1

CHAPTER

Simple Harmonic Motion

LEARNING OBJECTIVES

After completing this chapter, you will be able to:

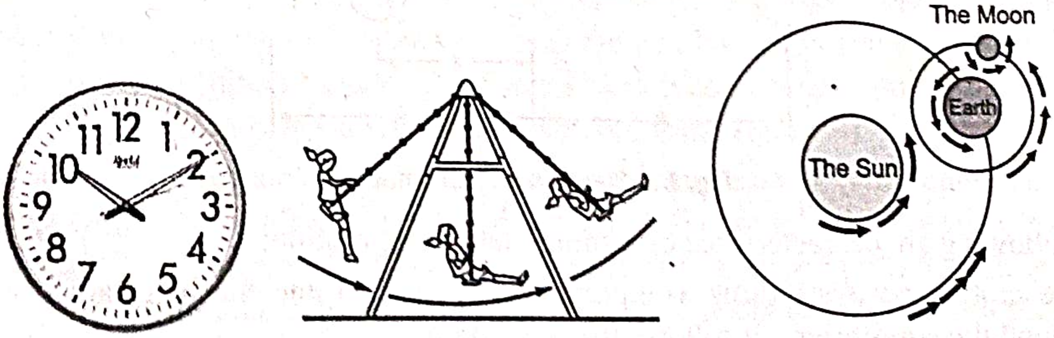
- Differentiate between S.H.M. and other types of periodic motion.
- Formulate differential equation for S.H.M. and obtain an equation for the displacement.
- Obtain expressions for various characteristics and energy of the S.H.M. oscillator.
- Distinguish Free Oscillators, Forced Oscillators, and Damped Oscillators and obtain relevant equations.
- Describe the resonance and sharpness of the resonance.
- Explain the terms Power dissipation and Quality Factor.

1.1 INTRODUCTION

If a body changes its position with time, then the body is said to be in ***motion***. The motion of the body is governed by the kinematic equations consisting of variables, the position of the x, velocity v, time t, and acceleration a. Constant forces cause bodies to move with constant acceleration. Such motions include rectilinear motion and uniform circular motion.

(*a*) **Periodic Motion:** Any motion, which repeats itself at regular intervals of time, is called a **periodic motion**. The particle performs the same set of movements again and again in a periodic motion and there is no equilibrium position and no restoring force.

We find many examples of periodic motion in our day-to-day life, the motions of the hands of a clock (Fig. 1.1a), the swinging on a swing (Fig. 1.1b), the fluttering flags in the breeze, planetary motion (Fig. 1.1c).



(a) (b) (c)

Fig. 1.1: Illustrations of Periodic motion

(*b*) **Oscillatory Motion:** Oscillatory motion is that motion in which a body moves to and fro or back and forth repeatedly about a mean position, in definite intervals of time. In such a motion, ***oscillatory*** or vibrational motion, the body is confined within well-defined limits (called extreme positions) on either side of the mean position. The mean position is also known as equilibrium position. (*a*) spring-mass system (Fig. 1.2a, b) simple pendulum (Fig. 1.2b, c) vibrating ruler (Fig. 1.2c), etc are examples of oscillating systems.

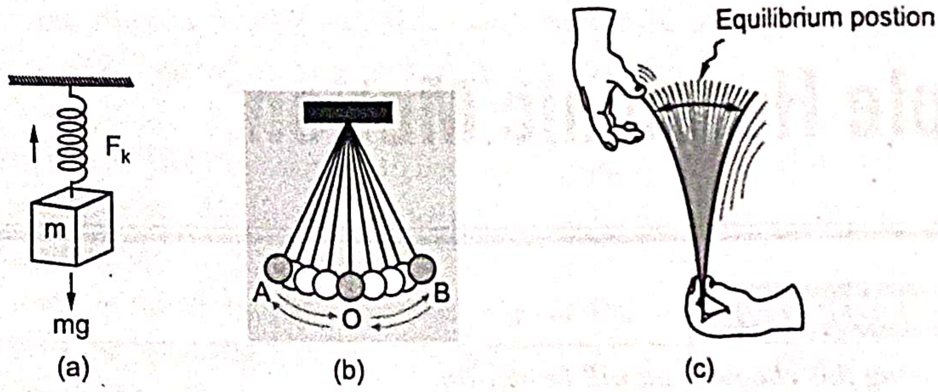


***Fig. 1.2:** Illustrations of Oscillatory motion*

The traditional variables x, v, t, and a describe the motion of an oscillating system but some new variables, amplitude, period, and frequency that describe the periodic nature of the motion are to be included.

Oscillatory motion occurs extensively in nature. Tree leaves fluttering in a gentle breeze, swinging in a swing, or the beating of the heart are familiar examples. Atoms in solids, electric and magnetic fields, large structures swaying due to earthquakes, etc., are all oscillatory.

Definition: *If a body in periodic motion executes to and fro motion about a fixed reference point it is said to have an* ***oscillatory motion.***

1.2 SIMPLE HARMONIC MOTION (S.H.M.)

Simple harmonic motion is a type of oscillatory motion. When an object moves to and fro along a line, the motion is ***simple harmonic motion.*** It involves repetitive movements back and forth in an equilibrium position. There is an equal maximum displacement on each side of this position. The time interval of each complete vibration is the same. The force causing motion is always directed toward the equilibrium position and is proportional to the distance from it.

That is, $F = -kx$,

where F is the force, x is the displacement, and k is a constant.

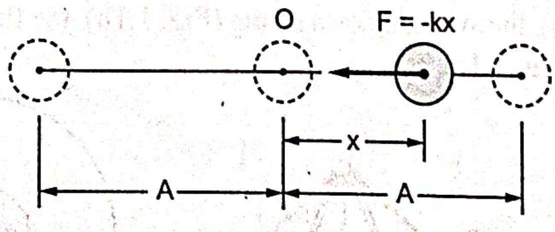


***Fig. 1.3:** Simple harmonic motion*

The following are characteristics of a simple harmonic motion:

(*i*) The motion always returns to equilibrium (Fig. 1.3) and the body always oscillates around the equilibrium. It follows the same path.

(*ii*) The force is *proportional* to *displacement* (the distance of the body from the fixed point).

Note:

(*i*) The force is directed towards an equilibrium position, proportional to the displacement, and it is also a restoring force because it restores the body to its equilibrium position.

(*ii*) The force restores equilibrium to the body since it brings it back to its original position.

(*iii*) S.H.M. is a non-uniformly accelerated motion because the acceleration varies with time.

(*iv*) Simple harmonic motions are not subject to the equations of motion with constant acceleration.

(*v*) As it has a fixed frequency and period, all simple harmonic motions are periodic. However, not all periodic motions are simple harmonic motions. Earth's motion around the sun, for example, is periodic, but not harmonic. An oscillatory motion is a simple harmonic motion.

Definition: The **Simple Harmonic Motion** occurs when a body moves such that its acceleration always follows a certain fixed point and varies directly as it moves away from the fixed point.

1.3 LINEAR SIMPLE HARMONIC MOTION (S.H.M.)

Definition: Linear simple harmonic motion is periodic *to* and *fro* motion along a straight line about equilibrium position under the force always directed towards the mean position and proportional to the distance from the mean position.

The nature of S.H.M. can be studied by obtaining expressions for acceleration, velocity, and displacement from the defining equation; this can be done by framing and solving the differential equation on $F = -kx$.

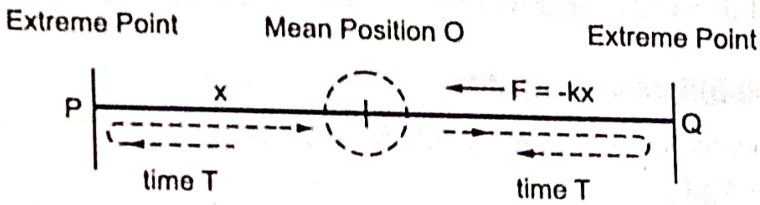


Fig. 1.4: *A particle executing simple harmonic motion.*

Let us consider a particle of mass (m) executing simple harmonic motion (Fig. 1.4) along a path POQ; the mean position is at O. Let the speed of the particle m be υ_o when it is at a distance x from O.

When the particle is at the equilibrium position O, it is in a *state of rest* (Fig. 1.4). When it is moved away from the equilibrium position, say to Q and released, a force $F = -(kx)$ comes into play. It pulls the particle back towards the equilibrium position. When the particle reaches the equilibrium position, the restoring force vanishes.

However, the **inertia** property causes the particle to overshoot the equilibrium position O and the motion continues towards position P. The particle stops somewhere, say at P, on the other side. But the particle is again pulled back towards the equilibrium position O by the restoring force which changed its direction now. The force F which continuously pulls back the particle to the equilibrium position O is known as the **restoring force** (Fig. 1.4).

At the equilibrium position O, the force $F = 0$ and due to inertia, the body continues forward. As the particle crosses the equilibrium position O, the restoring force comes into action but with a change in its direction. The particle stops at Q and the restoring force F pulls it back. The particle starts its journey towards the equilibrium position O, overshoots the equilibrium position, and continues its journey towards the extreme position P on the other side. The result is a continuing *oscillatory motion* of the particle back and forth along the line POQ.

1.3.1 Differential Equation of Linear S.H.M.

Let us consider a particle of mass m executing simple harmonic motion. If x be the displacement of the particle from equilibrium position at any instant t, the restoring force F acting on the particle is given by

$$F \propto x$$

or

$$F = -kx \qquad ...(1.1)$$

where k is the *force constant of proportionality*.

The negative sign is used to indicate that the direction of the force is opposite to the direction of increasing displacement.

Force constant k is defined as the ***restoring force per unit displacement***,

or

$$k = \frac{F}{x}$$

Its unit is Newton per metre.

If $\frac{d^2x}{dt^2}$ is the acceleration of the particle at time t, then we can write Eq. (1.1) as

$$m\frac{d^2x}{dt^2} = -kx$$

or

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

Writing $\frac{k}{m} = \omega^2$, we get $\frac{d^2x}{dt^2} + \omega^2 x = 0$ ...(1.2)

Eq. (1.2) is the differential equation of linear harmonic oscillator.

1.3.2 Solution of the Differential Equation

The differential equation representing S.H.M. is given by Eq. (1.2)

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Multiplying the above equation by $\left[2\frac{dx}{dt}\right]$, we get

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + \omega^2 2x\frac{dx}{dt} = 0$$

Integrating the above equation, we get

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2x^2 + C \qquad ...(1.3)$$

where C is constant of integration.

When the displacement is maximum, *i.e.*, at $x = A$, where A is the amplitude of the oscillating particle,

$$\frac{dx}{dt} = 0$$

i.e., the particle is momentarily at rest in the extreme position and begins its journey in the backward direction.

Substituting $x = A$ and $\frac{dx}{dt} = 0$ in Eq. (1.3), we get

$$C = A^2\omega^2$$

Substituting this value of C, in Eq. (1.3), we get

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2x^2 + A^2\omega^2 = \omega^2(A^2 - x^2)$$

or
$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$

This equation gives the velocity of the particle executing simple harmonic motion at a time t, when the displacement is y.

$$\therefore \qquad \frac{dx}{\sqrt{A^2 - y^2}} = \omega dt$$

Integrating, we get

$$\sin^{-1}\frac{x}{A} = \omega t + \phi$$

where ϕ is a constant of integration

$$x = A\sin(\omega t + \phi) \qquad \text{...(1.4)}$$

The term $(\omega t + \phi)$ in Eq. (1.4) represents the total phase of the particle at time t and ϕ is known as the initial phase or constant.

1.3.3 Simple Harmonic Motion is Sinusoidal or Co-sinusoidal

A simple harmonic motion can be also represented by the relation

$$y = A\cos(\omega t + \phi) \qquad \text{...(1.5)}$$

where y is the displacement at a time t, A is the amplitude, ω is the angular frequency and ϕ is the phase constant.

Starting with Eq. (1.4) we can derive all the characteristics of the linear S.H.M and the details worked out in the section 1.5 below.

1.4 ANGULAR SIMPLE HARMONIC OSCILLATIONS

Let us consider a circular disk suspended from a wire fixed to a ceiling. Rotating the disk will twist the wire. As the disk is released, the twisted wire exerts a restoring force on it, causing it to rotate past its equilibrium point, thereby twisting the wire in the opposite direction. This system is called a torsional oscillator (Fig. 1.5).

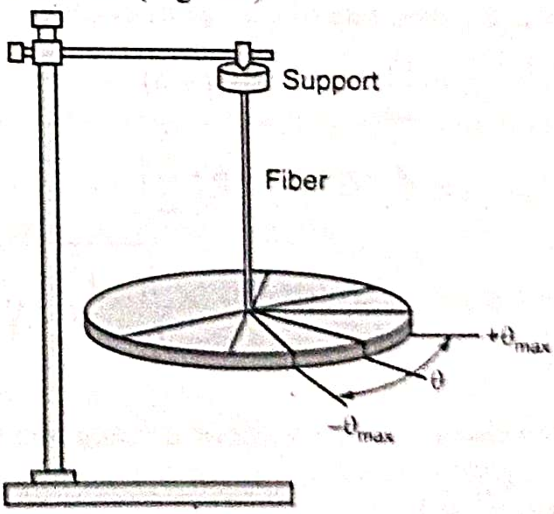


Fig. 1.5: A torsional oscillator

It has been found experimentally that the torque exerted on the disk is proportional to the angular displacement of the disk.

$$\tau = -\kappa\theta \qquad ...(1.6)$$

where κ is a proportionality constant, a property of the wire. It is torque per unit angular displacement.

The torque $\tau = I\alpha$ for any rotational motion, where I is the moment of inertia of the body and α is the angular acceleration.

Then, we can say that

$$\tau = I\alpha = -\kappa\theta \qquad ...(1.6a)$$

But

$$\alpha = \frac{d^2\theta}{dt^2} \qquad ...(1.7)$$

Also

$$\alpha = -\frac{k}{I}\theta \qquad ...(1.7a)$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \qquad ...(1.8)$$

If we substitute m for I, k for κ, and x for θ we see that this differential equation resembles simple harmonic differential equation.

1.5 CHARACTERISTICS OF S.H.M

1.5.1 Displacement

Definition: Displacement is the directed distance of the mass from its equilibrium position.

The displacement of a simple harmonic oscillator at any instant of time t is given by

$$y = A \sin(\omega t + \phi) \qquad ...(1.9)$$

- The displacement y varies periodically between the values $-A$ and $+A$.
- A is the maximum value of the displacement and is known as the **amplitude** of the oscillation. It may be noted that the amplitude A is constant.
- The quantity $(\omega t + \phi)$ is called the *phase angle* and ϕ is the initial phase, that is the phase at $t = 0$.

1.5.2 Velocity

Definition: The velocity is the time rate of change of displacement.

$$\therefore \text{ Velocity} \quad \upsilon = \frac{dy}{dt} = A\omega\cos(\omega t + \phi) \qquad ...(1.10)$$

$$= A\,\omega \sin\left(\omega t + \phi + \frac{x}{2}\right) \qquad ...(1.11)$$

As

$$\sin(\omega t + \phi) = \frac{y}{A}, \cos(\omega t + \phi) = \sqrt{1 - \frac{y^2}{A^2}} = \sqrt{\frac{A^2 - y^2}{A}}$$

$$\therefore \quad \upsilon = \omega\sqrt{A^2 - y^2} \qquad ...(1.12)$$

Maximum velocity: The velocity of the oscillator is maximum when in Eq. (1.11)

$$\sin\left(\omega t + \phi + \frac{\pi}{2}\right) = 1$$

$$\therefore \qquad \upsilon_{max} = A\omega \qquad ...(1.13)$$

The value of $\upsilon = \upsilon_{max} = A\omega$, when $y = 0$.

- The velocity varies harmonically with the same frequency ω.

Minimum velocity: From the Eq. (1.12), it is seen that when $y = A$, the value of $\upsilon = \upsilon_{min} = 0$. Thus υ varies periodically between the values $+\omega A$ and $-\omega A$.

- When the displacement y is greatest, the velocity is zero, and when it is least ($y = 0$), that is at the midpoint of the motion, the velocity is greatest.

1.5.3 Acceleration

Definition: **Acceleration** is defined as the time rate of change of velocity.

We can find the acceleration of the oscillating particle by differentiating the Eq. (1.10) for υ.

Now velocity $\qquad \upsilon = A\,\omega \cos(\omega t + \phi)$

$$\therefore \qquad \text{Acceleration} = \frac{d\upsilon}{dt} = \frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + \phi) \qquad ...(1.14)$$

or

$$\text{Acceleration, } a = A\,\omega^2 \sin(\omega t + \phi + \pi) \qquad ...(1.15)$$

Maximum acceleration: The acceleration of the oscillator is maximum when

$$\sin(\omega t + \phi + \pi) = 1$$

Thus,

$$\alpha_{max} = \left(\frac{d^2y}{dt^2}\right)_{max} = A\omega^2 \qquad ...(1.16)$$

- Acceleration varies periodically between the values $+\omega^2 A$ and $-\omega^2 A$.

Phase Relationship between Displacement, Velocity, and Acceleration:

Comparing Eq. (1.15) and Eq. (1.9), we find that the acceleration of a simple harmonic oscillator leads the displacement by π radian or (180°) in phase. Thus, the acceleration and displacement are in opposite phase.

$$a = -\omega^2 \qquad ...(1.17)$$

- The acceleration of a simple harmonic motion is proportional and opposite to the displacement. The displacement has its greatest positive value when the acceleration has its greatest negative value and vice versa.
- If the displacement is zero, acceleration is also zero.

1.5.4 Time Period

Definition: The **time period** T is the time taken for one complete oscillation.

From the Eq. (1.4) the displacement of the particle after a time $T = \frac{2\pi}{\omega}$ is the same. Hence T gives the period of the simple harmonic oscillator.

$$\therefore \qquad T = \frac{2\pi}{\omega} \qquad ...(1.18)$$

and frequency

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \qquad ...(1.19)$$

$\therefore$ The angular velocity of the harmonic oscillator $\omega = 2\,\pi\nu$

The acceleration of a simple harmonic oscillator is given by the relation

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

Neglecting the negative sign, we have

$$\omega^2 = \left(\frac{d^2y}{dt^2}\right)/(y) = \frac{\text{Acceleration}}{\text{Displacement}} \text{ and}$$

$$\therefore \qquad \omega = \sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

and

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}}$$

also

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{Acceleration}}{\text{Displacement}}}$$

According to Eq. (1.2) $\omega^2 = \frac{k}{m}$.

$$\therefore \qquad T = \frac{\ }{\ } = \pi\sqrt{\ } \qquad ...(1.20)$$

- Mass m and elastic constant k determine the period of motion. It *is independent of the oscillation amplitude.*
- Whatever the value of m and k, no matter how large or small the amplitude, the time for one complete oscillation remains the same.

1.5.5 Frequency

Definition: The frequency ν, is defined as the number of complete oscillations per second. The frequency, ν is the reciprocal of the time period.

$$\nu = \frac{1}{T}$$

$$\therefore \qquad \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad ...(1.21)$$

- The frequency and time period are independent of the amplitude A.

1.5.6 Phase

Phase indicates how the body is oscillating. The angle $(\omega t + \phi)$ is called the ***phase*** of the oscillation and ϕ is called the ***phase constant***.

- A phase is determined by the position and direction of motion of the body.
- Phase angle ϕ is determined by the initial displacement and velocity of the particle. The constant ϕ and amplitude A give the displacement at time $t = 0$.
- Based on the phase of the oscillation, we can compare the motion of two bodies (Fig. 1.6).

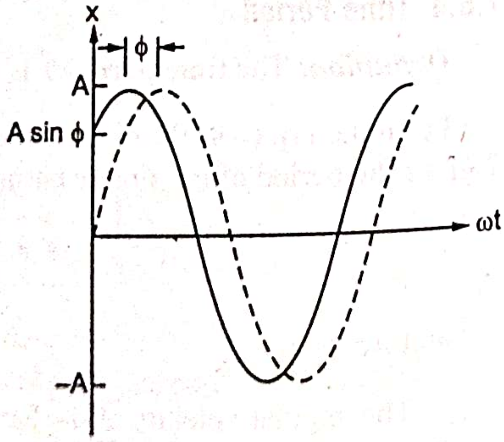


Fig. 1.6: Phase of oscillation

1.6 ENERGY OF A SIMPLE HARMONIC OSCILLATOR

Simple harmonic oscillators possess both potential energy and kinetic energy. The potential energy of an object arises from its displacement from its equilibrium position, whereas its mass (inertia) stores kinetic energy; therefore, the kinetic energy arises from its velocity. As the system oscillates, potential energy is continuously converted into kinetic energy and vice versa (Fig. 1.7). If there are no dissipative forces, the total energy is conserved.

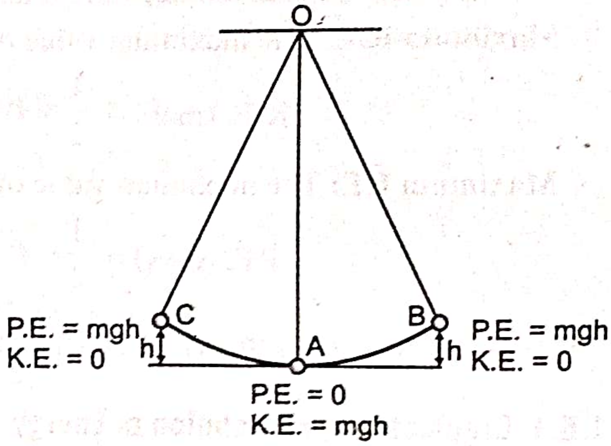


Fig. 1.7: Energy of a simple harmonic oscillator

Total Energy of a Simple Harmonic Oscillator at Any Time *t*:

$$E_{total} = \text{Kinetic Energy} + \text{Potential Energy.}$$

Kinetic energy:

The velocity at any instant $\upsilon = \frac{dy}{dt} = A\,\omega \cos(\omega t + \phi)$

$\therefore$ Kinetic energy of the oscillator $= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 = \frac{1}{2} m A^2 \omega^2 \cos^2(\omega t + \phi)$ ...(1.22)

Potential energy:

The potential energy is equal to the amount of work done in overcoming the restoring force from the mean position through a displacement y.

$$\text{Acceleration } a = \frac{d^2y}{dt^2} = -A\omega^2 \sin(\omega t + \phi)$$

Restoring force $= m\omega^2 y = sy$ where s is the force constant of proportionality or stiffness.

Hence total work done by the force through a displacement

$$y = \int_0^y ky\,dy = \frac{1}{2} ky^2 = \frac{1}{2} m\omega^2 y^2$$

As $$y = A \sin(\omega t + \phi)$$

or $$\text{P.E.} = \frac{1}{2} m\omega^2 A^2 \sin^2(\omega t + \phi) \qquad \text{...(1.23)}$$

Total Energy $$E = \text{K.E} + \text{P.E} = \frac{1}{2} my^2 + \frac{1}{2} ky^2$$

$$E = \frac{1}{2} mA^2\omega^2 \cos^2(\omega t + \phi) + \frac{1}{2} mA^2\omega^2 \sin^2(\omega t + \phi)$$

$$E = \frac{1}{2} mA^2\omega^2 \qquad \text{...(1.24)}$$

But $$\omega^2 = \frac{k}{m} \qquad (\therefore m\,\omega^2 = k)$$

Hence $$E = \frac{1}{2} kA^2$$

But $$\omega = 2\pi\nu. \qquad \text{...(1.25)}$$

$\therefore$ $$E = \frac{1}{2} mA^2 4\pi^2 n^2 = 2mA^2\pi^2\nu^2$$

- Total energy of the harmonic oscillator is a constant and proportional to the square of the amplitude. It does not depend on time and is a constant of the motion.

Maximum K.E: The maximum value of K.E for cos $(\omega t + \phi)$

$$\text{K.E. (max)} = \frac{1}{2}mA^2\omega^2$$

Maximum P.E: The maximum value of P.E for sin $(\omega t + \phi) = 1$

$$\text{P.E. (max)} = \frac{1}{2}mA^2\omega^2$$

$$\therefore \text{ K.E (max)} = \text{P.E (max)} = \text{Total energy } E = \frac{1}{2}mA^2\omega^2 = \text{constant}$$

1.6.1 Graphical Representation of Energy

Fig. 1.8 shows the kinetic energy, potential energy, and total energy of the oscillator plotted against the displacement y. The horizontal line represents the total energy E, which is constant and does not vary with the y. A potential energy curve is parabolic to the y and is symmetric about equilibrium, $y = 0$. Similarly, the kinetic energy curve is symmetrical around the point of equilibrium, $y = 0$ and parabolic to the y-axis.

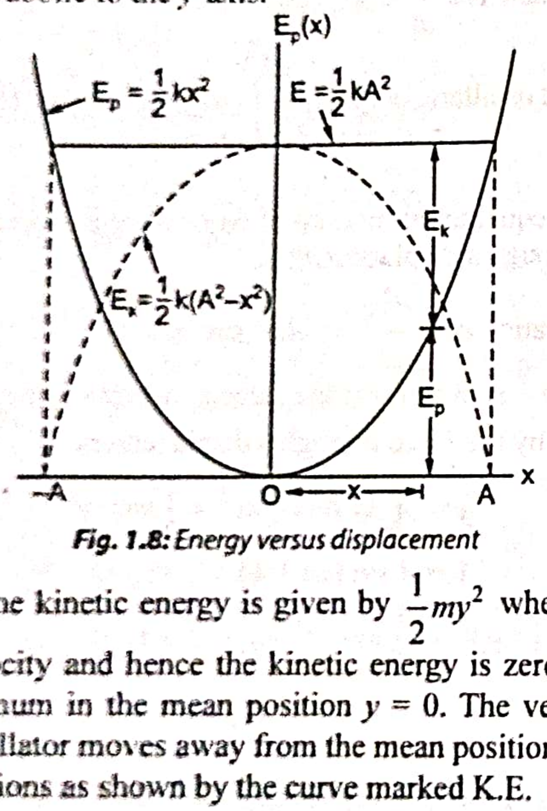


Fig. 1.8: Energy versus displacement

(i) **Kinetic energy:** The kinetic energy is given by $\frac{1}{2}my^2$ where k is the velocity of the oscillator. The velocity and hence the kinetic energy is zero at the extreme positions $y = \pm A$; and maximum in the mean position $y = 0$. The velocity and hence the K.E deceases as the oscillator moves away from the mean position and finally becomes zero at the extreme positions as shown by the curve marked K.E.

(ii) **Potential energy:** The potential is given by $\frac{1}{2}ky^2$ where k is the stiffness. For $y = 0$, P.E $= 0$ and for $y = \pm A$, the P.E is maximum $\frac{1}{2}kA^2$. Thus, potential energy goes on increasing as the oscillator moves away from its mean position and becomes maximum at the extreme positions $y = \pm A$.

(iii) **Total energy:** The total energy is given by $\frac{1}{2}kA^2$ which is a constant. This is represented by the straight line marked "Total energy" parallel to the displacement axis.

There is a constant energy transfer between the potential energy stored in the oscillator and the kinetic energy of the mass. The potential and kinetic energies have equal maximum values. In addition, the total energy of a particle at any point of executing S.H.M. remains conserved.

1.6.2 The Average Value of P.E. and K.E. for Complete Cycle

The average value of P.E. for complete cycle is given by

$$(E_{PE})_{avg} = \frac{1}{T}\int_0^T E_p dt$$

$$= \frac{1}{T}\int_0^T \frac{1}{2}M\omega^2 A^2 \sin^2 \omega t\, dt$$

$$= \frac{1}{2T}M\omega^2 A^2 \int_0^T \frac{(1-\cos 2\omega t)}{2} dt$$

$$= \frac{1}{4T}M\omega^2 A^2 \left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{1}{4T}M\omega^2 A^2 (T)$$

$$= \frac{1}{4}M\omega^2 A^2 \quad ...(1.26)$$

The average value of K.E. for complete cycle is given by

$$(E_{KE})_{avg} = \frac{1}{T}\int_0^T E_k dt = \frac{1}{T}\int_0^T \frac{1}{2}M\omega^2 A^2 \cos^2 \omega t\, dt$$

$$= \frac{1}{2T}M\omega^2 A^2 \int_0^T \frac{(1+\cos 2\omega t)}{2} dt$$

$$= \frac{1}{4T}M\omega^2 A^2 \left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{1}{4T}M\omega^2 A^2 (T)$$

$$(E_{KE})_{avg} = \frac{1}{4}M\omega^2 A^2 \quad ...(1.27)$$

Accordingly, the average values of kinetic energy and potential energy of the harmonic oscillator are equal and each equal to half of the total energy.

$$K_{average} = U_{average} = \frac{1}{2}E = \frac{1}{4}M\omega^2 A^2.$$

Example 1.1: *Find maximum amplitude of velocity if a particle executes S.H.M. of period 10 s and amplitude 5 cm.*

Solution: Here displacement amplitude $a = 5$ cm; Time period T = 10s.

$\therefore$ Angular frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{10} s^{-1}$

Maximum velocity $= a\omega = \frac{5 \times 2\pi}{10}$ cm/s

Example 1.2: *Estimate the displacement to amplitude ratio for a S.H.M. when K.E. is 90% of total energy.*

Solution: If m is the mass of the particle executing S.H.M., a is the amplitude and ω is the angular velocity, then

Total energy $= \frac{1}{2}ma^2\omega^2$

Let y be the displacement when K.E = 90% of total energy.

As K.E is 90% of total energy.

Potential energy = Total energy – Kinetic energy = 10% of total energy

Now potential energy $= \frac{1}{2}ma^2\omega^2$

$$\therefore \quad \frac{\frac{1}{2}m\omega^2 y^2}{\frac{1}{2}m\omega^2 a^2} = \frac{y^2}{a^2} = \frac{10}{100} = 0.1$$

$$\text{or} \quad \frac{\text{Displacement}}{\text{Amlitude}} = \frac{y}{a} = \sqrt{0.1} = 0.316$$

Example 1.3: *Calculate the total energy of the particle, if it has a mass 2.5 gm and frequency of vibration 10 Hz and an amplitude of 2 cm.*

Solution: Total energy, $E_T = \frac{1}{2}m\omega^2 a^2$.

$$\omega = 2\pi\nu = 2\pi \times 10 = 20\pi s^{-1}$$

$$a = 2 \text{ cm} = 0.02 \text{ m}$$

$$m = 2.5 \text{ gm} = 2.5 \times 10^{-3} \text{ kg}$$

$$\therefore \quad E_T = \frac{1}{2} \times (20\pi)^2 \times (2.5 \times 10^{-3})^2 = \frac{1}{2} \times 400(3.14)^2 \times 6.25 \times 10^{-6}$$

$$= 200 \times 9.86 \times 6.25 \times 10^{-6} = 12324.5 \times 10^{-6} = 12.3245 \times 10^{-3} \text{ J}$$

Example 1.4: *The amplitude of a simple harmonic oscillator is doubled. How does this affect the time period, total energy, and maximum velocity of the oscillator?*

Solution: When the displacement y of a simple harmonic oscillator is given by $y = A\sin(\omega t + \phi)$

Velocity $\quad \upsilon = \frac{dy}{dt} = A\omega\cos(\omega t + \phi)$

And acceleration $\quad a = \frac{d^2y}{dt^2} = -A\omega^2\sin(\omega t + \phi) = -\omega^2 y$

Now time period $T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{y}{\omega^2 y}} = \frac{2\pi}{\omega}$ (Ignoring the negative sign)

As $\frac{2\pi}{\omega}$ is a constant, the time period does not depend upon amplitude but remains constant maximum velocity $\upsilon_{max} = A\omega$ when $\cos(\omega t + \phi) = 1$

When amplitude A is doubled, the maximum velocity is also doubled.

Total energy $= \frac{1}{2}mA^2\omega^2$

When the amplitude A is doubled, the total energy becomes four times, because energy $\propto A^2$.

Example 1.5: *Calculate the displacement at which KE = PE for an oscillator characterized by $y = a \cos \omega t$.*

Solution: Let y be the displacement at which kinetic energy of the simple harmonic oscillator is equal to its potential energy.

Now $\quad y = a \cos \omega t \quad \therefore \quad \dfrac{dy}{dt} = -a\omega \sin \omega t$

The kinetic energy of a simple harmonic oscillator $= \dfrac{1}{2} m\upsilon^2 = \dfrac{1}{2} m(-a\,\omega \sin \omega t)^2$

$$\text{K.E.} = \frac{1}{2} ma^2\omega^2 \sin^2 \omega t = \frac{1}{2} ma^2\omega^2 (1 - \cos^2 \omega t)$$

$$= \frac{1}{2} m\omega^2 (a^2 - a^2 \cos^2 \omega t) = \frac{1}{2} m\omega^2 (a^2 - y^2)$$

$$\text{P.E.} = \frac{1}{2} sy^2$$

where $\quad s =$ stiffness and $\dfrac{s}{m} = \omega^2$ or $s = m\omega^2$

$\therefore \quad \text{P.E} = \dfrac{1}{4} m\omega^2 a^2$

When $\quad$ K.E = P.E.

$$= \frac{1}{2} m\omega^2 (a^2 - y^2) = \frac{1}{2} m\omega^2 y^2$$

Or $\quad a^2 - y^2 = y^2 \quad$ or $\quad a^2 = 2y^2$

$\therefore$ Displacement $y = \pm \dfrac{a}{\sqrt{2}}$ from the mean position

$$\text{P.E.} = \frac{1}{4} m\omega^2 a^2,$$

$$\text{K.E.} = \frac{1}{4} ma^2\omega^2 \text{ and}$$

Total energy $\quad = \dfrac{1}{2} m\omega^2 a^2$

i.e., half of the total energy is kinetic and half potential.

1.7 FREE OSCILLATIONS

Definition: Oscillations caused by a single initial deviation from equilibrium in a system are called **free oscillations**.

When a pendulum is displaced from its equilibrium position and left to the action of internal forces, it oscillates freely with the frequency given by

$$\upsilon = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad \text{...(1.28)}$$

It is independent of the mass of a simple pendulum that it has a period and frequency based on the length of the string and the acceleration due to gravity. So long as there is no resistance to the pendulum's motion, its frequency will remain the same.

The frequency with which the pendulum oscillates freely on its own is called its **natural frequency**. Therefore, if no resistance is offered to the motion of an oscillating body by air friction or other forces, the body will continue to oscillate at its natural frequency indefinitely, (Fig. 1.9). Such an oscillator is called an *ideal* oscillator.

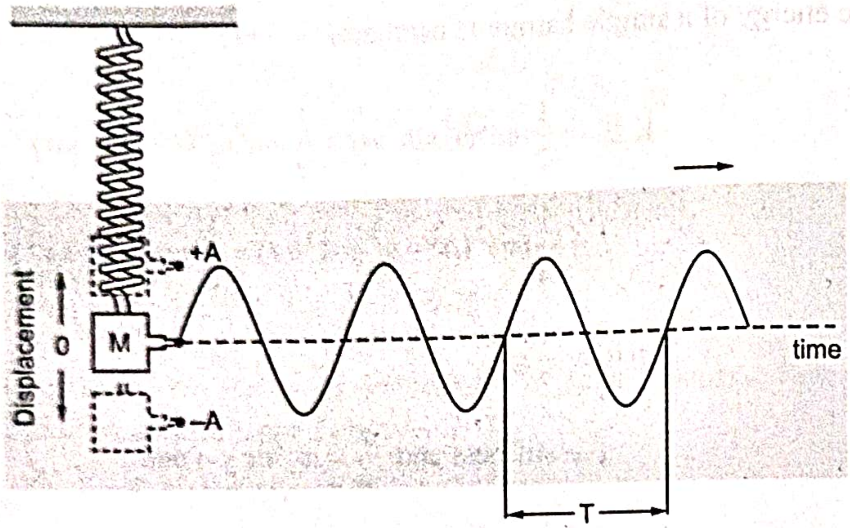


Fig. 1.9: Natural Frequency

- The period of oscillations of an ideal oscillator is independent of the amplitude.
- Frictionless systems do not dissipate energy, so the total mechanical energy and amplitude remain constant.

Example 1.6: *When a mass m = 0.6 kg is gently added to mass M= 2 kg of a vertical spring, it is further stretched by 4 cm. Calculate the period of oscillations if m is removed, and M is set into oscillations.*

Solution: Let x be the displacement of the spring with mass M.

Now $\quad F = kx, \quad \therefore \quad Mg = kx$

Or $\quad 2g = kx$

When an extra mass m (= 0.6 kg) is added, the spring is further stretched by 0.04 m.

Hence, $\quad (M + m) \times g = k(x + 0.04)$

or $\quad (2 + 0.06) \times g = kx + 0.04\,k = 2g + 0.04k$

$\therefore \quad 2.06 \times g = 2g + 0.04k \text{ or } k = \dfrac{0.6 \times 9.8}{0.04} = 14.7 \text{ N/m}$

Now $\quad T = 2\pi\sqrt{M/k} = 2\pi\sqrt{2/14.7} = 2.317$ s.

1.8 DAMPED OSCILLATIONS

An ideal pendulum oscillates between two spatial positions without diminishing in amplitude. Many oscillatory systems experience irreversible energy losses due to friction or viscous heat generation while oscillating. Whether the body is moving away from equilibrium or returning to it, friction opposes it. Its amplitude diminishes with time see Fig. 1.10 and this oscillation is called a ***damped harmonic oscillation.***

Definition: Damped harmonic oscillators are oscillatory systems for which the amplitude of oscillation decreases over time.

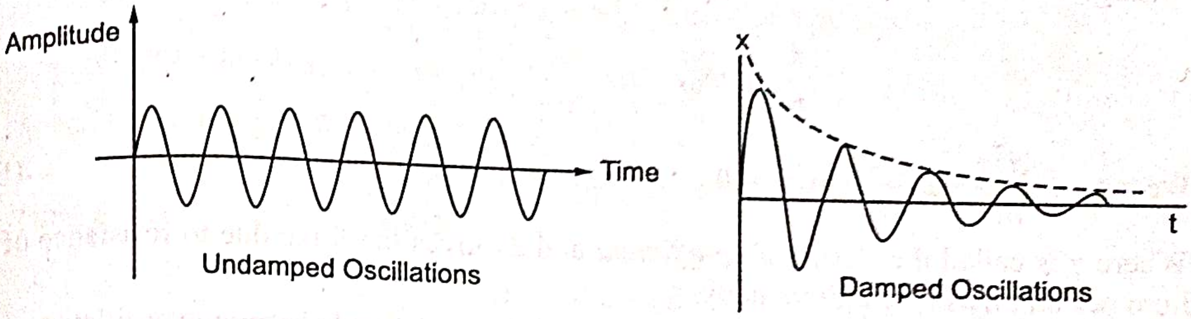


Fig. 1.10: *Damped Oscillations*

Definition: Damping is the phenomenon of decay in the amplitude of oscillations. The oscillation amplitude becomes zero when the initial energy of the oscillating body has been dissipated.

- Damped oscillations are *not sinusoidal* but are much more complex. The period depends on the amplitude. A pendulum immersed in water will exhibit damped oscillations, but if it is immersed in a viscous medium, such as oil, it will not exhibit any oscillations at all.
- Damped systems are described by ordinary differential equations of second order, including a term proportional to the first derivative of amplitude.

1.8.1 Equation of Motion of a Damped Harmonic Oscillator

Damping force is the resistance to motion. We may assume that in addition to the restoring force ($F = -kx$), there is a *damping force* that is opposed to the velocity. Friction and viscosity are such kinds of forces.

A damped system is subjected to the following two forces:

(*i*) A *restoring force* proportional to displacement x given by $-kx$ where k is a constant known as *stiffness constant*. The negative sign shows that the direction of the restoring force is opposite to that of displacement, and

(*ii*) A *frictional force* proportional to velocity given by $-b\frac{dx}{dt}$ where b is another constant known as *damping constant*. The negative sign shows that the retarding force acts opposite to the direction of the body.

Let m be the mass of the oscillating body and its acceleration $\frac{d^2x}{dt^2}$. The resultant force on the body is $\left(-b\frac{dx}{dt} - kx\right)$.

Therefore, the equation of motion of the body is

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx$$

or

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Dividing above equation by m, we get

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Putting
$$\frac{b}{m} = 2\gamma \text{ and}$$
$$\frac{k}{m} = \omega_0^2.$$

We get
$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \qquad ...(1.29)$$

Where γ is called the *damping co-efficient* and 2γ gives the force due to resistance of the medium per unit mass per unit velocity.

Eq. (1.29) is known as differential equation of a damped simple harmonic oscillator.

1.8.2 Solution of Equation of Damped S.H.M.

The differential equation representing damped harmonic motion is given by Eq. (1.29).
$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

where
$$2\gamma = \frac{b}{m} \text{ and } \omega_0 = \sqrt{\frac{k}{m}}.$$

To solve the above differential equation, let us assume the solution
$$x = Ce^{\lambda t} \qquad ...(1.30)$$

where C and λ are constants.

Differentiating Eq. (1.30) we get
$$\frac{dx}{dt} = C\lambda e^{\lambda t} \quad \text{and} \quad \frac{d^2x}{dt^2} = C\lambda^2 e^{\lambda t}$$

Substituting in Eq. (1.29), we get
$$C\lambda^2 e^{\lambda t} + 2\gamma C\lambda e^{\lambda t} + \omega_0^2 Ce^{\lambda t} = 0$$
$$Ce^{\lambda t}[\lambda^2 + 2\gamma\lambda + \omega_0^2] = 0$$
$$\therefore \quad \lambda^2 + 2\gamma\lambda + \omega_0^2 = 0 \qquad (\because Ce^{\lambda t} \neq 0)$$

This equation is quadratic in λ and its solution is
$$\lambda = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$
$$\therefore \quad \lambda_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2} \text{ and } \lambda_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$$

Therefore, the general solution of Eq. (1.29) is
$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$
$$= Ae^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + Be^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$$
$$= e^{-\gamma t}\left[Ae^{(\sqrt{\gamma^2 - \omega_0^2})t} + Be^{(-\sqrt{\gamma^2 - \omega_0^2})t}\right]$$

or
$$x = e^{-\gamma t}[Ae^{\alpha t} + Be^{-\alpha t}] \qquad ...(1.31)$$

where A and B are constants and $\alpha = \sqrt{\gamma^2 - \omega_0^2}$.

To Evaluate A and B:

The constants A and B are determined from the initial conditions.

Suppose at $t = 0$, $x = x_0$ and velocity $= \frac{dx}{dt} = 0$

Substituting in Eq. (1.31), we get

$$x_0 = A + B \quad \text{or} \quad A + B = x_0 \qquad \text{...(1.32)}$$

Differentiating Eq. (1.31), we get

$$\begin{aligned} \frac{dx}{dt} &= -\gamma e^{-\gamma t}(Ae^{\alpha t} + Be^{-\alpha t}) + e^{-\gamma t}(A\alpha e^{\alpha t} - B\alpha e^{-\alpha t}) \\ &= -\gamma e^{-\gamma t}(Ae^{\alpha t} + Be^{-\alpha t}) + e^{-\gamma t} \cdot \alpha(Ae^{\alpha t} - Be^{-\alpha t}) \end{aligned}$$

Substituting $t = 0$, and $\frac{dx}{dt} = 0$, we have

$$0 = -\gamma\,(A + B) + \alpha(A - B)$$

and from Eq. (1.31) $\alpha(A - B) = \gamma(A + B) = \gamma x_0$

$$\therefore \qquad A - B = \frac{\gamma}{\alpha} x_0 \qquad \text{...(1.33)}$$

Adding and subtracting Eq. (1.32) and Eq. (1.33), we get

$$A = \frac{\;}{\;}\left(\quad \frac{\;}{\;} \right)$$

$$B = \frac{x_0}{2}\left(1 + \frac{\gamma}{\alpha}\right)$$

Substituting these values in Eq. (1.31), we get

$$\begin{aligned} x &= e^{-\gamma t}\left[\frac{x_0}{2}\left\{\left(1 + \frac{\gamma}{\alpha}\right)e^{\alpha t} + \left(1 + \frac{\gamma}{\alpha}\right)e^{-\alpha t}\right\}\right] \\ &= x_0 e^{-\gamma t}\left[\frac{1}{2}\left\{e^{\alpha t} + \frac{\gamma}{\alpha}e^{\alpha t} + e^{-\alpha t} + \frac{\gamma}{\alpha}e^{-\alpha t}\right\}\right] \end{aligned}$$

$$\therefore \qquad x = x_0 . e^{-\gamma t}\left[\left(\frac{e^{\alpha t} + e^{-\alpha t}}{2}\right) + \frac{\gamma}{\alpha}\left(\frac{e^{\alpha t} + e^{-\alpha t}}{2}\right)\right] \qquad \text{...(1.34)}$$

where $\gamma = \frac{b}{2m}$, and $\alpha = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$

- Eq. (1.34) is the general solution of a damped harmonic oscillator.
- The equation gives the displacement (x) of a particle of mass m executing damped harmonic motion. The nature of motion depends upon the relative values of b and ω_0.

The following three important cases arise:

Case 1: Aperiodic or dead-beat motion: When $\gamma^2 > \omega_0^2$ the case of heavy damping, $\sqrt{\gamma^2 - \omega_0^2}$ is real, since $\gamma^2 - \omega_0^2$ is positive. Eq. (1.34) shows that the displacement x decreases from its initial value x_0 and it tends to zero when $t \to \infty$. In this case no oscillations occur. Hence such a motion is called ***aperiodic*** or ***over damped***. The particle passes its equilibrium position at the most once under very heavy damping before returning to rest asymptotically. The decrease of x with time t is shown by curve a in Fig. 1.11.

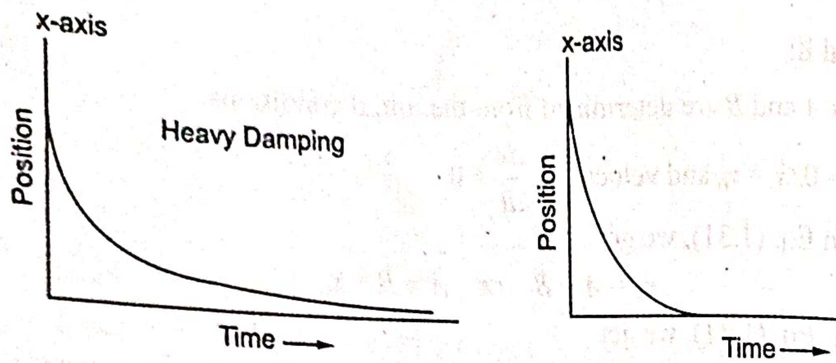


Fig. 1.11: (a) Heavily damped, (b) critically damped

As the damping increases, the time taken by the body to reach equilibrium also increases. For instance, if a door closing mechanism is heavily damped, when released from the open position the door gradually closes.

Definition: The system once displaced, returns to its equilibrium position quite slowly without any oscillations. The motion is termed as ***heavy*** or ***over-damped motion***.

Case 2: Critically damped motion: It arises when $\gamma^2 = \omega_0^2$ or $\alpha = \sqrt{\gamma^2 - \omega_0^2} = 0$.

If we substitute this value in Eq. (1.34), the second term has the indeterminate form $\frac{o}{o}$. Thus, in this case, Eq. (1.34) does not represent the solution *i.e.*, solution breaks down.

Let us, however, consider the case when $\sqrt{\gamma^2 - \omega_0^2} \neq 0$ but very small quantity, say h.

Thus, $$\sqrt{\gamma^2 - \omega_0^2} = h.$$

Substituting in Eq. (1.34), we get

$$x = x_0 e^{-\gamma t}\left[\left(\frac{e^{ht} + e^{-ht}}{2}\right) + \frac{\gamma}{\alpha}\left(\frac{e^{ht} + e^{-ht}}{2}\right)\right] = x_0 e^{-\gamma t}\left[\cosh(ht) + \frac{\gamma}{h}\sin h(ht)\right]$$

$$= x_0 e^{-\gamma t}\left[\left(1 + \frac{h^2 t^2}{2!} + \frac{h^4 t^4}{4!} + \ldots\right) + \frac{\gamma}{h}\left(ht + \frac{h^3 t^3}{3!} + \frac{h^5 t^5}{5!} + \ldots\right)\right]$$

Now since h is very small, the terms containing h^2 and higher powers are neglected.

$$\therefore \quad x = x_0 e^{-\gamma t}\left(1 + \frac{\gamma}{h} ht\right) = x_0(1 + \gamma t)e^{-\gamma t} \qquad \ldots(1.35)$$

Eq. (1.35) shows that at $t = 0$, $x = x_0$ and as t increases x decreases as shown by curve Fig. 1.15 (b). The motion is neither over damped nor oscillatory and said to be ***critically damped***. In other words, it is a transition stage between dead beat motion and the damped oscillatory motion.

Examples of critical damping: Many pointer-type instruments use critical damping to ensure that the pointer moves and reaches a stationary position quickly. They are called dead beats and are found in electrical meters. In the moving coil galvanometer, the ammeter, and the voltmeter, the current-carrying coil is wound on a metallic frame so that the induced eddy currents in the frame make the motion dead-beat. In the ballistic galvanometer, where a weak damping condition is required, the coil is wound onto a non-metallic frame. Another example is shock absorbers. A shock absorber is a spring with a sealed container of fluid. A bike with a shock absorber compresses the spring when it hits a bump.

Case 3: Oscillatory damped S.H.M.: When $\gamma^2 < \omega_0^2$ (light damping), $\gamma^2 < \omega_0^2 = \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$ is a negative quantity and hence $\sqrt{\gamma^2 - \omega_0^2}$ is an imaginary quantity.

Let $\sqrt{\gamma^2-\omega_0^2} = \omega = \sqrt{\omega_0^2-\gamma^2} = \sqrt{\frac{k}{}-\frac{b}{}}$

The general solution of Eq. (1.35) becomes

$$x = x_0 e^{-\gamma t}\left[\left(\frac{e^{i\omega' t}+e^{i\omega' t}}{2}\right)+\frac{\gamma}{\omega'}\left(\frac{e^{i\omega' t}+e^{i\omega' t}}{2i}\right)\right]$$

$$= x_0 e^{-\gamma t}\left[\cos\omega' t+\frac{\gamma}{\omega'}\sin\omega' t\right] \quad \text{...(1.36)}$$

Now, let $\frac{\gamma}{\omega'} = \cot\phi$

So that $\tan\phi = \frac{\omega'}{\gamma} = \frac{\sqrt{\omega_0^2-\gamma^2}}{\gamma}$

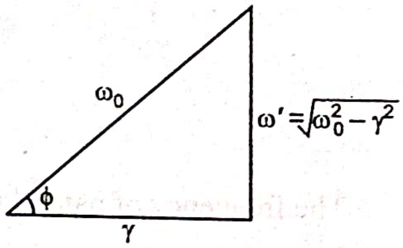


Fig. 1.12: *Oscillatory damped S.H.M*

$\therefore \quad \sin\phi = \frac{\omega'}{\omega_0} = \frac{\sqrt{\omega_0^2-\gamma^2}}{\omega_0}$

Hence, from Eq. (1.36), we get

$$x = x_0 e^{-\gamma t}\left[\cos\omega' t+\frac{\cos\phi}{\sin\phi}.\sin\omega' t\right] = \frac{x_0 e^{-\gamma t}}{\sin\phi}[\sin\phi\cos\omega' t+\cos\phi\sin\omega' t]$$

or $$x = \frac{x_0\omega_0 e^{-\gamma t}}{\omega'}\sin(\omega' t+\phi) = A_0 e^{-\gamma t}\sin(\omega' t+\phi)$$

or $$x = A\sin(\omega' t+\phi) \quad \text{...(1.37)}$$

But, $\omega_0 = \sqrt{\frac{k}{m}}$

$\gamma = \frac{b}{2m}$

and $\omega' = \sqrt{\omega_0^2-\gamma^2} = \sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$ and $A_0 = \frac{x_0\omega_0}{\omega t} = \frac{x_0\omega_0}{\sqrt{\omega_0^2-\gamma^2}}$

From Eq. (1.37), we come to the following conclusions:

(*i*) The motion of the particle is oscillatory, the displacement x varies as a sine curve as shown in Fig. 1.13.

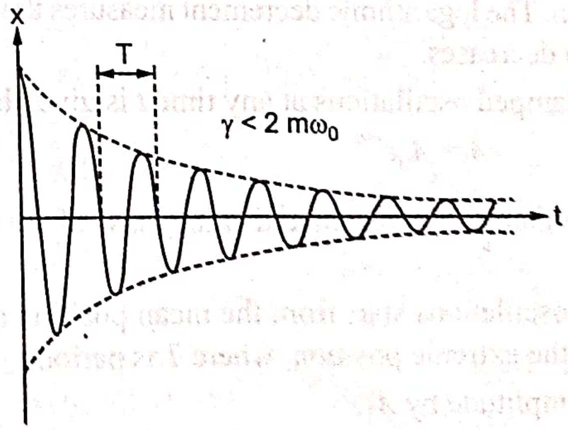


Fig. 1.13: *Damped Oscillatory motion*

(ii) The amplitude of the S.H.M. is

$$A = \frac{x_0\omega_0}{\sqrt{\omega_0^2-\gamma^2}}e^{-\gamma t} \quad ...(1.38)$$

Thus, amplitude A decreases exponentially with time (Fig. 1.13).

Therefore, the motion is called ***damped oscillatory***.

(iii) The periodic time t and the frequency ν of the damped S.H.M. are

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2-\gamma^2}} = \frac{2\pi}{\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}} \quad ...(1.39)$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$$

The frequency of natural undamped oscillations ($\gamma \to 0$) is

$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad ...(1.40)$$

Thus, ν is less than ν_0. The difference depends upon the damping factor b. Larger the value of b, higher is the difference between ν_0 and ν.

Week damping condition represents a simple harmonic motion with amplitude $Ae^{-\gamma/2t}$. The motion differs from the undamped motion. The amplitude of the oscillations is not constant but falls slowly with time over many oscillations, as shown in Fig. 1.13.

Definition: If ω is real, the damping force is weaker than the restoring force, and the oscillations are **underdamped, or weakly damped.**

Note:

(i) A damped oscillation is nearly sinusoidal, and its amplitude diminishes over time.

(ii) Weakly damped systems have angular frequencies near to the system's natural frequency.

(iii) A damped oscillation period is an interval between two consecutive maximum displacements.

(iv) Damped oscillations are practically unaffected by friction.

(v) A pendulum in air and an LCR circuit are examples of damped oscillations.

1.8.3 Logarithmic Decrement

From Fig. 1.13 and Eq. (1.37) it is seen that the amplitude goes on decreasing progressively in damped harmonic motion. The logarithmic decrement measures the rate at which the amplitude of a free damped vibration decreases.

The amplitude of the damped oscillations at any time t is given by

$$A = A_0 e^{-\gamma t}$$

where $\gamma = \frac{r}{2m}$, r being the damping constant and m the mass of the oscillator.

Let us assume that the oscillations start from the mean position and after a time $t = \frac{T}{4}$ the oscillating particle goes to the extreme position, where T is period.

Let us denote the first amplitude by A_1.

Then $$A_1 = A_0 e^{-\gamma T/4}$$

The particle reaches the mean position and then goes to the extreme position on the other side. Again, it comes back to the mean position and goes to the extreme position after a time T *i.e.,* $\left(T + \frac{T}{4}\right)$ from start.

Let the second amplitude be A_2, then $A_2 = A_0 e^{-\gamma\left(T + \frac{T}{4}\right)}$

Similarly, $$A_n = A_0 e^{-\gamma\left[(n-1)T + \frac{T}{4}\right]}$$

Hence $$\frac{A_1}{A_2} = \frac{A_2}{A_3} = \frac{A_3}{A_4} = \ldots\ldots\ldots = \frac{A_{n-1}}{A_n} = e^{\gamma T}$$

Taking natural logarithms (to the base e), we have

$$\log_e \frac{A_1}{A_2} = \log_e \frac{A_2}{A_3} = \ldots\ldots\ldots = \gamma T = \lambda(\text{say}) \tag{1.41}$$

Where λ is called the **logarithmic decrement** and from which we can find the value of A_0.

Definition: If two successive amplitudes of an oscillation are separated by one full-time period, the ***logarithmic decrement*** is the ratio between their amplitudes.

1.8.4 Energy of the Damped Oscillator

(*i*) Kinetic energy (*K*)

The displacement of a damped harmonic oscillator is $x(t) = Ae^{-\gamma t}\sin(\omega' t + \phi)$

The instantaneous velocity is

$$u = \frac{dx}{dt} = Ae^{-\gamma t}[-\gamma\sin(\omega' t + \phi) + \omega'\cos(\omega' t + \phi)] = Ae^{-\gamma t}\omega'\cos(\omega' t + \phi)$$

The approximation is done as $\gamma << \omega_0$.

Thus, kinetic energy is $K = \frac{1}{2}mu^2 = \frac{1}{2}mA^2 e^{-2\gamma t}\omega'^2\cos(\omega' t + \phi)$

(*ii*) Potential energy (*U*)

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 e^{-2\gamma t}\sin^2(\omega' t + \phi)$$

(*iii*) Total energy (*E*)

$$E = K + U = \frac{1}{2}mA^2 e^{-2\gamma t}\omega'^2\cos^2(\omega' t + \phi) + \frac{1}{2}kA^2 e^{-2\gamma t}\sin^2(\omega' t + \phi)$$

$$\therefore \quad E = \frac{1}{2}kA^2 e^{-2\gamma t}$$

$\because \gamma << \omega_0$ thus $\omega' = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0 = \sqrt{\frac{k}{m}}$ ...(1.42)

This shows that the energy of the oscillator decreases with time, the exponential decay of energy is shown below in Fig. 1.14.

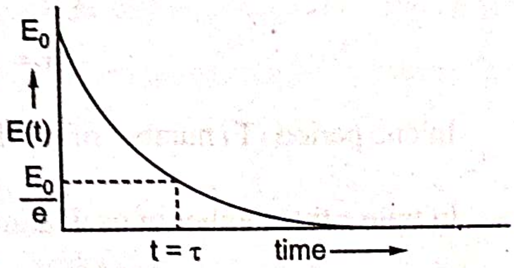


Fig. 1.14: Exponential decay of energy

1.8.5 Power Dissipation

It is the rate at which the energy is lost. Thus,

$$P = -\frac{dE}{dt} = -\frac{d}{dt}\left(\frac{1}{2}kA^2e^{-2\gamma t}\right) = 2\gamma\left(\frac{1}{2}kA^2e^{-2\gamma t}\right) = 2\gamma E \quad ...(1.43)$$

1.8.6 Relaxation Time

Relaxation time is the time taken for the total energy E to decay to $\frac{1}{e}$ of its initial value E_0.

Let τ be the relaxation time, then at $t = \tau, E = \frac{E_0}{\tau}$.

$$\therefore \quad \frac{E_0}{\tau} = E_0e^{-2\gamma\tau}$$

Thus, relaxation time
$$\tau = \frac{1}{2\gamma} \quad ...(1.44)$$

Eq. (1.43) may be expressed as $P = \frac{E}{\tau}$.

1.8.7 Quality Factor (Q)

Definition: Quality factor is defined as 2π times the ratio of the energy stored in the system to the energy lost per period.

$$\text{Quality Factor, } Q = 2\pi\frac{\text{Energy stored in system}}{\text{Energy loss per pariod}}$$

Energy stored in the system is E while the energy lost per period is PT.

$$\therefore \quad Q = 2\pi\frac{E}{PT} = 2\pi\frac{R}{\left(\frac{E}{\tau}\right)T} = \frac{2\pi\tau}{T} = \omega_0\tau \quad ...(1.45)$$

$$\frac{\omega_0\tau}{Q} = 1,$$

$$\tau = \frac{Q}{\omega_0} = \frac{TQ}{2\pi}$$

Thus, the energy falls to $\frac{1}{e}$ of its original value after $n = \frac{Q}{2\pi}$ cycles of free oscillation.

- It means that Q is related to the number of oscillations over which the energy fall to $\frac{1}{e}$ of its original value E_0, which is also called the ***relaxation time***.

This happens in time, $t = \tau$, where τ is given by

$$\frac{\omega_0\tau}{Q} = 1,$$

$$\tau = \frac{Q}{\omega_0} = \frac{TQ}{2\pi}$$

In one period (T) number of oscillations is = 1

In time τ the number of oscillations = n, then $n = \frac{\tau}{T} = \frac{Q}{2\pi}$ ...(1.46)

Thus, the energy falls to $\frac{1}{e}$ of its original value after $n = \frac{Q}{2\pi}$ cycles of free oscillation.

1.8.8 Relation between Relaxation Time, Mean Time and Quality Factor

Let N be the number of oscillations in (mean) time $\tau_m = \frac{1}{\gamma}$,

while n is the number of oscillations in (relaxation) time $\tau = \frac{1}{2\gamma}$.

$$\therefore \qquad \tau_m = 2\tau$$

In one period (T) number of oscillations is = 1

In time τ_m the number of oscillations = N

$$\therefore \qquad N = \frac{\ }{T} = \frac{\ }{T} = \frac{\ }{\ }$$

or

$$N = 2n$$

Example 1.7: *A 10.6 kg object oscillates at the end of a vertical spring. The spring constant is 2.05×10^4 N/m. If the effect of air resistance is represented by damping co-efficient b = 3 N s/m, what is the frequency of the damped oscillator.*

Solution:

Given that $m = 10.6$ kg, $k = 2.05 \times 10^4$ N/m and $b = 3$ N s/m

The frequency of undamped oscillator, $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.05 \times 10^{-4}\, N/m}{10.6\, kg}} = 44$ Hz

The angular frequency of damped oscillator is $\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$

$$= \sqrt{(44)^2 - \left(\frac{3}{2 \times 1.06}\right)^2} = \sqrt{1933.96 - 0.02} = 44 \text{ s}^{-1}$$

$$\therefore \quad \text{Frequency} \qquad \nu = \frac{\omega}{2\pi} = \frac{44}{2\pi} = 7 \text{ Hz.}$$

Example 1.8: *A pendulum with a length of 1.0 m is released from an initial angle of 15° and its amplitude is given by $A = A_0 e^{-\gamma t}$. Find the value of γ if the pendulum amplitude decreases by 5.5° due to friction after 1000 seconds?*

Solution:

Given that $\theta_0 = 15°$, $t = 1000$ sec, $\theta = 5.5°$, and $A = A_0 e^{\gamma t}$

$$\frac{A}{A_0} = \frac{A_0 e^{-\gamma 1000}}{A_0} = \frac{5.5}{15} \qquad \therefore \ln\left(\frac{5.5}{15}\right) = -1000\gamma$$

$$\therefore \qquad \gamma = 0.001 \text{ s}^{-1}$$

Example 1.9: *The Q-value of an under-damped harmonic oscillator of frequency 480 Hz is 80000. How many oscillations does it perform in the time in which its amplitude decays to $\frac{1}{e}$ of its initial value?*

Solution: Let τ_m be the mean time in which amplitude decays to $\frac{1}{e}$ of initial value.

Then, the number of oscillations in time τ_m is $N = \frac{Q}{\pi} = \frac{80000}{\pi} = 25464.8$

1.9 FORCED OSCILLATIONS

Definition: Forced oscillations are oscillations in which a body oscillates at a frequency different from its natural frequency under the influence of an external periodic force.

To illustrate, we could drag a swing to a certain height, then let it go, or we could repeatedly push the swing at any frequency. There are two frequencies here: the *natural frequency* ω_0 of the free oscillations and the *driving frequency* ω_f of the forced oscillations.

1.9.1 Equation for Forced Harmonic Oscillator

Let us consider a mass m oscillating under external periodic force F. Let x be the displacement at any instant. The forces that act on the body undergoing forced oscillations are as follows:

(*i*) **A restoring force** F_1, that is proportional to the displacement and oppositely directed;

$$F_1 = -kx.$$

(*ii*) **A damping force** F_2, that is proportional to the velocity but oppositely directed;

$$F_2 = -bv = -b\frac{dx}{dt}.$$

A damping force with magnitude proportional to velocity is called as a *linear damping force*.

(*iii*) **A driving force** (external periodic force), $F_3 = F_o \sin \omega_f t$. A driving force that depends sinusoidally on time is known as a *harmonic driving force*.

The total force acting on the body is given by $F = -kx - b\dfrac{dx}{dt} + F_o \sin \omega_f t$

According to Newton's second law this force must be equal to the product of the mass and acceleration. Therefore, the equation of the motion of the body is

$$ma = -kx - b\frac{dx}{dt} + F_o \sin \omega_f t$$

Rearranging the terms in the above equation, we get

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega_f t \qquad \text{...(1.47)}$$

Designating $\dfrac{b}{m} = 2\gamma, \dfrac{k}{m} = \omega_0^2$ and $\dfrac{F_0}{m} = f_0$, we write Eq. (1.47) as

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = f_0 \sin \omega_f t \qquad \text{...(1.48)}$$

Eq. (1.48) is the differential equation of the forced harmonic oscillator.

1.10 TRANSIENT AND STEADY-STATES

1.10.1 Transient States

Initially, the motion is complex. A driving frequency forces the system to vibrate with its natural frequency ω_0. As a result, the motion is initially a superposition of free damped oscillations and forced oscillations. This initial motion is termed transitory. It is also known as a **transitory state**. However, the transitory behavior decays exponentially.

Solving Eq. (1.48) gives the displacement of the oscillations by putting the external force $f_o \sin \omega_f t = 0$.

$$\therefore \quad \frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \text{...(1.49)}$$

The oscillatory solution of Eq. (1.49) is

$$x = A_0 e^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}\, t + \phi\right)$$

$$= A_0\, e^{-\gamma t} \sin(\omega' t + \phi) \qquad \text{...(1.50)}$$

where $$\omega' = \sqrt{\omega_0^2 - \gamma^2},\ A_0 = \frac{x_0\omega_0}{\sqrt{\omega_0^2 - \gamma^2}} \text{ and } \tan\phi_0 = \frac{\sqrt{\omega_0^2 - \gamma^2}}{\gamma} \qquad \text{...(1.51)}$$

The amplitude $A_0 e^{-\gamma t}$ decays with time due to the presence of the term $e^{-\gamma t}$. The quantity ω_0 is known as the *natural frequency* of the oscillator in the absence of dissipative as well as driving forces.

1.10.2 Steady-state

Once the free oscillations are over, only the forced oscillations remain, and the motion reaches a **steady state**. The oscillator oscillates at a constant amplitude. **Steady state part** gives the final behavior of the oscillator and is obtained from the trail method. The system will oscillate with the frequency of the impressed force when the transient effect dies away.

The solution of the ***steady state part*** can be given as

$$x = A \sin(\omega_f t - \phi) \qquad \text{...(1.52)}$$

where A is the amplitude under external force and ϕ is the phase difference between deriving force and the resulting displacement of the system.

Now from Eq. (1.46), we get

$$\frac{dx}{dt} = A\omega_f \cos(\omega_f t - \phi)$$

$$\frac{d^2x}{dt^2} = -\omega_f^2 A \sin(\omega_f t - \phi)$$

Using these values in Eq. (1.48)

$$-\omega_f^2 A \sin(\omega_f t - \phi) + 2\gamma A\omega_f \cos(\omega_f t - \phi) + \omega_0^2 A \sin(\omega_f t - \phi) = f_0 \sin[(\omega_f t - \phi) + \phi]$$

$$A(\omega_0^2 - \omega_f^2)\sin(\omega_f t - \phi) + 2\gamma A\omega_f \cos(\omega_f t - \phi) = f_0 \sin(\omega_f t - \phi)\cos\phi + f_0 \cos\phi\,(\omega_f t - \phi)\sin\phi$$

Above equation holds good for all values of t. We may equate the coefficients of sine and cosine of $(\omega_f t - \phi)$.

Thus, we get

$$A(\omega_0^2 - \omega_f^2) = f_0 \cos\phi \qquad \text{...(1.53)}$$

$$2\gamma\omega_f A = f_0 \sin\phi \qquad \text{...(1.54)}$$

Squaring and adding Eq. (1.53) and Eq. (1.54), we get

$$A^2\left[\left(\omega^2 - \omega_f^2\right)^2 + (2\gamma\omega_f)^2\right] = f_o^2$$

On simplification we get

$$A = \frac{f_0}{\sqrt{\left(\omega^2 - \omega_f^2\right)^2 + \left(2\gamma\omega_f\right)^2}}$$

$$\therefore \qquad A = \frac{F_o/m}{\sqrt{\left(\omega^2-\omega_f^2\right)^2+\left(2\gamma\omega_f\right)^2}} \qquad \text{...(1.55)}$$

Eq. (1.55) indicates that the forced oscillations are not damped but have a constant amplitude. Essentially, the external agent overcomes the damping forces and provides the energy necessary to maintain the oscillations.

Dividing Eq. (1.54) by Eq. (1.53), we get

$$\tan\phi = \frac{2\gamma\omega_f}{\left(\omega_0^2-\omega_f^2\right)^2} \qquad \text{...(1.56)}$$

$$x = \frac{F_o/m}{\sqrt{\left(\omega_0^2-\omega_f^2\right)+(2\gamma\omega_f)^2}}\sin(\omega_f t-\phi) \qquad \text{...(1.57)}$$

1.10.3 Complete Solution

The complete solution of Eq. (1.48) is

$$x = \frac{F_o/m}{\sqrt{\left(\omega_0^2-\omega_f^2\right)+(2\gamma\omega_f)^2}}\sin(\omega_f t-\phi)+A_o e^{-\gamma t}\sin(\omega' t+\phi) \qquad \text{...(1.58)}$$

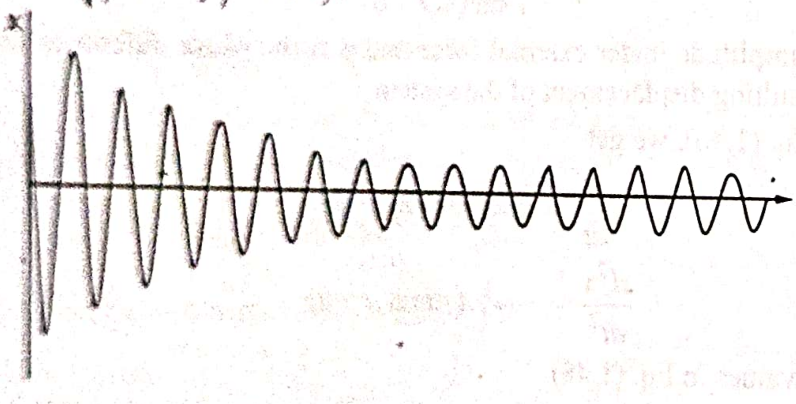


Fig. 1.15: Steady-state condition

Table 1.1 gives the distinction between free and forced oscillations.

Table 1.1: Distinction between free and forced oscillations

S. No.	Free oscillations	Forced oscillations
1.	Free oscillations occur when a body executes oscillations without being acted upon by external forces, and it occurs as a result of elastic forces and inertia.	Forced oscillations occur due to the action of a periodic force applied externally.
2.	Free oscillations decrease gradually as a result of damping forces.	Forced oscillations continue provided that the applied periodic force acts on the body.
3.	The frequency of free oscillations depends on the mass, and elasticity of the body.	The frequency of forced oscillations is equal to the frequency of the applied periodic force.

1.11 RESONANCE

Each driving frequency has its own amplitude. As the driving frequency increases, ω_f the amplitude increases until it reaches a maximum at ω_{max}. At higher frequencies, the amplitude gradually decreases. The oscillation amplitude increases when the frequency of the driving force is near the natural frequency ω_o of the oscillating system. The frequency of free oscillations depends on mass and elasticity.

Definition: The raise in amplitude near the natural frequency is called ***resonance*** and the frequency ω_0 is called the ***resonance frequency*** of the system.

Oscillations of large amplitude occur when the rate of energy transfer from the applied force to the forced oscillator is maximum. When a system is subjected to an external action that varies periodically with time and frequency, resonance occurs. The external force and the particle velocity are in phase at the resonance frequency. Thus, the power transfer to the oscillator is at its maximum. At frequencies above or below the resonance value, force and velocity are not in phase, resulting in a lower power transfer.

1.11.1 Amplitude Resonance

Based on the constants f_o and ω_f of the driving force, as well as the constants ω_0 and γ of the oscillator, the amplitude A of the forced oscillator depends. At some driving frequency amplitude, A becomes maximum, which is called amplitude resonance.

The following three different cases arise:

Case 1: At very ***low driving frequency*** ($\omega_f << \omega_0$), the driving force frequency is very much less than the natural frequency of oscillations. The amplitude A turns out to be

$$A = \frac{F_o/m}{\sqrt{\left[\left(\omega_0^2 - \omega_f^2\right)^2 + (2\gamma\omega_f)^2\right]}} = \frac{F_o/m}{\omega_o^2} = \frac{F_o/m}{k/m} = \frac{F_o}{k} \quad \text{...(1.59)}$$

- Eq. (1.59) shows that the amplitude, A depends only on the force constant, k and is independent of mass, damping, and driving frequency.

Case 2: At very ***high frequency*** ($\omega_f >> \omega_0$), the driving force frequency is very much greater than the natural frequency of oscillations.

We get
$$A = \frac{F_o/m}{\sqrt{\left[\left(\omega_0^2 - \omega_f^2\right)^2 + (2\gamma\omega_f)^2\right]}} = \frac{F_o/m}{\sqrt{\omega_f^4 + 4\gamma^2\omega_f^2}} = \frac{F_o/m}{\omega_f^2} = \frac{F_o}{m\omega_f^2} \quad \text{...(1.60)}$$

$4\gamma^2\omega_f^2$ can be neglected in comparison to ω_f^4 if the damping γ is very small.

- Eq. (1.60) shows that at very ***high frequency***, amplitude A depends upon mass and driving frequency.

Case 3: When ***driving frequency is equal to the natural frequency*** ($\omega_f = \omega_0$), amplitude will be a maximum. It is the case of ***amplitude resonance***.

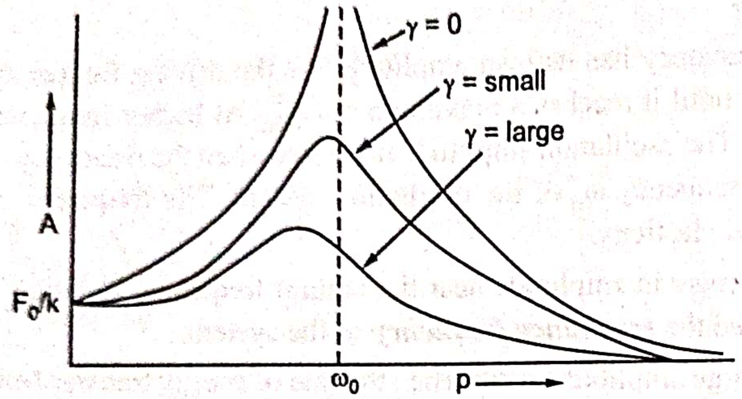


Fig. 1.16: Amplitude Resonance

At this stage the amplitude is given by

$$A = \frac{F_0/m}{\sqrt{\left[\left(\omega_0^2-\omega_f^2\right)^2+(2\gamma\omega_f)^2\right]}} = \frac{F_0/m}{2\gamma\omega_f} \quad \ldots(1.61)$$

- Eq. (1.61) shows that maximum amplitude is inversely proportional to the damping constant, γ.

1.11.2 Velocity Resonance

The velocity of the body depends on the constant f_0 and ω_0 of the driving force and the constant ω_0 and γ of the oscillator. A ***velocity resonance*** occurs when the amplitude of velocity reaches a maximum at a certain frequency.

The instantaneous velocity of the body is

$$u = \frac{dx}{dt} = \frac{f_0\omega_f}{\sqrt{\left(\omega_0^2-\omega_f^2\right)^2+4\gamma^2\omega_f^2}}\cos(\omega_f t-\theta).$$

When $\cos(\omega_f t-\theta) = 1$, the velocity will be maximum and this maximum value is known as "velocity amplitude" u_0.

Thus
$$u_0 = \frac{f_0\omega_f}{\sqrt{\left(\omega_0^2-\omega_f^2\right)^2+4\gamma^2\omega_f^2}} \quad \ldots(1.62)$$

Basing on driving frequency, following cases arise.

Case1: At very low driving frequency ($\omega_f \ll \omega_0$):

The velocity amplitude, $u_0 = \frac{f_0\omega_f}{\omega_0^2} = \frac{F_0\omega_f}{m\omega_0^2} = \frac{F_0\omega_f}{k}$

- At very low driving frequency the velocity amplitude depends on the spring constant k.

Case 2: At very high frequency ($\omega_f \gg \omega_0$):

The velocity amplitude

$$u_0 = \frac{f_0}{\omega_f}$$

$$= \frac{F_0/m}{\omega_f} = \frac{F_0}{m\omega_f}$$

- At very high driving frequency the velocity amplitude depends on mass and driving frequency.

Case 3: Velocity resonance frequency:

At certain frequency the velocity amplitude becomes maximum, and that frequency is called the *velocity resonance frequency*.

The velocity amplitude

$$u_0 = \frac{f_0\omega_f}{\sqrt{\left(\omega_0^2-\omega_f^2\right)^2+4\gamma^2\omega_f^2}} = \frac{f_0}{\sqrt{\left[\dfrac{\omega_0^2-\omega_f^2}{\omega_f}\right]^2+4\gamma^2}}$$

The velocity amplitude u_0 will be maximum when the denominator $\sqrt{\left[\dfrac{\omega_0^2-\omega_f^2}{\omega_f}\right]^2+4\gamma^2}$ is minimum, i.e., when

$$\left[\frac{\omega_0^2-\omega_f^2}{\omega_f}\right] = 0 \quad \text{when} \quad \omega_f = \omega_0$$

- This shows that the velocity resonance always occurs when driving frequency is equal to the natural undamped frequency of the body.

The velocity amplitude at the resonance is $u_0 = \dfrac{f_0}{2\gamma} = \dfrac{F_o/m}{b/m} = \dfrac{F_0}{b}$ ...(1.63)

This shows that velocity amplitude at resonance only depends on the damping constant. Dependence of the velocity of a forced oscillator on the driving frequency is shown below in Fig. 1.17.

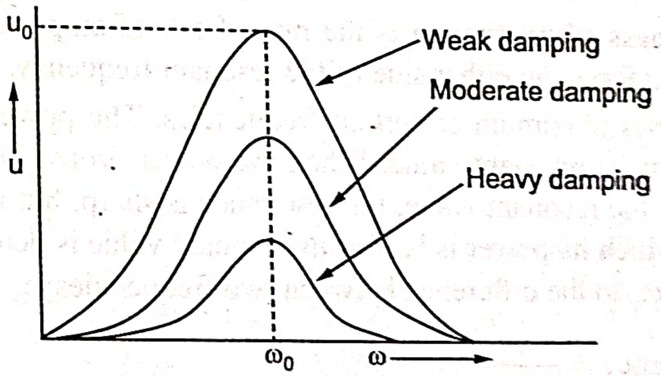


***Fig. 1.17:** Dependence of the velocity of a forced oscillator on the driving frequency*

Note:

- When damping is small, the amplitude of forced oscillations grows with increasing ω_f, and at $\omega_f = \omega_0$, the amplitude of the oscillations becomes equal to infinity. Also, the resonance curve has a sharp drop-off in amplitude, which is on either side of the resonance frequency.
- When damping is large, the amplitude reduces very slowly on either side of the resonant frequency.
- The resonance is flat if the oscillator responds to several nearby frequencies close to the resonant value.
- At resonance, the velocity is in phase with the applied force. Since the rate of work done on oscillator by the applied force is $F\upsilon$, this quantity is positive when F and υ are in phase and represents a favorable condition for transfer of energy to the oscillator.
- At resonance, the oscillating system absorbs energy from the external source.

Examples of the Phenomenon of Resonance:

(*i*) When tuning a radio receiver, the tuned circuit must match the frequency of the waves. We can hear clearly only when the resonance condition is met for a particular radio station.

(*ii*) Atoms absorb radiation in resonance when the frequency of the incident light waves matches their natural frequency.

(*iii*) In a cyclotron, particles are accelerated to high energies only if the frequency of the electric field accelerating the particles is equal to the frequency of the particle's revolution in a magnetic field perpendicular to the particle path.

(*iv*) During the march of soldiers across a bridge in St. Petersburg, the bridge collapsed due to resonance. A period of free oscillation of the bridge occurred synchronously with a period of marching and resonance, resulting in the swinging of the bridge with very high amplitude. As a result, the bridge collapsed.

(*v*) Another example was the Tacoma Narrows Bridge in Washington State. In 1940 Wind blowing through the Tacoma Narrows whipped up air puffs that shook the bridge at a frequency matching one of its natural vibrational frequencies. Over the next few hours, the amplitude grew to such an extent that the centre span collapsed due to resonance.

1.12 SHARPNESS OF RESONANCE

As the driving force increases or decreases from the resonant frequency, the amplitude drops from the maximum value. If there is no damping, there is no dissipation of energy. A real oscillator, however, loses energy due to frictional damping forces. The driving force must restore this lost energy. It must absorb power equal to the rate at which it dissipates energy to maintain its motion.

Definition: **Sharpness of resonance** is the rate of fall of amplitude with the change in frequency of the driving force, on either side of the resonant frequency.

The average power is maximum at certain frequencies. The power drops as the driving frequency deviates from its resonant value. When the power drops rapidly with a change in driving frequency from the resonant value, the resonance is sharp, but if the fall is small, it is flat. The frequency at which its power is half of its resonant value is determined by comparing the resonant frequency ω_0 to the difference between two frequencies ω_1 and ω_2 (Fig. 1.18).

$$\text{Sharpness of Resonance} = \frac{\omega_0}{\omega_2 - \omega_1} \qquad \text{...(1.64)}$$

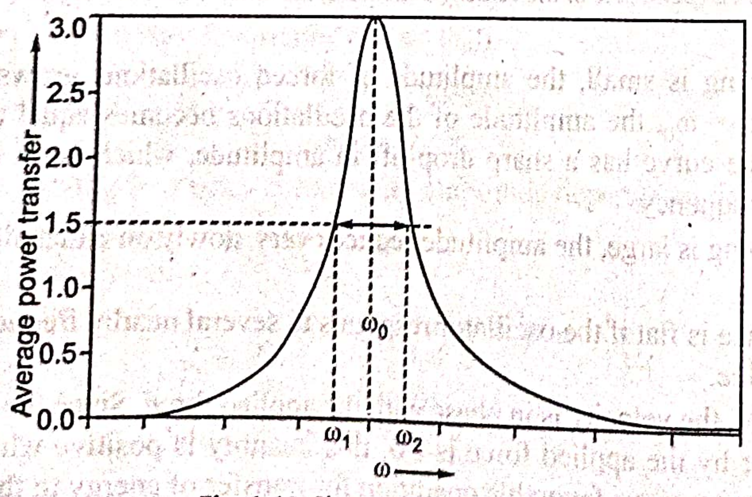


Fig. 1.18: Sharpness of Resonance

1.13 POWER DISSIPATION

By doing work against the damping force, oscillators absorb energy from the driving force. The instantaneous power P absorbed by the oscillator equals the energy per unit time.

$$P_{ab} = \frac{\text{energy}}{\text{time}} = \frac{F \cdot dx}{dt} = F \cdot \frac{dx}{dt}$$

Instantaneous power absorbed P_{ab}, therefore, equals the product of the instantaneous driving force and the instantaneous velocity.

So,
$$P_{ab} = F_0 \sin(\omega_f t)\frac{dx}{dt} = \frac{m f_0^2 \omega_f}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2}} \sin(\omega_f t).\cos(\omega_f t - \phi)$$

$$= \frac{m f_0^2 \omega_f}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2}} \sin\omega_f t \cos\omega_f t - \phi) + \sin^2\omega_f t \sin\phi$$

Now the average power absorbed is
$$\langle P_{ab}\rangle = \frac{m f_0^2 \omega_f}{\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2}\left(\frac{1}{2}\sin\phi\right) \qquad \ldots(1.65)$$

Note: The average values of the periodic function for one period $T = \frac{2\pi}{\omega_f}$ are

$$\frac{1}{T}\int_0^T \sin\omega_f t \cos\omega_f t \, dt = 0 \quad \text{and} \quad \frac{1}{T}\int_0^T \sin\omega_f^2 t dt = \frac{1}{2}$$

Since,
$$\tan\theta = \frac{2\gamma\omega_f}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 4\gamma^2\omega_f^2}}$$

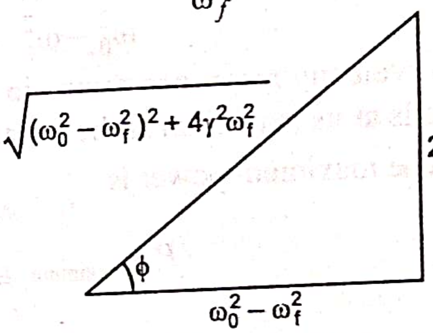


Fig. 1.19: Calculation of tan θ

The average power absorbed by the oscillator (average power supplied by the driving force) is

$$\langle P_{ab}\rangle = \frac{m f_0^2 \gamma\omega_f}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2}} \qquad \ldots(1.66)$$

By moving against the force of friction, the system can dissipate the power supplied by the driving force.

The instantaneous power dissipated through friction is given by

$$P_{\text{dis}} = |\text{instantaneous frictional force}| \times |\text{instantaneous velocity}|$$

$$= b\frac{dx}{dt} \times \frac{dx}{dt} = 2m\gamma\left(\frac{dx}{dt}\right)^2 = 2m\gamma\frac{f_0^2\omega_f^2}{\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + 4\gamma^2\omega_f^2}}\cos^2(\omega_f t - \theta)$$

Now the average of $\cos^2(\omega_f t - \phi)$ for one full period is $\frac{1}{2}$.

Therefore, the average power dissipated is

$$\langle P_{dis}\rangle = \frac{mf_0^2\gamma\omega_f}{\sqrt{\left(\omega_0^2-\omega_f^2\right)^2+4\gamma^2\omega_f^2}} \qquad \text{...(1.67)}$$

Thus,
$$\langle P_{ab}\rangle = \langle P_{dis}\rangle$$

- The driving force provides the same average power as the frictional force dissipates in the steady-state.
- Input power and dissipation power are not the same. The oscillator's power is not constant at any point in time.

Maximum Power Absorption

The average power can also be written as

$$\langle P_{ab}\rangle = \frac{mf_0^2\gamma\omega_f}{\sqrt{\left[\frac{\omega_0^2-\omega_f^2}{\omega_f}\right]^2+4\gamma^2}}$$

$\langle P_{ab}\rangle$ will be maximum when the denominator $\left[\frac{\omega_0^2-\omega_f^2}{\omega_f}\right]^2+4\gamma^2$ is a minimum, this occurs when

$$\omega_0^2-\omega_f^2=0 \quad \text{or} \quad \omega_f=\omega_0.$$

A velocity resonance occurs in this situation. Thus, the power transferred from the driving force is at its maximum at the frequency of velocity resonance.

The maximum power is

$$\langle P_{in}\rangle_{maximum} = \frac{mf_0^2}{4\gamma} \qquad \text{...(1.68)}$$

1.14 QUALITY FACTOR

The quality factor (Q factor) is a dimensionless parameter quantifying the quality of an oscillator. The less the damping the higher is the Q factor.

The quality factor is defined as $Q=\frac{\omega_0}{\omega_2-\omega_1}=\frac{\text{Resonance frequency}}{\text{bandwidth}}=\frac{\omega_0}{2\gamma}=\omega_0\tau$ ...(1.69a)

Quality factor also defined as

$$Q=2\pi\frac{\text{average energy stored in one period}}{\text{average energy lost in one period}}=2\pi\frac{E}{\langle P_{dis}\rangle T} \qquad \text{...(1.69b)}$$

where $\langle P_{dis}\rangle$ is the average power dissipated and T is the time period of oscillation.

Thus $\langle P_{dis}\rangle\times T$ is the average energy lost in one period. On solving for energy, we get the following expression of the quality factor.

$$Q=\frac{\omega_f^2+\omega_0^2}{4\gamma\omega_f}=\frac{1}{2}\left(\frac{\omega_f^2+\omega_0^2}{\omega_f^2}\right)(\omega_f\tau)$$

This is the exact expression for the quality factor of forced oscillator.

Near resonance, $\omega_f = \omega_0$, and so $Q = \frac{1}{2}\left(\frac{2\omega_0^2}{\omega_0^2}\right)(\omega_0\tau)$

Therefore, we find that $Q = \omega_0\tau$.

Example 1.10: *Find the amplitude of motion of spring with a force constant of 20 N/m attached to a 2.0 kg object and driven by a force F = (3.0N) sin (2πt). Assume that spring move without friction.*

Solution:

Given that $m = 2.0$ kg, $F = (3.0N)\sin(2\pi t)$ and $k = 20\frac{N}{m}, b = 0$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{2}} = 3.10 \text{ rad s and } \omega_f = \frac{2\pi}{T} = 2\pi \text{ rad/s}$$

The amplitude of the driven oscillator is $A = \frac{F_c}{m}\left(\omega_f^2 - \omega_0^2\right)^{-1} = \frac{3}{2}\left[4\pi^2 - (3.16)^2\right]^{-1}$

$$= 0.0509 \text{ m} = 5.09 \text{ cm.}$$

Example 1.11: *The spring extends 2.5 cm when a mass of 2 kg is hung from a spring of negligible mass and a spring constant of 800 N/m. Spring amplitude is 2 mm, and the damping constant is 0.5 s^{-1}. What is the amplitude of forced oscillations at $\omega_f = \omega_0$?*

Solution:

Given that $m = 2$kg, $k = 800$ N/m, displacement amplitude = 2 mm, $\gamma = 0.5$

The Amplitude of forced oscillation is $A = \dfrac{\frac{F_o}{m}}{\sqrt{\left(\omega_0^2 - \omega_f^2\right) + 4\gamma^2\omega_0^2}}$

At $\omega_f = \omega_0$, we get $A = \dfrac{\frac{F_o}{m}}{2\gamma\omega_0}$.

F_0 = spring constant × displacement amplitude = (800 N/m) (2 × 10^{-3}m) = 1.6N

and $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{800N/m}{2}} = 20$ rad/s.

$$\therefore \quad A = \frac{\frac{1.6}{2}}{2\times 0.5\times 20} = \frac{0.8}{20} = 0.04 \text{ m.4} = 4 \text{ cm.}$$

EVALUATE YOURSELF

I. Multiple Choice Questions

1. When an object moves back and forth repeatedly around a mean position, it is said to

   (*a*) Oscillate (*b*) revolve

   (*c*) rotate (*d*) motion

2. A ______ vibration has a reduced amplitude with each cycle of vibration

(a) Free
(b) forced
(c) damped
(d) none

3. In damped vibrations if x_1 and x_2 are the successive values of the amplitude on the same side of the mean position, then the logarithmic decrement is equal to

(a) x_1/x_2
(b) $\log (x_1/x_2)$
(c) $\log_e (x_1/x_2)$
(d) $\log (x_1.x_2)$

4. The degree of damping ______ with decreasing amplitude of resonant vibrations

(a) increases
(b) remains same
(c) decreases
(d) varies

5. Oscillations become damped due to

(a) Normal force
(b) friction
(c) tangential force
(d) parallel force

6. In S.H.M, object's acceleration depends upon

(a) Displacement from equilibrium position
(b) magnitude of restoring force,
(c) both (a) and (b)
(d) force exerted on it

7. Angular frequency of S.H.M. is equal to

(a) 2π
(b) $2\pi f$
(c) $2\pi/f$
(d) $2f$

8. For a resonating system it should oscillate

(a) bound
(b) only for some time
(c) freely
(d) for infinite time

9. If a small block oscillates on a smooth concave surface of radius R, the period of the small oscillation is

(a) $T = 2\pi \sqrt{(R/g)}$,
(b) $T = 2\pi \sqrt{(2R/g)}$
(c) $T = 2\pi \sqrt{(R/2g)}$
(d) None of these

10. When two mutually perpendicular S.H.M. of same frequency, amplitude and phase are superimposed gives

(a) Circular motion
(b) a linear S.H.M.
(c) elliptical motion
(d) None.

...ates S.H.M. with a frequency f. The frequency with which it's KE oscillates is

(*a*) $f/2$
(*b*) f
(*c*) $2f$
(*d*) $4f$

12. When the amplitude of a simple pendulum is decreased by 5%, what will be the percentage change in its time period?

(*a*) 6 %
(*b*) 3 %
(*c*) 1.5 %,
(*d*) 0 %

13. During one complete oscillation, a simple pendulum string performs ______ work.

(*a*) K.E+P.E of the pendulum
(*b*) K.E only
(*c*) P.E only
(*d*) Zero

14. Suppose a damped harmonic oscillator completes 100 oscillations with an amplitude of 1/3 of its initial value. What will the amplitude of the oscillator be upon completion of 200 oscillations?

(*a*) 1/5
(*b*) 2/3
(*c*) 1/6
(*d*) 1/9

15. Two SHMs are given by $y_1 = a\sin[(\pi/2)\,t + \phi]$ and $y_2 = b\sin\left[\left(\frac{2\pi t}{3}\right)+\psi\right]$. The phase difference between these after 1s is:

(*a*) $\pi/6$
(*b*) $\pi/2$
(*c*) $\pi/4$
(*d*) π

16. Which one of the following expressions correctly represents forced oscillations?

(*a*) $\frac{d^2x}{dt^2}+\omega^2x=0$
(*b*) $\frac{d^2x}{dt^2}=0$
(*c*) $\frac{d^2x}{dt^2}+2K\frac{dx}{dt}+\omega^2x=0$
(*d*) $\frac{d^2x}{dt^2}+2K\frac{dx}{dt}+\omega^2x=F\sin\omega_f t$

17. A spring oscillating in water is acted upon by an external force $b\cos\omega t$. After some time the frequency of the spring:

(*a*) Greater than ω
(*b*) less than ω
(*c*) Equal to ω
(*d*) decreasing exponentially

18. A damped simple harmonic oscillator of frequency f_1 is constantly driven by an ...l periodic force of frequency f_2. At the steady state, the oscillator frequency

19. In the case of forced simple harmonic vibrations, the body generally vibrates with:

(*a*) Natural frequency

(*b*) large amplitude

(*c*) the frequency of the external force with a small amplitude

(*d*) None

20. The maximum amplitude in the case of a forced oscillator occurs at the

(*a*) Natural frequency

(*b*) Frequency of the force

(*c*) Greater than the natural frequency

(*d*) Frequency less than the natural frequency of the oscillator

21. If the differential equation given by $\frac{d^2x}{dt^2}+2K\frac{dx}{dt}+\omega^2x=F_0\sin\omega_f t$ describe the oscillatory motion of body in a dissipative medium under the influence of a periodic force, then the state of maximum amplitude of the oscillation is a measure of:

(*a*) free vibration (*b*) damped vibration

(*c*) forced vibration (*d*) resonance

II. Fill in the Blanks

1. Any motion, which repeats itself at regular intervals according to a sinusoidal law, ______

2. The oscillations of a simple pendulum under a restoring force are directly proportional to ______

3. ______ is a non-uniformly accelerated motion.

4. In S.H.M since force is proportional to the displacement, the ______ is not constant but varies with time.

5. The restoring force F acting on the body is due to the stiffness of the spring and is given by ______ law.

6. In Hooks law, the negative sign indicates that the force F is opposite to the displacement.

7. ______ is the differential equation of simple harmonic motion.

8. ______ represents the state of the oscillation of the body by specifying the position and direction of motion of the body.

9. All simple harmonic oscillations are characterized by ______ potential well.

10. Forced oscillations are the oscillations in which the body oscillates with a frequency other than its __________ under the action of an external periodic force.
11. Forced oscillations occur due to the action of a ________ applied externally.
12. The amplitude of forced oscillations is small except in the vicinity of ________ frequency.
13. When damping is large, the _________ falls off very slowly on either side of the resonant frequency.
14. The frequency difference between two half-power points is called the _____ of the oscillator.

III. Short Answer Question

1. What is the criterion for the motion to be simple harmonic?
2. What are the two types of simple harmonic motion?
3. Explain linear S.H.M.
4. Explain angular S.H.M.
5. Define transient and steady states.
6. Discuss the concept of resonance.
7. Give some examples of simple harmonic oscillation.
8. Explain natural frequency?
9. Give reason for the energy dissipation in the case of a damped harmonic oscillator.
10. Why are damping devices often used on machinery? Give an example.
11. What are forced vibrations? Give two examples.
12. Why are the forced oscillations of a damped oscillator not damped?
13. Give some examples of common phenomena in which resonance plays an important role.
14. Define power dissipation.
15. Explain Quality Factor.

IV. Long Answer Questions

1. Derive the equation of simple harmonic oscillation and obtain its solution.
2. Define Kinetic energy, potential energy, total energy, and their time average values.
3. Describe Damped oscillation.
4. Define and explain Forced oscillations.
5. Explain sharpness of resonance.
6. Assuming the damping to be proportional to the velocity, write down the differential equation for a damped harmonic oscillator.
7. Discuss the under-damped, over-damped and critical damped motions of the oscillator.

Answers

Multiple Choice Questions

1. (a)	2. (c)	3. (b)	4. (c)	5. (b)
6. (c)	7. (b)	8. (c)	9. (a)	10. (b)
11. (c)	12. (d)	13. (d)	14. (d)	15. (a)
16. (d)	17. (c)	18. (b)	19. (c)	20. (d)
21. (d)				

Fill in the Blanks

1. Harmonic motion
2. Displacement
3. S.H.M. is a non-uniformly accelerated motion
4. Acceleration
5. Hooke's
6. The negative sign in the expression that the force F is opposite to the displacement
7. $\frac{d^2x}{dt^2}+\frac{k}{m}x=0$
8. Phase
9. Parabolic
10. Natural frequency
11. Periodic force
12. Resonance
13. Amplitude
14. Bandwidth