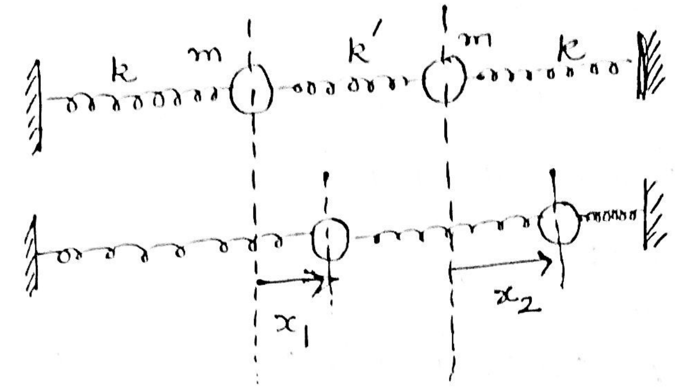
## Coupled oscillators & Normal modes

(Example of Two mass Three springs system)




→ Equilibrium condition

→ displaced system

$x_1$ & $x_2$ are instantaneous displacements of masses.

The equations of motion

For the first mass $\quad \dfrac{d^2x_1}{dt^2} = -\dfrac{k}{m}x_1 - \dfrac{k'}{m}(x_1 - x_2) \quad$ [∵ if $x_1 > x_2$] —— (1)

For the second mass $\quad \dfrac{d^2x_2}{dt^2} = -\dfrac{k}{m}x_2 - \dfrac{k'}{m}(x_2 - x_1) \quad$ [∵ if $x_2 > x_1$] —— (2)

Now adding equations (1) & (2)

$$\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} = -\frac{k}{m}(x_1 + x_2)$$

or

$$\frac{d^2}{dt^2}(x_1 + x_2) + \frac{k}{m}(x_1 + x_2) = 0 \quad \text{—— (3)}$$

Solution of the above equation

$$\boxed{x_1(t) + x_2(t) = C\cos\omega_1 t + D\sin\omega_1 t} \qquad \text{where } \omega_1 = \sqrt{\frac{k}{m}}$$

or we can write

or $\quad x_1(t) + x_2(t) = 2A_1\cos(\omega_1 t + \phi_1) \quad$ —— (4)

[∵ $2A_1\cos(\omega t + \phi_1) = 2A_1[\cos\omega_1 t \cdot \cos\phi_1 - \sin\omega_1 t \cdot \sin\phi_1]$

where $\cos\phi_1$ & $\sin\phi_1$ are constants depends on initial configuration

$= 2A_1[\cos\omega_1 t\, a_1 - \sin\omega_1 t\, a_2]$

$= C\cos\omega_1 t + D\sin\omega_1 t$ ]

Case(2) :

Now subtracting ② from ①

$$\frac{d^2}{dt^2}(x_1 - x_2) + \frac{(k+2k')}{m}(x_1 - x_2) = 0 \quad \text{——⑤}$$

Solution of this equation will be

$$x_1(t) - x_2(t) = E\cos\omega_2 t + F\sin\omega_2 t$$

where $\omega_2 = \sqrt{\frac{k+2k'}{m}}$.

or $$x_1(t) - x_2(t) = 2A_2\cos(\omega_2 t + \phi_2) \quad \text{——⑥}$$

Now adding (4) & (6)

$$x_1 + x_2 + x_1 - x_2 = 2A_1\cos(\omega_1 t + \phi_1) + 2A_2\cos(\omega_2 t + \phi_2)$$

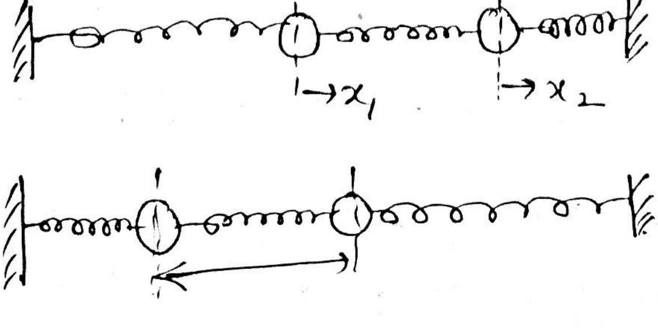
or $$x_1(t) = A_1\cos(\omega_1 t + \phi_1) + A_2\cos(\omega_2 t + \phi_2) \quad \text{——⑦}$$

Subtracting (6) from (4)

$$x_1 + x_2 - x_1 + x_2 = 2A_1\cos(\omega_1 t + \phi_1) - 2A_2\cos(\omega_2 t + \phi_2)$$

or $$x_2(t) = A_1\cos(\omega_1 t + \phi_1) - A_2\cos(\omega_2 t + \phi_2) \quad \text{——⑧}$$

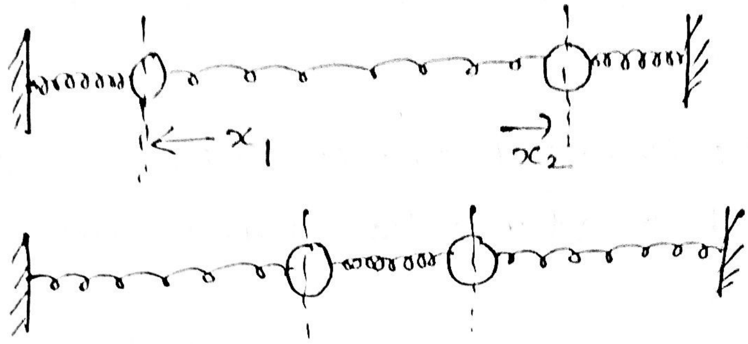
(Case 1): If we provide some kind of initial conditions where $A_2 = 0$ then $x_1(t) \pm x_2(t) = A\cos(\omega_1 t + \phi_1)$




Here $x_1$ & $x_2$ are in phase of same frequency and amplitude, This is called Normal Mode (first).

Case(2) : If we provide some kind of initial conditions when $A_1 = 0$

$$x_1(t) = -x_2(t) = A_2 \cos(\omega_2 t + \phi_2)$$




2nd normal mode both the masses will oscillate with same frequency and amplitude but are out of phase $180°$. $\omega_2 = \sqrt{\frac{k+2k'}{m}}$

The general solution is a linear superposition of these distinct frequencies $\omega_1$ & $\omega_2$. Hence these frequencies are known as normal frequencies.

It is very interesting to see here that individual solutions $x_1(t)$ and $x_2(t)$ do not show SHM or if shows ~~some~~ under some initial conditions. while the sum $x_1 + x_2$ and difference of these two $x_1 - x_2$ represents of SHM of distinct frequencies $\omega_1$ & $\omega_2$ respectively. Hence $x_1 + x_2$ and $x_1 - x_2$ are known as Normal coordinates.

BEATS: Let us consider some initial condition

$x_1(0) = 0$, $x_2(0) = A$ & $\dot{x}_1(0) = \dot{x}(0) = 0$

$$x_1(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$
$$x_2(t) = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2).$$

or

$$x_1(t) = a\cos\omega_1 t + b\sin\omega_1 t + c\cos(\omega_2 t) + d\sin\omega_2 t \quad \text{--- (i)}$$
$$x_2(t) = a\cos\omega_1 t + b\sin\omega_1 t - c\cos\omega_2 t - d\sin\omega_2 t \quad \text{--- (ii)}$$

at $t = 0$

$$0 = a\cos 0 + b\sin 0 + c\cos 0 + d\cos 0$$
$$\text{or} \quad a + c = 0$$

and
$$A = a\cos 0 + b\sin 0 - c\cos 0 - d\sin 0$$
$$a - c = A$$

or $\boxed{c = -\frac{A}{2} \text{ and } a = \frac{A}{2}}$

Now Differentiating (i) & (ii) w.r.t. ("t").

$$\dot{x}_1(t) = -a\omega_1 \sin\omega_1 t + b_1\omega_1 \cos\omega_1 t - c\omega_2 \sin\omega_2 t + d\omega_2 \cos\omega_2 t$$
$$\dot{x}_2(t) = -a\omega_1 \sin\omega_1 t + b\omega_1 \cos\omega_1 t + c\omega_2 \sin\omega_2 t - d\omega_2 \cos\omega_2 t$$

at $t = 0$

$$0 = -a\omega_1 \sin 0 + b\omega_1 \cos 0 - c\omega_2 \sin 0 + d\omega_2 \cos 0$$
$$\text{or} \quad b\omega_1 + d\omega_2 = 0$$

&
$$0 = -a\omega_1 \sin 0 + b\omega_1 \cos 0 + c\omega_2 \sin 0 - d\omega_2 \cos 0$$
$$\text{or} \quad b\omega_1 = d\omega_2$$

Hence $b\omega_1 + b\omega_1 = 0 \Rightarrow 2b\omega_1 = 0$ or $\boxed{b = 0 \;\&\; d = 0}$

Hence
$$\boxed{\begin{aligned} x_1(t) &= \frac{A}{2}(\cos\omega_1 t - \cos\omega_2 t) \\ x_2(t) &= \frac{A}{2}(\cos\omega_1 t + \cos\omega_2 t) \end{aligned}}$$

Let us make a substitution

$$\omega_1 = \frac{1}{2}(2\omega_1) = \frac{1}{2}(\omega_1 + \omega_2 + \omega_1 - \omega_2)$$

$$= \frac{\omega_2+\omega_1}{2} - \frac{(\omega_2-\omega_1)}{2}$$

and

$$\omega_2 = \frac{1}{2}(2\omega_2) = \frac{\omega_2+\omega_1}{2} + \frac{\omega_2-\omega_1}{2}$$

Hence

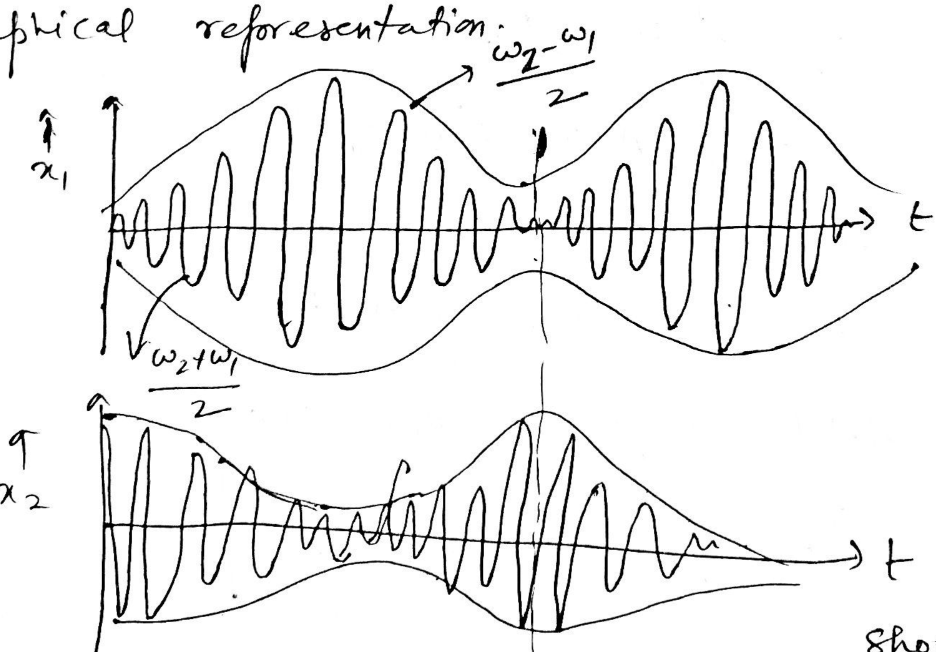
$$x_1(t) = \frac{A}{2}\left[\cos\left(\frac{\omega_2+\omega_1}{2}t - \frac{(\omega_2-\omega_1)t}{2}\right) - \cos\left(\frac{\omega_2+\omega_1}{2}t + \frac{(\omega_2-\omega_1)t}{2}\right)\right]$$

$$= \frac{A}{2}\Big[\cos\left(\frac{\omega_2+\omega_1}{2}\right)t\,\cos\left(\frac{\omega_2-\omega_1}{2}\right)t + \sin\left(\frac{\omega_2+\omega_1}{2}\right)t\,\sin\left(\frac{\omega_2-\omega_1}{2}\right)t$$

$$- \cos\left(\frac{\omega_2+\omega_1}{2}\right)t\,\cos\left(\frac{\omega_2-\omega_1}{2}\right)t + \sin\left(\frac{\omega_2+\omega_1}{2}\right)t\,\sin\left(\frac{\omega_2-\omega_1}{2}\right)t\Big]$$

$$= A\sin\left(\frac{\omega_2+\omega_1}{2}t\right)\sin\left(\frac{\omega_2-\omega_1}{2}\right)t$$

graphical reforesentation:




when we displace only ~~mass~~ one mass second mass also starts to oscillate by exchanging the energy between them in a manner shown in the diagram

These are known as beats.

One complete round ~~trip~~ of energy transfer from one particle to another and back from one to another is called a beat since $\frac{\omega_2 - \omega_1}{2} = \omega_m$ modulated frequency

$\frac{\omega_2 + \omega_1}{2} = \omega_a$ actual frequency

Hence $\omega_2 = \omega_m + \omega_a$

$\omega_1 = \omega_a - \omega_m$

so $x_1 = A[\cos(\omega_a - \omega_m)t + \cos(\omega_a + \omega_m)t]$

$= 2A \cos\omega_a t \cos\omega_n t$

$= (2A \cos\omega_m t) \cos\omega_a t = A_m \cos\omega_a t$

where $A_m = 2A \cos\omega_m t$

Similarly $x_2 = B_m \sin\omega_a t$ where $B_m = 2A \sin\omega_m t$

In one fact oscillation cycle, the ~~pendulum~~ ~~first~~ second mass is considered as a harmonic oscillation with frequency $\omega_a$ with constant amplitude $B_m$. Hence the Average energy

$$E_B = \frac{1}{2} m \omega_a^2 B_m^2$$

$$= 2m A^2 \omega_a^2 \sin^2 \omega_m t$$

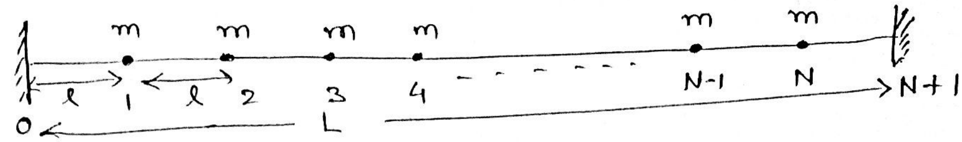
Similarly for particle 1.

$$E_A = 2m A^2 \omega_a^2 \cos^2 \omega_m t$$

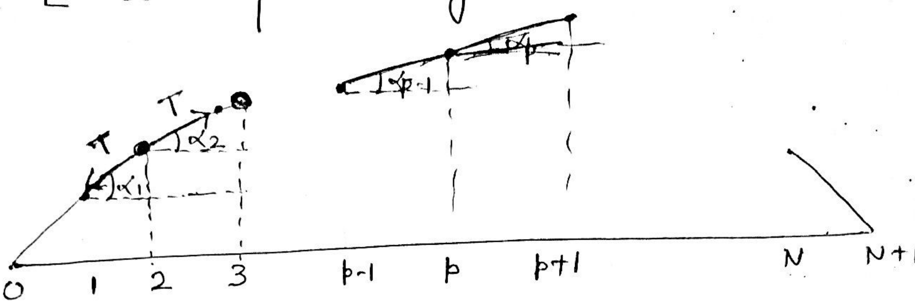
Thus the total energy of the system is

$$\boxed{E = E_A + E_B = 2m A^2 \omega_a^2}$$

# N-Coupled oscillators :




Let us consider N masses attached to a string of length L at equal lengths $l$.




The string is oscillating in a transverse direction. Let us consider the motion of three particles $p-1$, $p$, $p+1$ arbitrarily. The displacements correspoding to the particles is $y_{p-1}$, $y_p$ and $y_{p+1}$ respectively.

Equation of motion

In the vertical direction

$$F_{py} = m\ddot{y}_p = -T\sin\alpha_{p-1} + T\sin\alpha_p \quad \text{—— (1)}$$

In horizontal direction

$$F_{px} = -T\cos\alpha_{p-1} + T\cos\alpha_p \quad \text{—— (2)}$$

As we assume $\alpha$ is very small, Then

$$\sin\alpha_i \simeq \alpha_i$$

and $$\cos\alpha_i \simeq 1 - \frac{\alpha_i^2}{2} \simeq 1$$

Hence the force in the $x$-direction will be zero.

Hence

$$F_p = m\ddot{y}_p = -\frac{T}{l}(y_p - y_{p-1}) + \frac{T}{l}(y_{p+1} - y_p) \quad \text{—— (3)}$$

or

$$\frac{d^2y_p}{dt^2} = \frac{T}{ml}(y_{p+1} + y_{p-1} - 2y_p) \quad \text{—— (4)}$$

for N particles $p = 1$ to $N$, we will have N - Differential equations. Also at end points $y_0 = 0$ and $y_{N+1} = 0$.

Let $\omega_0^2 = \frac{T}{ml}$

or

$$\frac{d^2y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2 [y_{p+1} + y_{p-1}] = 0 \quad \text{—— (5)}$$

Normal Modes:

Case (i) for $p = N = 1$

$$y_{p-1} = y_{1-1} = y_0 = 0 \quad \text{and} \quad y_{p+1} = y_{N+1} = 0$$

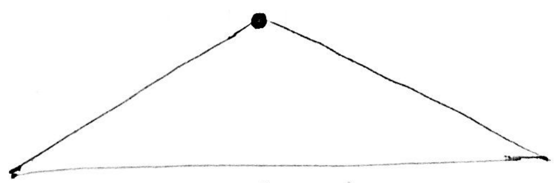
Hence

$$\frac{d^2y_1}{dt^2} + 2\omega_0^2 y_1 = 0 \quad \text{—— (6)}$$

The equation is the usual form of SHM.

Hence the angular frequency of oscillation is

$$\boxed{\omega = \sqrt{2}\,\omega_0 \quad \text{with} \quad \omega_0^2 = \frac{T}{ml} \quad \text{for} \quad N = 1}$$



A stretched string with $N = 1$

Case (ii) for $N = 2$, there are two differential equations for values of $p = 1$ and $p = 2$.

$$\frac{d^2y_1}{dt^2} + 2\omega_0^2 y_1 - \omega_0^2 y_2 = 0 \quad \text{—— (7)}$$

&

$$\frac{d^2y_2}{dt^2} + 2\omega_0^2 y_2 - \omega_0^2 y_1 = 0 \quad \text{—— (8)}$$

after adding and subtracting eqⁿ (7) & (8), we get

$$\left(\frac{d^2y_1}{dt^2} + \frac{d^2y_2}{dt^2}\right) + \omega_0^2 (y_1+y_2) = 0 \qquad \text{——— (9)}$$

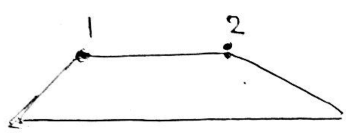
&

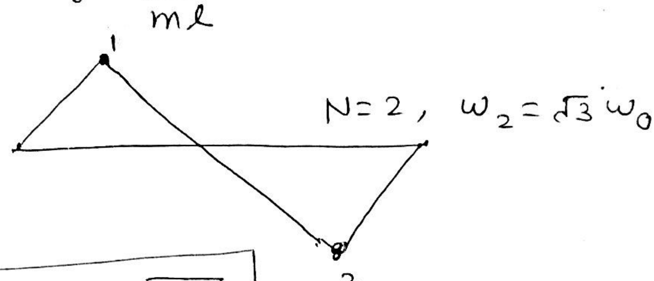
$$\frac{d^2y_1}{dt^2} - \frac{d^2y_2}{dt^2} + 3\omega_0^2 (y_1-y_2) = 0 \qquad \text{——— (10)}$$

where $\omega_1^2 = \omega_0^2$ & $\omega_2^2 = 3\omega_0^2$

$\Rightarrow$ $\boxed{\omega_1 = \omega_0 \ \& \ \omega_2 = \sqrt{3}\,\omega_0}$ for $N=2$.

$$\omega_0^2 = \frac{T}{ml}$$




or $\boxed{\omega_1 = \sqrt{\frac{T}{ml}}, \quad \omega_2 = \sqrt{\frac{3T}{ml}}}$ consequently $\omega_2 > \omega_1$.

<u>Case (iii)</u>: General solution for any N.

Suppose that each particle vibrate with the same frequency $\omega$ and amplitude $A_p$ where $p = 1, 2 \ldots N$.

Let the solution is

$$y_p = A_p \cos\omega t \qquad \text{--- (11)}$$

The general equation is

$$\frac{d^2y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

substituting the values

$$(-\omega^2 + 2\omega_0^2) y_p - \omega_0^2 (y_{p+1} + y_{p-1}) = 0$$

Therefore for $p = 1$

$$(-\omega^2 + 2\omega_0^2) A_1 - \omega_0^2 (A_2 + A_0) = 0$$

similarly for $p = 2$

$$(-\omega^2 + 2\omega_0^2) A_2 - \omega_0^2 (A_3 + A_1) = 0$$

$$\vdots \qquad \vdots$$

and in general

$$(-\omega^2 + 2\omega_0^2) A_p - \omega_0^2 (A_{p+1} + A_{p-1}) = 0 \quad \text{———} \quad (12)$$

or

$$\frac{A_{p+1} + A_{p-1}}{A_p} = \frac{(-\omega^2 + 2\omega_0^2)}{\omega_0^2} \quad \text{———} \quad (13)$$

for a particular value of $\omega$, the R.H.S. is constant.

Therefore the L.H.s must be a constant.

The value assigned to $A_p$ should be such that.

$$A_0 = 0 \quad \& \quad A_{N+1} = 0$$

Let us choose

$$A_p = c \sin p\theta \quad \text{———} \quad (14)$$

Hence $A_{p-1} + A_{p+1} = c\,[\sin(p-1)\theta + \sin(p+1)\theta]$

$$= 2c \sin p\theta \cos\theta$$

Hence

$$\frac{(A_{p-1} + A_{p+1})}{A_p} = 2\cos\theta \quad \text{———} \quad (15)$$

(independent of $p$).

The R.H.S. is constant if, when $p = 0$ or $p = N+1$,

then $A_p = 0$, so

$$A_p = c \sin p\theta$$

$$0 = c \sin (N+1)\theta$$

$$\Rightarrow \quad (N+1)\theta = n\pi \quad \text{or} \quad \theta = \frac{n\pi}{N+1}$$

Hence

$$\boxed{A_p = C \sin\left(\frac{n\pi p}{N+1}\right)} \qquad (16)$$

Now we have

$$\frac{A_{p+1} + A_{p-1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2} = 2\cos\left(\frac{n\pi}{N+1}\right)$$

$$\Rightarrow \quad \omega_n^2 = 2\omega_0^2\left[1 - \cos\left(\frac{n\pi}{N+1}\right)\right]$$

or

$$\omega_n^2 = 4\omega_0^2 \sin^2\left[\frac{n\pi}{2(N+1)}\right]$$

or

$$\boxed{\omega_n = 2\sqrt{\left(\frac{T}{ml}\right)} \sin\left(\frac{n\pi}{2(N+1)}\right)} \qquad (17)$$

(1) single oscillator $N=1,\ n=1,\ p=1$

$$\omega_1 = 2\omega_0 \sin\left(\frac{\pi}{4}\right) = \sqrt{2}\,\omega_0$$

$$A = C\sin\left(\frac{\pi}{2}\right) \Rightarrow A = C$$

(2) Two coupled oscillators

$N=2,\ p=1,\ p=2,\ n=1\ \&\ 2.$

for first Mode $n=1,\ p=1\ \&\ p=2$

$$\omega_1 = 2\omega_0 \sin\left(\frac{\pi}{6}\right) = \omega_0 \quad \& \quad A_1 = \frac{\sqrt{3}}{2}C$$

$$A_2 = C\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}C$$

for second Mode $n=2,\ p=1\ \&\ p=2$

$$\omega_2 = 2\omega_0 \sin\left(\frac{\pi}{3}\right) = \sqrt{3}\,\omega_0 \quad \& \quad A_1 = C\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}C$$

$$A_2 = C\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}C$$

Similarly we can do this for any number.