Persistent Excitation

Q1: Since we know that PE means:

$$\beta_0 I \leq W(k,N) = \frac{1}{N}\sum_{\tau=k}^{k+N-1} w(\tau)w^T(\tau) \quad \forall k \in \mathbb{N}_0 \tag{1}$$

By checking the special case when $w(k)$ is periodic, we can determine whether $W(k)$ is PE or not.

Suppose $N = 4, k = 0$.

$$W(0,4) = \frac{1}{4}\sum_{\tau=0}^{3} w(\tau)w^T(\tau) \tag{2}$$

$$w_1 = \begin{bmatrix} \sin\left(\frac{\pi}{4}k\right) \\ \cos\left(\frac{\pi}{4}k\right) \end{bmatrix} \tag{3}$$

If $\tau = 0$,

$$w(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w^T(0) = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{4}$$

If $\tau = 1$,

$$w(1) = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad w^T(1) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{5}$$

If $\tau = 2$,

$$w(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w^T(2) = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{6}$$

If $\tau = 3$,

$$w(3) = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad w^T(3) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \tag{7}$$

Thus,

$$W(0,4) = \frac{1}{4}\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{8}$$

Then the eigenvalues of $W(k,N)$ are $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2}$. All eigenvalues are greater than 0.

Since $W(k,N) = W(0,N), \forall k$, then $W(k,N)$ is independent of k. And since $W(k,N)$ is positive definite for $k = 0$, we can conclude that $W(k)$ is PE.

Q2: Suppose $N = 8, k = 0$, and

$$W_2(k) = \begin{bmatrix} \sin\left(\frac{\pi}{4}k\right) \\ \sin\left(\frac{\pi}{4}k\right) \end{bmatrix} \tag{9}$$

Similarly, we get

$$W_2(0,8) = \frac{1}{8}\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{10}$$

And its eigenvalues are $\lambda_1 = 1, \lambda_2 = 0$. Since there exists $\lambda_2 = 0$, $W_2(k)$ is not PE.

Linear Regression using Least Squares

Q1: Since $r(k) = \psi^T w(k)$, and $R(0,N)$ consisted of $r(k)$, $X(0,N)$ consisted of $w(k)$.

Thus, the equation is

$$R(0,N) = \psi^T X(0,N) = X(0,N)\psi \tag{11}$$

Q2: Normal equation:

$$X^T(0,N)X(0,N)\hat{\psi} = X^T(0,N)R(0,N) \tag{12}$$

We know that the linear equation $R(0,N) = X(0,N)\psi$ is solvable only if $X(0,N)$ is full rank.

Since

$$W(0,N) = \frac{1}{N}\sum_{\tau=0}^{N-1} w(\tau)w^T(\tau) \tag{13}$$

then

$$X^T(0,N)X(0,N) = NW(0,N) \tag{14}$$

If $X^T(0,N)X(0,N)$ is invertible, that means $W(0,N)$ is also invertible.

By persistent excitation (Q1), we know that $w_1(k)$ is PE, $W(0,N)$ is positive definite, all eigenvalues > 0, and $W(0,N)$ is invertible.

If $N \geq 2$,

$$X^T(0,2)X(0,2) = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} \\ 1 & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \tag{15}$$

$$\text{Rank}\left(X^T(0,2)X(0,2)\right) = 2, \quad \lambda_1 = \frac{2+\sqrt{2}}{2} > 0, \quad \lambda_2 = \frac{2-\sqrt{2}}{2} > 0 \Rightarrow \text{PE} \tag{16}$$

Thus, $N \geq 2$ makes it solvable, and

$$W(0,N) = \frac{1}{N}X^T(0,N)X(0,N). \tag{17}$$

Since we need to make the equation solvable, and we need $W(0,N)$ to be invertible, which means $w(k)$ is PE. $N \geq 2$ ensures that $w(k)$ is PE, and $W(0,N)$ is positive definite.

Thus, if $N \geq 2$, $W(0, N)$ is positive definite,

$$\Rightarrow w(k) \text{ is PE}$$
$$\Rightarrow \text{the equation is solvable}$$

If $N < 2$, $W(0, N)$ is not positive definite,

$$\Rightarrow \text{the equation is not solvable.}$$

Normal equation:

$$X^T(0, N)X(0, N)\hat{\psi} = X^T(0, N)R(0, N) \tag{18}$$

$$\Rightarrow W(0, N)\hat{\psi} = \frac{1}{N}X^T(0, N)R(0, N) \tag{19}$$

$$\Rightarrow \hat{\psi} = W^{-1}(0, N) \cdot \frac{1}{N}X^T(0, N)R(0, N) \tag{20}$$

Since $R(0, N) = X(0, N)\psi$,

$$\hat{\psi} = W^{-1}(0, N) \cdot \frac{1}{N}X^T(0, N)X(0, N)\psi \tag{21}$$

$$= W^{-1}(0, N) \cdot W(0, N) \cdot \psi \tag{22}$$

Since $W(0, N)$ is invertible,

$$\hat{\psi} = \psi = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{23}$$

Q3: If

$$w_2(k) = \begin{bmatrix} \sin\left(\frac{\pi}{4}k\right) \\ \cos\left(\frac{\pi}{4}k\right) \end{bmatrix} \tag{24}$$

we can get

$$W(0, N) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{Rank}(W(0, N)) = 1 \tag{25}$$

It's not full column rank, and $W_2(k)$ is not **PE**.
$W(0, N)$ is not invertible, the equation is not solvable.

MRAC Problem

Q1: We know

$$X_e(k) = X(k) - X_r(k),$$
$$X(k+1) = AX(k) + Bu(k),$$
$$X_r(k+1) = A_r X_r(k) + B_r r(k).$$

Then,

$$\begin{aligned}
X_e(k+1) &= X(k+1) - X_r(k+1) \\
&= AX(k) + Bu(k) - A_r X_r(k) - B_r r(k) \\
&= AX(k) + B(kX(k) + \hat{\varphi}^T(k)w(k)) - A_r X_r(k) - B_r \varphi^T w(k) \\
&= AX(k) + B(kX(k) + \hat{\varphi}^T(k)w(k)) - (A+Bk)X_r(k) - bB\varphi^T w(k) \\
&= A(X(k) - X_r(k)) + B(kX(k) - kX_r(k) + \hat{\varphi}^T(k)w(k) - b_r \varphi^T w(k)) \\
&= AX_e(k) + BkX_e(k) + B(\hat{\varphi}^T(k)w(k) - b_r \varphi^T w(k)) \\
&= (A+Bk)X_e(k) + B(\hat{\varphi}^T(k)w(k) - b_r \varphi^T w(k)).
\end{aligned}$$

Since $A_r = A + Bk$ is Schur stable, then $X_e(k)$ is AS.

$$\begin{aligned}
u(k) &= kX(k) + \hat{\varphi}^T(k)w(k) \\
r(k) &= \varphi^T w(k)
\end{aligned}$$

$\hat{\varphi}$ is estimating φ, it shows the relationship between $r(k)$ and $w(k)$.
Q2: Since

$$e(k) = B^T P X_e(k), \tag{26}$$

then

$$e(k+1) = B^T P X_e(k+1) \tag{27}$$

$$e(k+1) = B^T P(A+BK)X_e(k) + B(\hat{\varphi}^T(k) - b\varphi^T)w(k) \tag{28}$$

$$= B^T P(A+BK)X_e(k) + B^T PB(\hat{\varphi}^T(k) - b\varphi^T)w(k) \tag{29}$$

And it's a **Dynamic error model**.

Linear Regression and Static-EM

Q1: We define $\tilde{\varphi}(k) = \hat{\varphi}(k) - \varphi$, and $r(k) = \varphi^T w(k)$, then

$$\begin{aligned}
e(k) &= \hat{\varphi}^T(k)w(k) - r(k) \\
&= \hat{\varphi}^T(k)w(k) - \varphi^T w(k) \\
&= (\hat{\varphi}^T(k) - \varphi^T)w(k) \\
&= \tilde{\varphi}^T(k)w(k)
\end{aligned}$$

It's a linear combination of signals, which means it is a **static error model**.
Q2: Since we know the **Update Gradient law**:

$$\hat{\varphi}(k+1) = \hat{\varphi}(k) - \gamma(k)e(k)w(k), \tag{30}$$

where $\gamma(k)$ is a time-varying adaptation gain,

$$\gamma(k) = \frac{\gamma}{1 + ||W(k)||^2} \tag{31}$$

We know that $e(k) = \tilde{\varphi}^T(k)w(k)$, then

$$\begin{aligned}\tilde{\varphi}(k+1) &= \tilde{\varphi}(k) - \gamma(k)e(k)w(k) \\ &= \tilde{\varphi}(k) - \gamma(k)(\tilde{\varphi}^T(k)w(k))w(k) \\ &= \tilde{\varphi}(k) - \gamma(k)w(k)w^T(k)\tilde{\varphi}^T(k) \\ &= [I - \gamma(k)w(k)w^T(k)]\tilde{\varphi}^T(k)\end{aligned}$$

It's a **discrete LTV System**. Thus, $\tilde{\varphi}(k+1)$ dynamics is always **linear**.

However,

$$\tilde{\varphi}(k+1) = [I - \gamma(k)w(k)w^T(k)]\tilde{\varphi}^T(k) \tag{32}$$

If $w(k)$ is **not PE**, $W(k)$ is not invertible, and some entries of $\tilde{\varphi}(k)$ may not **converge**.

Q3: 1. Since $w_1(k)$ is PE, $\tilde{\varphi}(k)$ can **converge**, then $\hat{\varphi}(k) \to \varphi$, and also

$$e(k) = \hat{\varphi}^T(k)w(k) \to 0 \tag{33}$$

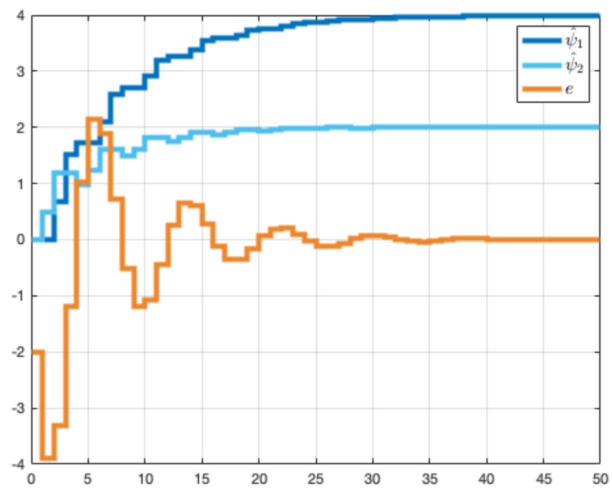

Figure 1: w1

2. Since $w_2(k)$ is **not PE**, and $W_2(k)$ has eigenvalues $\lambda_1 = 0$, that makes $\hat{\varphi}(k)$ **not converge**, and $\hat{\varphi}(k)$ cannot approach the true value φ. Since $w_2(k)$ is **not PE**, that can make it not converge,but it can still make φ be approaching to a stable value in some direction.Thus,

$$e(k) = \hat{\varphi}^T(k)w(k) \to 0. \tag{34}$$

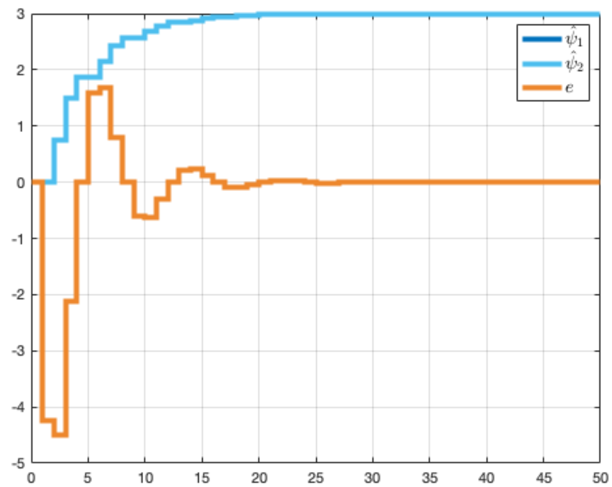

Figure 2: w2

Q4:
Least Squares:
Normal equation:

$$\hat{\varphi} = \left(X^T(0,N)X(0,N)\right)^{-1} X(0,N)^T R(0,N). \tag{35}$$

Pros:

1. It is faster than parameter adaptation. Once $w(k)$ is **PE**, which means X is full column rank, then
$$\hat{\varphi} = \varphi. \tag{36}$$

Cons:

1. It doesn't allow **real-time parameter adjustment** in a dynamic system.

Parameter Adaptation:
Pros:

1. It's **suitable for dynamic systems**. It also allows **real-time parameter adjustments**.

Cons:

1. It depends on whether $w(k)$ is **PE** or not. If the regressor is **not PE**, then the estimate may **not converge**.

2. By using the update law, the **cost of time** also depends on the **learning rate**.

Adaptation in MRAC and Dynamic-EM

Q1:

MRAC controller:

$$u(k) = Kx(k) + \hat{\varphi}^T(k)w(k) \tag{37}$$

where $K \in \mathbb{R}^{1\times 2}$ is the static gain matrix.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{38}$$

$$A_r = A + BK = \begin{bmatrix} 0 & 1 \\ -0.04 & 0.4 \end{bmatrix} \Rightarrow K = \begin{bmatrix} -0.04 & 0.4 \end{bmatrix} \tag{39}$$

$$B_r = bB = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} \Rightarrow b = 1.5 \tag{40}$$

$$u(k) = Kx(k) + \hat{\varphi}^T(k)w(k) \tag{41}$$

$$= \begin{bmatrix} -0.04 & 0.4 \end{bmatrix} x(k) + \hat{\varphi}^T(k)w(k), \quad x(k) \in \mathbb{R}^n \tag{42}$$

Suppose

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{43}$$

then

$$u(k) = -0.04x_1(k) + 0.4x_2(k) + \hat{\varphi}^T(k)w(k) \tag{44}$$

Q2:

Discrete-time Lyapunov equation

$$A_r^T P A_r - P = -I \tag{45}$$

If P exists, we need to prove that A_r is **Schur stable**, which means **all eigenvalues of A_r are less than 1**.

$$A_r = \begin{bmatrix} 0 & 1 \\ -0.04 & 0.4 \end{bmatrix} \tag{46}$$

$$\det(A - \lambda I) = 0 \tag{47}$$

$$\Rightarrow \lambda^2 - 0.4\lambda + 0.04 = 0 \tag{48}$$

$$\Rightarrow \lambda_1 = 0.2 < 1 \tag{49}$$

Thus, A_r is **Schur stable**. According to theorem **3.3.12**, there exists $P = P^T$, such that

$$A_r^T P A_r - P = I \tag{50}$$

where P is a symmetric positive definite matrix.

Suppose

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}, \quad A_r^T = \begin{bmatrix} 0 & -0.04 \\ 1 & 0.4 \end{bmatrix} \tag{51}$$

$$A_r^T P A_r - P = -I \tag{52}$$

$$\Rightarrow \begin{bmatrix} 0 & -0.04 \\ 1 & 0.4 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.04 & 0.4 \end{bmatrix} - \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{53}$$

By using the `dylap` function, we can get:

$$P = \begin{bmatrix} 1.0038 & -0.0362 \\ -0.0362 & 2.3510 \end{bmatrix} \tag{54}$$

Q3: Solve a difference equation with a Z-transform:

$$X(z) = (zI - A)^{-1} B U(z) \tag{55}$$

Taking inverse **Z-transform**:

$$x(k+1) = Ax(k) + Bu(k) \tag{56}$$

We have the filter:

$$H(z) = B^T P (zI - A_r)^{-1} B \tag{57}$$

then

$$W_a(k) = H(z)w(k) = B^T P (zI - A_r)^{-1} B w(k) \tag{58}$$

Thus,

$$Z(k+1) = AZ(k) + Bw(k) \tag{59}$$

$$\Rightarrow Z_i(k+1) = A_r Z_i(k) + Bw_i(k) \tag{60}$$

$$W_{a,i}(k) = B^T P Z_i(k) \tag{61}$$

Therefore,

$$Z_1(k+1) = A_r Z_1(k) + Bw_1(k) \tag{62}$$

$$Z_2(k+1) = A_r Z_2(k) + Bw_2(k) \tag{63}$$

$$W_a(k) = \begin{bmatrix} B^T P Z_1(k) \\ B^T P Z_2(k) \end{bmatrix} \tag{64}$$

Q4: Define augmented error:

$$e_a(k) = e(k) - \hat{y}(k) + \hat{\varphi}^T(k) W_a(k) \tag{65}$$

where

$$e(k) = B^T P X_e(k) \quad \text{(system error)} \tag{66}$$

$$\hat{y}(k) = B^T P \hat{X}_e(k) \quad \text{(estimate error)} \tag{67}$$

$$W_a(k) = H(z) I[w(k)] \quad \text{(augmented regressor)} \tag{68}$$

Then,

$$e_a(k) = B^T P X_e(k) - B^T P \hat{X}_e(k) + \hat{\varphi}^T(k) W_a(k) \tag{69}$$

$$= B^T P [X_e(k) - \hat{X}_e(k)] + \hat{\varphi}^T(k) W_a(k) \tag{70}$$

Since

$$\tilde{X}_e(k) = X_e(k) - \hat{X}_e(k) \tag{71}$$

then

$$e_a(k) = B^T P \tilde{X}_e(k) + \hat{\varphi}^T(k) W_a(k) \tag{72}$$

We also need to verify

$$e_a(k) = (\tilde{\varphi}(k) - b\varphi)^T W_a(k) + \varepsilon(k) \tag{73}$$

with $\varepsilon(k) \to 0$ by applying **Swapping Lemma**.
Since

$$\tilde{\varphi}(k) = \hat{\varphi}(k) - \varphi, \quad H(z) = B^T P (zI - A_r)^{-1} B \tag{74}$$

and

$$e(k) - \hat{y}(k) = H(z)[-b\varphi^T w(k)] \tag{75}$$

then

$$e_a(k) = B^T P \tilde{X}_e(k) + \hat{\varphi}^T(k) W_a(k) \tag{76}$$

$$= (\tilde{\varphi}(k) - b\varphi)^T W_a(k) + \varepsilon(k) \tag{77}$$

When $\varepsilon(k) \to 0$, if $W_a(k)$ is PE, then $\tilde{\varphi}(k)$ converges,

$$e_a(k) \to 0. \tag{78}$$

Q5:

Augmented error:

$$e_a(k) = (\hat{\varphi}(k) - b\varphi)^T W_a(k) + \varepsilon(k) \tag{79}$$

With $\varepsilon(k) \to 0$, suppose $\varepsilon(k) = 0$, then the augmented error simplifies to:

$$e_a(k) = (\hat{\varphi}(k) - b\varphi)^T W_a(k) \tag{80}$$

We know the **adaptation gradient law**:

$$\hat{\varphi}(k+1) = \hat{\varphi}(k) - \gamma(k) e_a(k) W_a(k) \tag{81}$$

$$\hat{\varphi}(k+1) = \hat{\varphi}(k) - \gamma(k)((\hat{\varphi}(k) - b\varphi)^T W_a(k)) W_a(k) \tag{82}$$

Define:

$$\tilde{\varphi}(k) := \hat{\varphi}(k) - b\varphi \tag{83}$$

$$\tilde{\varphi}(k+1) = \tilde{\varphi}(k) - \gamma(k)(\tilde{\varphi}^T(k) W_a(k)) W_a(k) \tag{84}$$

$$= \tilde{\varphi}(k) - \gamma(k) W_a(k) W_a^T(k) \tilde{\varphi}(k) \tag{85}$$

$$= (I - \gamma(k) W_a(k) W_a^T(k)) \tilde{\varphi}(k) \tag{86}$$

It's an **LTV system**. Thus, $\tilde{\varphi}(k) := \hat{\varphi}(k) - b\varphi$ has **linear dynamics**.
Q6: 1. Since $w_1(k)$ is PE, $\tilde{\varphi}(k)$ can **converge**, then $\hat{\varphi}(k) \to b\varphi$, and also

$$x(k) = x_r(k) \to 0 \tag{87}$$

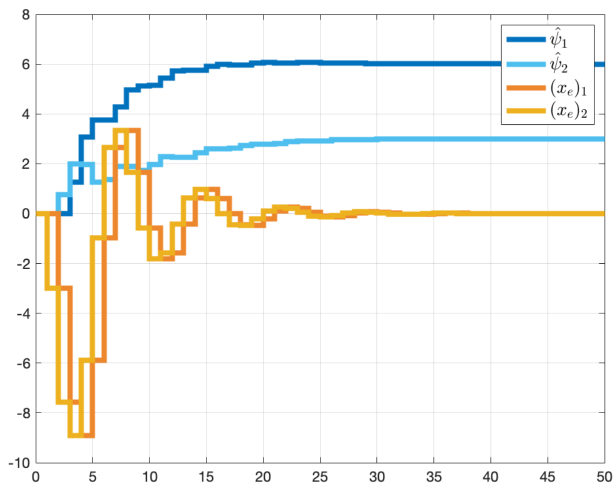

Figure 3: w1

Since $w_2(k)$ is not PE, $\tilde{\varphi}(k)$ can not **converge**, then $\hat{\varphi}(k)$ is not equal to $b\varphi$, and also

$$x(k) = x_r(k) \to 0 \tag{88}$$

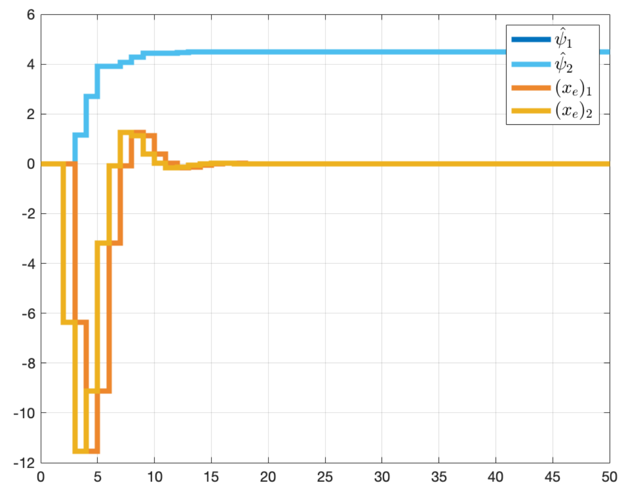

Figure 4: w2

Q7: The constant gain is 0.8.

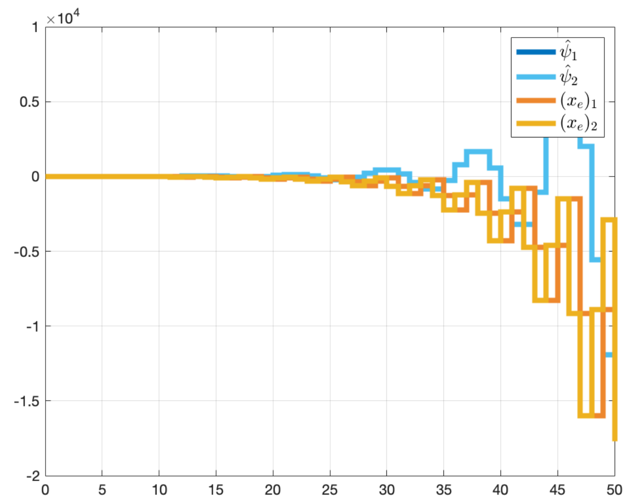

Figure 5: unstable constant gamma bar = 0.8

Q8:Stability Analysis

1. Negative Derivative of Lyapunov Function

$$V(\tilde{\psi}(k+1)) - V(\tilde{\psi}(k)) = \|\tilde{\psi}(k+1)\|^2 - \|\tilde{\psi}(k)\|^2 \tag{89}$$

If $V(k+1) > V(k)$, then the system is unstable.
2. Effect of $e_a(k)$

$$e_a(k) = \tilde{\psi}^T(k) w_a(k) \tag{90}$$

If $e_a(k) \neq 0$ and $\gamma(k)$ is too large, the parameter $\tilde{\psi}(k)$ may not converge.

- If g is too large (e.g., $g = 0.8$), it will cause $\gamma(k)$ to be too large, leading to excessively fast parameter updates, making the it unstable.