MAT137Y Tutorial 11 worksheet

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TOTAL POINTS

2 / 2

QUESTION 1

1 Q1+Q2 **2 / 2**

✓ **- 0 pts** Correct

- 1 pts Did not demonstrate thorough effort

- 2 pts No TA signature

- 1 pts Incorrect

- 2 pts Wrong document submitted

- 1 pts Did not demonstrate thorough understanding of the material

- 2 pts Wrong document submitted

MAT 137
Tutorial #11– Applied optimization problems+ L'Hôpital's Rule
Dec 6/7 , 2022
Due on Thursday, Dec 8 by 11:59pm via GradeScope

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- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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1. A farmer wants to hire workers to pick 1600 bags of beans. Each worker can pick 10 bags per hour and is paid \$1.00 per bag. The farmer must also pay a supervisor \$20 per hour while the picking is in progress. She has additional miscellaneous expenses of \$8 per worker (but not for the supervisor). How many workers should she hire to minimize the total cost? What will the cost per bag picked be?

Let the number of workers hired be x. Here x ∈ (0, ∞).
Let the total time taken be t in hours.
Since each worker picks up 10 bags per hour and 1600 bags are to be lifted, t = $\frac{160}{x}$ hr

The total cost c = $\left(\underbrace{1}_{\$1\text{ per bag}} \cdot \underbrace{10}_{10\text{ bags per hour}} \cdot t\right) \cdot x + \underbrace{20}_{\text{Supervisor rate per hour}} \cdot t + \underbrace{8}_{\text{miscellaneous expense per worker}} x$

Thus, c = $10 \cdot t \cdot x + 20t + 8x = 10 \times \frac{160}{x} \times x + 20 \times \frac{160}{x} + 8x$

$$\Rightarrow C = 1600 + \frac{3200}{x} + 8x$$

A value of x which minimizes the cost satisfies $\frac{dc}{dx} = 0$ or DNE

$$\Rightarrow \frac{dc}{dx} = \frac{-3200}{x^2} + 8 = 0 \Rightarrow x^2 - 400 \Rightarrow x = 20 \text{ (Since } x \text{ can't be negative)}$$

x = 0 is the only point where this derivative does not exist, at which the cost tends to ∞. The other end point in the range of x tends to ∞, the cost tends.to ∞ here again. Thus the end points and critical points other than 20 can not be the minima. Let us justify x as the minima using another test.

To justify this value of x as the minimum, let us check $\frac{d^2c}{dx^2}$ at $x = 20$,

$$\frac{d^2c}{dx^2} = \frac{6400}{x^3} = \frac{6400}{(20)^3} = \frac{4}{5} > 0$$

Thus by second derivative test, x = 20 minimizes the cost.

The total cost is c = $1600 + \frac{3200}{20} + 8(20) = 1920$

Since 1600 bags are picked up in total, the minimum cost per bag is $\frac{1920}{1600} = \$1.2/\text{bag}$

Therefore the farmer should hire 20 workers and the cost will come down to \$1.2/bag

2. Compute the following limits:

(a) $\lim_{x\to 1} \frac{(x-1)\sin x}{e^x \cos x}$

This function is continuous around 1, so we can say that

$$\lim_{x\to 1} \frac{(x-1)\sin x}{e^x \cos x} = \frac{0 \times \sin 1}{e \times \cos 1} = \boxed{0}$$

(b) $\lim_{x\to 0} \left(\frac{1}{x\sin x} - \frac{1}{x\tan x} \right)$

$$\lim_{x\to 0} \left(\frac{1}{x\sin x} - \frac{1}{x\tan x} \right) = \lim_{x\to 0} \frac{\tan x - \sin x}{x.\tan x.\sin x} = \lim_{x\to 0} \frac{1-\cos x}{x\sin x}$$

The numerator and denominator both tend to 0, applying L'Hospital rule:

$$\lim_{x\to 0} \frac{1-\cos x}{x.\sin x} = \lim_{x\to 0} \frac{\sin x}{\sin x + x\cos x}$$

Applying L'Hospital rule again:

$$\lim_{x\to 0} \frac{\sin x}{\sin x + x\cos x} = \lim_{x\to 0} \frac{\cos x}{\cos x - x\sin x + \cos x} = \frac{1}{1-0+1} = \boxed{\frac{1}{2}}$$

(c) $\lim_{x\to 1} (2-x)^{\tan(\pi x/2)}$

Let $y = (2-x)^{\tan(\pi x/2)}$. Taking $\ln()$ on both sides:

$\ln(y) = \tan\left(\frac{\pi x}{2}\right) \ln(2-x) = \frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)}$. Taking $\lim_{x\to 1}$ on both sides.

The numerator and denominator both tend to 0, applying L'Hospital rule:

$$\lim_{x\to 1} \ln(y) = \lim_{x\to 1} \frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1} \frac{\frac{-1}{2-x}}{-\frac{\pi}{2}\operatorname{cosec}^2\left(\frac{\pi x}{2}\right)} = \lim_{x\to 1} \frac{2}{\pi(2-x)} \times \sin^2\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \ln\left(\lim_{x\to 1} (2-x)^{\tan(\pi x/2)} \right) = \frac{2}{\pi(2-1)} \times \sin^2\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$\Rightarrow \lim_{x\to 1} (2-x)^{\tan(\pi x/2)} = \boxed{e^{2/\pi}}$$