# Contents

# 1 Introduction

The main objective of this assignment is to get familiar with the mathematical background of algorithms in logics, concerning demonstration, induction, and recognition of hasty generalization.

# 2 Exercise 1: Provement of Equivalence

**Requirement** Prove that:

$$\forall P, Q \in \mathbb{B}, (P \Rightarrow Q) \Leftrightarrow (\bar{P} \cup Q)$$

We demonstrate this equivalence in both directions. Namely, we will demonstrate the deduction from left to right, and then from right to left, to conclude this equivalence.

## 2.1 Deduction from left to right

In this subsection, we try to demonstrate:

$$\forall P, Q \in \mathbb{B}, (P \Rightarrow Q) \Rightarrow (\bar{P} \cup Q)$$

That is to say, our purpose is to demonstrate for all $P, Q \in \mathbb{B}$, if $P \Rightarrow Q$, then $\bar{P} \cup Q$.

Suppose we have two random booleans $P_0 \in \mathbb{B}$ and $Q_0 \in \mathbb{B}$ such that $P_0 \Rightarrow Q_0$. By the definition of $\Rightarrow$, we know that if $P_0$ is **True**, then $Q_0$ must also be **True**.

We distinct two cases:

- If $P_0$ is **True**, then $Q_0$ must also be **True** since we have $P_0 \Rightarrow Q_0$. This means that $\bar{P_0} \cup Q_0$ is **True**, because $Q_0$ is **True**.
- If $P_0$ is **False**, then $\bar{P_0}$ is **True**, thus $\bar{P_0} \cup Q_0$ must be **True**.

In either case, we have proved that if $P_0 \Rightarrow Q_0$, then $\bar{P_0} \cup Q_0$. Since we did not put restrictions on the choices of $P_0$ and $Q_0$, this should be the case for all $P, Q \in \mathbb{B}$.

We can thus conclude safely the deduction from left to right.

## 2.2 Deduction from right to left

In this subsection, we try to demonstrate:

$$\forall P, Q \in \mathbb{B}, (P \Rightarrow Q) \Leftarrow (\bar{P} \cup Q)$$

That is to say, our purpose is to demonstrate for all $P, Q \in \mathbb{B}$, if $\bar{P} \cup Q$, then $P \Rightarrow Q$. By the definition of $\Rightarrow$, we know that we should prove that if $P$ if **True**, then $Q$ is **True**.

Suppose we have two random booleans $P_0 \in \mathbb{B}$ and $Q_0 \in \mathbb{B}$ such that $\bar{P_0} \cup Q_0$. That is to say, at least one of $\bar{P_0}$ and $Q_0$ will be **True**.

We distinct two cases:

- If $P_0$ is **True**, then $\bar{P_0}$ is **False**. In order to have $\bar{P_0} \cup Q_0$ be **True**, $Q_0$ must be **True**.
- If $P_0$ is **False**, then $\bar{P_0}$ is **True**, we have no restrictions on $Q_0$, because $\bar{P_0} \cup Q_0$ is already **True** either the case.

We have proved that if $\bar{P_0} \cup Q_0$, then $Q_0$ must be **True** if $P_0$ is **True**, which is exactly the definition of $P_0 \Rightarrow Q_0$. Since we did not put restrictions on the choices of $P_0$ and $Q_0$, this should be the case for all $P, Q \in \mathbb{B}$.

We can thus conclude safely the deduction from right to left.

### 2.3 Conclusion of the equivalence

So far, we have proved the deduction from both sides.

We can thus conclude the equivalence:

$$\forall P, Q \in \mathbb{B}, (P \Rightarrow Q) \Leftrightarrow (\bar{P} \cup Q).$$

## 3 Exercise 2: Induction of Sequence

**Requirement** Prove using induction:

$$\forall n \in N, \sum_{i=0}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2$$

With the methodology of induction, we prove this proposition in 3 steps: initialization, iteration, and generalization.

### 3.1 Initialization

When $n = 0$, we have:

$$\sum_{i=0}^{0} i^3 = 0 = \frac{1}{4} 0^2 (0+1)^2$$

, which verifies that the proposition is valid for $n = 0$.

### 3.2 Iteration

Suppose that this proposition is true for any natural $N \in N$. That is to say, we suppose that we have $\sum_{i=0}^{N} i^3 = \frac{1}{4} N^2 (N+1)^2$. We want to demonstrate that this proposition is also valid for $N + 1$.

Here is the process of iterating $N$ to $N+1$:

$$\begin{aligned}
\sum_{i=0}^{N+1} i^3 &= \sum_{i=0}^{N} i^3 + (N+1)^3 \\
&= \frac{1}{4}N^2(N+1)^2 + (N+1)^3 \\
&= \left(\frac{1}{4}N^2 + N + 1\right)(N+1)^2 \\
&= \frac{1}{4}\left(N^2 + 4N + 4\right)(N+1)^2 \\
&= \frac{1}{4}(N+2)^2(N+1)^2 \\
\sum_{i=0}^{N+1} i^3 &= \frac{1}{4}(N+1)^2((N+1)+1)^2
\end{aligned}$$

We can conclude safely that if the proposition is valid for $N$, then it should also be valid for $N+1$.

### 3.3 Generalization

So far, we have demonstrated the validity of this proposition for $0$, and we have demonstrated that for any natural $N$ ranging from 0 to infinity, if the proposition if valid for $N$, then it is also valid for $N+1$. We have the valid rule for iteration, and we have a perfect initialization to start the iteration, therefore we can conclude safely that this proposition is valid for any natural number.

In conclusion, we have demonstrated that:

$$\forall n \in N, \sum_{i=0}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$$

## 4 Exercise 3: Recognition of False Logic

**Requirement** Find the error in the following proof:

- **Claim**: All horses are the same color.
- **Proof**: We prove that any collection of horses is monochromatic by induction on the number of horses in the collection.
- **Base Case**: Obviously, a set of one horse is a set of horses all with the same color.
- **Induction Hypothesis**: Assume that any set of $k$ horses are all the same color.
- **Inductive Step**: Consider a set of $k+1$ horses, and stand them all in a line. The first $k$ horses in the line form a set of $k$ horses, and so by the *Inductive Hypothesis*,

are all the same color. The same is true for the last $k$ horses in the line. Therefore the entire set consists of $k + 1$ horses of the same color.

The error lies in the **Inductive Step**. It is definitely NOT for sure that the top horse and the bottom horse, which are not included in the middle $k - 1$ horse overlap, are the same color!

This error can be obvious for the case where $k = 2$. The two only horses form 2 lines that are completely separated, so nobody could ever give the guarantee that these two horses are the same color. This has proven this proposition to be wrong at the very first base case where we have two horses, so the further inductive steps have no meaning.

In conclusion, this logic is false and we cannot prove the proposition in this way.

## 5 Conclusion

In this assignment, we learned the basic logics of mathematics and put them into use.