Unit 2 more continuity

Wednesday, October 12, 2022 9:56 AM

unit_2_03...

MAT137

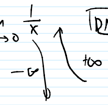
More Continuity

- Our test 1 will be on Friday, Oct 21 from 4:10 pm to 6 pm. Please read **Test 1 Information Page**.
- Watch video 2.19,2.20 and complete pre-class quiz 13
- Today we will discuss more continuity.

limits at infinity

if $\lim \frac{1}{x^2} = +\infty$ $\boxed{+\infty}$

$\lim_{x\to 0} \frac{1}{x}$ $\boxed{DNE}$



do not put DNE

$\lim_{x\to 0} \sin\frac{1}{x}$ $\boxed{DNE}$

Continuity

Continuity

If we have the following equality of limit and evaluation at point a

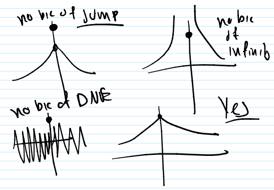
$$\underbrace{\lim_{x\to a} f(x)} = f(\lim_{x\to a} x) = \underbrace{f(a)}$$

then we say that f is *continuous* at $x = a$.

What is the epsilon-delta definition of continuity? We have the the regular algebraic operations preserve continuity:

$$\alpha f, f + gf \cdot g \text{ and } \frac{f(x)}{g(x)} \text{ if } g(a) \neq 0.$$

Powers x^α, trigonometric $\sin(x), \cos(x)$, exponentials e^x.



Discontinuities

Discontinuities

Consider the one-sided limits to a

$$L_+ = \lim_{x\to a^+} f(x) \text{ and } L_- = \lim_{x\to a^-} f(x).$$

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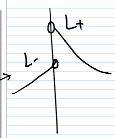
- If $L = L_+ = L_-$, then a is called *removable discontinuity* because we can just $f(a) := L$.
- If $L_+ \neq L_-$, then a is called *nonremovable discontinuity*.



Discontinuities

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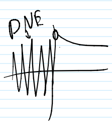
$$L_+ = \lim_{x\to a^+} f(x) \text{ and } L_- = \lim_{x\to a^-} f(x).$$

- If $L = L_+ = L_-$, then a is called *removable* ~~*discontinuity* be~~cause we can just $f(a) := L$.
- If $L_+ \neq L_-$, then a is called *nonremovable discontinuity*.
- Finally, if either of limits doesn't exist, then a is called *essential discontinuity*.



Examples/NonExamples

Classify the discontinuities of:

- $\begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

removable b/c

$L_- = \lim_{x\to 0^-} \frac{\sin x}{x} = 1 = \lim_{x\to 0^+} \frac{\sin x}{x} = L_+$

Examples/NonExamples

Classify the discontinuities of:

- $\begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$
- $\begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 1 & x = 0 \end{cases}$

Essential

$\lim_{x\to 0^-} \sin\frac{1}{x}$ DNE

b/c $x_1 = \frac{1}{\frac{\pi}{2} + 2k\pi} \to 0$

$y_1 = \frac{1}{\frac{3\pi}{2} + 2k\pi} \to 0$ as $k \to +\infty$.

$\sin(\frac{1}{x_1}) = 1$ and $\sin(\frac{1}{y_1}) = -1$

Discontinuities

Discontinuities

Consider the one-sided limits to a

$$L_+ = \lim_{x\to a^+} f(x) \text{ and } L_- = \lim_{x\to a^-} f(x).$$

- If $L = L_+ = L_-$, then a is called *removable discontinuity* because we can just $f(a) := L$.
- If $L_+ \neq L_-$, then a is called *nonremovable discontinuity*.
- Finally, if either of limits doesn't exist, then a is called *essential discontinuity*.

Examples/NonExamples

Classify the discontinuities of:

- $\begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$
- $\begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 1 & x = 0 \end{cases}$
- $\frac{x-1}{x^2-3x+2}$

removable

$\frac{x-1}{x^2-3x+2} = \frac{x-1}{(x-1)(x-2)}$

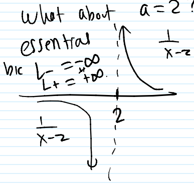
$\lim_{x\to 1} \frac{(x-1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{1}{x-2} = -1$

$\underset{L_-}{\parallel} \quad \underset{L_+}{\parallel}$

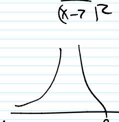
what about $a = 2$?

essential

b/c $L_- = -\infty$
$L_+ = +\infty$.



$\frac{1}{(x-2)^2}$



essential b/c $L_- = L_+ = +\infty$ is considered DNE.

Examples/NonExamples

Classify the discontinuities of:

$\lim_{x \to 1} \frac{(x-1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{1}{x-2} = -1$

$L_- = L_+$

$\frac{1}{x-2}$ 2

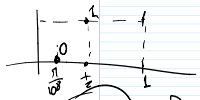
essential b/c $L_- = L_+ = \pm\infty$ is considered DNE.

Examples/NonExamples

Classify the discontinuities of:

- $\begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$
- $\begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 1 & x = 0 \end{cases}$
- $\frac{x-1}{x^2-3x+2}$
- The indicator

$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$



$a = 1$

Essential

1.2.10 book

Density result: any $x \in \mathbb{R}$ any $x \in \mathbb{Z}$

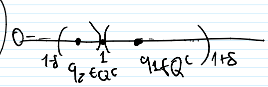
q_δ p_δ

$q_\delta \in \mathbb{Q}^c$ $x-\delta$ x $x+\delta$

$p_\delta \in \mathbb{Q}$

$\mathbb{Q} \ni \pi = 3.141\ldots$ 10--C

DNE $\lim_{x\to 1} \chi_{\mathbb{Q}}(x)$



$q_2 \in (1-\delta, 1)$ and $q_1 \in (1, 1+\delta)$

True or False? – Discontinuities

1. IF f and g have removable discontinuities at a THEN $f + g$ has a removable discontinuity at a

False

$\lim_{x\to a} f(x) = L_- = L_+ = L \Rightarrow \lim (f+g) = \lim f + \lim g = R + L$

$\lim_{x\to a} g(x) = R_- = R_+ = R$

$f(x) = \begin{cases} x + \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$g(x) = \begin{cases} -\frac{\sin x}{x}, & x \neq 0 \\ -1, & x = 0 \end{cases}$

$f+g = \begin{cases} x, & x \neq 0 \\ 0, & x = 0 \end{cases} = x$

$f+g$ continuous.

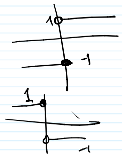
True or False? – Discontinuities

1. IF f and g have removable discontinuities at a THEN $f + g$ has a removable discontinuity at a
2. IF f and g have non-removable discontinuities at a THEN $f + g$ has a non-removable discontinuity at a

$a = 0$

$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x \leq 0 \end{cases}$

$g(x) = \begin{cases} -1, & x > 0 \\ 1, & x \leq 0 \end{cases}$

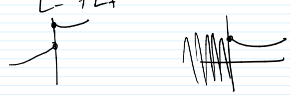


$f+g = \begin{cases} 0, & x > 0 \\ 0, & x \leq 0 \end{cases} = 0$ contin.

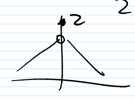
non-removable vs Essential

L_-, L_+ exist | L_- or L_+ DNE

$L_- \neq L_+$



$f(x) = \begin{cases} -x+1, & x > \\ x+1, & x < \\ 2, & x = \end{cases}$



$g(x) =$

Which one is the correct claim?

Claim 1?

(Assuming these limits exist)

$$\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right)$$

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Claim 1?

(Assuming these limits exist)

$$\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right)$$

Claim 2?

IF (A) $\lim_{x\to a} f(x) = L$, and (B) $\lim_{t\to L} g(t) = M$

THEN (C) $\lim_{x\to a} g(f(x)) = M$

A difficult example

one way

Hint: use floor

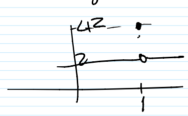
Construct a pair of functions f and g such that

$$\lim_{x\to 0} f(x) = 1$$
$$\lim_{t\to 1} g(t) = 2$$
$$\lim_{x\to 0} g(f(x)) = 42$$

$f(x) = 1$ $\lim_{x\to 0} f(x) = 1$ $\lim_{t\to 1} g(t) = 2$

$\lim_{x\to 0} g(f(x)) = g(1) = 42$

$g(x) = \begin{cases} 2, & x \neq 1 \\ 42, & x = 1 \end{cases}$



Behaviour of limits under composition

From previous examples, in general, it is not true that

$$\lim_{x\to a} f(g(x)) \neq f(\lim_{x\to a} g(x)) \quad (1)$$

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However, the statement does become true if f is continuous.

Limits for composition

Suppose that f, g are functions with

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- f is continuous

>0

<0

0

$-f(x)$

Behaviour of limits under composition

From previous examples, in general, it is not true that

$$\lim_{x\to a} f(g(x)) \neq f(\lim_{x\to a} g(x)) \tag{1}$$

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Limits for composition

Suppose that f, g are functions with

- $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = M$
- f is continuous

Then we can swap limit and function:

$$\lim_{x\to a} f(g(x)) = M.$$

Proof

1. Write down the formal definition $\lim_{x\to a} f(g(x)) = M$.

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Proof

1. Write down the formal definition $\lim_{x\to a} f(g(x)) = M$.
2. Write down what you know using different numbering eg. $\epsilon_1, \epsilon_2, \delta_1, \delta_2$
   - 2.1 $\lim_{x\to a} g(x) = L$
   - 2.2 $\lim_{x\to L} f(x) = M$
   - 2.3 f is continuous.
3. Reverse engineering: Start from $|f(g(x)) - M| \leq \epsilon$ and use change of variables $y = g(x)$.
4. Write down a complete direct proof.

Change of variables

change of variables

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change of variables

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- $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = M$
- Either f is continuous OR $g(x) \neq L$ near $x = a$ i.e. we can find interval I around a s.t. $g(x) \neq L$ for all $x \in I$.

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Then we can swap limit and function:

$$\lim_{x\to a} f(g(x)) = M.$$

Example

Use the change of variables theorem to compute the limit

$$\lim_{x\to 1} \frac{\sin(1-\cos(\frac{\pi x}{2}))}{1-\cos(\frac{\pi x}{2})}.$$