- Deadline to add/change courses: Wed., Sept 21
- Deadline of pre-calculus quiz: Thursday, Sept 22 at 11:59 pm.
- Deadline of tutorial 1 worksheet: Thursday, Sept 22 at 11:59 pm
- Today we will discuss proofs.
- Watch videos 1.14, 1.15 and complete pre-class quiz 5.

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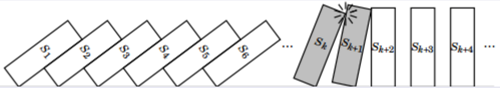

Proving a function is one-to-one

Definition

Let f be a function with domain D.
We say f is one-to-one when

- $\forall x_1, x_2 \in D,\ x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
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Exercise

Prove that $f(x) = 3x + 2$, with domain $\mathbb{R}$, is one-to-one.

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- Write the structure of your proof.

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Prove that $f(x) = x^2$, with domain $\mathbb{R}$, is not one-to-one.

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Let f be a function with domain D.

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4. Complete the proof.

DISproving a theorem

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2. Prove the theorem is false.

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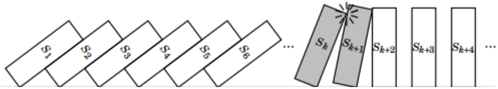

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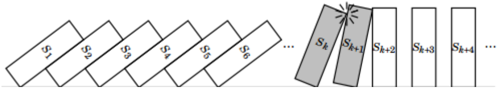

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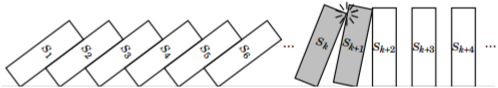

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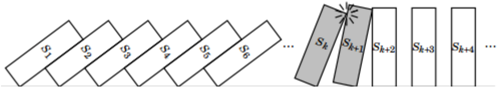

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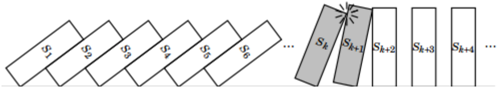

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That implies that all the $P(n), n \in \mathbb{N}$ are true.

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Variations on induction

Let S_n be a statement depending on a positive integer n.

In each of the following cases, which statements are guaranteed to be true?

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2. We have proven:
   - S_1
   - $\forall n \geq 3,\ S_n \implies S_{n+1}$
3. We have proven:
   - S_1
   - $\forall n \geq 1,\ S_n \implies S_{n+3}$
4. We have proven:
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Variations on induction 2

We want to prove

$$\forall n \geq 1,\ S_n$$

So far we have proven

- S_1
- $\forall n \geq 1,\ S_n \implies S_{n+3}.$

What else do we need to do?

Variations on induction 3

We want to prove

$$\forall n \in \mathbb{Z},\ S_n$$

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What is wrong with this proof by induction?

Theorem

$\forall N \geq 1$, *every set of* N *students in MAT137 will get the same grade.*

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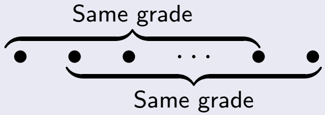
$\forall N \geq 1$, *every set of N students in MAT137 will get the same grade.*

Proof.

- **Base case.** It is clearly true for $N = 1$.
- **Induction step.**
  Assume it is true for N. I'll show it is true for $N + 1$.
  Take a set of $N + 1$ students. By induction hypothesis:
  - The first N students get the same grade.
  - The last N students get the same grade.



Hence the $N + 1$ students all get the same grade.

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For every $N \geq 1$, let

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What did we actually prove in the previous page?

- S_1 ?
- $\forall N \geq 1, \quad S_N \implies S_{N+1}$?

Induction exercise

Using induction prove the following
$P(n) =$"the inequality $2^n > n + 1$ for $n \geq 2$."

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