STAT 5703 **HW 1** Fall 2022

1. Use the fact that if c is a constant and $X_n \xrightarrow{d} c$, then $X_n \xrightarrow{P} c$, to prove that CLT implies the weak LLN. Note: justify every step!

2. Recall that both MM and MLE methods result in the following estimator for the population variance σ^2 of any distribution:
$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$
   a) Derive Bias($\hat{\sigma}^2$) (justify all steps).
   b) Recall that if, in particular, $X \sim$ Poisson(λ), then E(X) = Var(X) = λ, which means we can estimate λ with either $\bar{X}$ or $\hat{\sigma}^2$. We will now compare the two estimators.
      i. Use the MGF, or any other method, to derive the formula of the 4th central moment $\mu_4^0 = E[(X - E(X))^4]$ of $X \sim$ Poisson(λ).
      ii. Use the fact that, in general, $\mathrm{Var}(\hat{\sigma}^2) = \frac{\mu_4^0}{n} - \frac{\sigma^4}{n}$ to find SE($\hat{\sigma}^2$) in the Poisson case specifically.
      iii. Compare SE($\hat{\sigma}^2$) to SE($\bar{X}$) in the Poisson case.

3. Let $X_1, \ldots, X_n$ be an iid sample from the Exponential distribution with pdf $f(x;\lambda) = \lambda e^{-\lambda x}, x \geq 0$, where $\lambda > 0$ is unknown. In this exercise the parameter of interest, θ, is the pth quantile of the distribution, defined as $P(X < \theta) = p$, or $\theta = F^{-1}(p)$, where F^{-1} is the inverse of the cdf of the Exponential distribution. Note that in this case p is a known given number (like 0.9), but it depends on the unknown parameter λ.
   a) Derive the formula for θ as a function of p and the sample $X_1, \ldots, X_n$.
   b) Derive the formula for $\hat{\theta}_{MM}$, the MM estimator of θ.
   c) Derive the formula for $\hat{\theta}_{MLE}$, the MLE of θ.
   (Note the answers for b) and c) should be the same).
   d) Without using the general theoretical results about MLE from Lecture 4, justify why $\hat{\theta}_{MLE} \xrightarrow{P} \theta$.
   e) Again, without using the general results about MLE from Lecture 4, find the asymptotic distribution of $\hat{\theta}_{MLE}$. That is, you may use the general CLT and related results only.

4. Let $X_1, \ldots, X_n$ be an iid sample from the Geometric distribution with probability mass function $f(x;p) = p(1-p)^x, x = 0, 1, 2, \ldots$, where $0 < p < 1$ is unknown parameter.
   a) Derive the MM estimator of p.
   b) Derive the MLE of p.
   c) Derive the formula for the Fisher information $I_n(p)$.