With $dq = q_f - q_0$ and final time t_f:

$$q(t) = q_0 + P(t)dq \quad \text{with} \quad \begin{cases} P(0) = 0 & P(t_f) = 1 \\ \dot{q}(0) = 0 & \dot{q}(t_f) = 0 \\ \ddot{q}(0) = 0 & \ddot{q}(t_f) = 0 \end{cases}$$

$P(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

- Ok for position and velocity constraints

$P(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4 + p_5 t^5$

- Ok for position, velocity and acceleration constraints

Initial / final constraints for $\begin{cases} q(t) = q_0 + P(t)dq \\ P(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 \end{cases}$

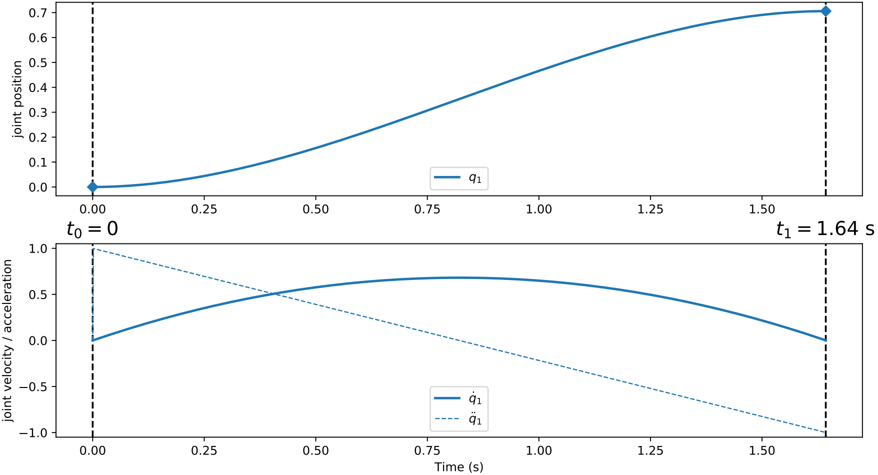
- Position $\begin{cases} P(0) = & p_0 & = 0 & \Rightarrow p_0 = 0 \\ P(t_f) = & p_0 + p_1 t_f + p_2 t_f^2 + p_3 t_f^3 & = 1 & \end{cases}$
- Velocity $\begin{cases} \dot{P}(0) = & p_1 & = 0 & \Rightarrow p_1 = 0 \\ \dot{P}(t_f) = & p_1 + 2p_2 t_f + 3p_3 t_f^2 & = 0 & \end{cases}$

$$\text{Solved to } P(t) = 3(t/t_f)^2 - 2(t/t_f)^3$$

Use constraints to find smallest t_f

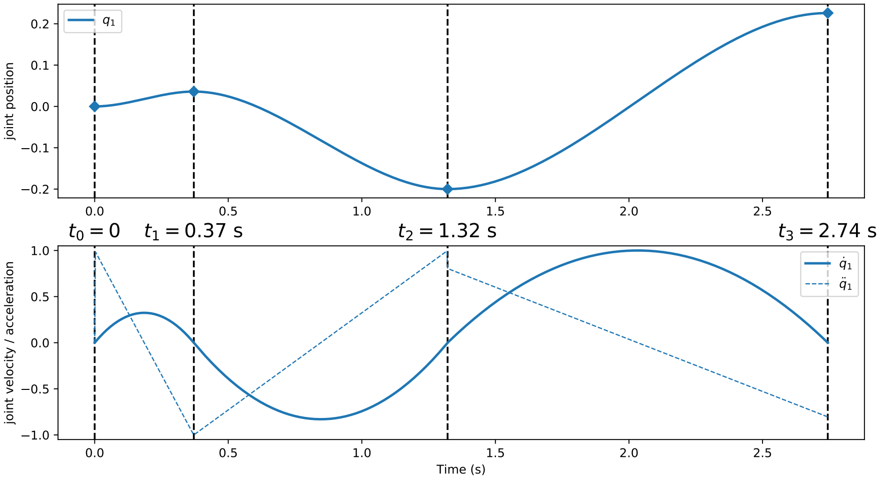
- $|\dot{q}(t)|_{\max} = \dfrac{3|dq|}{2t_f} \Rightarrow t_f \geq \dfrac{3|dq|}{2v_{\max}}$
- $|\ddot{q}(t)|_{\max} = \dfrac{6|dq|}{t_f^2} \Rightarrow t_f \geq \sqrt{\dfrac{6|dq|}{a_{\max}}}$

Polynomial point to point control - degree 3



Here max acceleration limits t_f
discontinuous acceleration

Polynomial point to point control - degree 3 with several via points



Max velocity or acceleration limits t_f, discontinuous acceleration

Arbitrary choice: $v_{f,k} = v_{0,k+1} = \frac{1}{2}\left(\frac{dq_k}{t_{f,k}} + \frac{dq_{k+1}}{t_{f,k+1}}\right)$

Initial / final constraints for

$$\begin{cases} q(t) = q_0 + P(t)dq \\ P(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + p_4 t^4 + p_5 t^5 \end{cases}$$

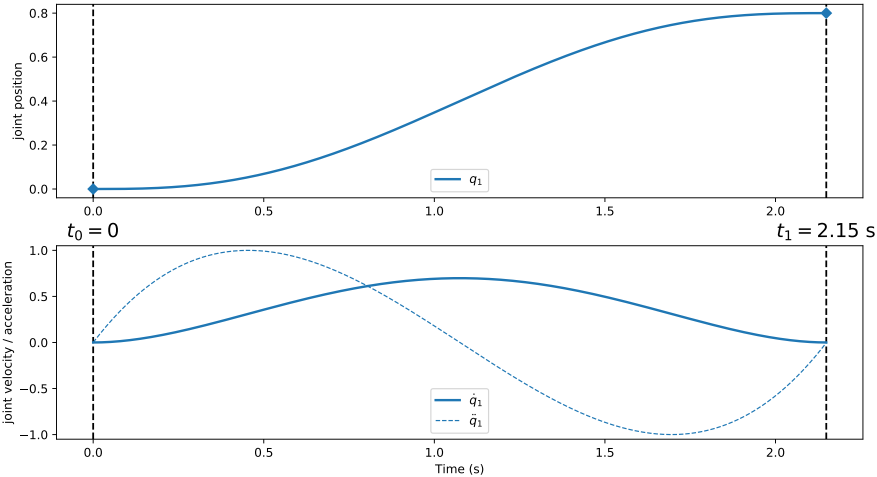
- Position $\begin{cases} P(0) = & p_0 & = 0 & \Rightarrow p_0 = 0 \\ P(t_f) = & p_0 + p_1 t_f + p_2 t_f^2 + p_3 t_f^3 + p_4 t_f^4 + p_5 t_f^5 & = 1 & \end{cases}$
- Velocity $\begin{cases} \dot{P}(0) = & p_1 dq & = 0 & \Rightarrow p_1 = 0 \\ \dot{P}(t_f) = & p_1 + 2p_2 t_f + 3p_3 t_f^2 + 4p_4 t_f^3 + 5p_5 t_f^4 & = 0 & \end{cases}$
- Acceleration $\begin{cases} \ddot{P}(0) = & 2p_2 & = 0 & \Rightarrow p_2 = 0 \\ \ddot{P}(t_f) = & p_2 + 6p_3 t_f + 12p_4 t_f^2 + 20p_5 t_f^3 & = 0 & \end{cases}$

Solved to $P(t) = 10(t/t_f)^3 - 15(t/t_f)^4 + 6(t/t_f)^5$

Use constraints to find smallest t_f

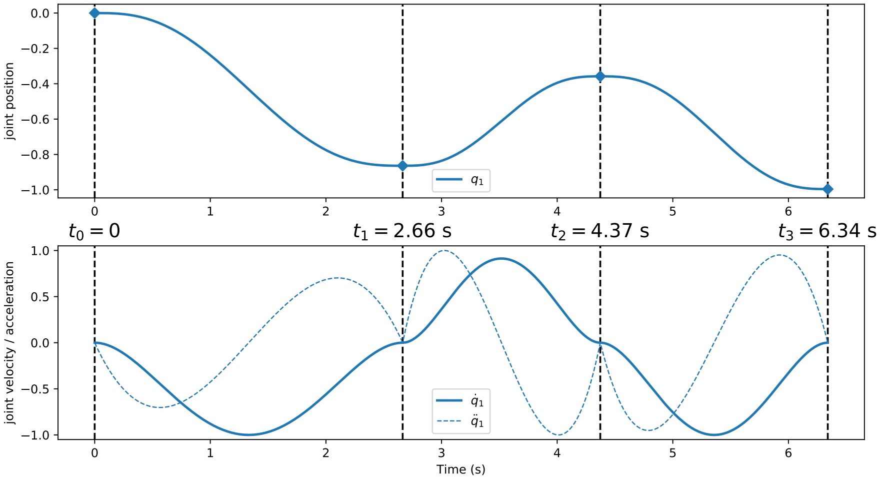
- $|\dot{q}(t)|_{\max} = \dfrac{15|dq|}{8t_f} \Rightarrow t_f \geq \dfrac{15|dq|}{8v_{\max}}$
- $|\ddot{q}(t)|_{\max} = \dfrac{10|dq|}{\sqrt{3}t_f^2} \Rightarrow t_f \geq \sqrt{\dfrac{10|dq|}{\sqrt{3}a_{\max}}}$

Polynomial point to point control - degree 5



Here max acceleration limits t_f, continuous acceleration

Polynomial point to point control - degree 5 with several via points



Max velocity or acceleration limits t_f, continuous acceleration

Arbitrary choice: $\begin{cases} v_{f,k} = v_{0,k+1} & = \frac{1}{2}\left(\frac{dq_k}{t_{f,k}} + \frac{dq_{k+1}}{t_{f,k+1}}\right) \\ a_{f,k} = a_{0,k+1} & = 0 \end{cases}$

Dealing with several joints

- Smallest t_f from worst joint (slowest / longest distance)

Initial / final constraints

- If possible: null velocity and acceleration
- If not: null velocity

With several via points?

- Unknown velocity / acceleration at via point
- Arbitrary choice not satisfactory
- Back to degree 3: splines

From q_0 to a sequence of via points $q_k^*, 1 \leq k \leq n$

- with $T_k = t_k - t_{k-1}$ for $k \in [1, n]$

n sub-pathes to generate with via-points constraints:

- Position $q_{k-1}(T_{k-1}) = q_k(0) = q_{k-1}^*$
- Velocity $\dot{q}_{k-1}(T_{k-1}) = \dot{q}_k(0)$
- Acceleration $\ddot{q}_{k-1}(T_{k-1}) = \ddot{q}_k(0)$

Number of actual constraints

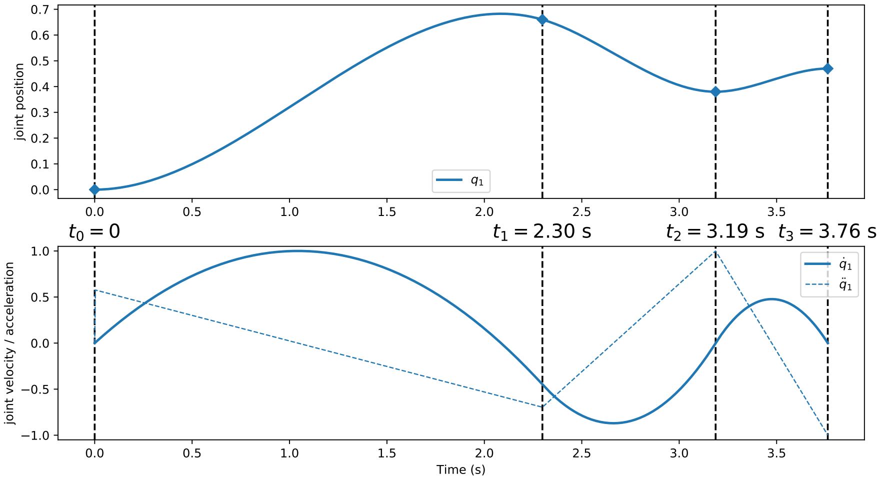
- Initial/final positions: 2
- Intermediary positions: $2(n-1)$
- Intermediary acceleration: $(n-1)$

Initial / final velocities: 2
Intermediary velocities: $(n-1)$

$4n$ constraints for n sub-pathes

- 1 sub-path = degree 3 polynomial
- $q_k(t) = p_{k,0} + p_{k,1}t + p_{k,2}t^2 + p_{k,3}t^3$
- No explicit intermediate velocities / acceleration
- Unknown: T_k, $1 \leq k \leq n$

Spline point to point control for several via points



Max velocity or acceleration limits t_k, continuous acceleration

Arbitrary choice: $v_0 = v_f = 0$

Assume T_k is given, equation for sub-path $k \in [1, n]$:

- $q_k(t) = p_{k,0} + p_{k,1}t + p_{k,2}t^2 + p_{k,3}t^3$ with $\begin{cases} q_k(0) & = q^*_{k-1} \\ q_k(T_k) & = q^*_k \\ \dot{q}_k(0) & = v_{k-1} \\ \dot{q}_k(T_k) & = v_k \end{cases}$

Solves to:

$$\begin{cases} p_{k,0} & = q^*_{k-1} \\ p_{k,1} & = v_{k-1} \\ p_{k,2} & = \frac{1}{T_k}\left(3\frac{q^*_k - q^*_{k-1}}{T_k} - 2v_{k-1} - v_k\right) \\ p_{k,3} & = \frac{1}{T_k^2}\left(2\frac{q^*_{k-1} - q^*_k}{T_k} + v_{k-1} + v_k\right) \end{cases}$$

Involves the knowledge of v_k...

- Based on acceleration continuity (not used yet)

Accelerations $\ddot{q}_k(T_k)$ and $\ddot{q}_{k+1}(0)$ should be equal ($k \in [1, n-1]$)

- $\ddot{q}_k(T_k) = 2p_{k,2} + 6p_{k,3}T_k$
- $\ddot{q}_{k+1}(0) = 2p_{k+1,2}$

Acceleration continuity equation:

$$T_{k+1}v_{k-1} + 2(T_{k+1} + T_k)v_k + T_k v_{k+1} = \frac{3}{T_k T_{k+1}}\left(T_k^2(q^*_{k+1} - q^*_k) + T_{k+1}^2(q^*_k - q^*_{k-1})\right)$$

In matrix form:

$$\begin{bmatrix} T_2 & 2(T_2+T_1) & T_1 & 0 & \dots & \dots & 0 \\ 0 & T_3 & 2(T_3+T_2) & T_2 & 0 & \dots & 0 \\ \vdots & \ddots & T_{k+1} & 2(T_{k+1}+T_k) & T_k & \ddots & 0 \\ 0 & \dots & \dots & 0 & T_n & 2(T_n+T_{n-1}) & T_{n-1} \end{bmatrix} \begin{bmatrix} v_0 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$n-1$ equations for $n+1$ unknown

- v_0 and v_n can be set arbitrarily

With known v_0 and v_n:

$$\begin{bmatrix} 2(T_2+T_1) & T_1 & 0 & \dots & \dots & 0 \\ T_3 & 2(T_3+T_2) & T_2 & 0 & \dots & 0 \\ \vdots & T_{k+1} & 2(T_{k+1}+T_k) & T_k & \ddots & 0 \\ 0 & \dots & \dots & 0 & T_n & 2(T_n+T_{n-1}) \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} c_1 - T_2 v_0 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} - T_{n-1} v_n \end{bmatrix}$$

$(n-1) \times (n-1)$ tri-diagonal matrix to invert: easy

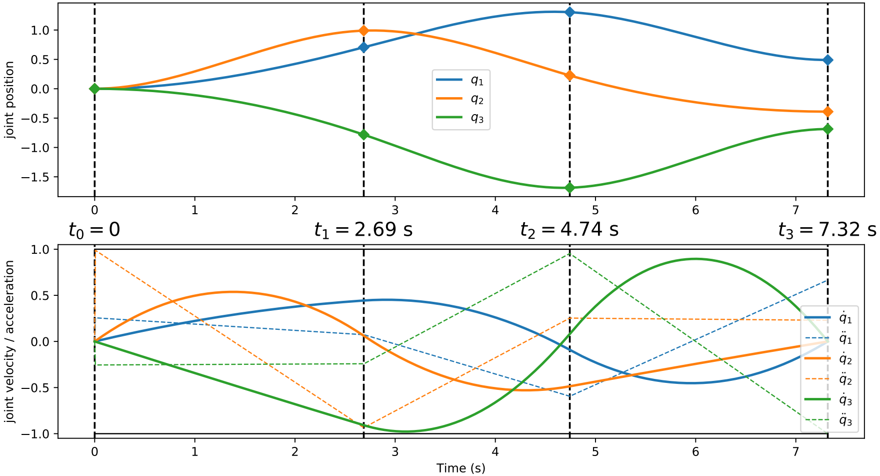
Unique solution from given sequence of $(T_1, \dots, T_n)$

- May not ensure minimum time travel
- May not ensure velocity / acceleration limits

Minimization of $\sum_{k=1}^{n} T_k$ under velocity and acceleration limits

For several joints: use same $(T_1, \dots, T_n)$

Spline point to point control for several via points and several joints



Max velocity or acceleration limits t_k, continuous acceleration

Arbitrary choice: $v_0 = v_f = 0$