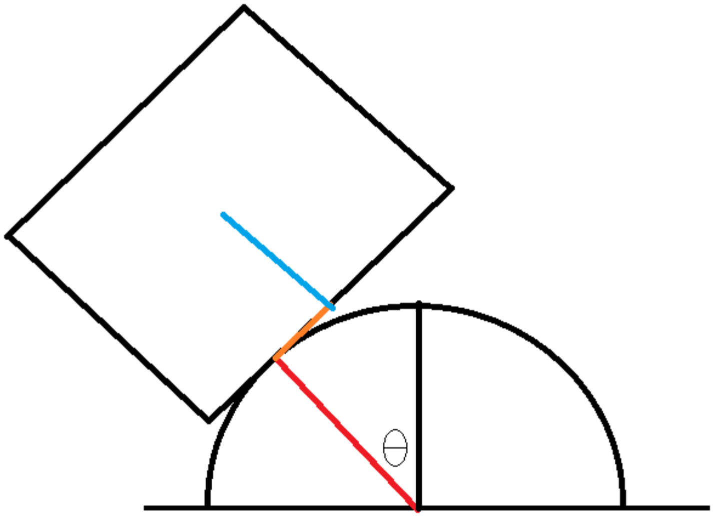
Stability of a cube on a hemisphere



A cube of length b rests on a hemisphere of radius R. Static friction is high so it cannot slip. Is it stable to tipping?

Let it tip an angle θ as shown. We want to find $U(\theta)$ and see if it is stable. Since $U = mgh$ this means we need to find the height to the centre of mass from the ground.

As shown, there are 3 components. The red and blue lines are parallel (excuse my poor drawing) while the orange line is perpendicular to both. The heights are:

$h_{red} = R\cos\theta$

$h_{orange} = (R\theta)\sin\theta$

$h_{blue} = \frac{b}{2}\cos\theta$

Why is the length of the orange line $R\theta$? It's not drawn to scale (I'm no artist), but the orange line should be the same length as the arc-length of the circle subtended by the angle θ since the cube tips without slipping.

If we take small angle approximations, we get $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$ and $\sin\theta \simeq \theta$.

So the potential energy is

$U(\theta) \simeq mg\left(R - \frac{R}{2}\theta^2 + R\theta^2 + \frac{b}{2} - \frac{b}{4}\theta^2\right) = mg\left(R + \frac{b}{2} + \frac{1}{4}\theta^2\left(2R - b\right)\right)$

This is stable if the θ^2 term is positive, which requires $2R > b$. This means the cube's length must be smaller than the sphere's diameter for this situation to be stable.