Black-Sholes-Merton Model

Assumptions

- Stock prices are lognormally distributed with μ and σ constant
- No transaction costs or taxes
- No dividend on the stock during the life of the option
- No riskless arbitrage opportunities
- Security trading is continuous
- Investors can borrow or lend at the same risk-free rate
- The short-term risk-free rate is constant

Inputs

- Stock price - S
- Strike price - K
- Exercise date
- Risk-free rate - r
- Volatility of the stock - σ

Risk Neutral Valuation

The ideas behind Black-Scholes are, the option price and stock price depend on the same underlying source of uncertainty, thus we can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty (because in any short period of time, the two prices are perfectly correlated). In a world where investors are risk neutral, the expected return on all investments are the risk-free interest rate. The is reason is that risk-neutral investors do not require a premium to induce them to take risks. Therefore, the expected payoff of the derivative can be discounted at the risk-free interest rate.

- The variable μ does not appear in the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- **Fat Tail:** The Black-Scholes model is based on normal distribution. If the distribution is instead a fat-tailed one, **the model will underprice options that are far out of the money**.

Pricing formula

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

$N(d_1)$ is a statistical measure of the option's delta

$N(d_2)$ is a statistical measure of the probability that the option will be exercised at expiration

Deriving the Formula using Risk Neutral Valuation

TODO

Example

S_0 = \$42, K = \$40, r = 10% per annum, σ = 0.2 per annum, T = 0.5 (6 month)

$$d_1 = \cdots = 0.7693$$

$$d_2 = \cdots = 0.6278$$

$$\ldots$$

$$c = 4.76, \quad p = 0.81$$

Implied Volatility

Suppose a non-dividend-paying stock is priced at x. The implied volatility is the value of σ when substituted into the equation, gives $c = x$. Unfortunately, it is impossible to invert the equation and calculate σ directly. However, an iterative search procedure can be applied to find σ.