CSC110 Lecture 16: Greatest Common Divisor, Revisited

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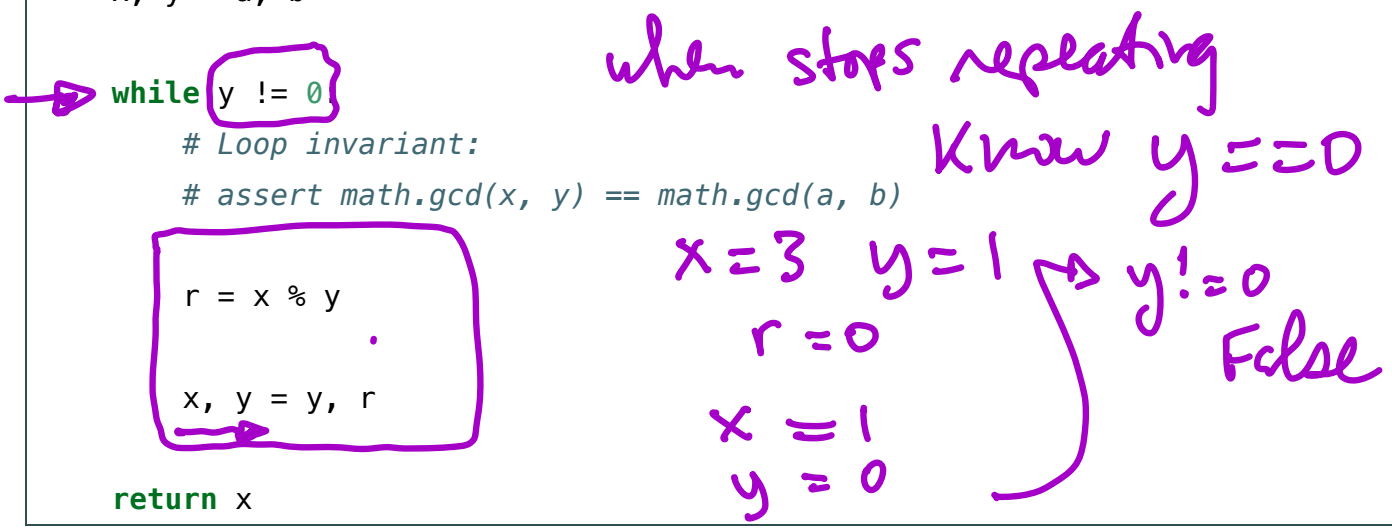
Exercise 1: The Euclidean Algorithm

Here is the code for the *Euclidean Algorithm* for calculating the greatest common divisor of two numbers

```
def euclidean_gcd(a: int, b: int) -> int:
    """Return the gcd of a and b.

    Preconditions:
    - a >= 0
    - b >= 0
    """
    x, y = a, b

    while y != 0:
        # Loop invariant:
        # assert math.gcd(x, y) == math.gcd(a, b)

        r = x % y

        x, y = y, r

    return x
```



1. Suppose we make the following function call:

```
>>> euclidean_gcd(50, 23)
```

Complete the following table to show the values of the loop variables x and y at the end of each loop

iteration when we call this function. We have completed the first row for you, and have given you more rows than necessary (part of this exercise is determining exactly when the loop will stop).

Iteration	x	y
0	50	23
1	23	4
2	4	3
3	3	1
4	1	0
5		
6		
7		

50%23 = 4

23%4 = 3

4%3 = 1

3%1 = 0

2. What does `euclidean_gcd(10, 0)` return? Is this correct (according to the definition of gcd)?

10 (initial x is 10, initial y is 0)

yes since 10 | 10 and 10 | 0

3. What does `euclidean_gcd(0, 0)` return? Is this correct (according to the definition of gcd)?

0 (initial x is 0, initial y is 0)

yes it is correct.

4. How can we modify this function to allow for negative values for `a` and `b`? (You don't need a formal proof of correctness here, but should briefly justify your response.)

since the gcd(a,b) ≥ 0

and since d|a ⇒ d|-a

we could return gcd(abs(a), abs(b))

set x,y to abs(a), abs(b) at start.

Exercise 2: Completing the Extended Euclidean Algorithm

We left off our discussion of the *Extended Euclidean Algorithm* in lecture with the following code:

```python
def extended_euclidean_gcd(a: int, b: int) -> tuple[int, int, int]:
    """Return the gcd of a and b, and integers p and q such that

    gcd(a, b) == p * a + b * q.

    Preconditions:
    - a >= 0
    - b >= 0

    >>> extended_euclidean_gcd(10, 3)
    (1, 1, -3)
    """
    x, y = a, b

    # Initialize px, qx, py, and qy
    px, qx = 1, 0  # Since x == a == 1 * a + 0 * b
    py, qy = 0, 1  # Since y == b == 0 * a + 1 * b

    while y != 0:
        # assert math.gcd(a, b) == math.gcd(x, y)  # Loop Invariant 1
        assert x == px * a + qx * b                # Loop Invariant 2
        assert y == py * a + qy * b                # Loop Invariant 3

        q, r = divmod(x, y)  # quotient and remainder when a is divided by b

        # Update x and y
        x, y = y, r

        # Update px, qx, py, and qy
        px, qx, py, qy = ...,  ..., ..., ...

    return (x, px, qx)
```

Remember that the key new **loop invariants** are:

```
x == px * a + qx * b
y == py * a + py * b
```

Now, let's investigate how to complete this function by filling in the `...` in the loop body.

1. Suppose we call `extended_euclidean_gcd(100, 13)`.

   Complete the **first row** in table below to show the *initial values* of all six loop variables at the very start of the loop (which we label "Iteration 0"). Leave the other rows blank for now; we'll get to them in future questions.

Iteration	x	px	qx	y	py	qy
0	100	1	0	13	0	1
1	13	0	1	9	1	-7
2	9	1	-7	4	-1	8
3	4	-1	-8	1	3	-23
4	1	3	-23	0	-13	100

function returns: 1, 3, -23 -b a

Before moving on, check that your values satisfy `x == px * 100 + qx * 13` and `y == py * 100 + qy * 13`. ✓ note: $1 = 3 \cdot 100 + (-23) \cdot 13$ r

2. Next, suppose we execute one iteration of the loop. Note that `divmod(100, 13) == (7, 9)`, i.e., 100 divided by 13 has quotient 7 and remainder 9.

   a. In the above table, fill in the values for `x` and `y` in the "Iteration 1" row. (end of iteration)

   b. Determine what numbers you should use to fill in for `px` and `qx` in "Iteration 1" to preserve the invariant `x == px * 100 + qx * 13`. Then, fill those entries.

   You may use the space below for rough work.

$13 = 0 \cdot 100 + 1 \cdot 13$

x q y r

$9 = 2 \cdot 4 + 1$

$3 \cdot 100 + -23 \cdot 13$

$1 = 9 - 2 \cdot 4$

$(1 \cdot 100 - 7 \cdot 13) - 2 \cdot (-1 \cdot 100 + 8 \cdot 13)$

   c. Does what you wrote generalize for all possible values of `x`, `y`, `a`, and `b`? Convince yourself that does, and then use this information to fill in the first two `...` in the reassignment statement fo `px` and `qx`. (Halfway done!) Since at start of loop body $y = py \cdot 100 + qy \cdot 13$

and next $x = oldy = oldpy \cdot 100 + oldqy \cdot 13$

You may use the space below for rough work.

next px next qx

d. Now let's look at `py` and `qy`. From the quotient-remainder theorem, you have the equation:

$$1 \cdot 100 = 7 \cdot 13 + 9$$

Use this information to fill in the values for `py` and `qy` for Iteration 1, while preserving the invariant

`y == py * 100 + qy * 13`.

You may use the space below for rough work.

the new y is r = 9

$9 = 1 \cdot 100 + (-7) \cdot 13$

x = q·y + r

r = x − q·y

px·a + qx·b

py·a + qy·b

e. Unfortunately, what you did in (e) doesn't generalize just yet. So, let's do another iteration.

First, using what you've learned fill in the "Iteration 2" row for `x`, `px`, `py`, `y`.

f. Now, you are given the following equations:

$$13 = 0 \cdot 100 + 1 \cdot 13$$
$$9 = 1 \cdot 100 - 7 \cdot 13$$
$$13 = 1 \cdot 9 + 4$$

Using these equations, perform a calculation to find coefficients p and q such that $4 = p \cdot 100 + q \cdot 13$. Then, use this to complete the "Iteration 2" row of the table.

$4 = 13 - 1 \cdot 9$

$= [0 \cdot 100 + 1 \cdot 13] - [1 \cdot 100 - 7 \cdot 13]$

$= -1 \cdot 100 + 8 \cdot 13$

g. Let's generalize what you did in the previous step. Given the following equations:

$$\texttt{x} = \texttt{px} \cdot 100 + \texttt{qx} \cdot 13$$
$$\texttt{y} = \texttt{py} \cdot 100 + \texttt{qy} \cdot 13$$
$$\texttt{x} = \texttt{q} \cdot \texttt{y} + \texttt{r}$$

Write r as a linear combination of 100 and 13.

the next y

$r = x - q \cdot y$ from QR Theorem

$= [px \cdot 100 + qx \cdot 13]$ subst. from known invariant

$- q[py \cdot 100 + qy \cdot 13]$

$= [px - q \cdot py] \cdot 100$ the next px

$+ [qx - q \cdot qy] \cdot 13$ the next py

h. Use your work in the previous part to complete the Extended Euclidean Algorithm by filling in the last two ... to update py and qy.

Congratulations, you've just completed a derivation of the Extended Euclidean Algorithm!

px, py, qx, qy =

qx, qy, (px - q*py), (qx - q*qy)