MAT137Y Tutorial 6 worksheet

Kaamil Shaikh, Raffaele Dengler, Fahia Mohamed, Shivesh Prakash

TOTAL POINTS

2 / 2

QUESTION 1

1 Q1+Q3 **2 / 2**

✓ **- 0 pts** **No major flaw in solution**

- 1 pts Major flaws present in the solutions, however there's significant progress and work in the worksheet. The solutions have demonstrated some degree of understanding and mastery.

- 2 pts Incomplete submission

- 2 pts No TA signature

MAT 137
Tutorial #6– Linear Approximation and Newton's method
October 25/26 , 2022
Due on Thursday, Oct 27 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

Academic integrity statement

I confirm that:

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- I understand that the collaboration of this worksheet is only among this group. I have not violated this rule while writing this worksheet.
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By signing this document, I agree that the statements above are true.

First Name	Last Name	UofT email	signature
SHIVESH	PRAKASH	SHIVESH.PRAKASH@MAIL.UTORONTO.CA	Sprakash
Kaamil Saveed	Shaikh	kaamil.shaikh@mail.utoronto.ca	[illegible]
Raffaele	Dengler	r.dengler@mail.utoronto.ca	Raffaele
Fahia	Mohamed	fahia.mohamed@mail.utoronto.ca	[illegible]

TA name: AUSTIN TA signature: [illegible]

1. In this problem we will estimate $\sqrt[3]{1.1}$. **For this question, you are not allowed to use a calculator.**

   (a) Consider the function $h(x) = \sqrt[3]{x}$. Notice that our goal is to compute $h(1.1)$. Write the equation of the line tangent to $y = h(x)$ at the point with x-coordinate 1. We will call this line L.

The point with x-coordinate 1 is $(1,1)$, the slope of the tangent at this point is $\rightarrow$

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt[3]{x}) = \frac{1}{3}x^{-2/3} = \frac{1}{3}(1)^{-2/3} = \frac{1}{3}$$

Thus the equation of the tangent is $y = \frac{1}{3}x + \frac{2}{3}$

   (b) Consider the point on L with x-coordinate 1.1 and the point on the graph of h with x-coordinate 1.1. Are their y-coordinates close to each other? You can use Desmos or Geogebra to graph h and L. Use this to obtain an approximate value for $\sqrt[3]{1.1}$.

From the graphs, the y-coordinates are very close to each other, almost equivalent. By our approximation the value of $\sqrt[3]{1.1}$ is $\frac{1}{3}(1.1) + \frac{2}{3} = 1.03$

   (c) Let's say we use the same method (with the same line L) to approximate $\sqrt[3]{1.05}$ and $\sqrt[3]{1.2}$. In which case would we get a smaller error?

From the graph, we estimate that we get a smaller error in $\sqrt[3]{1.05}$

This is coherent because 1.05 is closer to 1 compared to 1.2

   (d) Without using the above method, what do you think the approximate value of $\sqrt[3]{28}$ is? Now, using the same line L as above, what value do you get? You will see that this is a very bad approximation. Why didn't it work?

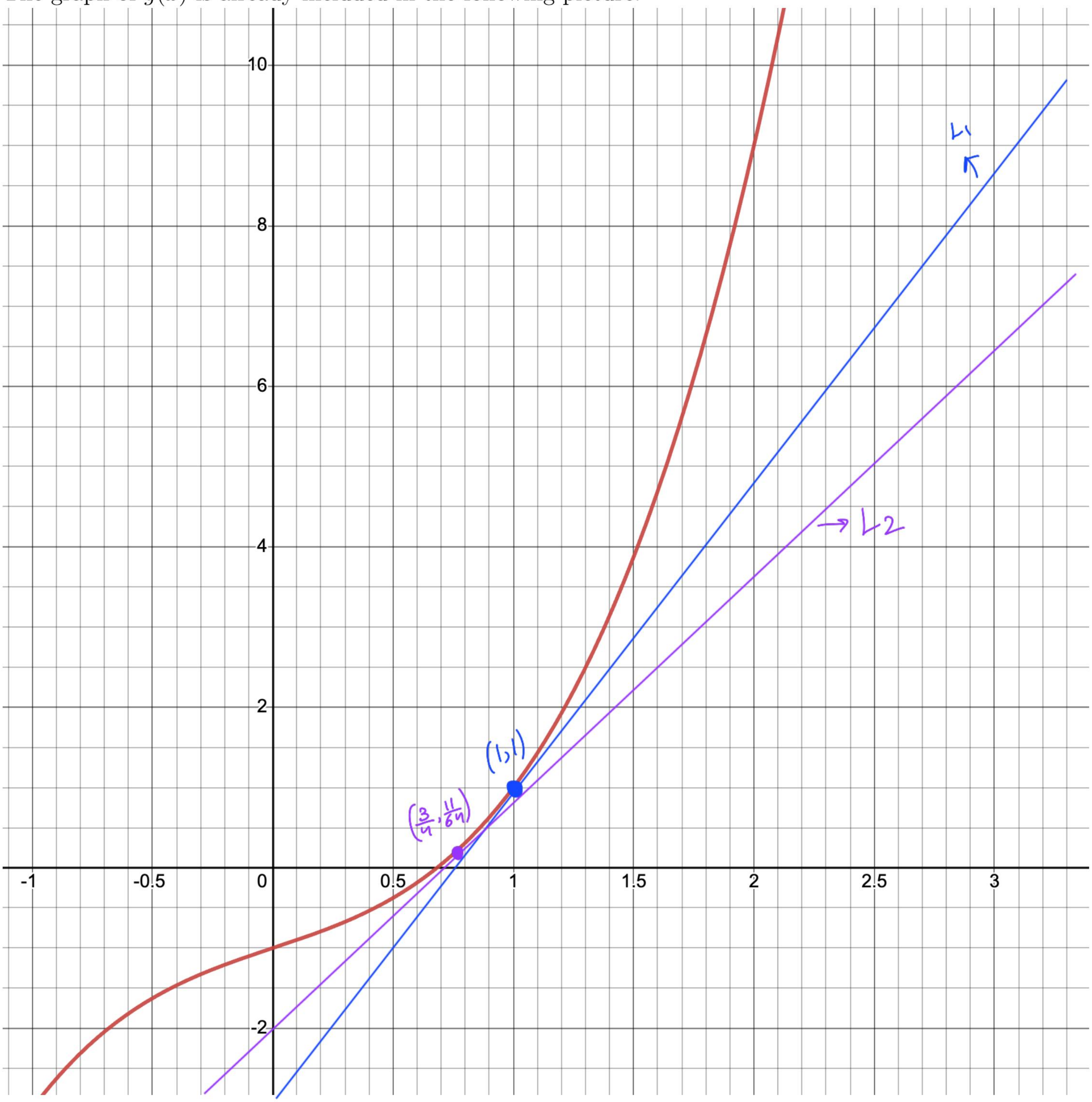
We estimate the value of $\sqrt[3]{28}$ to be slightly greater than 3, 3.02 approximately

Using line L, the value is 10. It does not work because our approximation works only for values near 1 and $28 >> 1$.

2. Now you will try the whole thing by yourself. Use a similar method to the previous problem to obtain an approximate value for $\sqrt{3.9}$. **This question is for your practice and you don't need to return your work.**

3. For this problem, we define the function $g(x) = x^3 + x - 1$. We want to find a number x such that $g(x) = 0$. In other words, we want to solve the equation $x^3 + x - 1 = 0$. You may use a calculator for this question.

The graph of $g(x)$ is already included in the following picture.



(a) Calculate $g(0)$ and $g(1)$. This guarantees there has to be some number $0 < x < 1$ such that $g(x) = 0$. Why? Hint: which theorem we can imply here.

$g(0) = -1$ and $g(1) = 1$. Since g is continuous on $[0,1]$, $g(0) < g(1)$ and $0 \in (-1,1)$, thus using Intermediate Value Theorem there exists $x \in (0,1)$ such that $g(x) = 0$.

(b) We are going to make a bunch of successive guesses for the solution to the equation. None of them will be exact, but each one will be better than the previous one. Our first guess is going to be $x_1 = 1$. Write the equation of the line tangent to $y = g(x)$ at the point with x-coordinate x_1. We will call it L_1. Draw this line on the picture in the previous page.

The point with x-coordinate 1 is $(1,1)$, the slope of the tangent at this point is $\Rightarrow$

$\frac{dy}{dx} = \frac{d}{dx}(x^3+x-1) = 3x^2+1 = 3(1)^2+1 = 4$

So, $y = 4x + b \iff 1 = 4(1) + b \iff b = -3$

Thus the equation of the tangent is $y = 4x - 3$

(c) We are looking for the point of the graph $y = g(x)$ that intersects the x-axis. Since this point is not too far from $(x_1, g(x_1))$, we can look for the point where the line L_1 intersects the x-axis instead. (Convince yourself that this makes sense!) Calculate this point. Call its x–coordinate x_2. This is our second guess.

The required point is $\left(\frac{3}{4}, 0\right)$. Thus x_2 is $\frac{3}{4}$.

(d) Calculate $g(x_2)$. Notice that $g(x_2)$ is not zero yet but it is closer to zero than $g(x_1)$. We are improving!

$$g(x_2) = \left(\frac{3}{4}\right)^3 + \frac{3}{4} - 1 = \frac{27}{64} + \frac{3}{4} - 1 = \frac{27+48-64}{64} = \frac{11}{64}$$

Thus $g(x_2) = \frac{11}{64}$, which is closer to zero than $g(x_1) = 1$.

(e) Now repeat the process you did in the last three steps, but starting with x_2 instead of with x_1. Write the equation of the line tangent to $y = g(x)$ at the point with x-coordinate x_2. We will call it L_2. Draw L_2 on the picture. Call the new value you obtain x_3. Calculate $g(x_3)$. Is it close enough to zero? Then obtain x_4. Is $g(x_4)$ close enough to 0?

The slope of tangent at x_2 is $\Rightarrow$ $m = 3x_2^2 + 1 = 3\left(\frac{3}{4}\right)^2 + 1 = \frac{43}{16}$

The equation of tangent at x_2 is $\Rightarrow$ $(y - g(x_2)) = m(x - x_2)$

$\Rightarrow y - \frac{11}{64} = \frac{43}{16}\left(x - \frac{3}{4}\right) \Rightarrow y = \frac{43x}{16} - \frac{59}{32} = L_2$

Let L_2 intersect x-axis at $x_3 \Rightarrow 0 = \frac{43x_3}{16} - \frac{59}{32} \Rightarrow x_3 = \frac{59}{86}$

Now $g(x_3) = \left(\frac{59}{86}\right)^3 + \frac{59}{86} - 1 = \frac{5687}{636056} \approx 0.00894$

This value is extremely close to 0. Nonetheless, let's find x_4.

The slope of tangent at x_3 is $\Rightarrow$ $m = 3x_2^2 + 1 = 3\left(\frac{59}{86}\right)^2 + 1 \approx 2.412$

The equation of tangent at x_3 is $\Rightarrow$ $(y - g(x_3)) = m(x - x_3)$

$\Rightarrow y - 0.00894 = 2.412\left(x - \frac{59}{86}\right) \Rightarrow y = 2.412x - 1.645 = L_3$

Let L_3 intersect x-axis at $x_4 \Rightarrow 0 = 2.412x_4 - 1.645 \Rightarrow x_4 \approx 0.68234$

Now $g(x_4) = (0.68234)^3 + 0.68234 - 1 = 0.00002823$

This is the closest value to 0 we got, thus the root to this equation is close to 0.68234

Keep track of your data:

n	x_n	$g(x_n)$
1	1	1
2	3/4	11/64
3	59/86	0.00894
4	0.68234	0.00002823
5		

4. Now you try the whole thing by yourself. Use a similar method to the previous problem to find a solution to $x^3 - x - 1 = 0$.This question is for your practice and you don't need to return your work.