CSC110 Lecture 25: Worst-Case Running Time Analysis

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Announcements and Today's Plan

Announcements

- Assignment 4 has been posted
  - Check out the A4 FAQ (+ corrections)
  - Additional TA office hours
  - Review advice on academic integrity
- Prep 10 (due next Monday) is the last prep (no preps in Weeks 11/12)
- The Term Test 3 Info Page has been posted.

Today you'll learn to...

- Analyse the worst-case running time of an algorithm.
- Identify algorithms for which worst-case analysis is appropriate.
- Identify built-in functions and data type operations for which worst-case analysis is appropriate.

Running time of `in` with a list

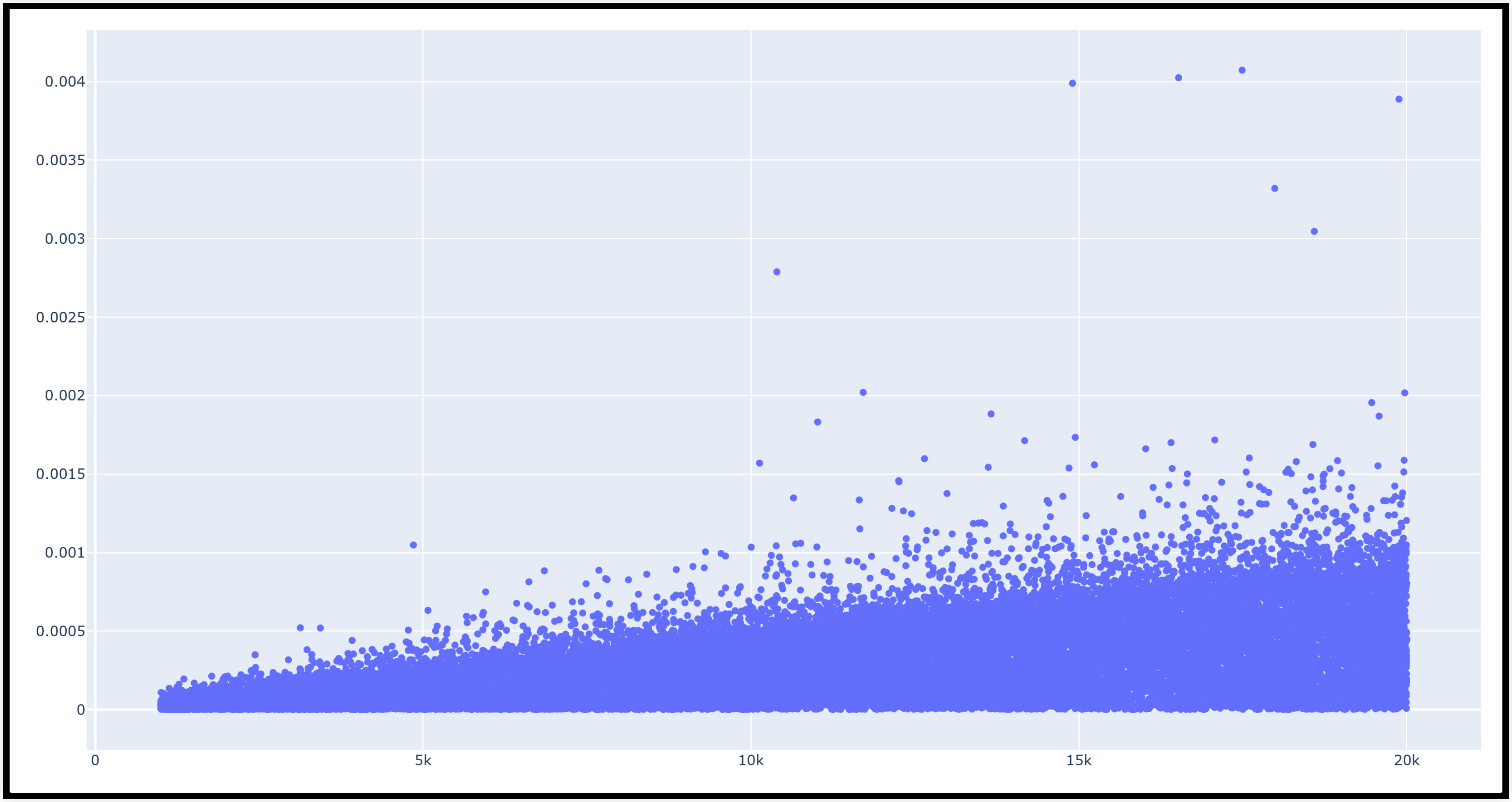
Consider the following statements:

```
>>> from timeit import timeit
>>> lst = list(range(0, 10000000))
>>> timeit('42 in lst', number=100, globals=globals())
???
>>> timeit('-1 in lst', number=100, globals=globals())
???
```

The running time of `in` with a `list` can vary significantly!

List length vs. time taken

`item in lst` for different `item`s



Evaluating `item in lst` requires searching

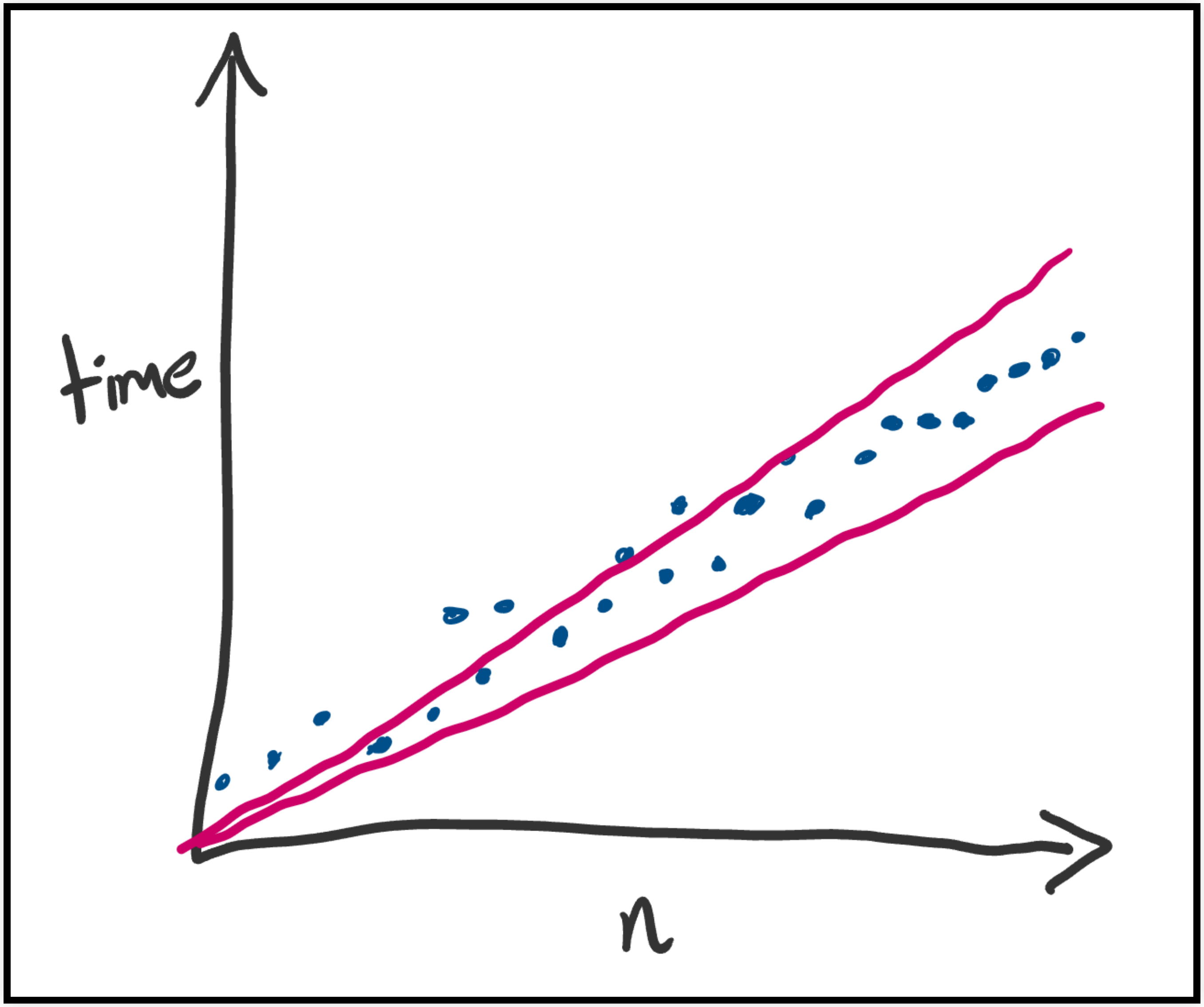
```
def search(lst: list[int], item: int) -> bool:
    """Return whether item is in lst."""
    for x in lst:
        if x == item:
            return True

    return False
```

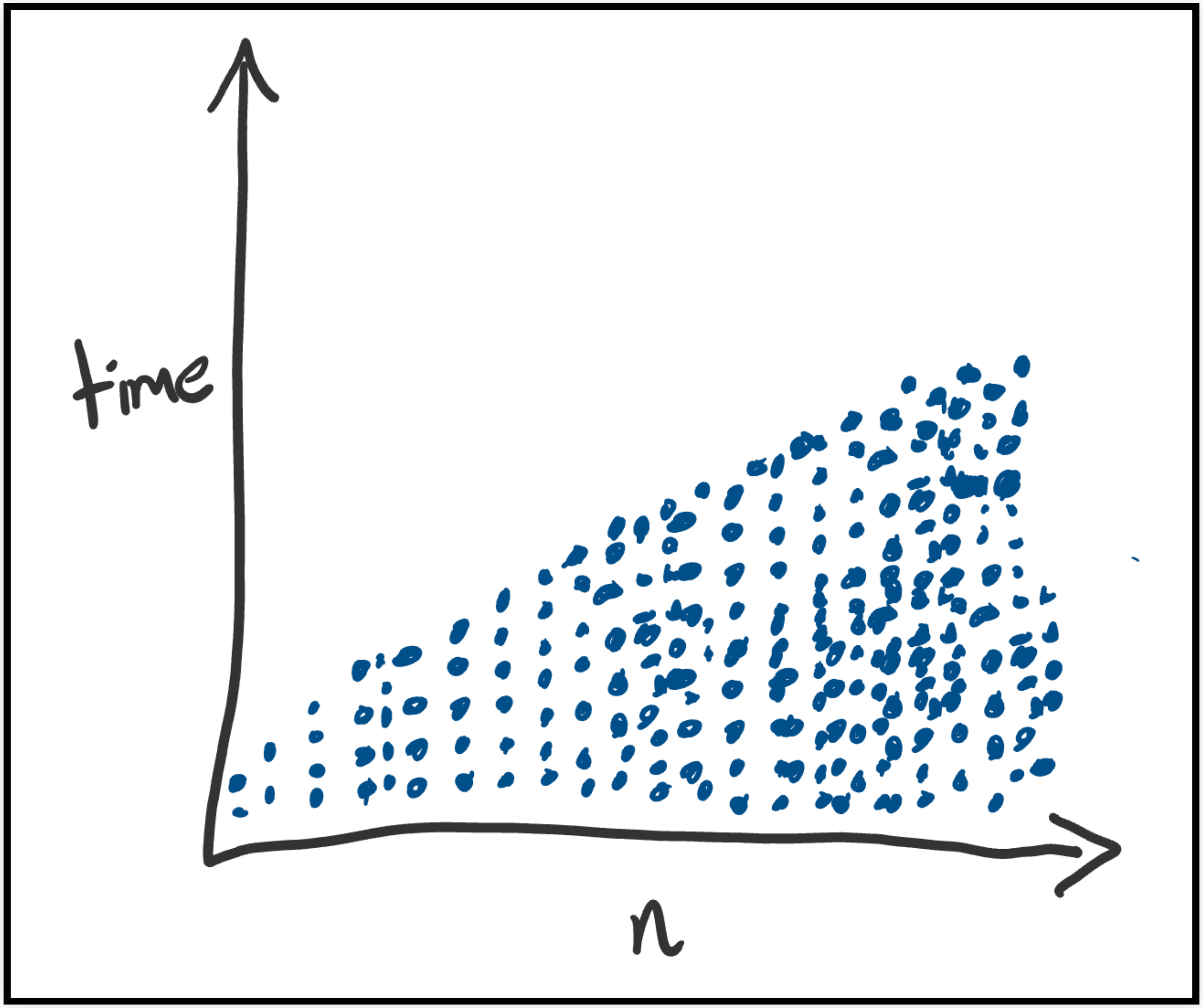
The running time of `search` doesn't just depend on the length of `lst`, it also depends on the elements in `lst` and the value of `item`.

Running time is not a function of input size: multiple inputs of the same size can have vastly different running times!

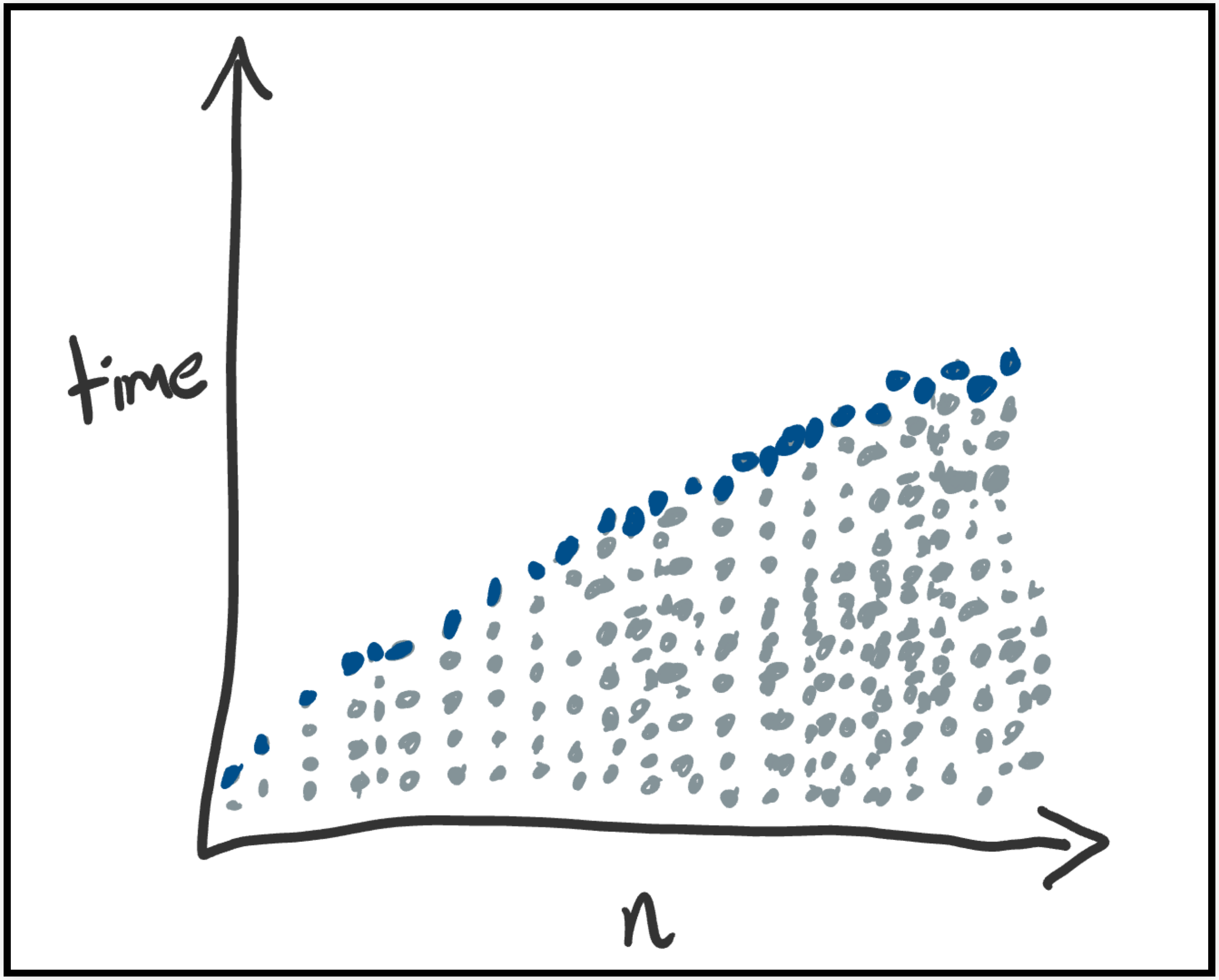
Running times from lectures before now...



Running times of `search`



Worst-case running time



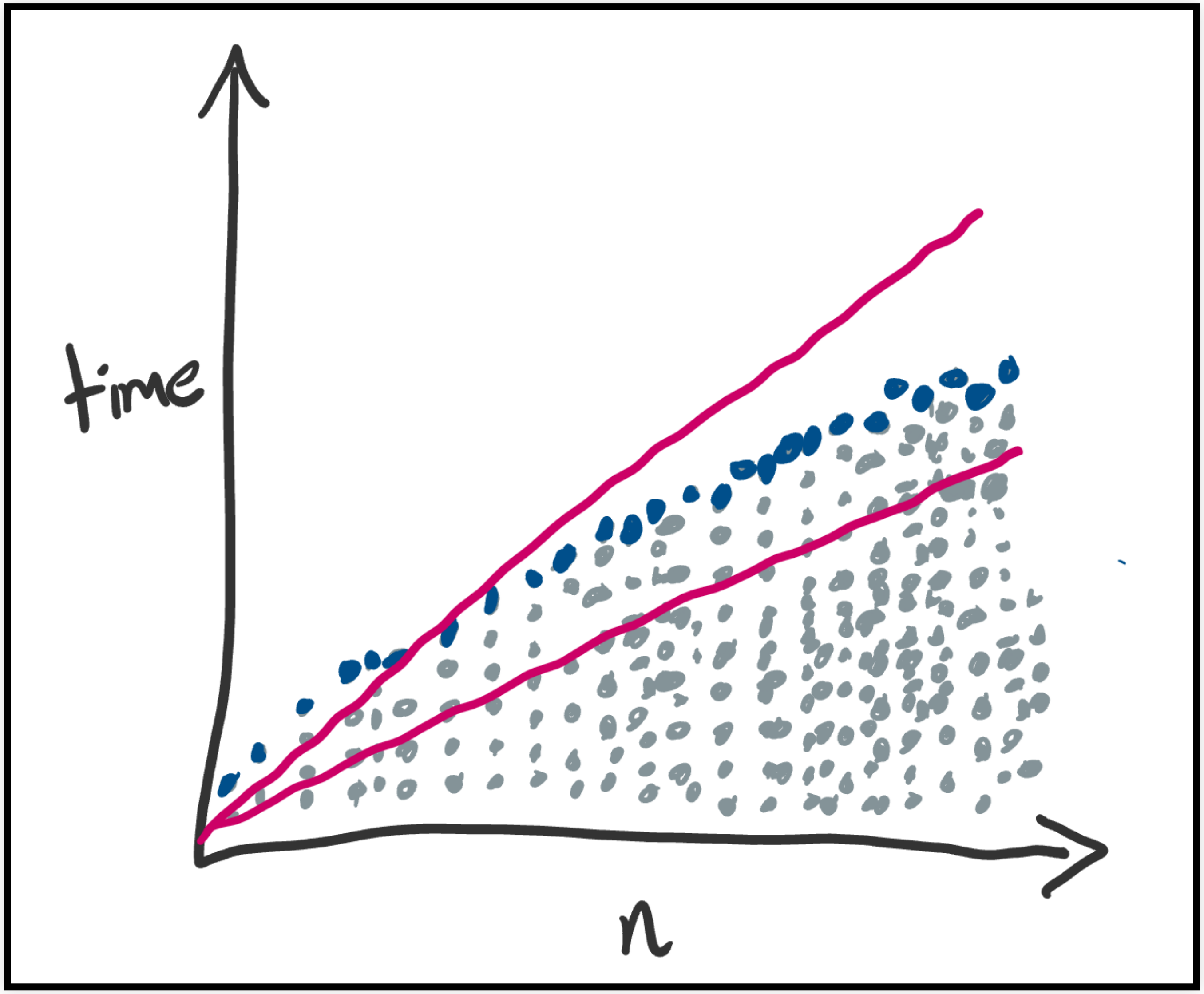
Let `func` be an algorithm, and $\mathcal{I}_n$ be the set of inputs to `func` of size n (where $n \in \mathbb{N}$).

We define the worst-case running time of `func` as:

$$WC_{\texttt{func}}(n) = \max \{\text{running time of } \texttt{func}(x) \mid x \in \mathcal{I}_n\}$$

$WC_{\texttt{func}}$ is a function, and so we can use Big-O/Omega/Theta to describe its growth rate!

Worst-case running time



Worst-case running-time analysis

Goal of a **worst-case running-time analysis** of `func`:

> *Find an elementary function f such that* $WC_{\texttt{func}} \in \Theta(f)$.

This means $WC_{\texttt{func}} \in \mathcal{O}(f)$ **and** $WC_{\texttt{func}} \in \Omega(f)$.

$WC_{\texttt{func}}(n)$ is the maximum of a set of numbers (running times).

How do we know what this maximum is?

Aside:

Let S be the set of ages (years) of the people in this room and let $M \in \mathbb{R}$.

M is an upper bound on $\max(S)$ whenever $\forall x \in S,\ x \leq M$.

120 is an upper bound on $\max(S)$ since we are all younger than 120.

So is 75.

If you discover that $18 \in S$, what do you then know about a lower bound on $\max(S)$?

$18 \leq \max(S)$

So 18 is a lower bound on $\max(S)$ since someone is 18.

Keep this aside in mind as we return to describing $WC_{\texttt{func}}$ for different Python `funcs`!

Goal: analyse the worst-case running time of `search`.

```
def search(lst: list[int], item: int) -> bool:
    """Return whether item is in lst."""
    for x in lst:
        if x == item:
            return True

    return False
```

Intuition: the maximum runtime occurs when `item` is not in `lst`. In this case, (roughly) n steps happen, where $n = \texttt{len(lst)}$.

In a worst-case running-time analysis, we don't try to compute $WC_{\texttt{func}}$ exactly, since it is hard in general to find an exact "maximum running time".

Instead, we find matching upper and lower bounds on the running time:

1. Find an elementary function f such that $WC_{\texttt{func}} \in \mathcal{O}(f)$
2. Then, show that $WC_{\texttt{func}} \in \Omega(f)$
3. Conclude that $WC_{\texttt{func}} \in \Theta(f)$

Finding an upper bound on the worst-case running time

f is an **upper bound** on $WC_{\texttt{func}}$ when

- $\forall n \in \mathbb{N},\ WC_{\texttt{func}}(n) \le f(n)$

i.e.,

- $\forall n \in \mathbb{N},\ \max\left\{\text{running time of } \texttt{func}(x) \mid x \in \mathcal{I}_n\right\} \le f(n)$

i.e.,

- $\forall n \in \mathbb{N},\ \forall x \in \mathcal{I}_n,\ \text{running time of } \texttt{func}(x) \le f(n)$

Finding an upper bound on the worst-case running time

$$\forall n \in \mathbb{N},\ \forall x \in \mathcal{I}_n,\ \text{running time of } \texttt{func}(x) \leq f(n)$$

To find an upper bound on the worst-case running time of `func`, we:

- Pick an **arbitrary** n
- Pick an **arbitrary** input `x` of size n
- Find an upper bound on the running time of `func(x)`.

```python
def search(lst: list[int], item: int) -> bool:
    """Return whether item is in lst."""
    for x in lst:
        if x == item:
            return True

    return False
```

Worst-case analysis (upper bound). Let $n \in \mathbb{N}$, and let `lst` be an **arbitrary** list of length n, and let `item` be an arbitrary `int`.

The for loop takes at most n iterations, and each iteration takes 1 step (constant time), for a total of at most n steps.

The `return False` either happens or doesn't; it takes at most 1 step.

The total running time is at most $n + 1$ steps, which is $\mathcal{O}(n)$.

Hence $WC_{\texttt{search}} \in \mathcal{O}(n)$.

Finding a lower bound on the worst-case running time

f is a **lower bound** on $WC_{\texttt{func}}$ when

- $\forall n \in \mathbb{N},\ WC_{\texttt{func}}(n) \geq f(n)$

i.e.,

- $\forall n \in \mathbb{N},\ \max\{\text{running time of } \texttt{func}(x) \mid x \in \mathcal{I}_n\} \geq f(n)$

i.e.,

- $\forall n \in \mathbb{N},\ \exists x \in \mathcal{I}_n,\ \text{running time of } \texttt{func}(x) \geq f(n)$

Finding a lower bound on the worst-case running time

$$\forall n \in \mathbb{N},\ \exists x \in \mathcal{I}_n,\ \textbf{running time of } \texttt{func}(x) \geq f(n)$$

To find a lower bound on the worst-case running time of `func`, we:

- Pick an **arbitrary** n
- Pick a **specific** input `x` of size n
- Find a lower bound on the running time of `func(x)`.
  - Or, usually we can find an exact running time of `func(x)`.

```python
def search(lst: list[int], item: int) -> bool:
    """Return whether item is in lst."""
    for x in lst:
        if x == item:
            return True

    return False
```

Worst-case analysis (lower bound).

Let $n \in \mathbb{N}$. Let `lst = [1, 2, ..., n]` and `item = 0`.

The for loop takes n iterations (the if condition is never `True`). Each iteration takes 1 step, for a total of n steps.

The `return False` executes and takes 1 step.

The total running time is $n + 1$ steps, which is $\Theta(n)$.

Hence $WC_{\texttt{search}} \in \Omega(n)$.

Putting it together

First, we proved that $WC_{\texttt{search}} \in \mathcal{O}(n)$.

Second, we found an input family (set of inputs, one for each $n \in \mathbb{N}$) whose running time is $\Theta(n)$. This told us that $WC_{\texttt{search}} \in \Omega(n)$.

Putting these two parts together, we can conclude that $WC_{\texttt{search}} \in \Theta(n)$.

Exercise 1: Worst-case running time analysis practice

Exercise 2: Lists vs. sets!

`any` and `all` revisited (briefly)

`any` and `all` are implemented using early returns:

- `any` can stop as soon as it encounters a `True`
- `all` can stop as soon as it encounters a `False`

Their worst-case running time is $\Theta(n)$, where n is the size of the input collection.

Demo: `any` and `all` with comprehensions

See Course Notes for details!

A trickier worst-case analysis

Definitions

A **palindrome** is a string that is the same when reversed.

- e.g., `'abba'`, `'davad'`, `'b'`.

A **prefix** of a string s is a string that appears at the beginning of s.

- e.g., `'abc'` is a prefix of `'abcdefg'`.

A **palindrome prefix** of a string s is a prefix of s that is a palindrome.

- e.g., `'abba'` is a palindrome prefix of `'abbaceb'`.

Problem: given a string `s`, return the length of the longest palindrome prefix of `s`.

- e.g., given `'abbaceb'`, return `4`.

```python
def palindrome_prefix(s: str) -> int:
    n = len(s)
    for prefix_length in range(n, 0, -1):  # goes from n down to 1
        # Check whether s[0:prefix_length] is a palindrome
        is_palindrome = ...

        # If a palindrome prefix is found, return the current lengt
        if is_palindrome:
            return prefix_length
```

```
def palindrome_prefix(s: str) -> int:
    n = len(s)
    for prefix_length in range(n, 0, -1):  # goes from n down to 1
        # Check whether s[0:prefix_length] is a palindrome
        is_palindrome = all(s[i] == s[prefix_length - 1 - i]
                            for i in range(0, prefix_length))

        # If a palindrome prefix is found, return the current lengt
        if is_palindrome:
            return prefix_length
```

We can show that the worst-case running time is $\mathcal{O}(n^2)$, where n is the length of `s`. (Exercise!)

To prove a matching lower bound, we need to find an input family whose runtime is $\Theta(n^2)$.

Finding a "maximum" input family

Let $n \in \mathbb{N}$.

1. Attempt 1: Let $s =$ '$aaa \dots a$' repeated n times.
    - `all` call takes n steps, and then returns `True`
    - The for loop only iterates once!
    - $\Theta(n)$ running time

2. Attempt 2: Let $s = abcabcabc \dots$ (abc repeated for n characters).
    - The for loop iterates n times (since no prefix is a palindrome).
    - But the `all` call only takes 1 or 2 steps before returning `False`.
    - So again, $\Theta(n)$ running time!

See Course Notes for a discussion of a "good enough" input family!

Summary

Today you learned to...

- Analyse the worst-case running time of an algorithm.
- Identify algorithms for which worst-case analysis is appropriate.
- Identify built-in functions and data type operations for which worst-case analysis is appropriate.

Homework

- Readings:
  - From today: 9.8
  - Next week: Chapter 10
- Assignment 4 due next week!
- Prep 10 has been released!
  - Prep 10 is the last prep (no preps in Weeks 11/12)
- Term Test 3 Info Page has been posted.