, which we may rewrite as

$$q_\theta\left(s\right)=\int_{-\infty}^{+\infty}\mathscr{F}\left[\mu\right]\left(\nu_s\cos\theta,\nu_s\sin\theta\right)\left|\nu_s\right|\mathrm{e}^{\imath 2\pi\left(x\nu_s\cos\theta+y\nu_s\sin\theta\right)}\mathrm{d}\nu_s=\int_{-\infty}^{+\infty}\mathscr{F}\left[p_\theta\right]\left(\nu_s\right)\left|\nu_s\right|\mathrm{e}^{\imath 2\pi\nu_s s}\mathrm{d}\nu_s$$

where $s=x\cos\theta+y\sin\theta$. But this is a mere filtering operation, where the (non-integrable) filter has frequency response $\left|\nu_s\right|$. This filter is generally called the Ram-Lak filter, which stands for Ramachandran–Lakshminarayanan filter. Rewriting — and recalling that $\imath 2\pi\nu\mathscr{F}\left[g\right]\left(\nu\right)=\mathscr{F}\left[g'\right]\left(\nu\right)$ — yields

$$\begin{aligned}q_\theta\left(s\right)&=\int_{-\infty}^{+\infty}\imath 2\pi\nu_s\mathscr{F}\left[p_\theta\right]\left(\nu_s\right)\frac{\left|\nu_s\right|}{\imath 2\pi\nu_s}\mathrm{e}^{\imath 2\pi\nu_s s}\mathrm{d}\nu_s\\&=\int_{-\infty}^{+\infty}\mathscr{F}\left[p'_\theta\right]\left(\nu_s\right)\frac{\operatorname{sign}\left(\nu_s\right)}{\imath 2\pi}\mathrm{e}^{\imath 2\pi\nu_s s}\mathrm{d}\nu_s\end{aligned}$$

and $\mathscr{F}^{-1}\left[\frac{\operatorname{sign}}{\imath 2\pi}\right]\left(s\right)=\int_{-\infty}^{+\infty}\frac{\operatorname{sign}(\nu_s)}{\imath 2\pi}\mathrm{e}^{\imath 2\pi\nu_s s}\mathrm{d}\nu_s=\frac{1}{2\pi^2 s}$. Finally, we can write the filtering operation

$$q_\theta\left(s\right)=\frac{1}{2\pi^2 s}*p'_\theta\left(s\right)=\mathscr{H}\left[p'_\theta\right]\left(s\right)$$

where $\mathscr{H}$ denotes the Hilbert transform (see the notion of an *analytic signal* in signal processing).

Back-projection

Once we have the inverse transform given the angle — the inner integral of (3), — we may back-transform to the original imaging plane by computing the outer integral of (3) with respect to θ. Writing this as a function of q_θ, the integral reads

$$\mu(x, y) = \int_0^\pi q_\theta(x\cos\theta + y\sin\theta)\,\mathrm{d}\theta \ .$$

This is indeed a back-projection, since we had $s = x\cos\theta + y\sin\theta$ and we integrate over all θ, i.e., we *integrate* $q_\theta(s)$ over all s in the (θ, s)-plane for which — remember the sinogram? — we have

$$\theta \mapsto s = x\cos\theta + y\sin\theta = \sqrt{x^2+y^2}\cos\left(\theta - \operatorname{sign}(y)\arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right)\right) \ .$$

Most often, the back-projection is carried out with a linear interpolation in the (s, θ) plane to recover the values.

Exercise 2 (Sampling in the (θ, s)-plane)**.** A famous phantom is the Shepp–Logan phantom, which we will use here as well. It models a subject's head and has the advantage of having the highly absorbing skull bones that mask the inner structures which makes the reconstruction non-trivial. When using the Fourier slice theorem — and hence back-projection — without resampling, the rays must be emitted by a flat emitter geometry and a flat receiver geometry (s coordinates correspond to the received ray) in parallel with the emitter, turning around a point on a line orthogonal to the receiver, crossing its midpoint. Change the number of parallel rays (sampling rate of s), and the number of angles in the acquisition (sampling rate of θ) for the Radon transform with filtered back-projection. What can you conclude on the reconstruction quality? Is there a sort of Nyquist frequency for θ as a function of s and the number of pixels that are imaged?

Filtered back-projection

The combination of the filtering operation with the back-projection is called — suspense — the filtered back-projection approach. The filtering limits the noise introduced by the high frequencies in the simple back-projection, which is due to the Ram-Lak filter. Different filters are available in the standard filtered back-projection algorithms: raised cosine, Hamming, Hanning, …

Exercise 3 (Influence of filters on the back-projection of noisy acquisitions)**.** Speckle noise + filters By adding noise to the Radon transformed image in the (θ, s)-plane, study the influence on the reconstruction by filtered back-projection using different filters. The noise should be chosen such that the image stays positive in all pixels (remember that we measure an intensity due to absorbance, which is always positive). Some possibilities are: speckle noise, Poisson noise, salt&pepper noise.

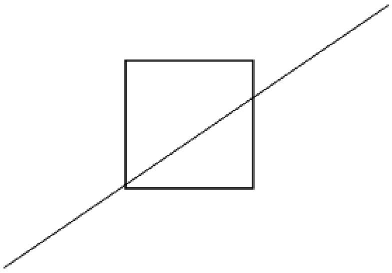
Python tip: In the main module `skimage` of the scikit-image package you'll find the function `random_noise` in the submodule `utils`.
Matlab tip: You'll find different types of noise in the `imnoise` function.

Algebraic reconstruction techniques

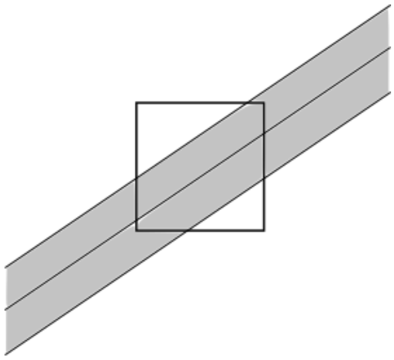
Consider again the line $\ell_\theta(s)$ that is traversing the imaged medium. Considering that we are reconstructing in a pixel space, we have that each line traverses a pixel with a given length. The length can be computed in different ways, depending on whether we give the ray a physical width or not (see figure 2).

In the case of a line integral as in figure 2a the absorbance is modelled as the length of the ray segment traversing the pixel (δz) — see also equation (1) — times the constant absorbance in the pixel, yielding a sum rather than an integral for the total absorbance in equation (2). Similarly for the ray integral, where here the surface swept out by the ray is used rather than the distance travelled trough the pixel. Let us denote the segment length or the surface swept out by the ith ray in pixel j by $w_{j,i}$ and the normalised absorbance in the jth pixel by x_j. We then have, for an imaging problem covering a total of n pixels based on m observations of absorbance:

$$\log\left(\frac{I(s\vec{e}_z + z_0\vec{e}_z)}{I(s\vec{e}_z + z\vec{e}_z)}\right) = y_i = \sum_{j=1}^{n} w_{j,i}x_j \qquad \text{or} \qquad \boldsymbol{y} = \boldsymbol{W}^t\boldsymbol{x}$$



(a) Line integral.



(b) Ray integral.

Figure 2: Integration of the absorbance (constant over a pixel), given (a) a line integral and (ba) a finite ray width integral.

which needs to be solved[2] for $\boldsymbol{x}$ ($\cdot^t$ is the transpose operator). The measurements y_i depend on the chosen values of (θ_i, s_i) in the (θ, s)-plane, in other words, we have a bijective map $i \leftrightarrow (\theta_i, s_i)$. The measurement matrix $\boldsymbol{W}$ is a thin matrix, i.e., having many rows (n pixels) and few columns (m measurements), which makes the problem ill-defined. It is easily observed that many of the values $w_{j,i}$ are actually zero, since a given ray only crosses a limited amount of pixels in the image.

Reformulating the problem

Since we have many solutions to the above problem we will reformulate the problem, which will allow us to incorporate additional constraints (physiological or biological priors, for instance). The problem is to find

$$\arg\min_{\boldsymbol{x}} \|\boldsymbol{y} - \boldsymbol{W}\boldsymbol{x}\|_2^2 = \arg\min_{\boldsymbol{x}} \sum_{i=1}^{m} \left|y_i - \boldsymbol{w}_i^t \boldsymbol{x}\right|^2$$

where $\boldsymbol{w}_i$ is the ith column of the matrix $\boldsymbol{W}$. If we consider only a single term in the above objective function, i.e., $\left|y_i - \boldsymbol{w}_i^t \boldsymbol{x}\right|^2$, then we have that the first- and second-order conditions of optimality imply

$$\begin{cases} \nabla_{\boldsymbol{x}} \left|y_i - \boldsymbol{w}_i^t \boldsymbol{x}\right|^2 = 2\left(y_i - \boldsymbol{w}_i^t \boldsymbol{x}\right) \boldsymbol{w}_i = 0 \\ \nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{x}}^t \left|y_i - \boldsymbol{w}_i^t \boldsymbol{x}\right|^2 = -2\boldsymbol{w}_i \boldsymbol{w}_i^t \geq 0 \end{cases}$$

Since the Hessian is ill-conditioned, we clearly have many solutions to the problem. Hence, we will concentrate on iterative updates of the form

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \eta \left(\nabla_{\boldsymbol{x}} \nabla_{\boldsymbol{x}}^t\right)^{-} \nabla_{\boldsymbol{x}} \equiv \boldsymbol{x}^{(k)} - \eta \left(\boldsymbol{w}_i \boldsymbol{w}_i^t\right)^{-} \left(y_i - \boldsymbol{w}_i^t \boldsymbol{x}\right) \boldsymbol{w}_i \;\; .$$

Now we have $\boldsymbol{w}_i \boldsymbol{w}_i^t = \frac{\boldsymbol{w}_i \boldsymbol{w}_i^t}{\|\boldsymbol{w}_i\|^2} \|\boldsymbol{w}_i\|^2 = \|\boldsymbol{w}_i\|^2 \prod_{\boldsymbol{w}_i}$ where $\prod_{\boldsymbol{w}_i}$ is the projector operator on the one-dimensional subspace spanned by $\boldsymbol{w}_i$. Hence, $\left(\boldsymbol{w}_i \boldsymbol{w}_i^t\right)^{-} \boldsymbol{w}_i = \frac{\prod_{\boldsymbol{w}_i}(\boldsymbol{w}_i)}{\|\boldsymbol{w}_i\|^2}$ which finally gives us

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k)} - \frac{\eta}{\|\boldsymbol{w}_i\|^2} \left(y_i - \boldsymbol{w}_i^t \boldsymbol{x}^{(k)}\right) \boldsymbol{w}_i$$

with $0 < \eta$ being a relaxation parameter. Hence starting with an initial estimate $\boldsymbol{x}^{(0)} = \boldsymbol{x}_0$, we update $\boldsymbol{x}$ iteratively by using $i = k \mod m$, as such circularly iterating through the m observations.

The interpretation of this equation is as follows: for the ith line integral of the current estimate $\boldsymbol{x}^{(k)}$ represented by $\boldsymbol{w}_i^t \boldsymbol{x}^{(k)}$, look at the yet unexplained variance of the associated observed absorbance y_i given by $\left|y_i - \boldsymbol{w}_i^t \boldsymbol{x}^{(k)}\right|^2$. Update the current estimate $\boldsymbol{x}^{(k)}$ along $\boldsymbol{w}_i$ proportional to the current error $y_i - \boldsymbol{w}_i^t \boldsymbol{x}^{(k)}$, which updates only the pixels on the path ray's path.

Depending on whether a single pixel or (weighted) group of pixels are considered, the algorithm vary from algebraic reconstruction technique (ART) over simultaneous algebraic reconstruction technique (SART), to simultaneous iterative reconstruction technique (SIRT).

[2]It is common in statistical problems to note $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}$ where $\boldsymbol{\beta}$ is the unknown vector and $\boldsymbol{X}$ is the design matrix. In this terminology we have $\boldsymbol{W}$ in our problem that is the design matrix.

Exercise 4. Use an algebraic reconstruction technique with constraints on the boundaries (zero attenuation outside the phantom).What can you say about the algorithm’s convergence speed with respect to the filtered back-projection? What about the quality of reconstruction? What then is the advantage of this family of techniques?