MAT 137
Tutorial #3– The definition of limit
Oct 4-5, 2022
Due on Thursday, Oct 6 by 11:59pm via GradeScope

- **Submissions are only accepted by Gradescope**. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
- **Submit your polished solutions using only this template PDF.** You will submit a single PDF with your full written solutions. If your solution is not written using this template PDF (scanned print or digital) then you will receive zero. Do not submit rough work. Organize your work neatly in the space provided.
- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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Let f be a function. Let $a, L \in \mathbb{R}$. Assume that f is defined on some open interval around a, except maybe at a. As you know, the definition of the statement $\lim_{x \to a} f(x) = L$ is

(1) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\quad 0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon$.

Below is a list of seven other statements. Write formal, rigorous proofs for these statements:

(a) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\quad 0 < |x - a| < \delta \implies |f(x) - L| \leq \varepsilon$.
What does this statement mean? Hint: what's the difference between (1) and (a)? can we say statement (1) implies this statement (a)? How about the $(a) \implies (1)$? are the epsilons in this statement and in the definition of $\lim_{x \to a} f(x) = L$ necessary to be the same?

observe that (1) $\Rightarrow$ (a), since $|f(x) - L| < \varepsilon \Rightarrow |f(x) - L| \leq \varepsilon$

WTS: (a) $\Rightarrow$ (1)

assume (a)

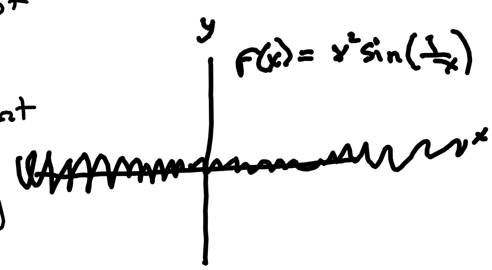
Let $\varepsilon_1 = \frac{\varepsilon}{2}$, since $0 < \frac{\varepsilon}{2} < \varepsilon$,

statement (a) is true for all ε meaning it's true for ε_1. This implies statement (1) for the original ε

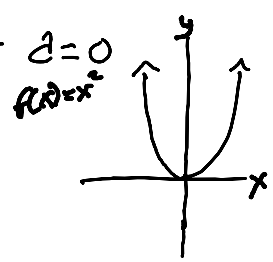
Since $|f(x) - L| \leq \frac{\varepsilon}{2} \Rightarrow |f(x) - L| < \varepsilon$.

(b) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\quad 0 < |x - a| < \delta \implies 0 < |f(x) - L| < \varepsilon$. Compare this statement with the definition of $\lim_{x \to a} f(x) = L$. What does this statement mean? Hint: Can you find a function $f(x)$ that satisfies this statement? Compare $f(x) = x$ and $f(x) = x^2 \sin(\frac{1}{x})$. Use Desmos to graph these two functions. (https://www.desmos.com/calculator) Do these two functions satisfy this statement?

The difference between the two is that here $f(x)$ cannot be exactly equal to L as $x \to a$.

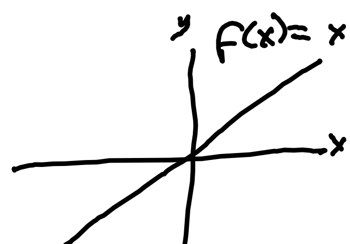


$f(x) = x^2$ is a function which satisfies this statement at $a = 0$



$f(x) = x^2 \sin\left(\frac{1}{x}\right)$ does not satisfy the statement because $\sin\frac{1}{x}$ oscillates between -1 and 1 as $x \to 0$.

$f(x) = x$ satisfies this statement.



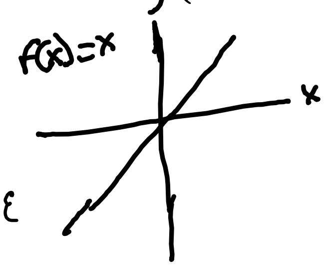
(c) For every $\varepsilon \geq 0$, there exists $\delta > 0$ such that $0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$. Can you find a function satisfy this statement? If yes, give one example. If no, can you explain why. Hint: can you write down the negation of this statement? The negation is true or false?

No function can satisfy this statement because when $\varepsilon = 0$, $|f(x) - L| < 0$ which is a contradiction and can never be true based on the definition of absolute values. The negation of the statement:

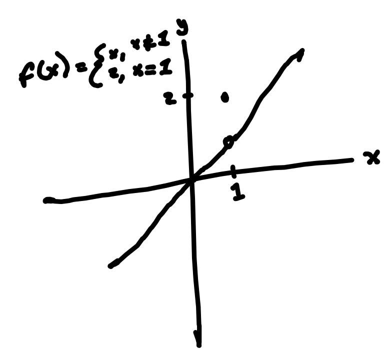
$\exists \varepsilon \geq 0 \ \forall \delta > 0$, $0 < |x-a| < \delta$ and $|f(x) - L| \geq \varepsilon$

for $\varepsilon = 0$ the negation is always true, therefore the statement is false

(d) For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-a| < \delta \implies |f(x) - L| < \varepsilon$.
Compare this statement with the definition of $\lim_{x\to a} f(x) = L$. What's the difference? What does this statement mean? Hint: sketch the graphs of the functions $f(x) = x$ and $f(x) = \begin{cases} x, & x \neq 1 \\ 2, & x = 1 \end{cases}$. Let $a = 1$ and $L = 1$. Check if they are satisfies (1) and (d).



The difference between the two statements is that even when x is exactly a, $f(x)$ tends to L. This statement implies the definition of a limit but also says that for $\forall \varepsilon > 0$, $|x-a| = 0 \Rightarrow |f(x) - L| < \varepsilon$ when $|x-a| = 0$, $x = a$, meaning $|f(a) - L|$ must be less than ε, $\forall \varepsilon > 0$.



Since $|f(a) - L|$ is always greater than or equal to 0 by the definition of an absolute value this means that $|f(a) - L| = 0$, therefore $f(a) = L$.
Therefore the statement (d) $\Rightarrow \lim_{x\to a} f(x) = f(a) = L$.

$f(x) = x$ satisfies the statement (d), but the second function does not satisfy the statement $a = 1$ because $f(1) = 2 \neq \lim_{x\to 1} f(x) = 1$

(e) For every $\varepsilon > 0$, there exists $\delta \geq 0$ such that $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$. Can you find a function satisfies this statement? If yes, give one example. If no, can you explain why. Hint: what happens if I take $\delta = 0$?

$\forall \varepsilon > 0$, fix $\delta = 0$,
$0 < |x-a| < \delta$ is false, this implies the statement
$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ is vacuously true.
An example is $f(x) = \frac{1}{x}$ as $x \to 0$.
$\forall \varepsilon > 0$, δ can be 0, therefore the statement is vacuously true for $f(x) = \frac{1}{x}$ as $x \to 0$.

These two questions are for your practice and you don't need to return your work.

(f) For every $\delta > 0$, there exists $\varepsilon > 0$ such that $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$.
What does this statement mean? Hint: let $a = 0, L = 1$ and $f(x) = x$. For every $\delta > 0$, can you find the corresponding ε?

(g) There exists $\delta > 0$ such that for every $\varepsilon > 0$, $\quad 0 < |x-a| < \delta \implies |f(x) - L| < \varepsilon$. What does this statement mean?