MAT137Y Problem Set 2

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TOTAL POINTS

25 / 30

QUESTION 1

1 Q1 **4.5 / 5**

- **- 0 pts** Correct
- **- 0.5 pts** f needs to be defined for all $$x\in \mathbb{R}$$, even when limit does not exist
- **- 0.5 pts** Part h) is not satisfied
- ✓ **- 0.5 pts according to b) the limit of f should not exist when $$a\in{\pm4, 2}$$ and exist everywhere else**
- **- 0.5 pts** part d not satisfied
- **- 0.5 pts** part g) not satisfied
- **- 0.5 pts** For part j $$\lim_{x\to4}f(1-e^{f(x)})$$ needs to be $$2$$
- **- 5 pts** please upload your problem 1 here
- **- 0.5 pts** part c not satisfied
- **- 0.5 pts** part d not satisfied
- **- 0.5 pts** e) not satisfied
- **- 0.5 pts** part f not satisfied
- **- 0.5 pts** part i) not satisfied
- **- 0.5 pts** Part j) is not satisfied
- **- 1 pts** The graph you drew is not a function, does not pass vertical line test at x = 3 for example.
- **- 0.5 pts** See comment on page
- **- 1 pts** please see comment on page
- **- 3.5 pts** please see comment on page
- **- 4 pts** Too blurry cannot read
- **- 3 pts** See comment on page

QUESTION 2

2 Q2(a) **1 / 1**

- ✓ **- 0 pts Correct**
- **- 1 pts** Incorrect domain. See solutions.
- **- 0.5 pts** Missing justification
- **- 0.5 pts** Justification incorrect
- **- 1 pts** No solution
- **- 0.5 pts** Notation error
- **- 0.5 pts** Small error
- **- 0.5 pts** Messy or hard to follow
- **- 0 pts** Incorrect page
- **- 0 pts** Page sideways

QUESTION 3

3 Q2(b) **0.5 / 1**

- **- 0 pts** Correct
- ✓ **- 0.5 pts Incorrect or missing justification**
- **- 0.5 pts** Minor error in solution
- **- 1 pts** Incorrect set $$I$$
- **- 0 pts** Wrong document
- **- 0 pts** Wrong question
- **- 0 pts** Not using template
- **- 0 pts** Oriented wrong

1 You have not shown $$f = g$$ on this domain.

QUESTION 4

4 Q2(c) **3 / 4**

- **- 0 pts** Correct
- **- 0.5 pts** Communication is not entirely clear.
- **- 1 pts** Chosen $$\delta$$ does not work.
- ✓ **- 1 pts Chosen $$\delta$$ is not the minimum of two numbers.**
- **- 1 pts** Chosen $$\delta$$ does not bound $$\sqrt{x}+3$$ correctly.
- **- 1 pts** Incorrect algebra for bounding $$|\sqrt{x}+1-4|$$.
- **- 0 pts** Wrong Question.
- **- 4 pts** Missing solution.

2 What if $$\varepsilon$$ is large, so that $$|x-9|<\delta$$ doesn't guarentee that $$x \geq 0$$?

QUESTION 5

5 Q2(d) **1 / 2**

- 0 pts Correct

✓ **- 1 pts Incorrect delta or no delta specified. Need to explicitly define $$\delta$$ to be equal to a value in any delta-epsilon proof.**

- 1 pts doesn't explicitly justify that the range for x rests in the interval where two functions are equal

- 1 pts Incorrect algebra

- 0.5 pts Notation error

- 0.5 pts incorrect range for x

- 0.5 pts Proof structure incorrect

- 0.5 pts Small error

- 2 pts No solution submitted

- 0 pts Incorrect page

QUESTION 6

6 Q3(a) **3 / 3**

✓ **- 0 pts Correct**

- 3 pts No solution/ Choose True while the statement is False

- 0.5 pts Improper notation/Symbols

- 2 pts You counter example does not satisfy the requirement/Your function is not properly defined

- 0.5 pts Your function is not defined on all of real numbers

- 1 pts Did not properly show why limit of $$f(g(x))$$ exists

- 1 pts you function is not properly defined on discontinuous point

- 1 pts You did not properly explain why your counter example works

- 1 pts You did not define your function precisely

- 2 pts No counter examples/Your argument is incorrect

- 1 pts You did not properly define your function $$g$$ or $$f$$

QUESTION 7

7 Q3(b) **2 / 3**

- 0 pts Correct

✓ **- 1 pts Does not clearly state the theorems and properties used (i.e. specific limit laws, continuity, etc).**

- 2 pts Proof is incorrect or insufficient.

- 1 pts Highlighted part needs to be justified or is incorrectly justified.

- 0 pts Wrong question

- 0.5 pts Highlighted part needs to be justified or is incorrectly justified.

- 1 pts Error in the argument.

- 3 pts Missing solution

- 1 pts Missing work

- 3 pts Said the statement is False.

- 0.5 pts Error in the highlighted part.

3 You need to mention that you use continuity of $$|x|$$ to bring the limit inside the absolute value.

QUESTION 8

8 Q4(a) **3 / 3**

✓ **- 0 pts Correct**

- 0 pts Submitted at wrong place

- 0 pts Not using template

- 0 pts Similar to the Chegg solution

- 1 pts Ambiguous counterexample

- 2 pts Wrong counterexample

- 2 pts No or wrong explanation

- 2 pts Unidentifiable

- 3 pts Wrong answer

- 3 pts Blank or wrong submission

QUESTION 9

9 Q4(b) **3 / 4**

- 0 pts Correct

- 1 pts Incorrect or major errors in proof structure.

You must start by fixing an arbitrary $$L$$, take a fixed $$\epsilon$$, fix an arbitrary $$M>0$$, and construct $$x$$ according to M in that order.

- 0.5 pts $$L$$ must be fixed in the real numbers.

- 4 pts Incorrect document

- 0 pts Wrong question

- 1 pts Did not separate proof into two cases according to the value of $$L$$ or the two cases given are incorrect.

- **3 pts** Only selected TRUE. No attempt at a written proof or proof is completely incorrect.

✓ **- 0.5 pts A proof by cases needs a conclusion regarding $$|f(sin(x)) - L|| \geq \epsilon$$ in each case.**

- **0.5 pts** Minor error in proof

✓ **- 0.5 pts Did not directly define $$x$$ to depend on $$M$$**

- **1 pts** Incorrect argument or missing a major part of the argument.
- **0.5 pts** Click here to replace this description.
- **0.5 pts** Did not directly define $$x$$ to depend on $$X$$
- **4 pts** Selected FALSE when the correct answer is true.
- **2 pts** Correct however did not use a DIRECT PROOF as required in the problem.
- **0.5 pts** Minor error with proof structure.
- **0.5 pts** When choosing $$x$$, did not justify why it must be that $$x > M$$. Please see the solutions for a correct justification.
- **3 pts** Did not prove directly AND proof is incorrect. Please view the solutions for a direct proof.
- **0 pts** Not using the given template
- **4 pts** Did not submit the question.
- **0.5 pts** Did not take $$\epsilon$$ to be anything or choice of $$\epsilon$$ does not work.
- **0.5 pts** Some errors in proof structure.
- **4 pts** No proof and no TRUE / FALSE selection
- **4 pts** Completely illegible.
- **3.5 pts** Some signs of understanding.

4

5 In this case $$M = K$$ as you have defined

QUESTION 10

10 Q4(c) 3 / 3

✓ **- 0 pts Perfect solution.**

- **1 pts** Invalid counterexample.
- **1 pts** Explanation is incorrect.
- **3 pts** Statement is False.

6 Need more explanation

QUESTION 11

11 Q5(a) 1 / 1

✓ **- 0 pts Correct**

- **1 pts** Incorrect example.
- **0.5 pts** Minor error in solution.
- **0 pts** Wrong document
- **0 pts** Wrong question
- **0 pts** Incorrect use of template

7 Good!

QUESTION 12

12 Signature 0 / 0

✓ **- 0 pts Correct**

- **1 pts** No signature
- **1 pts** Repeated submission or forget to add your group member

MAT 137Y: Calculus with proofs
Assignment 2
Due on Thursday, Oct 27 by 11:59pm via GradeScope

Instructions

This assignment has purposely been made shorter because of your upcoming midterm exam. However, working through this assignment will help you on the upcoming exam because many of the topics overlap! This problem set is based on Unit 2: Limits and Continuity. Please read the Problem Set FAQ for details on submission policies, collaboration rules, and general instructions. Remember you can submit in pairs or individually.

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

Full Name: Shivesh Prakash

Student number: 1008693790

Full Name: Rishit Dagli

Student number: 1008840685

I confirm that:

- I have read and followed the policies described in the Problem Set FAQ.
- I have read and understand the rules for collaboration on problem sets described in the Academic Integrity subsection of the syllabus. I have not violated these rules while writing this problem set.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while writing this assessment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Sprakash..

2) Rishit Dagli

1. (*Note:* Before you attempt this problem, solve Problem 1 and 3 from Practice problems for Unit 2 on Quercus or Problem 2.1-7 in the textbook. Otherwise you may find this question difficult.)

   Sketch the graph of a function f that satisfies all 10 conditions below simultaneously. For this question, you do not need to prove or explain your answer, as long as the graph is correct and very clear.

   (a) The domain of f is $\mathbb{R}$.

   (b) $\lim_{x\to a} f(x)$ does not exists when $a \in \{-4, 2, 4\}$, and the limit exists for all other $a \in \mathbb{R}$.

   (c) $\lim_{x\to 0} f(x) = 0$

   (d) $\lim_{x\to 0} \lfloor f(x) \rfloor = 0$

   (e) $\lim_{x\to -4} f(x) = \infty$

   (f) $\lim_{x\to 4} f(x) = -\infty$

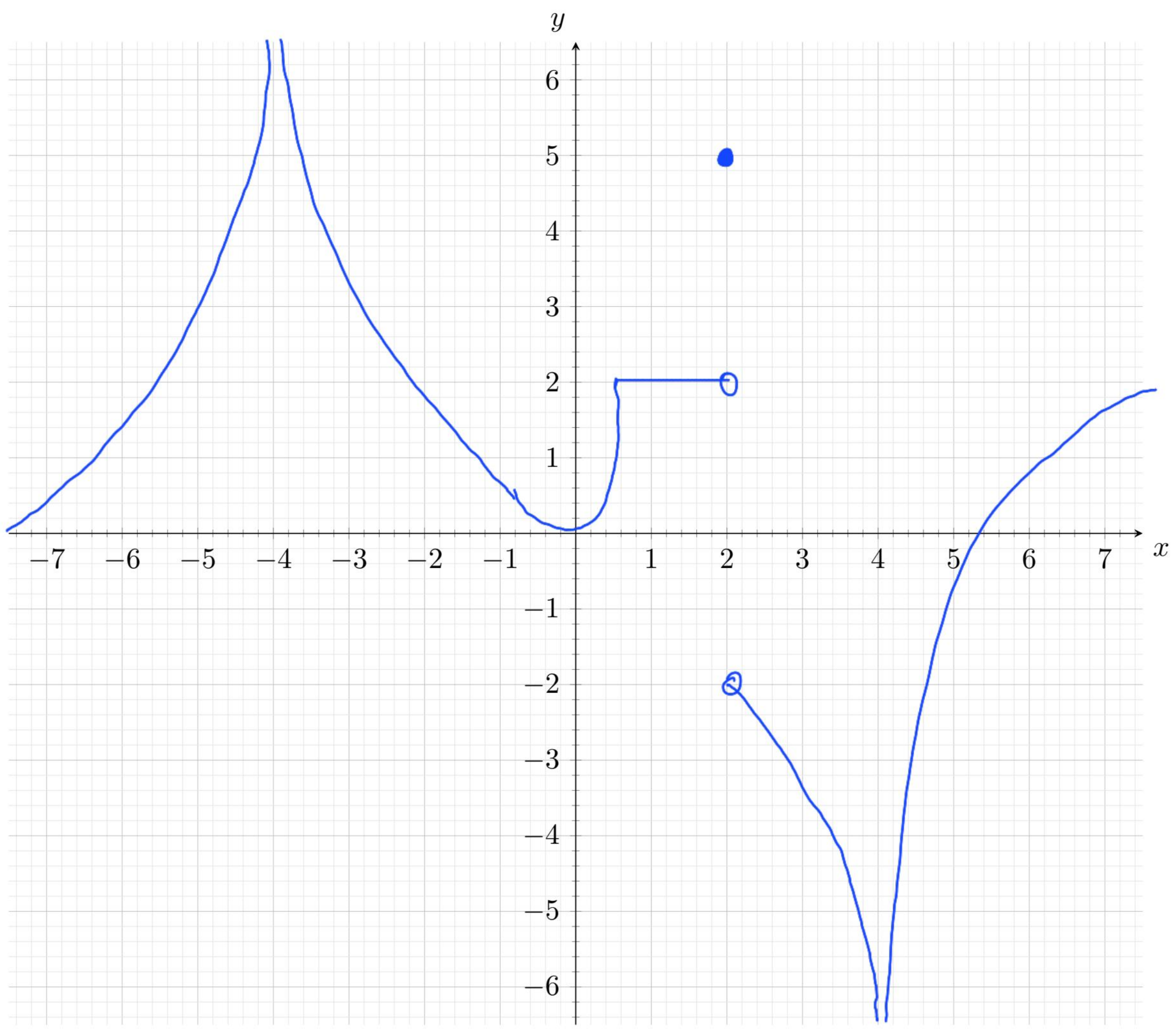
   (g) $\lim_{x\to 2} [f(x)]^2 = 4$

   (h) $\lim_{x\to 2^-} f(f(x)) = 5$

   (i) $\lim_{x\to \infty} f(x) = 2$

   (j) $\lim_{x\to 4} f(1 - e^{f(x)}) = 2$

   To clarify, we want one single function f that satisfies all the conditions in all the parts, all at once. Make your graph tidy and unambiguous.



2. Let $f(x) = \frac{x-1}{\sqrt{x}-1}$

(a) What is the largest possible domain of f?

The function contains $\sqrt{x}$ so $x \geq 0$ and the denominator cannot be 0 so $x \neq 1$ therfore the largest possible domain of f is $[0, 1) \cup (1, \infty)$

(b) Let $g(x) = \sqrt{x} + 1$. Find the largest subset $I \subset \mathbb{R}$ such that for all $x \in I$ we have $f(x) = g(x)$.

$f(x) = g(x)$ whenever both f and g are defined. Thus, $x \geq 0$ and $x \neq 1$.
Therefore, $I = [0, 1) \cup (1, \infty)$.

(c) Prove, directly from the epsilon-delta definition of limit, that

$$\lim_{x \to 9} g(x) = 4$$

Don't use any of the limit laws or other theorems.

WTS: $\lim_{x \to 9} g(x) = 4$
$\forall \epsilon > 0 \ , \exists \delta > 0 \ , 0 < |x - 9| < \delta \implies |g(x) - 4| < \epsilon$

Proof.
Let ϵ be an arbitrary real such that $\epsilon > 0$
Now choose, $\delta = 3\epsilon$
Assume $0 < |x - 9| < 3\epsilon$
$\implies \frac{0}{3} < \frac{|x-9|}{3} < \epsilon$
We can write that:

$$\begin{aligned} |x - 9| &= |\sqrt{x} + 3| \cdot |\sqrt{x} - 3| \\ &= |\sqrt{x} + 1 - 4| \cdot |\sqrt{x} + 3| \\ &= |g(x) - L| \cdot |\sqrt{x} + 3| \end{aligned}$$

$\implies |g(x) - L| = \frac{|x-9|}{\sqrt{x}+3}$
Now, $\forall x$ in the domain of $\sqrt{x}$, $\sqrt{x} + 3 \geq 3$
Thus, $|\sqrt{x} + 3| \geq 3$
We can rewrite our original statement as: $\frac{|x-9|}{|\sqrt{x}+3|} < \epsilon$
Which shows, $|g(x) - L| < \epsilon$

Thus, we have shown that $\lim_{x \to 9} g(x) = 4$.

$\square$

(d) Prove that

$$\lim_{x \to 9} f(x) = 4$$

You may need to use some results from (b) and (c).

WTS: $\lim_{x \to 9} f(x) = 4$
$\forall \epsilon > 0\ ,\ \exists \delta > 0\ ,\ 0 < |x - 9| < \delta \implies |f(x) - 4| < \epsilon$

Proof.

Drawing from the results of (b), $x \to 9 \implies f(x) = g(x)$.
Let ϵ be an arbitrary real such that $\epsilon > 0$
Now choose, $\delta = 3\epsilon$
Assume $0 < |x - 9| < 3\epsilon$
$\implies \frac{0}{3} < \frac{|x-9|}{3} < \epsilon$
We can write that:

$$\begin{aligned} |x - 9| &= |\sqrt{x} + 3| \cdot |\sqrt{x} - 3| \\ &= |\sqrt{x} + 1 - 4| \cdot |\sqrt{x} + 3| \\ &= |f(x) - L| \cdot |\sqrt{x} + 3| \end{aligned}$$

$\implies |f(x) - L| = \frac{|x-9|}{|\sqrt{x}+3|}$
Now, $\forall x$ in the domain of $\sqrt{x}$, $\sqrt{x} + 3 \geq 3$
Thus, $|\sqrt{x} + 3| \geq 3$
We can rewrite our original statement as: $\frac{|x-9|}{|\sqrt{x}+3|} < \epsilon$
Which shows, $|f(x) - L| < \epsilon$

Thus, we have shown that $\lim_{x \to 9} f(x) = 4$. $\square$

3. Let $a \in \mathbb{R}$. Let f and g be functions defined on $\mathbb{R}$. Is each of the following claims true or false? Prove your answer. Hint: often times, the easiest way to prove something is false is by providing a counter example and and proving that counter example satisfies the required conditions. It your answer is true, the proof should be a short, "one-line" proof using the properties of limits you already know (review section 2. 10 and section 2.12). You don't need to use the epsilon-delta definition in this question.

(a) IF $\lim_{x \to a} f(x)$ does not exist and $\lim_{x \to f(a)} g(x)$ does not exist,
THEN $\lim_{x \to a} g(f(x))$ does not exist.

False

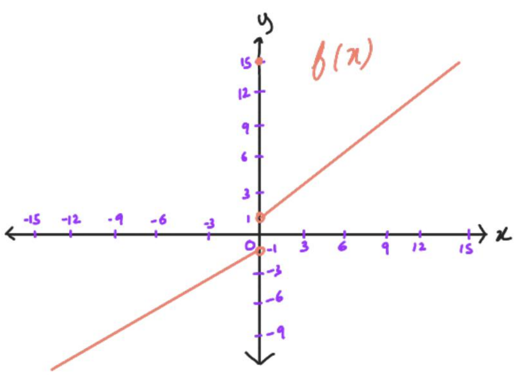
We show this false by a counterexample by taking:

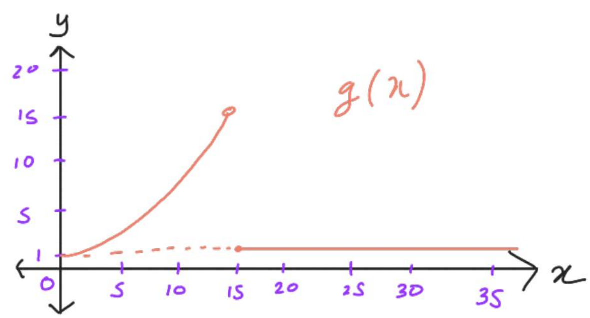
$$g(x) = \begin{cases} \sqrt{x^2+1} & , x < 15 \\ 1 & , x \geq 15 \end{cases}$$

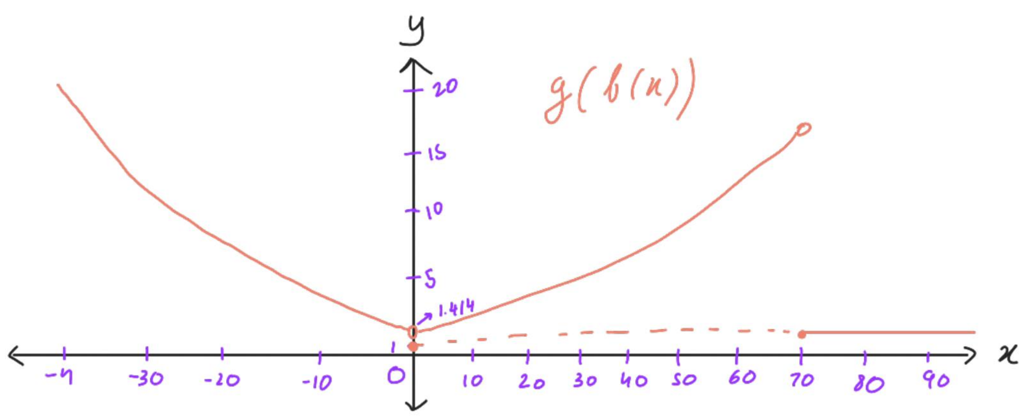
$$f(x) = \begin{cases} \frac{1}{5}x - 1 & , x < 0 \\ 15 & , x = 0 \\ \frac{1}{5}x + 1 & , x > 0 \end{cases}$$

$$g(f(x)) = \begin{cases} \sqrt{(\frac{1}{5}x-1)^2+1} & , x < 0 \\ 1 & , x = 0 \\ \sqrt{(\frac{1}{5}x+1)^2+1} & , 70 > x > 0 \\ 1 & , x > 70 \end{cases}$$

Take $a = 0$, now $f(a) = 15$. We can see from these equations that, $\lim_{x \to 0} f(x)$ does not exist and $\lim_{x \to 15} g(x)$ does not exist however $\lim_{x \to 0} g(f(x))$ exists.







(b) IF $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, and $\lim_{x\to a} |f(x) - g(x)| = 0$,
THEN $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

True

Proof.

By limit laws, we already know,

$$\textit{if } \lim_{x\to a} f(x) \textit{ exists and } \lim_{x\to a} g(x) \textit{ exists then } \lim_{x\to a} f(x) + g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, and $\lim_{x\to a} |f(x) - g(x)| = 0$.
Now, using this limit law and the definition of absolute value, we can say that:

$$\boxed{\lim_{x\to a} |f(x) - g(x)| = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \textit{ or } \lim_{x\to a} g(x) - \lim_{x\to a} f(x)}$$

3

In both cases, $\lim_{x\to a} g(x) - \lim_{x\to a} f(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = 0$
Thus we can say that, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

$\square$

4. Let $a \in \mathbb{R}$. Let f and g be functions defined on $\mathbb{R}$. Is each of the following claims true or false? Prove your answer. If it is true, prove it directly from the epsilon-delta definition of a limit. Hint: often times, the easiest way to prove something is false is by providing a counter example and and proving that counter example satisfies the required conditions.

   (a) IF $f(x)$ is continuous at a and $g(x)$ is not continuous at a,
   THEN $f(x)g(x)$ is not continuous at a.

False

   A counterexample would be $f(x) = 0$, $g(x) = \lfloor x \rfloor$ and $a = 0$
   We can see that $f(x)$ is continuous at 0 while $g(x)$ is not continuous at 0.
   Let ϵ be an arbitrary real number such that $\epsilon > 0$ and fix $\delta = 0.1$
   Now $\forall x$ such that $|x - 0| < 0.1$
   If $|x| < 0.1$ then $|f(x)g(x) - f(0)g(0)| < \epsilon \implies 0 < \epsilon$
   which is always true so $f(x)g(x)$ is continuous at $x = 0$.
   Thus, the statement is False.

(b) IF f is continuous on $\mathbb{R}$, $f(-1) = 10$ and $f(1) = 20$, THEN $\lim_{x\to\infty} f(\sin(x))$ does not exist.

True

Proof.

Let us assume f is continuous on $\mathbb{R}$, $f(-1) = 10$ and $f(1) = 20$
WTS: $\forall L \in \mathbb{R}, \exists \epsilon > 0$ such that $\forall K \in \mathbb{R}, \exists x \in \mathbb{R}$ such that $x > K$ and $|f(sin(x)) - L| \geq \epsilon$
Let $L \in \mathbb{R}$.
Take $\epsilon = 5$
Let $K \in \mathbb{R}$

Case A: $10 \notin (L - \epsilon, L + \epsilon)$
I choose any $x \in \mathbb{R}$, greater than K such that $Sinx = -1$. Then $f(sin(x)) = 10$.

4 5

Case B: $20 \notin (L - \epsilon, L + \epsilon)$
I choose any $x \in \mathbb{R}$, greater than K such that $Sinx = 1$. Then $f(sin(x)) = 20$.

Either way, it satisfies $x > K$ and $|f(sin(x)) - L| \geq \epsilon$

$\square$

(c) IF f is continuous on $\mathbb{R}$, $f(-2) = 10$ and $f(2) = 20$, THEN $\lim_{x\to\infty} f(\sin(x))$ does not exist.

False

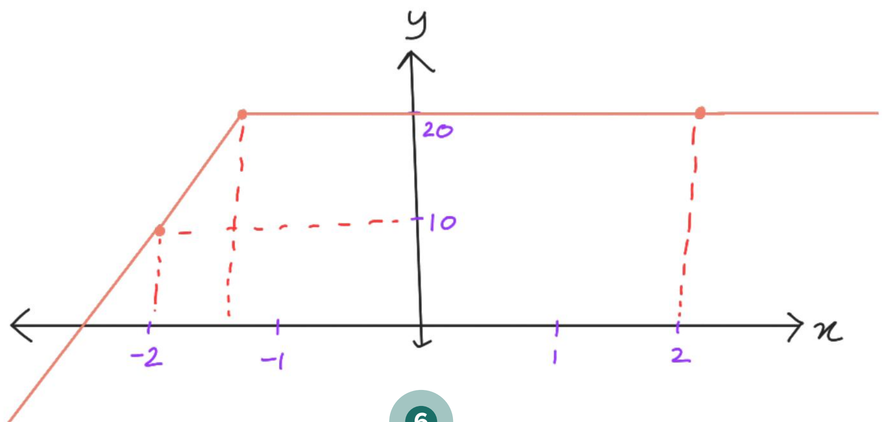
Let us assume f is continuous on $\mathbb{R}$, $f(-2) = 10$ and $f(2) = 20$

WTS: $\lim_{x\to\infty} f(\sin x)$ can exist

As $x \to \infty$ $\sin x$ oscillates between -1 and 1 but $f(x)$ is constant between -1 and 1.

From the below graph we see that $f(x)$ is constant at 20 as x varies between -1 and 1 so $f(\sin x)$ is constant at 20.

Therefore, $\lim_{x\to\infty} f(\sin x)$ exists



6

5b) is quite challenging. You can try it however we will not require you to return your work. 5b) is not counted for credits.

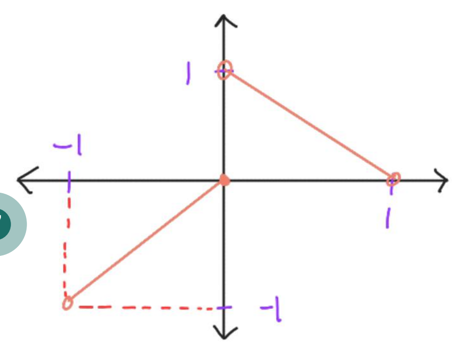
5. We will try to prove the following theorem.

   Theorem 1. *Let $a, b \in \mathbb{R}$ and $a < b$. Let f be a one-to-one continuous function on an interval (a, b). Then*
   f is strictly increasing on (a, b), i.e. $\forall x_1, x_2 \in (a, b), x_1 < x_2 \implies f(x_1) < f(x_2)$
   OR
   f is strictly decreasing on (a, b), i.e. $\forall x_1, x_2 \in (a, b), x_1 < x_2 \implies f(x_1) > f(x_2)$.

   (a) What happens when we drop continuity property? Find an example of one-to-one non-continuous function defined on $(-1, 1)$ which increases on $(-1, 0]$ and decreases on $(0, 1)$. No justification is necessary.

   If we drop the continuity property this theorem fails, an example of which is shown in the graph below.

$$f(x) = \begin{cases} x, & x \in (-1, 0] \\ 1 - x, & x \in (0, 1) \end{cases}$$



   (b) Prove this following theorem. Hint: you may need to use the Intermediate Value Theorem in your proof.