MAT137Y Tutorial 7 worksheet

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TOTAL POINTS

2 / 2

QUESTION 1

1 Q1+Q2 **2 / 2**

✓ **+ 2 pts** **Mostly correct.**

- 1 pts Good effort.

- 2 pts No effort.

- 2 pts No signature.

MAT 137
Tutorial #7– Linear Approximation and Newton's method
Nov 1/2 , 2022
Due on Thursday, Nov 3 by 11:59pm via GradeScope

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- **Show your work and justify your steps** on every question, unless otherwise indicated. Put your final answer in the box provided, if necessary.

We recommend you write draft solutions on separate pages and afterwards write your polished solutions here. You must fill out and sign the academic integrity statement below; otherwise, you will receive zero.

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1. Let $a \in \mathbb{R}$. Let f be a function that is differentiable at a. Define the function g by $g(x) = x^2 f(x)$. Prove that g is differentiable at a and $g'(a) = 2af(a) + a^2 f'(a)$.
Write a proof directly from the definition of the derivative. Do not use any differentiation rules, e.g. quotient rule or chain rule.

(a) Write out the definition of $f'(a)$ as a limit. Does $f'(a)$ exist?

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Since f is differentiable, by definition $f'(a)$ exists.

(b) Prove that $\lim_{x \to a} f(x) = f(a)$. The proof should be short like one or two sentences.

Since f is differentiable, f is continuous. $\lim_{x \to a} f(x) = f(a)$ is the definition of continuity of f at a, therefore it is true if f is differentiable.

(c) Prove that $g(x)$ is defined on an interval centered at a. Hint: can you prove $f(x)$ is defined near a and at a? Write out the epsilon-delta definition of (b) and you will get some idea.

$$\lim_{x \to a} f(x) = f(a) \iff \left(\forall \varepsilon > 0, \exists \delta > 0 \text{ st. } 0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\right)$$

WTS: $\exists \delta > 0$ st. $|x-a| < \delta \Rightarrow g(x)$ is defined

Since f is differentiable at a, it is continuous at a. Since f is continuous at a:

$\forall \varepsilon_1 > 0, \exists \delta_1 > 0$ st. $|x-a| < \delta_1 \Rightarrow |f(x) - f(a)| < \varepsilon_1$

Let $\delta = \delta_1$

Since $|f(x) - f(a)| < \varepsilon_1$, $f(x)$ must be defined.

Therefore $|x-a| < \delta_1 \Rightarrow f(x)$ is defined

Since x^2 is defined $\forall x \in R$, $|x-a| < \delta_1 \Rightarrow x^2 f(x)$ is defined

Therefore $\exists \delta > 0$ st $|x-a| < \delta \Rightarrow g(x)$ is defined

∎

(d) Prove that $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$ exists and satisfies the desired formula.
Hint: use the same trick in the proof of the product rule and quotient rule. Remember to check that all the limits exist before applying limit laws.

We know f(x) is differentiable at a and $g(x) = x^2.f(x)$

WTS: $g'(a) = 2a\, f(a) + a^2.f'(a)$ and thus it exists

Let $h(x) = x^2$

By definition of derivative, $g'(a) = \lim_{x \to a} \frac{g(x)-g(a)}{x-a} = \lim_{x \to a} \frac{h(x).f(x) - h(a).f(a)}{x-a}$

$$= \lim_{x \to a} \frac{h(x).f(x) - h(a).f(x) + h(a).f(x) - h(a).f(a)}{x-a}$$

$$= \lim_{x \to a} \left[\left(\frac{h(x)-h(a)}{x-a} \right) f(x) + \left(\frac{f(x)-f(a)}{x-a} \right) h(a) \right]$$

Observe that $\lim_{x \to a} \frac{h(x)-h(a)}{x-a} = h'(a)$ and $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$

Since x^2 is a polynomial and all polynomials are differentiable, $h'(a)$ exists. Since f is differentiable at a, $f'(a)$ exists.

Since all the limits in our expression for $g'(a)$ exist, we are using limit laws of sum and product consecutively.

Now, $g'(a) = h'(a).f(a) + f'(a).h(a)$

Since $h(x) = x^2$, $h(a) = a^2$ and $h'(a) = 2a$

Therefore, $g'(a) = 2a\, f(a) + a^2.f'(a)$ and thus it exists.

Hence proved.

$\blacksquare$

2. Find an expression for $\frac{dy}{dx}$ by differentiation implicitly: $e^x \sin(y) = x + \sin(xy) - e$.
Then find the tangent line to this curve at $(e, 0)$.

$$e^x \sin(y) = x + \sin(xy) - e$$

differentiating both sides with respect to x

$$\frac{d}{dx}\left(e^x \sin(y)\right) = \frac{d}{dx}(x) + \frac{d}{dx}\left(\sin(xy)\right) + \frac{d}{dx}(-e)$$

Using Chain rule and product rule of differentiation

$$e^x \sin(y) + e^x \cos(y) \cdot \frac{dy}{dx} = 1 + \cos(xy)\left(y + x \cdot \frac{dy}{dx}\right) + 0$$

$$\Rightarrow \frac{dy}{dx}\left(e^x \cdot \cos(y) - x\cos(xy)\right) = 1 + y \cdot \cos(xy) - e^x \cdot \sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + y \cdot \cos(xy) - e^x \cdot \sin y}{e^x \cdot \cos(y) - x\cos(xy)} = f'(x, y)$$

Now $f'(e, 0) = \dfrac{1 + 0 - 0}{e^e - e} = \dfrac{1}{e^e - e}$

Thus the tangent to the curve in slope-point form is $\rightarrow$

$$(y - 0) = \frac{1}{e^e - e}(x - e) \Rightarrow y = \frac{x}{e^e - e} - \frac{1}{e^{e-1} - 1}$$

This is the required equation of the tangent.