Notation we use

$y \in \mathbb{R}^n$: outcome vector

$X = [1, x_1, \cdots, x_p] \in \mathbb{R}^{n \times (p+1)}$: design matrix with an intercept

$\beta = (\beta_0, \beta_1, \cdots \beta_p)^T$: coefficients

$\varepsilon \sim N(0, \sigma^2 I_n)$: homoskedastic Gaussian errors

$p$: number of markers excluding the intercept.

$df_2 = n - p - 1$: residual degrees of freedom

$R$: the constraint matrix that encodes the linear hypothesis $H_0: R\beta = r$

Under the normal-error linear model

$$\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$$

Applying the linear map $R$ gives

$$R\hat{\beta} \sim N(R\beta, \sigma^2 R (X^T X)^{-1} R^T)$$

Therefore, $R\hat{\beta} - r \sim N(R\beta - r, \sigma^2 R (X^T X)^{-1} R^T)$

The standardized quadratic form

$$T^2 = \frac{(R\hat{\beta} - r)^T [R(X^T X)^{-1} R^T]^{-1} (R\hat{\beta} - r)}{\sigma^2}$$

Under $H_0$, $T^2 \sim \chi^2_q$ (central) with $q = \text{rank}(R)$

Under $H_1$, $T^2 \sim \chi^2_q(\lambda)$ (non-central) with noncentrality parameter

$$\lambda = \frac{(R\beta - r)^T [R(X^T X)^{-1} R^T]^{-1} (R\beta - r)}{\sigma^2}$$

The $\lambda$ is the key link between true effect and factor

1° Single parameter $H_0: \beta_j = 0$

Set $R = e_j^T$ (a $1 \times p$ row vector with a 1 in position $j$, 0 elesuhere)

$r = 0$

$R\beta - r = e_j^T \beta - 0 = \beta_j$ (a scalar)

$R(X^T X)^{-1} R^T = e_j^T (X^T X)^{-1} e_j$. Because $e_j^T A e_j = A_{jj}$ for any matrix $A$, this equals $(X^T X)^{-1}_{jj}$ (a scalar)

Its inverse is $[R(X^T X)^{-1} R^T]^{-1} = ((X^T X)^{-1}_{jj})^{-1}$

Plug these into NCP formular:

$$\lambda_j = \frac{(\beta_j)^2}{\sigma^2} \cdot ((X^T X)^{-1}_{jj})^{-1} = \frac{\beta_j^2}{\sigma^2 (X^T X)^{-1}_{jj}}$$

We now prove: $\frac{1}{(X^T X)^{-1}_{jj}} = x_j^T M_{-j} x_j = \tilde{x}_j^T x_j$

where $M_{-j} = I - X_{-j} (X_{-j}^T X_{-j})^{-1} X_{-j}^T$ is the residual-marker on the space orthogonal to $X_{-j}$

and $\tilde{x}_j := M_{-j} x_j$ is $x_j$ after partialling out the other regressors.

The Gram matrix in block form by splitting off column $j$:

$$X^T X = \begin{bmatrix} X_{-j}^T X_{-j} & X_{-j}^T x_j \\ x_j^T X_{-j} & x_j^T x_j \end{bmatrix}$$

The Schur complement of the top-left block is

$$S = x_j^T x_j - x_j^T X_{-j} (X_{-j}^T X_{-j})^{-1} X_{-j}^T x_j$$

$$= x_j^T (I - X_{-j} (X_{-j}^T X_{-j})^{-1} X_{-j}^T) x_j$$

$$= x_j^T M_{-j} x_j$$

A standard block-inverse identity tells us that the bottom right element of $(X^T X)^{-1}$ equals $S^{-1}$. Therefore

$$(X^T X)^{-1}_{jj} = \frac{1}{x_j^T M_{-j} x_j} \Rightarrow \frac{1}{(X^T X)^{-1}_{jj}} = x_j^T M_{-j} x_j$$

Define $\tilde{x}_j := M_{-j} x_j$. Then, $x_j^T M_{-j} x_j = \tilde{x}_j^T \tilde{x}_j$

So the NCP can be written as $\lambda_j = \frac{\beta_j^2}{\sigma^2} \tilde{x}_j^T \tilde{x}_j$

If the column is mean-centered, $\tilde{x}_j^T \tilde{x}_j = \sum_{i=1}^{n} \tilde{x}_{ij}^2 \approx n \, Var(\tilde{x}_j)$

Hence, the approximation $\lambda_j \approx n \frac{\beta_j^2}{\sigma^2} Var(\tilde{x}_j)$

Consider to collinearity, we take the relation of VIF into account.

If $x_j$ is standardized so $Var(x_j) = 1$, and $R_j^2$ is the $R^2$ from regressing $x_j$ on $X_{-j}$.

then $Var(\tilde{x}_j) = 1 - R_j^2 = \frac{1}{VIF_j}$

$$\Rightarrow \lambda_j \approx n \cdot \frac{\beta_j^2}{\sigma^2} \cdot \frac{1}{VIF_j}$$

$\Rightarrow$ Therefore, showing directly that higher collinearity inflates the required $n$ for fixed effect size.

The noncentral $t$-statistic for $\beta_j$ has

$$\delta_j = \frac{\beta_j}{\sigma \sqrt{(X^T X)^{-1}_{jj}}} \Rightarrow \delta^2 = \frac{\beta_j^2}{\sigma^2 (X^T X)^{-1}_{jj}} = \lambda_j$$

2° Group of $q$ coefficients

$H_0: \beta_S = 0$

Let $S$ index the $q$ coefficients being tested jointly; let $R$ pick those $q$ coordinates

So $R\beta = \beta_S$, a $q \times 1$ vector and $r = 0$

with $R$ as a $q \times p$ selector.

$R(X^T X)^{-1} R^T = (X^T X)^{-1}_{S,S}$

the $q \times q$ principal submatrix of $(X^T X)^{-1}$ for indices $S$.

Hence, $\lambda_S = \dfrac{\beta_S^T [(X^T X)^{-1}_{S,S}]^{-1} \beta_S}{\sigma^2}$

Split $X = [X_{-S}, X_S]$. The residual-marker for the complement is $M_S = I - X_{-S}(X_{-S}^T X_{-S})^{-1} X_{-S}^T$

The block Schur complement identity generalizes: $[(X^T X)^{-1}_{S,S}]^{-1} = X_S^T M_{-S} X_S$.

Define the residualized block $\tilde{X}_S := M_{-S} X_S$. Then

$$\lambda_S = \frac{1}{\sigma^2} \beta_S^T \tilde{X}_S^T \tilde{X}_S \beta_S \approx n \frac{\beta_S^T \Sigma_{\tilde{X}_S} \beta_S}{\sigma^2}$$

Where $\Sigma_{\tilde{X}_S}$ is the sample covariance of $\tilde{X}_S$