Adaptive Control and Reinforcement Learning

Lab 1

Excitation and Convergence in Adaptive Systems

Purpose

This lab provides hands on experience concerning adaptive systems and the connection between excitation of the regressor, convergence of the parameter estimate, and stability. You will simulate the time evolution of two standard error models: Static-Em and Dynamic-EM.

Introduction

In the real world, one generally has a nominal model of some system (e.g., robot) with associated model parameters which are unknown or uncertain. Adaptive control provides a systematic and provable way to compensate for such (deterministic) uncertainties. The fundamental tool from which our investigation begins is the standard gradient algorithm

$$\widehat{\psi}(k+1) \;=\; \widehat{\psi}(k) - \gamma(k)e(k)w(k)$$

where $\widehat{\psi}(k) \in \mathbb{R}^q$ is an estimate of an unknown constant parameter $\psi \in \mathbb{R}^q$, $w(k) \in \mathbb{R}^q$ is called the regressor in adaptive control, $\gamma(k) \in \mathbb{R}$ is a time-varying adaptation gain which is scaled by the constant gain $\overline{\gamma} \in \mathbb{R}$, and $e(k) \in \mathbb{R}$ is an error signal. Define the parameter estimation error $\widetilde{\psi}(k) := \widehat{\psi}(k) - \psi$. The simplest error model (Static-EM) corresponds to

$$e(k) \;=\; \widetilde{\psi}^T(k)w(k)$$

When a linear time-invariant (LTI) plant is involved, the simplest error model (Dynamic-EM) becomes

$$\begin{aligned} x(k+1) &= Ax(k) + B\widetilde{\psi}^T(k)w(k) \\ e(k) &= Cx(k) \end{aligned}$$

where $x(k) \in \mathbb{R}^n$. In such a case one must construct an augmented error $e_a(k)$ and regressor $w_a(k)$, which results in a slight modification of the standard gradient algorithm

$$\widehat{\psi}(k+1) \;=\; \widehat{\psi}(k) - \gamma(k)e_a(k)w_a(k)$$

Notice that the error $e(k)$ (or $e_a(k)$) and gain $\gamma(k)$ are scalar, but we need to recover a vector ψ of unknown parameters. Extra information must be provided by the vector regressor $w(k)$ (or $w_a(k)$). The amount of "information" in the regressor is characterized by a notion called persistence of excitation, which determines the convergence and stability properties of the parameter estimate $\widehat{\psi}(k)$.

Preparation

Persistent Excitation

A fundamental concept in adaptive control that determines the convergence and stability of adaptive laws is persistent excitation. Intuitively, the requirement of persistent excitation for adaptation tells us that only parameters excited by a regressor can be learned. Mathematically, a regressor $w(k) \in \mathbb{R}^q$ is said to be persistently exciting (PE) if there exists a constant $\beta_0 > 0$ and some number of steps $N \in \mathbb{N}$ such that

$$\beta_0 I \preccurlyeq W(k,N) := \frac{1}{N} \sum_{\tau=k}^{k+N-1} w(\tau)w^T(\tau) \quad \forall \; k \in \mathbb{N}_0$$

where $W(k,N) \in \mathbb{R}^{q \times q}$. The matrix inequality $\beta_0 I \preccurlyeq W(k,N)$ means that the smallest eigenvalue of the Gram matrix $W(k,N)$ satisfies $\beta_0 \leq \lambda_{\min}(W(k,N))$. The definition of PE tells us that excitation of $w(k)$ is quantified by the eigenvalues of $W(k,N)$.

While PE is an extremely useful property, checking the PE condition can be very difficult since $W(k,N)$ generally depends on k and the matrix inequality must hold for all $k \in \mathbb{N}_0$. To help us compute $W(k,N)$ we consider the special case when $w(k)$ is periodic. Observe that if $w(k)$ is periodic then $w(k)w^T(k)$ is periodic. Choosing $N \in \mathbb{N}$ as the period of $w(k)w^T(k)$, we have that the summation

$$\frac{1}{N} \sum_{\tau=k}^{k+N-1} w(\tau)w^T(\tau)$$

used to compute $W(k,N)$ is independent of k. As a result, the PE condition boils down to checking that $W(k,N)$ is positive definite for any (not every) k.

Q1. Is the regressor $w_1(k) = [\sin(0.25\pi k) \;\; \cos(0.25\pi k)]^T \in \mathbb{R}^2$ PE? Compute the eigenvalues of $W(k,N)$.

Hint: *try* $N = 4$ *and set* $k = 0$.

Q2. Is the regressor $w_2(k) = [\sin(0.25\pi k) \;\; \sin(0.25\pi k)]^T \in \mathbb{R}^2$ PE? Compute the eigenvalues of $W(k,N)$.

Linear Regression using Least-Squares

The simplest problem in both adaptive control and supervised machine learning is that of linear regression. Linear regression consists of identifying a parameter ψ that satisfies the linear relationship

$$r(k) = \psi^T w(k)$$

for some input-output pairs $(r(k), w(k))$. Despite solving the same problem, adaptive control and machine learning take fundamentally different approaches stemming from their different perspectives of the problem. In machine learning, linear regression is posed as an optimization problem with respect to a given dataset. In adaptive control, linear regression is posed as a control problem to be solved using specific measurements. Here we review the most basic machine learning approach to linear regression, with the adaptive control approach implemented in **`Experiment`**.

Q1. Suppose we collect the dataset $\{ (r(k), w(k)) \in \mathbb{R} \times \mathbb{R}^2 \}_{k=0}^{N-1}$ of input-output pairs and define the following quantities:

$$R(0,N) := \begin{bmatrix} r(0) \\ \vdots \\ r(0+N-1) \end{bmatrix} \text{ and } X(0,N) := \begin{bmatrix} w^T(0) \\ \vdots \\ w^T(0+N-1) \end{bmatrix}$$

Determine the (linear) equation relating $R(0,N)$, $X(0,N)$, and ψ.

Q2. To obtain an estimate $\widehat{\psi}$ we solve for ψ using the optimization problem

$$\min_{\widehat{\psi} \in \mathbb{R}^2} \|X(0,N)\widehat{\psi} - R(0,N)\|^2$$

which is known as a least-squares problem. Suppose $w(k) = w_1(k) = [\sin(0.25\pi k) \quad \cos(0.25\pi k)]^T$ and $\psi = [4 \quad 2]^T$.

- For what $N \in \mathbb{N}$ is the above solvable?
- Show that you can express the Gram matrix $W(0,N)$ it terms of $X(0,N)$.

Hint: *write out $X^T(0,N)X(0,N)$ and relate it to $W(0,N)$.*

- Based on your findings from the previous two steps, how does this N relate to persistent excitation?
- What value $\widehat{\psi}$ do you recover?

Hint: *use the normal equations to solve for $\widehat{\psi}$.*

Q3. Suppose $w(k) = w_2(k) = [\sin(0.25\pi k) \quad \sin(0.25\pi k)]^T$. Can you recover ψ using least-squares? If so, compute ψ. If not, provide a mathematical justification.

Error Model for an MRAC Problem

Model Reference Adaptive Contol (MRAC) is a control design technique historically used in aerospace applications to control an aircraft. The idea is to specify a reference model having the desired behaviour and then design a controller that makes the aircraft behave like the reference model. Our goal is to show that we can convert an MRAC problem to an error model of adaptive control, as we studied in class. The process of identifying error models will be a recurring theme that one should get familiar with.

To set up the MRAC problem, consider a plant and a reference model (respectively)

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ x_r(k+1) &= A_r x_r(k) + B_r r(k) \end{aligned}$$

where the matrices $A,\ A_r \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$ are known, and the states $x(k),\ x_r(k) \in \mathbb{R}^n$ are measurable. Suppose that the reference signal $r(k) \in \mathbb{R}$ is unknown to us but we have access to a regressor $w(k) \in \mathbb{R}^q$ such that $r(k) = \psi^T w(k)$ for some unknown parameter $\psi \in \mathbb{R}^q$. In other words, we know of some basis functions for the reference signal. Provided that A_r is Schur stable, the goal is to track the reference model (i.e., $x(k) \to x_r(k)$).

Q1. For the MRAC problem to be solvable we require that the matching conditions hold; that is, there exists $K \in \mathbb{R}^{1\times n}$ and $b \in \mathbb{R}$ such that

$$A_r = A + BK \text{ and } B_r = bB$$

Since the goal is to have the state $x(k)$ track the state $x_r(k)$, we define the tracking error state $x_e(k) := x(k) - x_r(k)$. Derive the closed-loop dynamics for the state $x_e(k)$ by substituting the controller $u(k) = Kx(k) + \hat{\psi}^T(k)w(k)$, where $\hat{\psi}(k) \in \mathbb{R}^q$ is a parameter estimate.

Hint: *compute* $x_e(k+1)$.

What parameter is $\hat{\psi}(k)$ estimating?

Q2. Since the state $x_e(k)$ is a vector in $\mathbb{R}^n$ we need to define a scalar output to obtain an error model. Consider an error $e(k) = B^T P x_e(k)$ where $P \in \mathbb{R}^{n\times n}$ is some matrix. What error model do you have?

Hint: *your error model is made up of the state* $x_e(k)$ *with output* $e(k)$.

Experiment

Simulation will be performed for $k \in \{\, 0, \ \ldots, T \,\}$ with the regressors $w_1(k), \ w_2(k)$ defined in **`Preparation`**. Recall MATLAB indexing starts at 1.

```
% Simulation Time
T = 50;
kk = 0:1:T;

% Regressors
w1 = @(k) [ sin(0.25 * pi * k) ; cos(0.25 * pi * k) ];
w2 = @(k) [ sin(0.25 * pi * k) ; sin(0.25 * pi * k) ];

% Unknown Parameter (DO NOT USE IN YOUR DESIGN!)
psi = [ 4 ; 2 ];

% Constant Gain \bar{\gamma}
g = 0.5;
```

IMPORTANT: Adaptive control concerns the control of unknown systems. Therefore, it is of paramount importance to be aware of what measurements are available and, equally important, what variables are unknown. When designing a controller or implementing an adaptive control scheme, one must only use measurements that are available to the controller. A controller or adaptive law that uses unknown variables directly (such as including ψ) will merit ZERO marks since it is impossible to implement!

Linear Regression and Static-EM

Recall from **`Preparation`** that linear regression consists of finding an unknown parameter ψ such that $r(k) = \psi^T w(k)$. To cast this as an adaptive control problem, let $\hat{\psi}(k) \in \mathbb{R}^q$ denote our estimate of ψ and define the scalar error

$$e(k) = \hat{\psi}^T(k) w(k) - r(k)$$

The control problem is to drive $e(k) \to 0$, without knowing ψ, when only measuring $r(k)$ and $w(k)$ at time k. In particular, we need to design an adaptation law for $\hat{\psi}(k)$.

Q1. Define $\tilde{\psi}(k) := \hat{\psi}(k) - \psi \in \mathbb{R}^q$ and express the error $e(k)$ in terms $\tilde{\psi}(k)$. What error model do you have?

Q2. Write out the adaptation law for your error model. When you substitute in your expression for $e(k)$ from **Q1** are the $\tilde{\psi}(k)$ dynamics linear?

Q3. Simulate your adaptation law for $w(k) = w_1(k)$, which is PE. Plot both $\hat{\psi}(k)$ and $e(k)$. Does $\hat{\psi}(k) \to \psi$? Does $e(k) \to 0$?

Hint: *the measurements are the regressor $w(k)$ and the reference $r(k)$. You cannot use ψ directly to implement your adaptation law.*

```
% States + Initial Conditions
psih = NaN(2, length(kk));  psih(:, 1) = [ 0 ; 0 ];

% Error (for plotting purposes)
e = NaN(1, length(kk));

% Simulate Dynamics
for idx = 1:(length(kk) - 1)
    % Measurements (w, r)
    w = w1(kk(idx));  % Regressor
    r = psi.' * w;    % Reference

    % TODO: Adaptation Law

end

% Display recovered value
psih(:, end)
% Plot
plot_staticEM(kk, psih, e)
```

Repeat for $w(k) = w_2(k)$, which is not PE.

```
% States + Initial Conditions
psih = NaN(2, length(kk));  psih(:, 1) = [ 0 ; 0 ];
```

```
% Error (for plotting purposes)
e = NaN(1, length(kk));

% Simulate Dynamics
for idx = 1:(length(kk) - 1)
    % Measurements (w, r)
    w = w2(kk(idx));  % Regressor
    r = psi.' * w;    % Reference

    % TODO: Adaptation Law

end

% Display recovered value
psih(:, end)
% Plot
plot_staticEM(kk, psih, e)
```

Q4. Compare and contrast solving for ψ using least-squares versus parameter adaptation. Present at least one pro and one con for each method.

Adaptation in MRAC and Dynamic-EM

Consider the MRAC problem presented in **`Preparation`**. Suppose you are given the following matrix values:

```
% Plant
sys.A = [ 0 1 ; 0 0 ];          % Known
sys.B = [ 0 ; 1 ];              % Known

% Reference Model
sys.Ar = [ 0 1 ; -0.04 0.4 ];   % Known
sys.Br = [ 0 ; 1.5 ];           % Unknown
```

Based on our identification of the MRAC problem with Dynamic-EM in **`Preparation`**, we will walk through all the steps needed to construct appropriate error signals and filters to devise an adaptation law to solve our MRAC problem. We will see that our MRAC controller enables the plant to track the reference model.

Q1. Do the matching conditions hold? If so, find the static gain $K \in \mathbb{R}^{1\times 2}$. What is the unknown $b \in \mathbb{R}$?

```

```

Write out the MRAC controller $u(k) \in \mathbb{R}$ you need to implement.

Hint: *see* ***`Preparation`***.

Q2. To obtain the error signal $e(k) = B^T P x_e(k)$, we need the matrix $P > 0$ solving the discrete-time Lyapunov equation

$$A_r^T P A_r - P = -I$$

State why such a matrix P exists and compute it.

Q3. Given $w(k) \in \mathbb{R}^2$ write out the dynamics to compute the augmented regressor $w_a(k) \in \mathbb{R}^2$ using intermediate states $Z_1(k),\ Z_2(k) \in \mathbb{R}^2$.

Hint: *You need to apply the filter $H(z) = B^T P(zI - A_r)^{-1} B$ component-wise on $w(k)$.*

Q4. Consider the filter state

$$\begin{aligned} \hat{x}_e(k+1) &= A_r \hat{x}_e(k) + B\hat{\psi}^T(k) w(k) \\ \hat{y}(k) &= B^T P \hat{x}_e(k) \end{aligned}$$

What is the formula for the augmented error $e_a(k) \in \mathbb{R}$?

Hint: *use only signals that you can measure or construct (you cannot measure ψ).*

From your formula for the augmented error above, verify that $e_a(k) = (\hat{\psi}(k) - b\psi)^T w_a(k) + \varepsilon(k)$ with $\varepsilon(k) \to 0$ by applying the Swapping Lemma.

Hint: *show that $e(k) - \hat{y}(k) = H(z)[-b\psi^T w(k)]$ with the transfer function $H(z) = B^T P(zI - A_r)^{-1} B$ by computing the dynamics of $\tilde{x}_e(k) := x_e(k) - \hat{x}_e(k)$.*

Hint: *from* ***Q3*** *we know that* $w_a(k) = H(z)I[w(k)]$.

Q5. By constructing an augmented error which can be expresed as $e_a(k) = (\widehat{\psi}(k) - b\psi)^T w_a(k) + \varepsilon(k)$ with $\varepsilon(k) \to 0$, we have reduced Dynamic-EM to Static-EM. Write out the parameter adaptation dynamics for $\widehat{\psi}(k)$. If we set $\varepsilon(k) = 0$, are the $\widetilde{\psi}(k) := \widehat{\psi}(k) - b\psi$ dynamics linear?

Note: *we do not build* $e_a(k)$ *by computing* $(\widehat{\psi}(k) - b\psi)^T w_a(k) + \varepsilon(k)$ *since* $b\psi \in \mathbb{R}^q$ *and* $\varepsilon(k) \in \mathbb{R}$ *are unknown.*

Q6. Simulate the MRAC controller with $w(k) = w_1(k)$, which is PE. Plot $\widehat{\psi}(k)$ and $x_e(k)$. Does $\widehat{\psi}(k) \to b\psi$? Does $x(k) \to x_r(k)$?

Hint: *the measurements are the regressor* $w(k)$, *the plant state* $x(k)$, *and the reference model state* $x_r(k)$. *You cannot use* ψ, B_r, *or* b *to implement your adaptive controller.*

```
% States + Initial Conditions
x = NaN(2, length(kk));        x(:, 1) = [ 0 ; 0 ];
xr = NaN(2, length(kk));    xr(:, 1) = [ 0 ; 0 ];
psih = NaN(2, length(kk));    psih(:, 1) = [ 0 ; 0 ];
% TODO: Add more states as needed


% Simulate Dynamics
for idx = 1:(length(kk) - 1)
    % Measurements (w, x, xr)
    w = w1(kk(idx));    % Regressor

    % TODO: Controller
    u = NaN;

    % Update Plant
    x(:, idx + 1) = sys.A * x(:, idx) + sys.B * u;

    % Update Reference Model
    xr(:, idx + 1) = sys.Ar * xr(:, idx) + sys.Br * psi.' * w;
end
```

```
% Display recovered value
psih(:, end)
% Plot Simulation
plot_dynamicEM(kk, psih, x - xr)
```

Repeat for $w(k) = w_2(k)$, which is not PE.

```
% States + Initial Conditions
x = NaN(2, length(kk));        x(:, 1) = [ 0 ; 0 ];
xr = NaN(2, length(kk));    xr(:, 1) = [ 0 ; 0 ];
psih = NaN(2, length(kk));    psih(:, 1) = [ 0 ; 0 ];
% TODO: Add more states as needed


% Simulate Dynamics
for idx = 1:(length(kk) - 1)
    % Measurements (w, x, xr)
    w = w2(kk(idx));    % Regressor

    % TODO: Controller
    u = NaN;

    % Update Plant
    x(:, idx + 1) = sys.A * x(:, idx) + sys.B * u;

    % Update Reference Model
    xr(:, idx + 1) = sys.Ar * xr(:, idx) + sys.Br * psi.' * w;
end

% Display recovered value
psih(:, end)
% Plot Simulation
plot_dynamicEM(kk, psih, x - xr)
```

Q7. Suppose $w(k) = w_1(k)$. Find a constant gain $\overline{\gamma} \in \mathbb{R}$ such that the dynamics are unstable. Provide a plot of the relevant dynamics.

```
% States + Initial Conditions
x = NaN(2, length(kk));        x(:, 1) = [ 0 ; 0 ];
xr = NaN(2, length(kk));    xr(:, 1) = [ 0 ; 0 ];
psih = NaN(2, length(kk));    psih(:, 1) = [ 0 ; 0 ];
% TODO: Add more states as needed
```

```
% Simulate Dynamics
for idx = 1:(length(kk) - 1)
    % Measurements (w, x, xr)
    w = w1(kk(idx));    % Regressor

    % TODO: Controller
    u = NaN;

    % Update Plant
    x(:, idx + 1) = sys.A * x(:, idx) + sys.B * u;

    % Update Reference Model
    xr(:, idx + 1) = sys.Ar * xr(:, idx) + sys.Br * psi.' * w;
end

% Plot Simulation
plot_dynamicEM(kk, psih, x - xr)
```

Q8. Using the Lyapunov function $V(\widetilde{\psi}(k)) = \|\widetilde{\psi}(k)\|^2$ and ignoring exponentially decaying terms, explain where the stability argument breaks down for your choice of $\overline{\gamma}$ in **Q7**.

Hint: *ignoring exponentially decaying terms we have* $e_a(k) = \widetilde{\psi}^T(k)w_a(k)$.

Submission

Export your completed MATLAB livescript as a PDF or HTML. Ensure that all plots and answers are clear.

Group:

Helper Functions

```
function plot_staticEM(k, psih, e)
    % Colours
    blue = '#0072BD';
    lblue = '#4DBEEE';
    orange = '#ED872D';

    % Plot
    figure

    stairs(k, psih(1, :), 'Color', blue, 'LineWidth', 4)
```

```
    hold on
    stairs(k, psih(2, :), 'Color', lblue, 'LineWidth', 4)
    stairs(k, e, 'Color', orange, 'LineWidth', 4)
    hold off

    legend({'$\hat{\psi}_1$', '$\hat{\psi}_2$', '$e$'}, 'Interpreter',
'latex', 'FontSize', 14)
    grid on
end

function plot_dynamicEM(k, psih, xe)
    % Colours
    blue = '#0072BD';
    lblue = '#4DBEEE';
    orange = '#ED872D';
    yellow = '#EDB120';

    % Plot
    figure

    stairs(k, psih(1, :), 'Color', blue, 'LineWidth', 4)
    hold on
    stairs(k, psih(2, :), 'Color', lblue, 'LineWidth', 4)
    stairs(k, xe(1, :), 'Color', orange, 'LineWidth', 4)
    stairs(k, xe(2, :), 'Color', yellow, 'LineWidth', 4)
    hold off

    legend({'$\hat{\psi}_1$', '$\hat{\psi}_2$', '$(x_e)_1$', '$(x_e)_2$'},
'Interpreter', 'latex', 'FontSize', 14)
    grid on
end
```