MAT137Y Tutorial 5 worksheet

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TOTAL POINTS

2 / 2

QUESTION 1

1 Q1+Q2 **2 / 2**

- ✓ **- 0 pts** Correct
- **- 1 pts** Several errors or lack of explanation
- **- 2 pts** No signature
- **- 2 pts** Blank
- **- 2 pts** Fake signature
- **- 1 pts** Incomplete submission
- **- 2 pts** No effort shown

MAT 137
Tutorial #5– Continuity and limit computations
October 18/19 , 2022
Due on Thursday, Oct 20 by 11:59pm via GradeScope

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- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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1. Show that f is continuous at x_0 if and only if $\lim_{h\to 0}[f(x_0+h) - f(x_0)] = 0$.

   Hint: write down the formal $\varepsilon-\delta$ definitions of "f is continuous at x_0" and $\lim_{h\to 0}[f(x_0+h) - f(x_0)] = 0$.

WTS: $\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t. $\forall x \in R$, $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon_1 \iff$

$\forall \varepsilon_2 > 0, \exists \delta_2 > 0$ s.t. $\forall h \in R$, $0 < |h| < \delta_2 \Rightarrow |f(x_0+h) - f(x_0)| < \varepsilon_2$

1 $\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ s.t. $\forall x \in R$, $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon_1$

2 $\forall \varepsilon_2 > 0, \exists \delta_2 > 0$ s.t. $\forall h \in R$, $0 < |h| < \delta_2 \Rightarrow |f(x_0+h) - f(x_0)| < \varepsilon_2$

Case (i): Assume 1, Let $\varepsilon_2 > 0$ and $h \in R$. Fix $\varepsilon_1 = \varepsilon_2$. Now, by 1 we can see $\exists \delta_1$ s.t $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon_2$. Now, fix $\delta_2 = \delta_1$ and assume $0 < |h| < \delta_2$. Since the ranges defined by $|h|$ and $|x - x_0|$ are same we can say that $0 < |h| < \delta_2 \Rightarrow h = x - x_0$ Using that implication, $0 < |h| < \delta_2 \Rightarrow x = x_0 + h$ but because of the implication above and 1 $0 < |h| < \delta_2 \Rightarrow |f(x) - f(x_0)| < \varepsilon_2$. By this implication, since $0 < |h| < \delta_2 \Rightarrow |f(x_0+h) - f(x_0)| < \varepsilon_2$ and thus 1 $\Rightarrow$ 2

Case (ii): Assume 2, Let $\varepsilon_1 > 0$ and $x \in R$. Fix $\varepsilon_2 = \varepsilon_1$. Now, by 2 we can see $\exists \delta_2$ s.t $0 < |h| < \delta_2 \Rightarrow |f(x_0+h) - f(x_0)| < \varepsilon_1$. Now, fix $\delta_1 = \delta_2$ and assume $0 < |x - x_0| < \delta_1$ Since the ranges defined by $|h|$ and $|x - x_0|$ are same we can say that $0 < |x - x_0| < \delta_1 \Rightarrow h = x - x_0$ Using that implication, $0 < |x - x_0| < \delta_1 \Rightarrow x - x_0 = h$ but because of the implication above and 2 $0 < |x - x_0| < \delta_1 \Rightarrow |f(x_0+h) - f(x_0)| < \varepsilon_1$. By this implication, since $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon_1$ and thus 2 $\Rightarrow$ 1

Since 1 $\Rightarrow$ 2 and 2 $\Rightarrow$ 1, 1 $\iff$ 2 hence proved.

$\blacksquare$

2. Calculate the following limits:
Note: L'Hôpitals' Rule is not allowed to use for any limits in this worksheet.

(a) $\lim_{x\to 3} \frac{2-x}{x-3}$ $\boxed{DNE}$

$LHL = \lim_{x\to 3^-} \frac{2-x}{x-3} \approx \frac{2-2.99}{2.99-3} \approx \infty$

$RHL = \lim_{x\to 3^+} \frac{2-x}{x-3} \approx \frac{2-3.01}{3.01-3} = -\infty$

Since the limit does not approach the same value from the left and right, the limit does not exist.

(b) $\lim_{x\to 3} \frac{2-x}{(x-3)^2} = \boxed{-\infty, DNE}$

$LHL = \lim_{x\to 3^-} \frac{2-x}{(x-3)^2} \approx \frac{2-2.99}{(2.99-3)^2} \approx -\infty$

$RHL = \lim_{x\to 3^+} \frac{2-x}{(x-3)^2} \approx \frac{2-3.01}{(3.01-3)^2} \approx -\infty$

Thus the limit tends to $-\infty$ and does not exist.

(c) $\lim_{x\to 1} \frac{\sin x}{x} = \boxed{\sin 1}$

Since $\frac{\sin x}{x}$ is continuous around 1, $\lim_{x\to 1} \frac{\sin x}{x} = \sin 1$

(d) $\lim_{x\to\infty} \frac{\sin x}{x} = \boxed{0}$

$\sin x$ lies between $[-1, 1]$, so $-1 \le \sin x \le 1$. Since x is positive, $\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

Using squeeze theorem, $\lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to-\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$. Now as $x\to\infty$, $\frac{1}{x}\to 0$ and $\frac{-1}{x}\to 0$

$\Rightarrow 0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0 \Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$

(e) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \boxed{\frac{1}{2}}$

Rationalising the numerator,

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0} \frac{x+1-1}{x\sqrt{x+1}+x}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

(f) $\lim_{x\to -\infty} \frac{\sqrt{x^2+1}+2x}{5x} = \boxed{\frac{1}{5}}$

Normalising with respect to x,

$$\lim_{x\to -\infty} \frac{\frac{1}{x}\left(|x|\sqrt{1+\frac{1}{x^2}}+2x\right)}{5}$$

As $x \to -\infty$, $|x| = -x$

$$= \lim_{x\to -\infty} \frac{\frac{1}{x}\left(-x\sqrt{1+\frac{1}{x^2}}+2x\right)}{5} = \lim_{x\to -\infty} \frac{-\sqrt{1+\frac{1}{x^2}}+2}{5}$$

As $x \to -\infty$

$\frac{1}{x^2} \to 0$

$$= \frac{-1+2}{5} = \frac{1}{5}$$

More questions for practice (you are not required to return your work):

3. Calculate the following limits:

(a) $\lim_{x\to 1} \left(x^2 + 2^x\right)$

(b) $\lim_{h\to 2} \dfrac{h^3 - 5h^2 + 3h + 6}{h^3 - h^2 - 3h + 2}$

(c) $\lim_{t\to 0} \dfrac{t}{\sin(2t)}$

(d) $\lim_{x\to 0} \dfrac{\sin(2x)}{\sin(3x)}$

(e) $\lim_{z\to 0} \dfrac{\sin(2z^2)}{\cos(3z)\ \sin^2(5z)}$

(f) $\lim_{x\to 3} \dfrac{\tan(x-3)}{2x-6}$

(g) $\lim_{x\to 0} \dfrac{2e^x}{\sin(2e^x)}$

(h) $\lim_{t\to 0} \dfrac{1-\cos(3t)}{t^2}$

(i) $\lim_{y\to 1} \dfrac{\sqrt{y+4} - \sqrt{4y+1}}{\sqrt{y} - 1}$

(j) $\lim_{x\to 0} \dfrac{\sin(1-\cos x)}{x\tan(\pi x)}$

(k) $\lim_{u\to 2} \dfrac{1}{2-u}\left(\sqrt{\dfrac{u+2}{u-1}} - 2\right)$

(l) $\lim_{x\to\infty} \left[x + \sqrt{x^2 - x}\right]$

(m) $\lim_{x\to -\infty} \left[x + \sqrt{x^2 - x}\right]$

Hint: The answers to Questions 3l and 3m are different.
Answer: (a) 3, (b)-1, (c) $\frac{1}{2}$,(d) $\frac{2}{3}$,(e) $\frac{2}{25}$, (f)$\frac{1}{2}$, (g)$\frac{2}{\sin 2}$, (h) $\frac{9}{2}$, (i)$\frac{-3}{\sqrt{5}}$, (j)$\frac{1}{2\pi}$, (k)$\frac{3}{4}$,
(l)∞(There is no indeterminate form to begin with. It is $\infty + \infty$.),
(m) $\frac{1}{2}$ (Multiply and divide by the conjugate first. Make sure you get $1/2$ and not $-1/2$.)

4. Calculate the following limits (Hint: you may need to use Squeeze Theorem):

(a) $\lim_{x\to 0} \sin(x^2)\sin(\dfrac{1}{x})$

(b) $\lim_{x\to 1^-} \sqrt{1-x^2}\cos(\dfrac{1}{(x-1)^2})$

(c) $\lim_{x\to\infty} \dfrac{x^2(2+\sin^2 x)}{x+10}$

(d) $\lim_{x\to -\infty} \dfrac{5x^2 - \sin(3x)}{x^2+2}$

Answer: (a)0, (b) 0, (c)∞, (d)5.