MAT137Y Test 1

Shivesh Prakash

TOTAL POINTS

37 / 40

QUESTION 1

1 Q1(a) **1 / 1**

+ **1** Point adjustment

QUESTION 2

2 Q1(b) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 3

3 Q1(c) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 4

4 Q1(d) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 5

5 Q1(e) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 6

6 Q2(a) **0.5 / 0.5**

✓ **- 0 pts** Correct

QUESTION 7

7 Q2(b) **0.5 / 0.5**

✓ **- 0 pts** Correct

QUESTION 8

8 Q2(c) **0.5 / 0.5**

✓ **- 0 pts** Correct

QUESTION 9

9 Q2(d) **0.5 / 0.5**

✓ **- 0 pts** Correct

QUESTION 10

10 Q2(e) **0.5 / 0.5**

✓ **- 0 pts** Correct

QUESTION 11

11 Q2(f) **0.5 / 0.5**

✓ **- 0 pts** Correct

QUESTION 12

12 Q3(a)-(1) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 13

13 Q3(a)-(2) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 14

14 Q3(a)-(3) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 15

15 Q3(a)-(4) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 16

16 Q3(b) **2 / 2**

✓ **- 0 pts** Correct

QUESTION 17

17 Q4(a) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 18

18 Q4(b) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 19

19 Q4(c) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 20

20 Q4(d) **1 / 1**

✓ **- 0 pts** Correct

QUESTION 21

21 Q5(a) **1 / 1**

✓ **- 0 pts** Correct

- 1 pts Incorrect final answer.

- 0.5 pts Mostly clear and correct answer, has notational issues.

QUESTION 22

22 Q5(b) **1 / 1**

✓ **- 0 pts** Perfect.

- 1 pts The correct answer is $$(-\infty,0]$$.

- 0.5 pts If you wrote $$x\in (-\infty,0]$$ or $$x:(-\infty,0]$$ or $$x= (-\infty,0]$$ or something to this effect, then note that literally speaking, your answer is wrong. Taken literally, these are statements about a free variable $$x$$. A statement is not an interval.

The question says "Express the set ... in interval notation." So the final answer should just be an interval, i.e $$(-\infty,0]$$, and nothing else.

- 0.5 pts If you wrote $$A=(-\infty,0]$$ or $$A:(-\infty,0]$$ or $$B=(-\infty,0]$$ or $$B:(-\infty,0]$$, then your set is correct, but take note that the Question 5b doesn't introduce the symbols $$A,B$$. So, your final answer ideally shouldn't include such unnecessary notation (unless you explicitly **first** define $$A$$ (or $$B$$) to be the set in question, and **then** say $$A=(-\infty,0]$$).

This is like a wild pikachu appearing.

QUESTION 23

23 Q5(c) **1 / 2**

- 0 pts Correct

✓ **- 1 pts** **The correct answer is $$1.8\leq L\leq 2.2$$ (or $$L\in [1.8,2.2]$$). 1 point is deducted for answers such as:**
- $$(1.8,2.2)$$ or $$(1.8,2.2]$$, or $$[1.8,2,2)$$
- "$$L$$ is between $$1.8$$ and $$2.2$$." This is imprecise because it doesn't specify which (if any) endpoints are included.
- using a random letter such as $$x$$ or $$y$$, and having wrong inequality signs (e.g $$1.8<x<2.2$$). First, it should both be weak inequalities $$\leq$$. Second, the question explicitly defines $$L$$ to be the limit, so saying things about $$x$$ or $$y$$ doesn't directly answer the question.
- $$L=(1.8,2.2)$$ (wrong interval and notation). Note that $$L$$ is the limit of a function, i.e it is a number, so it cannot be equal to a set of real numbers.

- 2 pts The correct answer is $$1.8\leq L\leq 2.2$$ (or $$L\in [1.8,2.2]$$).

- 0.5 pts Right answer expressed with incorrect notation. For example:

- writing $$L=[1.8,2.2]$$. Note that $$L$$ is defined to be the limit of a function, so it is a real number. A real number is not equal to a set of real numbers, so using $$=$$ is wrong, the correct symbol is $$\in$$.
- Writing $$1.8\leq x\leq 2.2$$ or $$1.8\leq y\leq 2.2$$. The question explicitly defines $$L$$ to be the limit of the function, so saying things about randomly introduced letters such as $$x$$ or $$y$$ doesn't directly answer the question (unless you first **explicitly** define $$x:=L$$ (or $$y:=L$$), which is quite silly, but if you did this, then your answer would be logically correct, and you get full credit).

QUESTION 24

24 Q6(a) **2 / 2**

✓ **- 0 pts** Correct

- 1 pts One major error.

- 2 pts Two or more major errors.

- 0.5 pts Minor error / missing minor detail

QUESTION 25

25 Q6(b) **4 / 5**

+ **2 pts** Correct $$\delta$$ chosen. If did not use the inequality $$||a|-|b|| \leq |a-b|$$ in the proof then $$\delta$$ should be the minimum of two numbers to ensure $$x$$ is positive.

✓ + **1 pts** **Mostly correct algebra to go from $$0<|x-1|<\delta$$ to $$0<|2|2x|-4|<\epsilon$$.**

✓ + **1 pts** **Correct algebra: Either used triangle inequality or showed that $$x$$ is positive via a bound.**

✓ + **1 pts** **Correct proof structure for $$\epsilon-\delta$$ proof.**

✓ + **1 pts** **$$\delta$$ chosen is partially correct. If did not use the inequality $$||a|-|b|| \leq |a-b|$$ in the proof then $$\delta$$ should be the minimum of two numbers to ensure $$x$$ is positive.**

+ **0 pts** Please see the solution.

+ **0 pts** Click here to replace this description.

1 This < does not directly follow from your assumptions.

QUESTION 26

26 Q7(a) **2 / 2**

✓ - **0 pts** **Correct**

- **1 pts** There is a typo or small mistake present in this submission. Please see the posted solutions.

- **2 pts** This is not the formal definition. Please see the posted solutions.

- **2 pts** Blank

QUESTION 27

27 Q7(b) **3 / 4**

- **0 pts** Correct

✓ - **1 pts** **Did not pick correct $$M$$ in the assumption that $$\lim_{x \to a} f(x) = - \infty $$) i.e. $$M = - \frac{1}{\epsilon} $$ (negative is important in order for the inequalities for |f(x)| to correctly work out later)**

- **1 pts** Did not correctly use assumption to pick $$\delta$$ such that $$f(x) < - \frac{1}{\epsilon}$$

- **1 pts** Did not correctly use this $$\delta$$ to show that when $$0 < | x - a | < \delta $$ we have $$ |f(x) | > \frac{1}{\epsilon} $$ i.e. $$ | \frac{1}{f(x)} | < \epsilon $$

- **1 pts** Did not have a correct general proof structure (either a sufficient proof was not written at all, or you did not correctly introduce variables etc)

- **0.5 pts** Misc calc errors

- **4 pts** Your solution is not on the right track please see sample solutions

2 you said above $$M \in \mathbb{R}$$; if you want this inequality to work you need to explicitly say $$ M < 0 $$

QUESTION 28

28 Q8 **5 / 5**

✓ - **0 pts** **Well done!**

- **1 pts** Did not check base case

- **1 pts** Did not correctly state induction step (i.e. assume $$P(n)$$ is true for $$n\in \mathbb{N}$$, want to show $$P(n+1)$$ is true)

- **1 pts** Did not properly state where the induction hypothesis is used

- **2 pts** Incorrect/incomplete algebra in proving induction step

- **1 pts** Algebraic error in proving induction step

- **4 pts** Only checked base case; any other work shown is incorrect or does not contribute to the proof

- **1 pts** Not enough explanation. Please indicate what you are assuming and what you are proving.

- **5 pts** Blank / Insufficient work

- **5 pts** Too many errors, please read the official solution.

- **3 pts** Did everything correctly up to stating inductive step

- **0 pts** Used sum formula correctly, not induction

University of Toronto
Faculty of Arts and Science

MAT137Y Test 1

D. Àlvarez, X. Cui, I. Gaiur, B. Khesin, T. Kojar, J. Siefken, A. Zalloum

October 21, 2022
Duration: 110 minutes
No Aids Permitted

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Please make sure your information above are correct.

This exam contains 10 **one-sided pages** (including this cover page) and is printed on 10 sheets of paper. There are 8 problems.

Instructions

- **Keep this booklet closed**, until an invigilator announces that the test has begun. However, you may fill out your information in this page before the test starts.
- Please place your **student ID** in a location on your desk that is easy for an invigilator to check without disturbing you during the test.
- **Turn your cell phone completely off** (not just in vibrate mode) and store it with your other belongs.
- **No aids are permitted on this examination.** Examples of illegal aids include but are not limited to textbooks, notes, calculators, cellphones, or any electronic device.
- If you need **space for your draft work**, you can use the back of each page as your scratch papers. Note that we will not scan the back of each page.
- **Write clearly** and concisely by using **dark pencil or pen.** Work scattered all over the page without a clear ordering will receive very little credit.
- For questions with a boxed area, ensure your answer is completely **inside the box**.
- **If you need more space**, use the blank pages at the end of the exam and clearly indicate on the question page when you have done this. **Do not tear any pages off this exam.**

Question:	1	2	3	4	5	6	7	8	Total
Points:	5	3	6	4	4	7	6	5	40
Score:									

1. For each of the following questions, only your final answer will be graded.

No justification is necessary.

Evaluate the following limits. If a limit is ∞ or $-\infty$, clearly state this. If the limit does not exist and it's not ∞ or $-\infty$, put "DNE" in the box.

(1a) (1 point) $\lim_{x\to 3}\left(\frac{1}{x}-\frac{1}{3}\right)\left(\frac{1}{x-3}\right)$

Final Answer

$\frac{-1}{9}$

(1b) (1 point) $\lim_{x\to 2^+}\frac{\sqrt{x^2-4}}{x-2}$

Final Answer

$+\infty$

(1c) (1 point) $\lim_{x\to 1}(x-1)^2\cos\left(\frac{1}{x-1}\right)$

Final Answer

0

(1d) (1 point) $\lim_{x\to -\infty}\frac{4x^2+\sqrt{x^6+x}}{x^3-2}$

Final Answer

-1

(1e) (1 point) $\lim_{x\to 0}\frac{\sin^2(5x^{10})}{x^{20}}$

Final Answer

25

For each of the following questions, only your final answer will be graded. Only bubble in exactly one answer for each question below.

No justification is necessary.

2. (3 points) Let A, B, C be disks (filled in circles) in the xy-plane. You know the following:

   - A is centered at $(0,0)$ and has radius 1.
   - B is centered at $(0,0)$ and has radius 2.
   - C is centered at $(2,0)$ and has radius 2.

   Classify the following statements as **TRUE**, **FALSE**, or **NO TRUTH VALUE** if the statement cannot be classified as true or false.

   (a) $\forall x \in A, x \in B$.

   ● True ○ False ○ No Truth Value

   (b) $\forall x \in B, x \in A$.

   ○ True ● False ○ No Truth Value

   (c) $\exists x \in C$ such that $x \in A$.

   ● True ○ False ○ No Truth Value

   (d) $x \in A \cap C \implies x \in B$.

   ● True ○ False ○ No Truth Value

   (e) $A \cap C$.

   ○ True ○ False ● No Truth Value

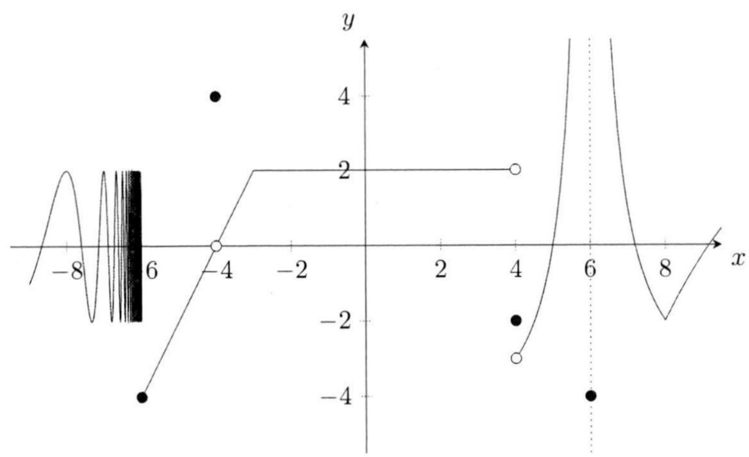
   (f) $x \in B \cap C \implies x \in A \cap C$.

   ○ True ● False ○ No Truth Value

For each of the following questions, only your final answer will be graded.

No justification is necessary.

3. Consider the graph of $y = f(x)$ for $-9 \leq x \leq 9$.



(3a) (4 points) Evaluate each limit. If a limit is ∞ or $-\infty$, clearly state this. If the limit does not exist and it's not ∞ or $-\infty$, put "DNE" in the box.

1) $\lim_{x\to 6}\left(\frac{1}{f(x)}\right)$

Final Answer

0

2) $\lim_{x\to -4^-} \lfloor f(x) \rfloor$ Here, $\lfloor x \rfloor$ is the largest integer less or equal to x.

Final Answer

-1

3) $\lim_{x\to -6^-} f(f(x))$

Final Answer

2

4) $\lim_{x\to 1}\left(\frac{f(x+5)}{f(x-1)}\right)$

Final Answer

$+\infty$

(3b) (2 points) For which value(s) of $x \in (-9, 9)$ is the function $g(x) = (x+4)f(x)$ **not** continuous?

Final Answer

$x \in \{-6, 4, 6\}$

For each of the following questions, only your final answer will be graded.

No justification is necessary.

4. (4 points) For each statement, determine whether it is TRUE or FALSE.

a)Let f be a function defined on $\mathbb{R}$. If f^2 is continuous at a, then f is continuous at a.

○ True ● False

b)If there exists $x \in \mathbb{R}$ s.t. $x^2 + 1 = 0$, then $x > 0$.

● True ○ False

c)Let f, g, and h be functions defined on $\mathbb{R}$ and let $a \in \mathbb{R}$. If

- $\forall x \in \mathbb{R}$, $f(x) \leq g(x) \leq h(x)$,
- $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} h(x) = \infty$,

then $\lim_{x \to a} g(x)$ exists or is ∞.

○ True ● False

d) Let f and g be functions defined on $\mathbb{R}$ and let $a \in \mathbb{R}$.
If $\lim_{x \to a} f(x)g(x) = 0$, then $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$.

○ True ● False

5. (5a) (1 point) We say a set A is *realistic* if $\exists a \in A$ such that $\forall x \in A, x \leq a$. Find a realistic set.

Final Answer

$A = \{1, 2\}$

(5b) (1 point) Express the following set $\{x \in \mathbb{R} : \forall a > 0, \exists b \in \mathbb{R}\ s.t.\ b < x < a\}$ in interval notation.

Final Answer

$(-\infty, 0]$

(5c) (2 points) Let f be a function defined on $\mathbb{R}$. Assume we know that

$$\forall x \in \mathbb{R}, 0 < |x - 1| < 0.1 \implies |f(x) - 2| < 0.2$$

If the limit $L = \lim_{x \to 1} f(x)$ exists then what are the possible values of L?

Final Answer

$1.8 < L < 2.2$

6. Let $f(x) = |x|$ with domain $\mathbb{R}$.

(6a) (2 points) Write the formal definition of $\lim_{x\to 1} 2f(2x) = 4$.

$\forall \varepsilon > 0, \exists \delta > 0$ st. $(\forall x \in R)$ $0 < |x-1| < \delta \Rightarrow |2f(2x) - 4| < \varepsilon$

(or)

$\forall \varepsilon > 0, \exists \delta > 0$ st. $(\forall x \in R)$ $0 < |x-1| < \delta \Rightarrow \big|2|2x| - 4\big| < \varepsilon$

(6b) (5 points) Use the formal ε-δ definition of the limit to prove

$$\lim_{x\to 1} 2f(2x) = 4.$$

Do not use any limit laws or any other theorems.

- Want to show:

$\forall \varepsilon > 0, \exists \delta > 0$ st. $(\forall x \in R)$ $0 < |x-1| < \delta \Rightarrow \big|2|2x| - 4\big| < \varepsilon$

- Let $\varepsilon > 0$.
- Take $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{3}\}$
- Assume $0 < |x-1| < \delta$. Since $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{3}\} \Rightarrow$

$0 < |x-1| < \frac{1}{2}$ and $0 < |x-1| < \frac{\varepsilon}{3}$

$\Rightarrow \frac{1}{2} < x < \frac{3}{2}$ $(x \neq 1)$

$\Rightarrow 1 < 2x < 3$. Thus $|2x| = 2x$ from the definition of absolute value

Thus, second statement becomes $|4x - 4| < \varepsilon$

$\Rightarrow |4|\,|x-1| < \varepsilon \Rightarrow |x-1| < \frac{\varepsilon}{4} < \frac{\varepsilon}{3} = \delta$

Which is true from our assumption. Since $\varepsilon > 0$ and $\delta > 0$

Hence proved, $\lim_{x\to 1} 2f(2x) = 4$.

7. Let $a \in \mathbb{R}$. Let f be a function defined on $\mathbb{R}$.

(7a) (2 points) Write the formal definition of $\lim_{x\to a} f(x) = -\infty$.

$\forall M \in \mathbb{R}, \exists \delta > 0$ st. $(\forall x \in \mathbb{R})\ 0 < |x-a| < \delta \Rightarrow f(x) < M$

(7b) (4 points) Write a proof for the following theorem:

Theorem If $\lim_{x\to a} f(x) = -\infty$, then

$$\lim_{x\to a} \frac{1}{f(x)} = 0\,.$$

Write a formal proof directly from the ε-δ definitions of limit. Do not use the limit laws.

- Want ~~to~~ show:

(① $\forall M \in \mathbb{R}, \exists \delta_1 > 0$ st. $0 < |x-a| < \delta_1 \Rightarrow f(x) < M$) $\Rightarrow$ (② $\forall \varepsilon > 0, \exists \delta_2 > 0$ st. $0 < |x-a| < \delta_2 \Rightarrow \left|\frac{1}{f(x)}\right| < \varepsilon$)

- Let $M \in \mathbb{R}$. Let $\varepsilon > 0$. Assuming ① to be true with $\delta = \delta_1$, where $\delta_1 > 0$.
- Taking $\delta = \delta_2$ in ②, where $\delta_2 > 0$
- Take $\varepsilon = \left|\frac{1}{M}\right|$. Notice ε is still arbitrary and >0.
- Now, Take $\delta_1 = \delta_2$. So, $0 < |x-a| < \delta_1 \Leftrightarrow 0 < |x-a| < \delta_2$

Therefore, $(0 < |x-a| < \delta_1 \Leftrightarrow 0 < |x-a| < \delta_2) \Rightarrow f(x) < M$

Since $f(x) \to -\infty$, $f(x) < M \Leftrightarrow |f(x)| > |M| \Leftrightarrow \left|\frac{1}{f(x)}\right| < \left|\frac{1}{M}\right| = \varepsilon$

Thus, $(0 < |x-a| < \delta_1 \Leftrightarrow 0 < |x-a| < \delta_2) \Rightarrow \left(f(x) < M \Leftrightarrow \left|\frac{1}{f(x)}\right| < \varepsilon\right)$

This is what we wanted to show, hence proved.

$\lim_{x\to a} f(x) = -\infty \Rightarrow \lim_{x\to a} \frac{1}{f(x)} = 0$

8. (5 points) Prove that the following statement

$$P(n) : \sum_{k=1}^{n} 2(3k-2) = 2 \cdot 1 + 2 \cdot 4 + 2 \cdot 7 + \cdots + 2(3n-2) = 3n^2 - n$$

is true for all positive integer n.
Hint: induction may work well for this question.

- Verifying Base case →

$P(1) = \sum_{k=1}^{1} 2(3k-2) = 2(3-2) = 2$

$P(1) = 3(1)^2 - 1 = 3 - 1 = 2$

Thus base case $P(1)$ is true.

- Proving $P(n+1)$ using $P(n)$ is true → (where n is a positive integer)

Assume $P(n)$ is true =>

$P(n) = \sum_{k=1}^{n} 2(3k-2) = 2\cdot1 + 2\cdot4 + \cdots + 2(3n-2) = 3n^2 - n$

Adding $6n+2$ on both sides

$2\cdot1 + 2\cdot4 + \cdots + 2(3n-2) + 6n+2 = 3n^2 - n + 6n + 2$

$\Rightarrow 2\cdot1 + 2\cdot4 + \cdots + 2(3n-2) + 2(3n+1) = 3n^2 + 6n + 3 \quad - n - 1$

$\Rightarrow 2\cdot1 + 2\cdot4 + \cdots + 2(3n-2) + 2(3n+3-2) = 3(n^2 + 2n + 1) - (n+1)$

$\Rightarrow 2\cdot1 + 2\cdot4 + \cdots + 2(3n-2) + 2(3(n+1)-2) = 3(n+1)^2 - (n+1)$

$\Rightarrow \sum_{k=1}^{n+1} 2(3k-2) = 3(n+1)^2 - (n+1) \iff P(n+1)$ is true

Since $P(1)$ is true and $P(n) \Rightarrow P(n+1)$, Using induction hypothesis $P(n)$ is true for all positive integers.

Hence Proved.

Do not tear this page off. This page is for additional work and will not be marked, unless you clearly indicate it on the original question page.

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