3. **[5 marks] Asymptotic Notation.**

Recall that the **ceiling function** $\lceil x \rceil$ is defined as the smallest integer $\geq x$. For example, $\lceil 7.5 \rceil = 8$ and $\lceil 0.001 \rceil = 1$. Consider the following statement:

$$\forall k \in \mathbb{R}^+, \ \lceil k \cdot n \rceil \in \mathcal{O}(n)$$

(a) **[1 mark]** Rewrite the above statement with the definition of Big-O expanded.

$\forall k \in R^+, \exists c, n_0 \in R^+, \forall n \in N, \ n \geq n_0 \Rightarrow \lceil k.n \rceil \leq cn$

(b) **[4 marks]** Prove the above statement, *without* using any properties of Big-O/Omega/Theta. You may use the following property of the ceiling function: $\forall x \in \mathbb{R}, \ \lceil x \rceil < x + 1$.

Fix $k \in R^+$

Let $c = k+1 \in R^+$

Let $n_0 = 1 \in R^+$

Fix $n \in N$

WTS: $n \geq n_0 \Rightarrow \lceil k.n \rceil \leq cn$

Assume $n \geq n_0 = 1$,

We know that $\forall x \in R, \lceil x \rceil < x+1$,

Since $kn \in R$, $\lceil kn \rceil < kn+1$

Since $n \geq 1$, $kn+1 \leq kn+n$

$\lceil kn \rceil < kn+1 \leq kn+n = (k+1)n = c.n$

$\Leftrightarrow \lceil kn \rceil < cn$ Hence proved.

You may continue your proof on the next page.

Use this page to continue your proof and/or for rough work (clearly indicate what should be graded).