MAT137Y Problem Set 1

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TOTAL POINTS

12.5 / 30

QUESTION 1

1 Q1-part 1-(A) 1 / 2

- **1 pts** Wrong Submission formats: 1) didn't use the given template (e.g. submitted a long pdf file with all pages); 2) the questions were not in the correct place (pages were in the wrong order or the pages needed to be flipped) 3) missing several pages 4) adding some extra pages. If your submission is one of these cases, your submission will be deducted by 1 point. This punishment is done in Q1-part1-(A). No repeated punishments.
- **0 pts** Correct
- **0.3 pts** Use words and connectors to explain. A proof should be read as usual prose even if it has mathematical symbols.
- **0.3 pts** Explanation: Shaded part still needs explanation OR read the comment.

✓ **- 1 pts Some ideas were developed, but logic in the proof fails. See posted solutions.**

- **0.3 pts** There are still cases to review (or mention why you are not considering them). Definition asks to check for every element in $$A$$.
- **0.2 pts** False statement.
- **1.8 pts** Logic in the proof fails to prove the statement. See posted solutions.
- **1.5 pts** $$A$$ is a blow-up set. Make sure that you understand the definition.
- **0 pts** Not using given template
- **2 pts** You picked an element in A that works. But you don't show evidence of knowing how this works.
- **2 pts** Not answered.

1 P and Q statements are wrong picked.

QUESTION 2

2 Q1-part 1-(B) 2 / 2

✓ **- 0 pts Correct**

- **0.5 pts** You should make it clear that you are picking **one** **arbitrary** $$x$$ in the set, it is not clear from what you wrote, you should use quantifiers properly.
- **0.5 pts** You should make clear that you are choosing y according to x, you should use quantifiers properly
- **0.5 pts** Using unquantified/undefined variable
- **0.5 pts** You need to show why element $$y$$ is in the set $$B$$
- **0.5 pts** The shaded part is not a proper notation/ is extra weird wording
- **0.5 pts** You did not explain logic properly in the shaded part **or** did not properly use words to explain what you are doing **or** your logic is weird
- **0 pts** Wrong Question
- **2 pts** Your argument does not properly show that B is a step-down. Please refer to the solutions.
- **0.5 pts** Your choice of y does not work
- **1 pts** Click here to replace this description.

QUESTION 3

3 Q1-part 2-(a) 3 / 4

- **0 pts** Correct

✓ **- 1 pts Insufficient explanation for why $$A$$ is a blow up**

- **1 pts** Insufficient explanation for why $$A$$ is not a step down
- **1 pts** Incorrect counter example
- **2 pts** Insufficient explanation for why counter example works
- **3 pts** No counter example
- **4 pts** Statement is false, but answer says it's true
- **4 pts** Empty or wrong submission
- **0 pts** Wrong Question

- **4 pts** Illegible

QUESTION 4

4 Q1-part 2-(b) 0 / 3

- **0 pts** Correct

✓ - **3 pts** **Not stating it is true**

- **1 pts** No or wrong statement for negation
- **1 pts** No or inadequate explanation
- **0 pts** Wrong place
- **2 pts** You should prove the statement for any step-down set $$A$$, not just for an example of $$A$$.
- **3 pts** Wrong or missing submission
- **3 pts** Can't find the answer
- **1 pts** Don't have to be an integer
- **0.5 pts** Not for all $$y\in A$$, only for those $$y$$ that satisfies $$x-y>0$$
- **1 pts** This doesn't imply $$y\in A$$.
- **0 pts** not using given template
- **2 pts** Cannot read
- **1 pts** improper use of quantifiers
- **0.5 pts** Weird notation

QUESTION 5

5 Q1-part 2-(c) 0 / 4

- **0 pts** Correct

✓ - **4 pts** **statement is false, please check solution key**

- **2 pts** invalid counter example, if you take min element of A, then the blow up implication holds vacuously
- **2 pts** your set is step down, you must find a counter example that is not blow up + not step down
- **3 pts** need counter example set A and provide corresponding proofs
- **3 pts** see comment on page, or check answer key
- **2 pts** see comment on page, or check answer key
- **0 pts** Repeated submission (You should submit once and add your partner in your group).
- **1 pts** see comment on page, or check answer key
- **0.5 pts** minor error, see comment on page, or solution key

2 statement is false; consult answer key for correct sol.tuions and proofs

QUESTION 6

6 Q1-part 2-(d) 0 / 3

- **0 pts** Correct
- **1 pts** It is not clear what is your logic in the shaded part
- **2 pts** You did not argue correctly how you conclude not step down from blow up
- **0 pts** Wrong question
- **2 pts** It is not really the contrapositive of b since we require A is blow up and $$A \subset \mathbb{Z}$$
- **2 pts** You cannot provide one particular example, you are proving for all set A

✓ - **3 pts** **You did not answer questions correctly**

QUESTION 7

7 Q1-part 2-(e) 3 / 3

✓ - **0 pts** **Correct**

- **1 pts** Correct explanation but don't have a correct detailed proof
- **2 pts** Used examples for A and B instead of generic step-down sets
- **2 pts** Incorrect explanation or missing proof.
- **3 pts** Blank or irrelevant work
- **3 pts** Said the statement is False
- **0 pts** Wrong question
- **0.5 pts** Highlighted part is wrong.
- **1 pts** Error in the proof
- **1 pts** Missing explanation
- **1 pts** Highlighted part needs to be justified
- **0 pts** Not using the correct template
- **0.5 pts** Highlighted notation is incorrect
- **0.5 pts** Small error in the proof
- **1 pts** Very difficult to read

I recommend that you either handwrite your solutions or use LaTeX to typeset them.

QUESTION 8

8 Q1-part 2-(f) 0 / 5

- **0 pts** Correct
- **5 pts** No Answer

✓ **- 5 pts Selected "True" when "False" is the correct answer.**

- **1 pts** Insufficent proof form.
- **2 pts** No proof of $$ A \cup B $$ is not blow up
- **0 pts** Wrong Question
- **1 pts** Did not prove $$A$$ and/or $$B$$ are blow ups.
- **1 pts** Incorrect counter example
- **5 pts** Did not submit the correct document.
- **1 pts** Incorrect explaination for $$A$$ and/or $$B$$ being blow ups
- **1 pts** Insufficient proof for showing $$A \cup B$$ is not a blow up
- **0 pts** Wrong
- **0.5 pts** Wrong order of fixing variables
- **0.5 pts** No mention of the case $$y = x$$ when proving a set is blow up.
- **0.5 pts** Fixing a variable and then immediately choosing a value for it (re-defining it).
- **2 pts** Incorrect proof for $$A \cup B$$ is not a blow up
- **1 pts** No counter example
- **1 pts** Incorrect counter example. If $$A \cup B$$ is finite, then $$\exists x \in A \cup B$$ such that $$\forall y \in A \cup B,~ x <= y $$, hence it is vacuously true that $$A \cup B$$ is a blow up.
- **3 pts** Incorrect counter example. If $$A \cup B$$ is finite, then $$\exists x \in A \cup B$$ such that $$\forall y \in A \cup B,~ x <= y $$, hence it is vacuously true that $$A \cup B$$ is a blow up.
- **1 pts** Incorrect proof for $$A$$ and/or $$B$$ being a blow up
- **0.5 pts** Steps are unclear and/or do not follow a logical progression.
- **0 pts** Not using given template
- **0.5 pts** Minor error in proof of $$A$$ and/or $$B$$ being blow up.
- **2 pts** No counter example
- **1 pts** Incorrect counter example
- **5 pts** Did not submit the question
- **1 pts** Insufficient proof for $$A$$ and or $$B$$ being blow up
- **0 pts** Page overflow
- **5 pts** Illegible
- **0.5 pts** Minor error in the proof for $$A \cup B$$ is not a blow up
- **5 pts** No selection
- **1 pts** No attempt to prove $$A \cup B$$ is not a blow up
- **3 pts** No counter example
- **1 pts** Selected TRUE when FALSE was the correct selection
- **0.5 pts** Incorrect proof type
- **0.5 pts** Did not correctly justify $$A$$ and or $$B$$ being blow up sets
- **3 pts** Incorrect counter example, $$A$$ and $$B$$ are equal
- **3 pts** In correct counter example, $$A \cup B$$ has a minimum in the set

3 This statement is actually False. For a correct proof please view the solutions

QUESTION 9

9 Q2 3.5 / 4

- **1 pts** Incorrect polynomial/no polynomial
- **0.5 pts** Base case error/lack of explanation/did not show LHS = RHS
- **1 pts** Did not check base case or it is incorrect.
- **1 pts** No induction hypothesis/ incorrect

✓ **- 0.5 pts Did not state where they used induction hypothesis.**

- **0.5 pts** Did not fix n∈Z+ arbitrarily at the beginning of the induction step.
- **0.5 pts** Did not fix k∈Z+ arbitrarily at the beginning of the induction step.
- **0.5 pts** Induction step has insufficient explanation/small error.
- **1 pts** Induction step is incorrect.
- **0 pts** Correct, nice job!
- **0 pts** Submitted wrong question/wrong format.
- **0 pts** Small algebra error.

- **0 pts** Scan quality needs to be better.
- **0 pts** This should be n/n+1.
- **1 pts** You do not need to prove the induction hypothesis is true, you assume it is true.
- **0 pts** If you are typing your answer, please use latex. This is very hard to read.
- **0.5 pts** This part is hard to read. Please write your proofs neatly and clearly indicate how you're going from one step to the next.
- **1 pts** Base case incorrect.
- **3 pts** Incorrect polynomial/no polynomial. But structure of induction almost correct.
- **4 pts** Illegible.
- **4 pts** No submission.
- **3.5 pts** Incorrect polynomial, and structure of proof mostly incorrect.
- **4 pts** Completely incorrect/no polynomial given
+ **0.5 pts** Some understanding of the structure of proof by induction but major errors.
- **4 pts** Not using the induction method.
+ **1 pts** Correct formula.

4

QUESTION 10

10 Front Page-signature checks 0 / 0

✓ **- 0 pts Correct**

- **1 pts** Repeated submission (You should only submit once and add your partner in your group).
- **1 pts** No signature or the signature is digital

MAT 137Y: Calculus with proofs
Assignment 1
Due on Thursday, Sept 29 by 11:59pm via GradeScope

Instructions

This problem set is based on Unit 1: Logic, sets, and notations. Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions. Remember you can submit in pairs or individually.

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: Shivesh Prakash

Student number: 1008693790

Full Name: _____

Student number: _____

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) 

2) _____

1. In this problem, we will deal with subsets $A \subseteq \mathbb{R}$. Let's define two new concepts.

   (i) We say that A is *blow-up* if

   $$\exists x \in A \ s.t. \ \forall y \in A, x - y > 0 \implies x - y \ is \ an \ odd \ integer.$$

   (ii) We say that A is *step-down* if

   $$\forall x \in A, \exists y \in A \ s.t. \ x - y \geq 1 \ AND \ x - y \ is \ an \ even \ integer.$$

   (Part 1) To help you understand the definitions, here are two sets

   $$A = \{-2, 0, 3, 3.5, 4\} \text{ and } B = \{2k : k \in \mathbb{Z}\}.$$

   (A) Prove that A is a blow-up set by using the definition.

The definition says- $\exists x \in A \ s.t. \ \forall y \in A, x-y > 0 \Rightarrow x-y$ is an odd integer

Lets say this statement is P and this is Q.

Now this definition is of the form P=>Q . Let's check the P statement for A. P says there exists an element in A which is greater than all elements, including itself. This is clearly false because even the greatest element 4 is not greater than itself. Thus P is false. So the definition of the form P=> Q is true. Furthering this argument, any right closed set is blow-up and obviously any right open set is blow up because a greatest element does not exist. Thus any set is blow-up. Hence proved A is blow-up.

   (B) Prove that B is a step-down set by using the definition.

The definition says - $\forall x \in A, \exists y \in A \ s.t. \ x-y \geq 1$ AND $x-y$ is an even integer

Lets say this statement is M and this is N.

Now this definition is of the form M and N. Let's check statement M for Set B. For all fixed arbitrary x in A let us assume y = x - 2 which always exists in B because B is a set of all even numbers. Here x - y = 2 which is ≥ 1, so M is true and x - y =2 is even so N is true. Thus the definition M and N is true, therefore B is step-down. Hence proved. By this proof any left open (to -inf) set of continuous integers is also step-down. The empty set { } is also step-down because both M and N are vacuously true.

(Part 2) Below are six claims. Which ones are true and which ones are false? If a claim is true, prove it. If a claim is false, provide a counterexample and a justification of how the counterexample shows the claim is false.

(a) If A is blow-up, then A is step-down.

This statement is ○ True ● False

As proved in Part 1 (A), any set is blow-up, so the claim states that every set is step-down. A counter example would be A = {1, 2} which is blow-up but not step-down because there is no y in A for any x in A such that x - y ≥ 1 and is even. Thus this claim is false, proven by the counter example A = {1, 2}.

(b) If A is step-down, then A is not blow-up.

This statement is ○ True ● False

As proved in Part 1 (A), any set is blow-up, so the claim states that a step-down set does not exist. A counter example would be A = { 2k : k ∈ ℤ }, this set is step-down as proved in Part 1 (B) and it is also blow-up (from Part 1 (A)). Thus this claim is false, proven by the counter example A = { 2k : k ∈ ℤ }.

(c) If A is not blow-up, then A is step-down.

This statement is ● True ○ False

As proved in Part 1 (A), any set is blow-up. So the first part of the claim is vacuous, as there exists no A to check the second part of the claim. The claim is thus true.

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(d) If A is blow-up and $A \subset \mathbb{Z}$, then A is not step-down.

This statement is ○ True ● False

As proved in Part 1 (A), any set is blow-up, so the claim states that no set of integers is step-down. A counter example would be A = { 2k : k ∈ ℤ }, this set is step-down as proved in Part 1 (B) and it is also blow-up (from Part 1 (A)). This set is also a subset of integers. Thus this claim is false, proven by the counter example A = { 2k : k ∈ ℤ }.

(e) If we have two sets $A, B \subset \mathbb{R}$ and $A \neq B$, A and B are both step-down, then $A \cup B$ is also step-down.

This statement is ● True ○ False

It is given that A and B are step-down individually, that means for all x in A, there exists y in A such that x - y ≥ 1 and x- y is an even integer. Now all elements of A exist in A∪ B, so all those X's have their corresponding y's in A∪ B, thus they all satisfy the step-down definition. The same argument can be used to show that all elements of B is exist in A∪ B and satisfy the step-down definition. So in this way all the elements of set A∪ B satisfy the step-down definition. Hence proved set A∪B is step-down if A and B are step-down individually.

(f) If we have two sets $A, B \subset \mathbb{R}$ and $A \neq B$, A and B are both blow-up, then $A \cup B$ is also blow-up.

This statement is ● True ○ False

As proved in Part 1 (A), any set is blow-up, so the claim states that the union of two blow-up sets is blow-up. Now, since the union of any sets A and B gives a valid set, it must be blow-up from the hypothesis in Part 1 (A). Hence proved A∪ B is blow-up.

2. Define $f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)}$, where $n \in \mathbb{Z}^+$.
Here, $\mathbb{Z}^+$ is the set of all positive integers. Find a rational polynomial that is equal to $f(n)$.

$$f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Justify this equality by induction. Hint: compute $f(1), f(2), f(3), f(4)$ by hand first and then make a conjecture.

$f(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \frac{4-3}{3\cdot 4} + \cdots + \frac{(n+1)-n}{n\cdot(n+1)}$

$\Rightarrow f(n) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{(n+1)} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{(n+1)} = \frac{1}{1} - \frac{1}{(n+1)} = \boxed{\frac{n}{n+1}}$

Justification by induction →

To justify: $f(n) = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Lets check for $n=1$ first, Left side $= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$. Right side $= \frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$ This case is satisfied. Let x be an arbitrary number, assume that $\sum_{i=1}^{x} \frac{1}{i(i+1)} = \frac{x}{x+1}$

If we show that this result holds for $(x+1)$ the statement will be justified.

Now, $\sum_{i=1}^{x+1} \frac{1}{i(i+1)} = \sum_{i=1}^{x} \frac{1}{i(i+1)} + \frac{1}{(x+1)(x+2)} = \frac{x}{x+1} + \frac{1}{(x+1)(x+2)} = \frac{x(x+2)+1}{(x+1)(x+2)}$

$= \frac{x^2+2x+1}{(x+1)(x+2)} = \frac{(x+1)^2}{(x+1)(x+2)} = \frac{x+1}{x+2} = \frac{(x+1)}{(x+1)+1}$

So we have shown that the statement holds for $n=1$ and it holds for $(x+1)$ assuming it to be true for some arbitrary x. Thus by mathematical induction this statement is proven.