MAT 137
Tutorial #2– Proofs
September 27-28, 2022
Due on Thursday, Sept 29 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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I confirm that:

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- I understand that the collaboration of this worksheet is only among this group. I have not violated this rule while writing this worksheet.
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By signing this document, I agree that the statements above are true.

First Name	Last Name	UofT email	signature
Fahia	Mohamed	Fahia.mohamed@mail.utoronto.ca	
Raffaele	Dengler	R.dengler@mail.utoronto.ca	Raffaele
Kaamil Javeed	Shaikh	Kaamil.Shaikh@mail.utoronto.ca	KShaikh
Shivesh	Prakash	shivesh.prakash@mail.utoronto.ca	

TA name: Austin TA signature:

Recall some proof techniques:

- To prove "$\exists x \ldots$", say what x is.
- To prove "$\forall x \ldots$", begin by fixing a generic x.
- To prove "$P \implies Q$", assume P is true and show that Q is true.

Write formal, rigorous proofs for these statements:

1. In tutorial 1, we have defined several concepts. Let A be a non-empty subset of $\mathbb{R}$.
   We say A is *excited* if $\exists a \in \mathbb{R}$ such that $\forall x \in A, x \geq a$.
   We say A is *happy* if $\exists a \in A$ such that $\forall x \in A, x \geq a$.
   Below are three claims. Which ones are true and which ones are false? If a claim is true, prove it. If a claim is false, provide a counterexample and a justification of how the counterexample shows the claim is false.

   (a) If A is happy, then A is excited.

   Assume A is happy,
   therefore, $\exists a \in A$ s.t. $\forall x \in A, x \geq a$
   Since $A \subseteq \mathbb{R}$ and $A \neq \emptyset$
   $a \in \mathbb{R}$, therefore $\exists a \in \mathbb{R}$ s.t. $\forall x \in A, x \geq a$
   by definition, A is excited ■

   (b) If A is excited, then A is happy.

   Let $S = \{2^{-x} : x \in \mathbb{N}\}$
   Let $b \in S$
   by definition, $b = 2^{-k}$ for some natural number k
   Since $\forall x \in \mathbb{R}$, $2^{x} > 0$ and $-k \in \mathbb{R}$
   $b = 2^{-k} \geq 0$
   fix $c = 0$
   therefore, $\exists c \in \mathbb{R}$ s.t. $\forall b \in S$, $b \geq c$
   Consider $\frac{b}{2}$
   $\frac{b}{2} = \frac{2^{-k}}{2} = 2^{-k-1} = 2^{-(k+1)}$
   Since $k \in \mathbb{N}$, $k+1 \in \mathbb{N}$
   therefore $\frac{b}{2} \in S$. by the definition of S.
   Since $b > 0$, $\frac{b}{2} < b$ Fix $d = \frac{b}{2}$, therefore,
   $\forall b \in S$, $\exists d \in S$ s.t. $d < b$ which is the negation of the definition of a happy set ■

(c) If A and B are both happy and non-empty with $A \neq B$, then $A \cup B$ is happy.

Assume A and B are both happy and non-empty

by definition $\exists a \in A$ s.t. $\forall x \in A$, $x \geq a$

$\exists b \in B$ s.t. $\forall y \in B$, $y \geq b$

Since $A \neq B$, either $a > b$ or $a < b$

If $a < b$ then $\forall x \in A \cup B$, $x \geq a$

If $a > b$ then $\forall x \in A \cup B$, $x \geq b$

Since $a \in A$ and $b \in B$, $a \in A \cup B$ and $b \in A \cup B$

therefore, $A \cup B$ is happy ■

2. For every positive number $x > 0$ and for every natural number $n \geq 2$,

$$(1+x)^n > 1+nx.$$

Hint: Use induction.

Base case: Since $x > 0$, $x^2 > 0$

let $P(m)$ be the statement that $(1+x)^m > 1+mx$

Let $m = 2$

$P(2)$: $(1+x)^2 > 1+2x$

$= 1+2x+x^2 > 1+2x$

since x^2 is positive, $1+2x+x^2 > 1+2x$, therefore $P(2)$ is true.

Induction hypothesis: Let k be an arbitrary natural number ≥ 2, assume $P(k) = (1+x)^k > 1+kx$ is true

WTS: $P(k+1)$ is true

$P(k+1) \Rightarrow (1+x)^{k+1} > 1+(k+1)x$

multiplying $(1+x)$ on both sides of $P(k)$ since $1+x > 0$

$(1+x)^k(1+x) > (1+kx)(1+x) \Rightarrow (1+x)^{k+1} > 1+kx+x+kx^2 =$

$1+(k+1)x+kx^2 > 1+(k+1)x$, since $x^2 > 0$ and $k > 0$, therefore $P(k) \Rightarrow P(k+1)$

thus $P(2)$ is true and $P(k+1)$ is true

Where k is any natural number ≥ 2.

Therefore, $(1+x)^n > 1+nx$ ■