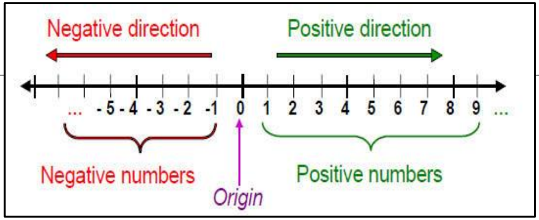
Dual Quaternion Examples

CMPUT 307

Dual Quaternion Skinning

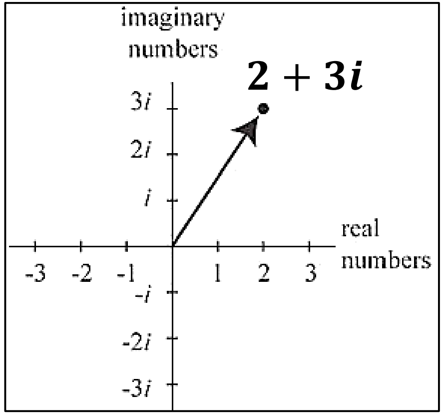
Real numbers



A complex number is a **two-dimensional** extension of the real numbers.

Complex numbers

By definition:
$i^2 = -1$

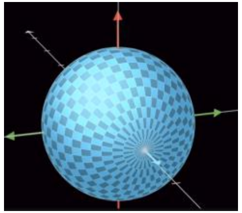


Quaternions

In1843 mathematician William Rowan Hamilton was out walking along the Royal Canal in Dublin with his wife when the solution in the form of the following equation suddenly occurred to him

"$i^2 = j^2 = k^2 = ijk = -1$"

He added two more imaginary dimensions to the complex plane.



$$\underbrace{4}_{\text{Real part}} + \underbrace{6i + 1j + 9k}_{\text{Imaginary part}}$$

Real part: Scalar
Imaginary part: Vector

A quaternion is a **four-dimensional** extension of the complex numbers. They have pragmatic utility describing rotation in 3D and even quantum mechanics.

Multiplication of basis elements

A feature of quaternions is that multiplication of two quaternions is noncommutative.

The products of basis elements are defined by $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -\mathbf{1}$, and:

$$\begin{aligned} \boldsymbol{ij} &= \boldsymbol{k}, & \boldsymbol{ji} &= -\boldsymbol{k} \\ \boldsymbol{jk} &= \boldsymbol{i}, & \boldsymbol{kj} &= -\boldsymbol{i} \\ \boldsymbol{ki} &= \boldsymbol{j} & \boldsymbol{ik} &= -\boldsymbol{j} \end{aligned}$$

These multiplication formulas are equivalent to $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathbf{1}$

In fact, the equality ijk = -1 results from

$$(\boldsymbol{ij})\boldsymbol{k} = \boldsymbol{k}^2 = -\mathbf{1}$$

The converse implication results from manipulations similar to the following. By right-multiplying both sides of $-1 = \boldsymbol{ijk}$ by $-\boldsymbol{k}$, one gets

$$\boldsymbol{k} = (\boldsymbol{ijk})(-\boldsymbol{k}) = (\boldsymbol{ij})(-\boldsymbol{k}^2) = \boldsymbol{ij}$$

Quaternion Multiplication

$$(\boldsymbol{w_1} + \boldsymbol{x_1 i} + \boldsymbol{y_1 j} + \boldsymbol{z_1 k})(\boldsymbol{w_2} + \boldsymbol{x_2 i} + \boldsymbol{y_2 j} + \boldsymbol{z_2 k}) =$$
$$(\boldsymbol{w_1 w_2} - \boldsymbol{x_1 x_2} - \boldsymbol{y_1 y_2} - \boldsymbol{z_1 z_2}) + (\boldsymbol{w_1 x_2} + \boldsymbol{x_1 w_2} + \boldsymbol{y_1 z_2} - \boldsymbol{z_1 y_2})\boldsymbol{i} +$$
$$(\boldsymbol{w_1 y_2} + \boldsymbol{y_1 w_2} + \boldsymbol{z_1 x_2} - \boldsymbol{x_1 z_2})\boldsymbol{j} + (\boldsymbol{w_1 z_2} + \boldsymbol{z_1 w_2} + \boldsymbol{x_1 y_2} - \boldsymbol{y_1 x_2})\boldsymbol{k}$$

Multiplication rule written in terms of the dot product and the cross product.

$$\overrightarrow{v_1} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{y_1} \\ \boldsymbol{z_1} \end{bmatrix} \qquad \overrightarrow{v_2} = \begin{bmatrix} \boldsymbol{x_2} \\ \boldsymbol{y_2} \\ \boldsymbol{z_2} \end{bmatrix}$$

It is more common to represent the quaternion as two components, the vector component (x, y and z) and the scalar component (w)

$$(\boldsymbol{w_1} + \boldsymbol{x_1 i} + \boldsymbol{y_1 j} + \boldsymbol{z_1 k})(\boldsymbol{w_2} + \boldsymbol{x_2 i} + \boldsymbol{y_2 j} + \boldsymbol{z_2 k}) = (\boldsymbol{w_1}, \overrightarrow{v_1})(\boldsymbol{w_2}, \overrightarrow{v_2}) =$$
$$(\boldsymbol{w_1 w_2} - \overrightarrow{v_1} \cdot \overrightarrow{v_2}, \boldsymbol{w_1}\overrightarrow{v_2} + \boldsymbol{w_2}\overrightarrow{v_1} + \overrightarrow{v_1} \times \overrightarrow{v_2})$$

Dual Numbers

Similar to complex number that consists of two parts known as the real part and dual or complex part.

$$\mathcal{Z} = a + b\mathcal{E}$$

Except that: $\mathcal{E}^2 = 0$ but $\mathcal{E} \neq 0$

where $\mathcal{E}$ is the dual operator, a is the real part and b is the dual part.

The dual operator $\mathcal{E}$ is added to distinguish the real and dual components.

The real part of a dual calculation is independent of the dual parts of the inputs.

The dual part of a multiplication is a “cross” product of real and dual parts.

Dual Quaternion

Dual quaternions is composed of two quaternions, one responsible for orientation, and the other responsible for translation. Combining the algebra operations associated with quaternions with the additional dual number $\boldsymbol{\varepsilon}$, we can form the dual quaternion arithmetic.

Can be written as $\hat{q} = w + ix + jy + kz$ where w is the scalar part(dual number), (x, y, z) is the vector part (dual vector) and i, j, k are the usual quaternion units.

A dual quaternion can also be considered as an 8-tuple, or as the sum or two ordinary quaternions, $\hat{q} = q_0 + q_\varepsilon \cdot \varepsilon$

To transform a point using dual quaternion we use: **P = $\boldsymbol{q}$P$\boldsymbol{q^*}$** (where $\boldsymbol{q^*}$ denotes conjugate)

Dual quaternion skinning blends the dual quaternion of each bone by the blending weights

$$q = \frac{\sum_{i=1}^{n} w_i q_i}{\left\|\sum_{i=1}^{n} w_i q_i\right\|}$$

https://cs.gmu.edu/~jmlien/teaching/cs451/uploads/Main/dual-quaternion.pdf

Dual Quaternion

There are 8 elements, the 4 quaternion elements **(real, i, j and k)** and their duals **(ε, iε, jε and kε).** This gives dual quaternions a 8x8 multiplication table as shown here:

Multiplication table for dual quaternion units

×	1	i	j	k	ε	εi	εj	εk
1	1	i	j	k	ε	εi	εj	εk
i	i	−1	k	−j	εi	−ε	εk	−εj
j	j	−k	−1	i	εj	−εk	−ε	εi
k	k	j	−i	−1	εk	εj	−εi	−ε
ε	ε	εi	εj	εk	0	0	0	0
εi	εi	−ε	εk	−εj	0	0	0	0
εj	εj	−εk	−ε	εi	0	0	0	0
εk	εk	εj	−εi	−ε	0	0	0	0

Dual Quaternion Multiplication

D1 = (1, 2i, j, k) + (1, 2i, j, k) ε

D2 = (1, i, 3j, k) + (1, i, j, k) ε

D1 · D2 DQ part

D1 DQ part: (1, 2i, j, k)
D2 DQ part: (1, i, 3j, k)

Use the distribution property of product:

1 · (1, i, 3j, k) = (1, i, 3j, k)
2i · (1, i, 3j, k) = (2i, -2, 6k, -2j)
j · (1, i, 3j, k) = (j, -k, -3, i)
k · (1, i, 3j, k) = (k, j, -3i, -1)

So:
(1, 2i, j, k) · (1, i, 3j, k) = (-5, i, 3j, 7k)

Dual Quaternion Multiplication

D1 = (1, 2i, j, k) + (1, 2i, j, k) ε

D2 = (1, i, 3j, k) + (1, i, j, k) ε

D1 · D2 dual DQ part

D1 DQ part: (1, 2i, j, k)
D2 dual DQ part: (1, i, j, k) ε
Similarly:
(1, 2i, j, k) · (1, i, j, k) ε = (-3, 3i, j, 3k) ε

D1 dual DQ part: (1, 2i, j, k) ε
D2 DQ part: (1, i, 3j, k)
Similarly:
(1, 2i, j, k) ε · (1, i, 3j, k) = (-5, i, 3j, 7k) ε

D1 dual DQ part: (1, 2i, j, k) ε
D2 dual DQ part: (1, i, j, k) ε
(1, 2i, j, k) $\boldsymbol{\varepsilon}$ · (1, i, j, k) $\boldsymbol{\varepsilon}$ = 0

Dual Quaternion Calculation

D1 = (1, 2i, j, k) + (1, 2i, j, k) ε

D2 = (1, i, 3j, k) + (1, i, j, k) ε

Result:

D1 · D2 = (-5, i, 3j, 7k) + (-8, 4i, 4j, 10k) ε

Another way to do the computation

Multiplication rule written in terms of the dot product and the cross product.

$$\overrightarrow{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \qquad \overrightarrow{v_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

It is more common to represent the quaternion as two components, the vector component (x, y and z) and the scalar component (w)

$$(w_1 + \overbrace{x_1\boldsymbol{i} + y_1\boldsymbol{j} + z_1}^{\overrightarrow{v_1}}\boldsymbol{k})(w_2 \overbrace{+ x_2\boldsymbol{i} + y_2\boldsymbol{j} + z_2}^{\overrightarrow{v_2}}\boldsymbol{k}) = (w_1, \overrightarrow{v_1})(w_2, \overrightarrow{v_2}) =$$

$$(w_1 w_2 - \overrightarrow{v_1} \cdot \overrightarrow{v_2}, w_1\overrightarrow{v_2} + w_2\overrightarrow{v_1} + \overrightarrow{v_1} \times \overrightarrow{v_2})$$

- D1 = (1, 2i, j, k) + (1, 2i, j, k) ε
- D2 = (1, i, 3j, k) + (1, i, j, k) ε

- Result: (-5, i, 3j, 7k) + (-8, 4i, 4j, 10k) ε

Numerical details on next slide

Numerical details

$$\textcircled{1}\quad \{(1, 2i, j, k) + (1, 2i, j, k)\varepsilon\} \cdot \{(1, i, 3j, k) + (1, i, j, k)\varepsilon\}$$

D1 D2

$$\omega_1 = 1 \quad \underbrace{\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}}_{\vec{v}_1} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \omega_2 = 1 \quad \underbrace{\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}}_{\vec{v}_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \qquad \omega_3 = 1 \quad \underbrace{\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}}_{\vec{v}_3} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \qquad \omega_4 = 1 \quad \underbrace{\begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}}_{\vec{v}_4} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(\omega_1, \vec{v}_1) \cdot (\omega_3, \vec{v}_3) = (\omega_1\omega_3 - \vec{v}_1 \cdot \vec{v}_3,\ \omega_1 \vec{v}_3 + \omega_3 \vec{v}_1 + \vec{v}_1 \times \vec{v}_3)$$

$$= (1-6,\ i+3j+k + 2i+j+k - 2i - j + 5k)$$

$$= \boxed{(-5,\ i+3j+7k)} \quad \cdots \text{A}$$

$$(\omega_1, \vec{v}_1) \cdot (\omega_4, \vec{v}_4)\,\varepsilon = (\omega_1\omega_4 - v_1 \cdot v_4,\ \omega_1 v_4 + \omega_4 v_1 + \vec{v}_1 \times \vec{v}_4)\varepsilon$$

$$= (1-4,\ i+j+k + 2i+j+k + -j+k)\,\varepsilon$$

$$= \boxed{(-3,\ 3i+j+3k)\,\varepsilon} \quad \cdots \text{B}$$

$$(\omega_2, \vec{v}_2) \cdot (\omega_3, \vec{v}_3)\,\varepsilon = (\omega_2\omega_3 - \vec{v}_2\vec{v}_3,\ \omega_2 \vec{v}_3 + \omega_3 \vec{v}_2 + \vec{v}_2 \times \vec{v}_3)\varepsilon$$

$$= (1-6,\ i+3j+k + 2i+j+k + -2i-j+5k)$$

$$= \boxed{(-5,\ i+3j+7k)\,\varepsilon} \quad \cdots \text{C}$$

$$\text{A} + \text{B} + \text{C} = (-5,\ i+3j+7k) + (-8,\ 4i+4j+10k)\,\varepsilon$$

Dual Quaternion for Only Rotation or Only Translation

The dual-quaternion can represent a pure rotation just as a quaternion by setting the dual part to zero.

$$q_r = \cos\left(\frac{\theta}{2}\right) + \sin(\frac{\theta}{2})(x * \boldsymbol{i} + y * \boldsymbol{j} + z * \boldsymbol{k}) + \varepsilon \cdot 0$$

$$= \left(\cos\left(\frac{\theta}{2}\right), n_x\, sin\left(\frac{\theta}{2}\right), n_y \sin\left(\frac{\theta}{2}\right), n_z\, sin\left(\frac{\theta}{2}\right)\right) + \varepsilon \cdot 0$$

$(x * \boldsymbol{i} + y * \boldsymbol{j} + z * \boldsymbol{k})$ specifies the axis for rotation.

To represent a pure translation with no rotation, the real part can be set to identity with the dual part representing translation.

$$q_t = (1,0,0,0) + \frac{\varepsilon}{2}\left(0, t_x, t_y, t_z\right)$$

A point with coordinates (x, y, z) in DQ would be (1, 0, 0, 0) $+(\boldsymbol{0}, \boldsymbol{x} * \boldsymbol{i}, y * \boldsymbol{j}, z * \boldsymbol{k})\varepsilon$

Example 1

Rotate the vector **P** (0, 2, 0) = 2**j** , 90 degrees counter-clockwise about a vertical axis (parallel with k), then apply a translation (2,2,2). This is a simple example and you can easily get the result (0,2,2).

Let's see how it's done using dual quaternion.

Rotation part: $q_r = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(x * \boldsymbol{i} + y * \boldsymbol{j} + z * \boldsymbol{k}) = (\frac{\sqrt{2}}{2}, \boldsymbol{0i}, \boldsymbol{0j}, \frac{\sqrt{2}}{2}\boldsymbol{k})$

Translation part: $q_t = (1{,}0\boldsymbol{i}, 0\boldsymbol{j}, 0\boldsymbol{k}) + (0, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})\varepsilon$

Dual quaternion: $q = q_t q_r = \left(\frac{\sqrt{2}}{2}, \boldsymbol{0}, \boldsymbol{0}, \frac{\sqrt{2}}{2}\boldsymbol{k}\right) + (-\frac{\sqrt{2}}{2}, \sqrt{\boldsymbol{2}}\boldsymbol{i}, \boldsymbol{0}, \frac{\sqrt{2}}{2}\boldsymbol{k})\ \varepsilon$

P: $(\boldsymbol{1}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}) + (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{2j}, \boldsymbol{0})\varepsilon$

Conjugate: $q^* = \left(\frac{\sqrt{2}}{2}, \boldsymbol{0}, \boldsymbol{0}, \frac{-\sqrt{2}}{2}\boldsymbol{k}\right) + (\frac{\sqrt{2}}{2}, \sqrt{\boldsymbol{2}}\boldsymbol{i}, \boldsymbol{0}, \frac{\sqrt{2}}{2}\boldsymbol{k})\ \varepsilon$

P'= $qPq^* = (\boldsymbol{1}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}) + (\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{2j}, \boldsymbol{2k})\varepsilon$

Example 2

Rotate the vector **P1** (1, 2, 0) =(1, 0, 0, 0)+(0, i , 2j, 0) ε , 60 degrees rightward about a vertical axis (parallel with k).

The axis of the quaternion must be vertical, and must point down in order to represent a rightward rotation by the right hand rule (because when you point your right thumb down, your fingers curl round to the right) thus q=(0,0,-1)=-k

Rotation part: $q_r = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(x * \boldsymbol{i} + y * \boldsymbol{j} + z * \boldsymbol{k})$

$$q_r = \left(\cos\left(\frac{60°}{2}\right), 0, 0, -sin\left(\frac{60°}{2}\right)\right)$$

$$q_r = q = (0.87\ ,\ 0,\ \ 0, -0.5) = 0.87 - 0.5\text{k}$$

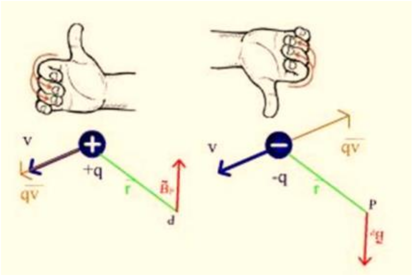
P2 = $\boldsymbol{q}$ P1 $\boldsymbol{q}^*$

$\text{P2} = (0.87 - 0.5\text{k}\ ) * ((1, 0, 0, 0)+(0, \text{i}\ , 2\text{j}, 0)\ \varepsilon) * (0.87 + 0.5\text{k}\ )$

$\text{P2} = (1, 0, 0, 0) + (0.87\text{i} + 1.74\text{j} - 0.5\text{ki} - \text{kj})\varepsilon * (0.87 + 0.5\text{k}\ )$

$\text{P2} = (1, 0, 0, 0) + (1.87\text{i}+1.24\text{j}) * (0.87 + 0.5\text{k}\ )$

$\text{P2} = (1, 0, 0, 0) + (2.25\text{i}+0.14\text{j})\ \varepsilon$



Dual Quaternion Skinning

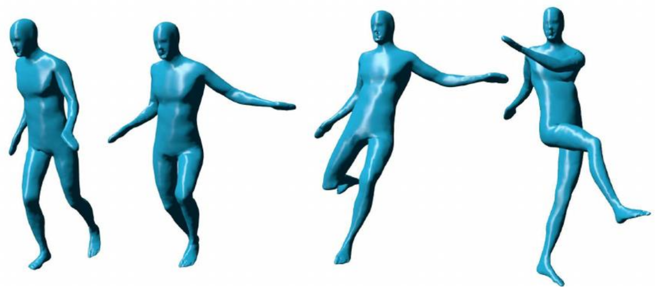
Combining the rotational and translational transforms into a single unit quaternion to represent a rotation followed by a translation we get:

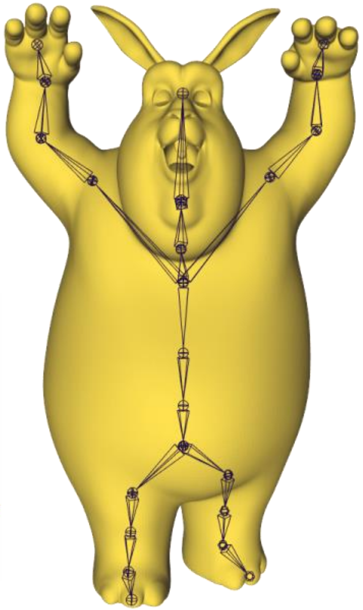
$$q = q_t q_r$$

Applying the dual quaternion q to a vertex v

$$\boldsymbol{v}' = \boldsymbol{q}\boldsymbol{v}\boldsymbol{q}^*$$

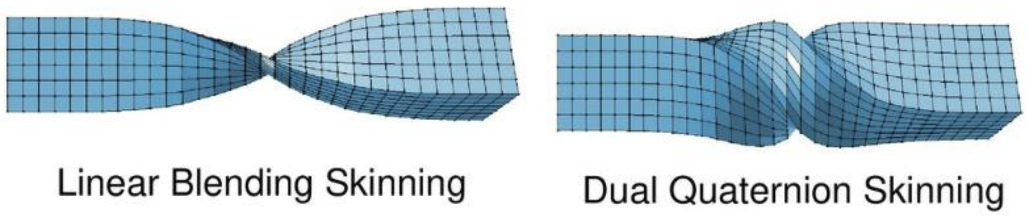
Where $\boldsymbol{q}^*$ is a conjugate of $\boldsymbol{q}$





Dual Quaternion Skinning

Joint collapse and candy wrap can be avoided using dual quaternion skinning



Homogeneous Matrix VS DQ

6 vertices:

P1: (0, 2.5, 0) P2: (1, 2.5, 0) P3: (2, 2.5, 0)

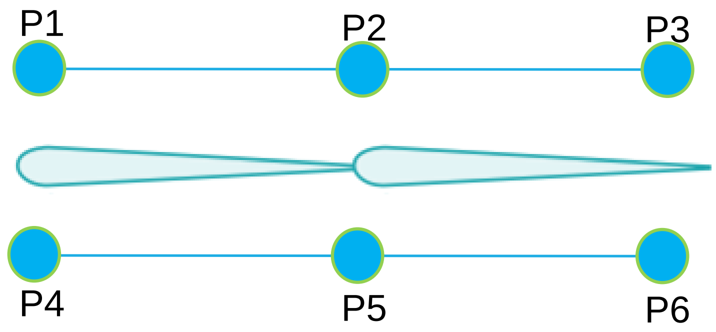
P4: (0, -2.5, 0) P5: (1, -2.5, 0) P6: (2, -2.5, 0)

2 joints:

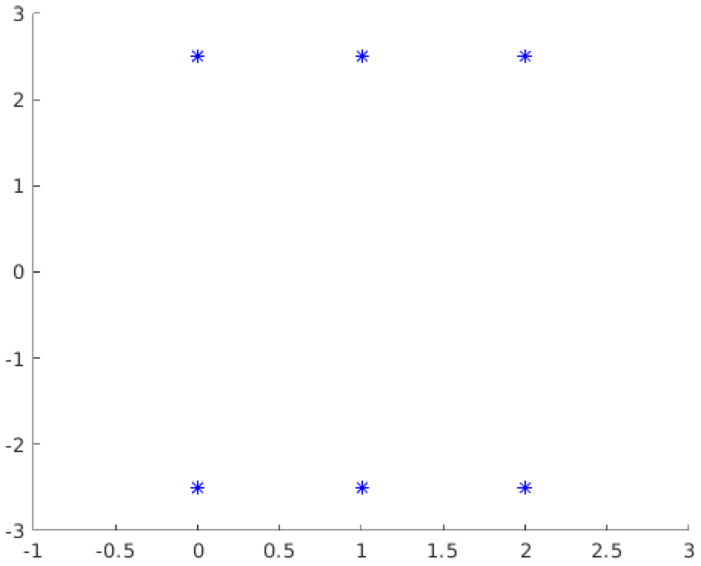
J1: (0, 0, 0) J2: (2, 0, 0)

Weight of every vertices to joints:

$$W_{ij}=\begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \\ 0 & 1 \\ 1 & 0 \\ 0.5 & 0.5 \\ 0 & 1 \end{bmatrix}$$ where i refers to the vertices and j refers to the joints.



Homogeneous Matrix VS DQ



Homogeneous Matrix VS DQ

Now we rotate J1 by 180 degrees, axis of rotation is x axis.

Homogeneous transformation matrix for J1:

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous transformation matrix for J2:

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Matrix VS DQ

Now we do linear blending using homogeneous matrix :

For every point:

$P_i' = \left(\sum_{j=1}^{n} w_{ij} M_j\right) P_i$

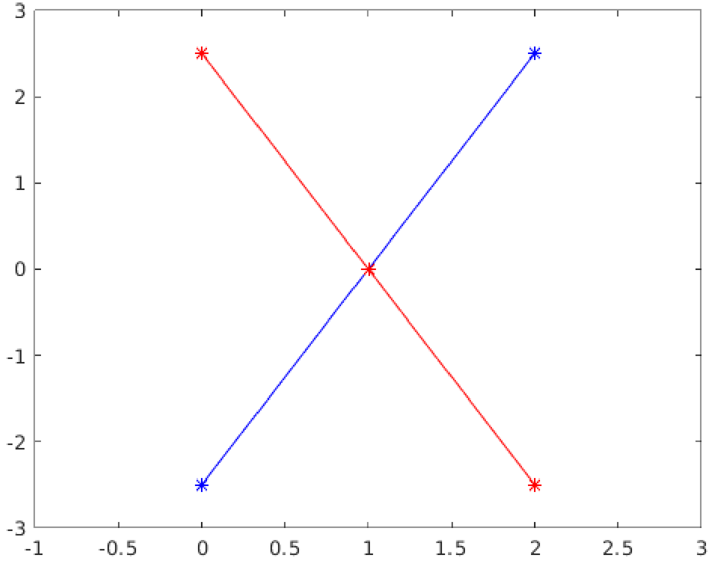
So after the transformation:

P1: (0, -2.5, 0) P2: (1, 0, 0) P3: (2, 2.5, 0)

P4: (0, 2.5, 0) P5: (1, 0, 0) P6: (2, -2.5, 0)

P2 and P5 collapsed to the same point, which will cause the candy wrapper effect.

Homogeneous Matrix VS DQ



Homogeneous Matrix VS DQ

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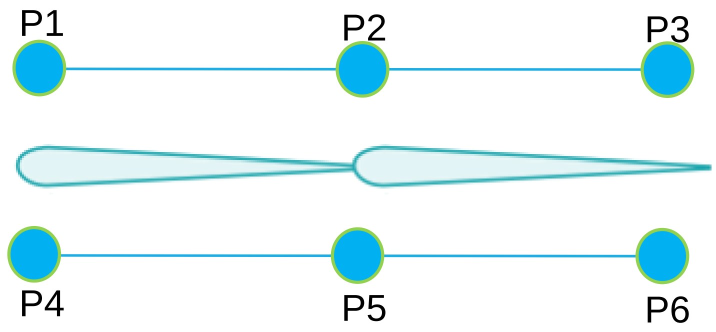
2 joints:

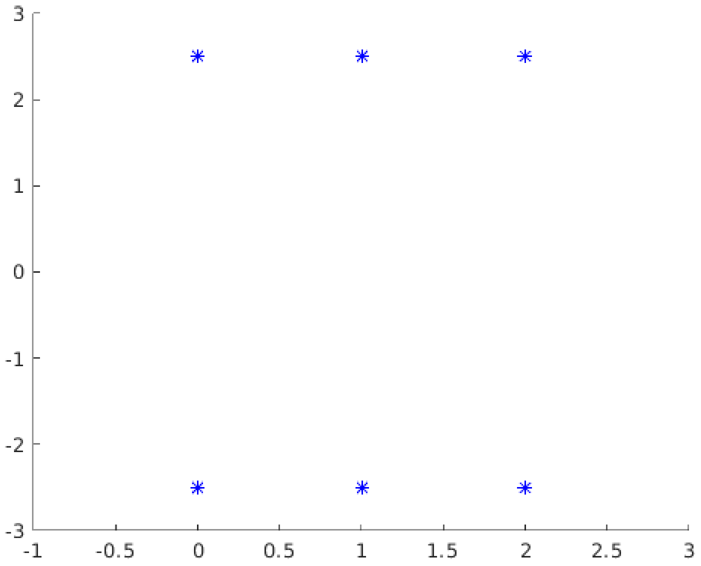
J1: (0, 0, 0) J2: (2, 0, 0)

Weight of every vertices to joints:

$$W_{ij}=\begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \\ 0 & 1 \\ 1 & 0 \\ 0.5 & 0.5 \\ 0 & 1 \end{bmatrix}$$ where i refers to the vertices and j refers to the joints.



Homogeneous Matrix VS DQ



Homogeneous Matrix VS DQ

Now we rotate J1 by 180 degrees, axis of rotation is x axis.

Dual quaternion for J1:

$Q_1 = (0, i, 0, 0) + 0\varepsilon$

Dual quaternion for J2:

$Q_2 = (1, 0, 0, 0) + 0\varepsilon$

Homogeneous Matrix VS DQ

Now we do linear blending using homogeneous matrix :

For every point, the transformation DQ:

$$Q_i = \frac{\sum_{j=1}^{n} w_{ij} Q_j}{\left\| \sum_{j=1}^{n} w_{ij} Q_j \right\|}$$

For rotations, $\left\| \sum_{j=1}^{n} w_{ij}\, Q_j \right\|$ is the norm of the first part of $\sum_{j=1}^{n} w_{ij}\, Q_j$.

$$P_i' = Q_i P_i Q_i^*$$

So after the transformation:

P1: (0, -2.5, 0) P2: (1, 0, 2.5) P3: (2, 2.5, 0)

P4: (0, 2.5, 0) P5: (1, 0, -2.5) P6: (2, -2.5, 0)

Distance between P2 and P5 remains the same, so there's no candy wrapper effect.

DQ Example

② $\theta = 180^\circ \Rightarrow \frac{\theta}{2} = 90^\circ$ x-axis $\Rightarrow (x, y, z) = (1, 0, 0)$

$Q_1 = (\cos 90^\circ,\ 1 \times \sin 90^\circ,\ 0 \times \sin 90^\circ,\ 0 \times \sin 90^\circ) = (0, i, 0, 0)$

$Q_2 = (1, 0, 0, 0)$

$0.5 Q_1 + 0.5 Q_2 = (0.5,\ 0.5 i,\ 0,\ 0) = (0.5 + 0.5 i)$

$\| 0.5 Q + 0.5 Q_2 \| = \sqrt{0.25 + 0.25} = \sqrt{.5} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$$Q = \frac{(0.5 + 0.5 i)}{\frac{1}{\sqrt{2}}} = \sqrt{2}\left(\frac{1}{2} + \frac{1}{2} i\right) = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$Q^* = \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$\boxed{v = (1, 2.5, 0)}$ in DQ format $P = (1, 0, 0, 0) + (0, i, 2.5j, 0)\varepsilon$

$\Rightarrow P = \{1 + (i + 2.5j)\varepsilon\}$

$$P' = QPQ = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\{1 + (i + 2.5j)\varepsilon\}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)(i + 2.5j)\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\varepsilon$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{i}{\sqrt{2}} + \frac{2.5}{\sqrt{2}} j - \frac{1}{\sqrt{2}} + \frac{2.5k}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\varepsilon$$

$$= 1 + \left(\frac{i}{2} + \cancel{\frac{2.5}{2} j} - \cancel{\frac{1}{2}} + \frac{2.5k}{2} - \cancel{\frac{2.5j}{2}} + \cancel{\frac{1}{2}} + \frac{2.5k}{2} + \frac{i}{2}\right)\varepsilon$$

$$= 1 + \underbrace{(i + 2.5k)}\varepsilon$$

$\Rightarrow v' = (1, 0, 2.5)$

Similarly, when $v = (1, -2.5, 0)$

$v' = (1, 0, -2.5)$