Algorithm & Data Structure Analysis

Lecture 2: Integer Arithmetics

Overview

- School method addition
- School method multiplication

Example of addition

Let $a = 1709$ and $b = 2530$
Compute $a + b$

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a	1	7	0	9	
					$+$
b	2	5	3	0	
					$+$
carries	1	0	0	0	
sum	4	2	3	9	

Number representation

An integer d can be expressed as

$$d = \sum_{i=0}^{n-1} d_i B^i$$

where $B > 1$ and $0 \leq d_i < B$.

B is called base or radix

Examples

$$d = \sum_{i=0}^{n-1} d_i B^i$$

$B = 10$ (decimal) $\quad 1 \cdot 10^3 + 7 \cdot 10^2 + 0 \cdot 10^1 + 9 \cdot 10^0 = 1709_{10}$

Examples

$$d = \sum_{i=0}^{n-1} d_i B^i$$

$B = 2$ (binary) $\quad 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 11_{10}$

$B = 10$ (decimal) $\quad 1 \cdot 10^3 + 7 \cdot 10^2 + 0 \cdot 10^1 + 9 \cdot 10^0 = 1709_{10}$

Examples

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$B = 2$ (binary) $\quad 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 11_{10}$

$B = 10$ (decimal) $\quad 1 \cdot 10^3 + 7 \cdot 10^2 + 0 \cdot 10^1 + 9 \cdot 10^0 = 1709_{10}$

$B = 16$ (hexadecimal) $\quad A \cdot 16^3 + D \cdot 16^2 + 5 \cdot 16^1 + A \cdot 16^0 = 44378_{10}$

Examples

$$d = \sum_{i=0}^{n-1} d_i B^i$$

$B = 2$ (binary)	$1{\cdot}2^3 + 0{\cdot}2^2 + 1{\cdot}2^1 + 1{\cdot}2^0 =$	11_{10}
$B = 8$ (octal)	$5{\cdot}8^3 + 3{\cdot}8^2 + 4{\cdot}8^1 + 7{\cdot}8^0 =$	2791_{10}
$B = 10$ (decimal)	$1{\cdot}10^3 + 7{\cdot}10^2 + 0{\cdot}10^1 + 9{\cdot}10^0 =$	1709_{10}
$B = 16$ (hexadecimal)	$A{\cdot}16^3 + D{\cdot}16^2 + 5{\cdot}16^1 + A{\cdot}16^0 =$	44378_{10}

Primitive operations

Assume that we have:

- Addition of three 1-digit

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  - ex. (B=10): $9 + 9 + 9 = 27_{10}$
- Multiplication of 1-digit by 1-digit
  - obtaining a 2-digit result
  - ex. (B=10): $9 \times 9 = 81_{10}$
  - ex. (B=16): $F \times F = E1_{16}$

School method addition

- Let $a = a_{n-1} \cdots a_0$ and $b = b_{n-1} \cdots b_0$ be two n-digit integers in base B

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$$
\begin{array}{lcclr}
 & a_{n-1} \cdots a_1 & a_0 & & \text{first operand} \\
 & b_{n-1} \cdots b_1 & b_0 & + & \text{second operand} \\
c_n & c_{n-1} \cdots c_1 & 0 & + & \text{carries} \\
\hline
s_n & s_{n-1} \cdots s_1 & s_0 & & \text{addition result}
\end{array}
$$

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 & a_{n-1} \cdots a_1 & a_0 & \text{first operand} \\
 & b_{n-1} \cdots b_1 & b_0 \; + & \text{second operand} \\
c_n & c_{n-1} \cdots c_1 & 0 \; + & \text{carries} \\
\hline
s_n & s_{n-1} \cdots s_1 & s_0 & \text{addition result}
\end{array}
$$

- $c_n \cdots c_0$ is a sequence of carries where $c_0 = 0$ and $c_{i+1} \cdot B + s_i = a_i + b_i + c_i$ for $0 \leq i < n$

School method addition (cont.)

Let a and b be n-digit integers
Compute $s = a + b$

$c \leftarrow 0$
for $i = 0$ to $n - 1$ **do**
    $(c_{i+1} s_i) \leftarrow a_i + b_i + c_i$
end for
$s_n \leftarrow c_n$

School method addition (cont.)

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end for
$s_n \leftarrow c_n$

Theorem

The school method addition of two n-digit integers requires exactly n primitive operations resulting in an $(n+1)$-digit integer.

Example of multiplication

Let $a = 1709$ and $b = 25$ in base 10
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$$\begin{array}{rrrrl} 1 & 7 & 0 & 9 & \\ & & 2 & 5 & \times \\ \hline \end{array}$$

multiplicand
multiplier

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Compute $a \times b$

$$\begin{array}{cccccl} 1 & 7 & 0 & 9 & & \text{multiplicand} \\ & & 2 & 5 & \times & \text{multiplier} \\ \hline 8 & 5 & 4 & 5 & & \text{partial product } a \cdot b_0 \end{array}$$

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Compute $a \times b$

$$
\begin{array}{rrrrcr}
 & 1 & 7 & 0 & 9 & & \text{multiplicand} \\
 & & & \color{blue}{2} & 5 & \times & \text{multiplier} \\
\hline
 & 8 & 5 & 4 & 5 & & \text{partial product } a \cdot b_0 \\
\color{blue}{3} & \color{blue}{4} & \color{blue}{1} & \color{blue}{8} & & + & \text{partial product } a \cdot \color{blue}{b_1}
\end{array}
$$

Example of multiplication

Let $a = 1709$ and $b = 25$ in base 10
Compute $a \times b$

$$
\begin{array}{ccccccl}
 & 1 & 7 & 0 & 9 & & \text{multiplicand} \\
 & & & 2 & 5 & \times & \text{multiplier} \\
\hline
 & 8 & 5 & 4 & 5 & & \text{partial product } a \cdot b_0 \\
3 & 4 & 1 & 8 & & + & \text{partial product } a \cdot b_1 \\
\hline
4 & 2 & 7 & 2 & 5 & & \text{add aligned partial products}
\end{array}
$$

School method multiplication

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- Compute $p = a \cdot b$ where $p = p_{2n-1} \cdots p_0$ is an $2n$-digit integer

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Procedure:

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Procedure:

1. Multiply an n-digit integer a by a 1-digit integer b_j obtaining a partial product p_j for $0 \leq j \leq n-1$

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Procedure:

1. Multiply an n-digit integer a by a 1-digit integer b_j obtaining a partial product p_j for $0 \leq j \leq n-1$
2. Add the aligned partial product $p_j \cdot B^j$

Partial product

Compute $p_j = a \cdot b_j$

$$\begin{array}{rrrrrl} a_{n-1} & \cdots & a_3 & a_2 & a_1 & a_0 \qquad \text{multiplicand} \\ b_{n-1} & \cdots & b_3 & b_2 & b_1 & b_0{}^{\times} \qquad \text{multiplier} \\ \hline \end{array}$$

Partial product

Compute $p_j = a \cdot b_j$

$$
\begin{array}{ccccccccl}
 & a_{n-1} & \cdots & a_3 & a_2 & a_1 & a_0 & & \text{multiplicand} \\
 & b_{n-1} & \cdots & b_3 & b_2 & b_1 & b_0 & \times & \text{multiplier} \\
\hline
c_{n-1} & c_{n-2} & \cdots & c_2 & c_1 & c_0 & 0 & & \text{carries} \\
 & d_{n-1} & \cdots & d_3 & d_2 & d_1 & d_0 & & a \cdot b_0
\end{array}
$$

Partial product

Compute $p_j = a \cdot b_j$

$$
\begin{array}{cccccccl}
 & & a_{n-1} & \cdots & a_3 & a_2 & a_1 \quad a_0 & \text{multiplicand} \\
 & & b_{n-1} & \cdots & b_3 & b_2 & b_1 \quad b_0 \times & \text{multiplier} \\
\hline
c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 & 0 & \text{carries} \\
 & d_{n-1} & d_{n-2} & \cdots & d_2 & d_1 & d_0 & a \cdot {\color{blue} b_1}
\end{array}
$$

Partial product

Compute $p_j = a \cdot b_j$

$$\begin{array}{cccccccr}
 & & a_{n-1} & \cdots & a_3 & a_2 & a_1 \quad a_0 & \text{multiplicand} \\
 & & b_{n-1} & \cdots & b_3 & b_2 & b_1 \quad b_0^{\times} & \text{multiplier} \\
\hline
c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 & 0 & \text{carries} \\
 & d_{n-1} & d_{n-2} & \cdots & d_2 & d_1 & d_0 & a \cdot b_1
\end{array}$$

- Number of primitive operation

Partial product

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 & & a_{n-1} & \cdots & a_3 & a_2 & a_1 & a_0 & \text{multiplicand} \\
 & & b_{n-1} & \cdots & b_3 & b_2 & b_1 & b_0{}^{\times} & \text{multiplier} \\
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 & d_{n-1} & d_{n-2} & \cdots & d_2 & d_1 & d_0 & & a \cdot b_1
\end{array}$$

- Number of primitive operation
  - n multiplications (a has n digits)

Partial product

Compute $p_j = a \cdot b_j$

$$
\begin{array}{cccccccr}
 & & a_{n-1} & \cdots & a_3 & a_2 & a_1 & a_0 \qquad \text{multiplicand} \\
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\end{array}
$$

- Number of primitive operation
  - n multiplications (a has n digits)
  - $n + 1$ additions (carries and alignment)

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Compute $p_j = a \cdot b_j$

$$
\begin{array}{cccccccc|r}
 & & a_{n-1} & \cdots & a_3 & a_2 & a_1 & a_0 & \text{multiplicand} \\
 & & b_{n-1} & \cdots & b_3 & b_2 & b_1 & b_0 \times & \text{multiplier} \\
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c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_1 & c_0 & 0 & & \text{carries} \\
 & d_{n-1} & d_{n-2} & \cdots & d_2 & d_1 & d_0 & & a \cdot b_1
\end{array}
$$

- Number of primitive operation
  - n multiplications (a has n digits)
  - $n+1$ additions (carries and alignment)
  - total of $2n+1$

School method multiplication (cont.)

Let a, b be n-digit integers in base B
Compute $p = a \cdot b$

$p \leftarrow 0$
for $i = 0$ to $n - 1$ **do**
    $p \leftarrow p + a \cdot b_i \cdot B^i$
end for

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$p \leftarrow 0$
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  $p \leftarrow p + a \cdot b_i \cdot B^i$
end for

Theorem

The school method multiplication of two n-digit integers requires $3n^2 + 2n = \Theta(n^2)$ primitive operations.

Multiplication analysis

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- $2n + 1$ operations for one partial product p_j

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- In total, $3n^2 + 2n$ operations

Multiplication analysis

- $2n + 1$ operations for one partial product p_j
  $\Longrightarrow 2n^2 + n$ for $0 \leq j < n$
- n summations of $(n + 1)$-digit numbers
  $\Longrightarrow n^2 + n$ operations
- In total, $3n^2 + 2n$ operations
  $\Longrightarrow \Theta(n^2)$

Summary

- Addition can be done using n primitive operations
- School method multiplication requires $\Theta(n^2)$ primitive operations
- Questions: Faster multiplication algorithms?
- Reading: Algorithms and Data Structures
  - Chapter 1.1: Addition
  - Chapter 1.2: Multiplication: The School Method

Next week

- Recursive multiplication (Tuesday)
- Karatsuba multiplication (Wednesday)
- Reading: Algorithms and Data Structures
  - Chapter 1.4: A Recursive Version of the School Method
  - Chapter 1.5: Karatsuba Multiplication